QM Spring 95B

Consider a quantum-mechanical system consisting of a single spin-1 degree of freedom. Let the state vectors $|\alpha\rangle$, for $\alpha = 0, \pm 1$, be normalized eigenvectors of the z-component $\hat{S}_z$ of the angular momentum operator $\hat{\mathbf{S}}$, so that $\hat{S}_z|\alpha\rangle = \hbar\alpha|\alpha\rangle$ for $\alpha = 0, \pm 1$. [The caret symbol ($\hat{}$) denotes an operator.] You may use without proof the following matrix representations of the operator $\hat{\mathbf{S}}$:

$$\langle\alpha|\hat{S}_x|\alpha'\rangle = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \langle\alpha|\hat{S}_y|\alpha'\rangle = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \langle\alpha|\hat{S}_z|\alpha'\rangle = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Suppose that the hamiltonian operator $\hat{H}$ is given by

$$\hat{H} = \hbar^{-1}\Omega(\mathbf{N}\cdot\hat{\mathbf{S}})^2,$$

where $\mathbf{N}$ is an arbitrary unit vector and Ω is an arbitrary positive constant (having the dimensions of an angular frequency).

(a) Give all the possible outcomes of a measurement of the energy of this system.

(b) Suppose that $\mathbf{N} = (\mathbf{e}_x + \mathbf{e}_y)/\sqrt{2}$, where $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ forms a cartesian basis. Find a complete orthonormal set of stationary state-vectors, expressing them in terms of $\{|\alpha\rangle\}$.

(c) Suppose that $\mathbf{N} = \mathbf{e}_z$ and that the system is prepared at time $t = 0$ in the state $|\Psi\rangle$ given by

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |-1\rangle\right).$$

Calculate the amplitude that, at the subsequent time $t = T$, the system is again found to be in the state $|\Psi\rangle$.

(d) Suppose that the system is prepared at a certain time in the state

$$|\Theta\rangle = \frac{1}{\sqrt{14}}(3|1\rangle + 2|0\rangle + |-1\rangle).$$

Immediately thereafter, the quantity $(\hat{S}_z)^2$ is measured, and the largest possible result is found. Give the normalized state vector immediately after the measurement.

(e) Immediately after the measurement of $(\hat{S}_z)^2$ described in part (d) the quantity $\hat{S}_z$ is measured. Give the probability that the result $\hbar$ is obtained.

(f) Suppose that $\mathbf{N} = \mathbf{e}_z$ but that Ω now varies with time as $\Omega(t) = \tilde{\Omega}\exp(-t/\tau)$, where $\tilde{\Omega}$ and τ are constants. Suppose further the system is again prepared at time $t = 0$ in the state $|\Psi\rangle$, given in part (c). Calculate the probability that, at time $t = \infty$, the system is found to be in the state $|\Psi\rangle$. Briefly explain whether or not the z-component of angular momentum is conserved as the system evolves in time.