QM Spring 02 B

This problem is one-dimensional. Ignore motion along y and z axes.

A molecule consists of two nuclei of mass m_1, m_2, located at x_1, x_2, and bound together by a 1-d harmonic oscillator potential with spring constant k. The Hamiltonian is

$$\hat{H} = -\frac{\hbar^2}{2m_1}\frac{\partial^2}{\partial x_1^2} - \frac{\hbar^2}{2m_2}\frac{\partial^2}{\partial x_2^2} + \frac{1}{2}k(x_1 - x_2)^2.$$

The two-particle eigenstates $\psi_{\{P,n\}}(x_1, x_2)$ are labelled by the total momentum $P = p_1 + p_2$ of the two particles, and by the harmonic oscillator quantum number n of the relative vibrational mode.

a) Make a change of variables so as to express the Hamiltonian in terms of the center of mass co-ordinate R and the relative co-ordinate $r = x_2 - x_1$. What are the allowed vibrational energies?

b) Write down an explicit expression for the normalized wavefunction $\psi_{\{P,n=0\}}(x_1, x_2)$ in terms of $\hbar$, m_1, m_2, x_1, x_2 and k. (Use box normalization, *i.e.* periodic boundary conditions for x_1 and x_2 with period L much larger than the size of the molecule.)

Immediately before $t = 0$ the molecule is in its ground state *i.e.* $\psi(x_1, x_2, t = 0_-) = \psi_{\{0,0\}}$. At $t = 0$ the nucleus of mass m_2 emits a gamma ray and recoils with momentum I. Immediately after $t = 0$ the wavefunction is therefore $\psi(x_1, x_2, t = 0_+) = \psi_{\{0,0\}}(x_1, x_2)e^{iIx_2/\hbar}$. (Ignore any change in m_2 due to the emission of the gamma ray.)

c) Write down, in terms of $\psi(x_1, x_2, t = 0_+)$ and $\psi_{\{I,0\}}(x_1, x_2)$, the expression that gives the *amplitude* for the molecule to end up with center-of-mass momentum I, but no vibrational energy.

d) Using your explicit formula for $\psi_{\{P,n=0\}}$, evaluate the expression from the previous part and determine the *probability*, P_0, that the molecule ends up with center-of-mass momentum I, but no vibrational energy.

HINT: The ground-state wavefunction for $\hat{H} = -\frac{\hbar^2}{2M}\frac{\partial^2}{\partial x^2} + \frac{1}{2}kx^2$ is $\psi_0(x) = \left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\alpha x^2}$, where $\alpha = \sqrt{Mk}/\hbar$. Also

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}ax^2 + ibx}\, dx = \frac{1}{\sqrt{2\pi a}} \exp\{-\tfrac{1}{2}b^2/a\}.$$