1. This problem is concerned with a nonrelativistic Hydrogen atom in a uniform, external electric field, $\mathbf{E}$. Let the field point along the z axis, $\mathbf{E} = E\hat{\mathbf{z}}$. The perturbing Hamiltonian is

$$H' = -eEz = -eEr\cos\theta \,.$$

You will need the $n = 1$ and $n = 2$ states of the Hydrogen atom,

$$\begin{aligned} < r,\theta,\phi|100 > &= \frac{2}{a^{3/2}} e^{-r/a} Y_0^0(\theta,\phi) \\ < r,\theta,\phi|200 > &= \frac{1}{\sqrt{2}a^{3/2}}\left(1 - \frac{r}{2a}\right) e^{-r/2a} Y_0^0(\theta,\phi) \\ < r,\theta,\phi|21m > &= \frac{1}{\sqrt{24}a^{3/2}} \frac{r}{a} e^{-r/2a} Y_1^m(\theta,\phi) \quad (m = 0, \pm 1) \end{aligned}$$

where a is the Bohr radius.

(a) Show that the shift in the energy of the ground state is zero in first-order perturbation theory.

(b) The four $n = 2$ states are degenerate, so degenerate perturbation theory must be used. Label the four states as

$$\begin{aligned} |1> &= |200> \\ |2> &= |210> \\ |3> &= |211> \\ |4> &= |21-1> \end{aligned}$$

As a first step, calculate the 4×4 matrix $< i|H'|j >$. Be aware that all but one of the matrix elements (and its complex conjugate) are zero, for reasons of symmetry or by direct calculation. Express them in terms of a, e, and E.

(c) Use this matrix to find the energy shifts in the presence of the external electric field. Express your result in terms of a, e, and E.

(d) Sketch a qualitative graph of the $n = 2$ energy levels versus the magnitude of the external electric field. Indicate any degeneracies on your graph.