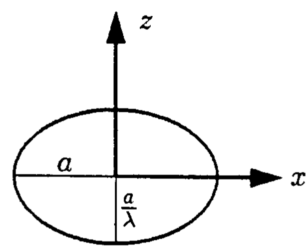
QM Spring 95A

Consider a quantum mechanical particle of mass m, position operator $\hat{\mathbf{r}}$ and momentum operator $\hat{\mathbf{p}}$, in three dimensions. [The caret symbol ($\hat{}$) denotes an operator.] The particle has kinetic energy $|\hat{\mathbf{p}}|^2/2m$ and is confined to a sphere of radius a by potential well $\hat{V} = V(\hat{\mathbf{r}})$, where

$$V(\mathbf{r}) = \begin{cases} 0, & r \leq a; \\ \infty, & r > a. \end{cases}$$

(a) Find the energy and normalized eigenfunction of the ground state of the particle in this potential well.

The spherical well is now replaced by an ellipsoid of revolution about the z-axis, with maximum radius a; the radius (in the z-direction) of the well from the center is now a/λ, where $\lambda > 1$, as shown in the figure. Denote the new potential function by $\hat{W}$.



(b) Write down an explicit expression for the potential $\hat{W}$ and the hamiltonian $\hat{H}$ describing the distorted system, and explain why it is not possible to treat the distortion directly as a perturbation of the original spherically symmetric potential.

However, energy eigenstates of the particle can be determined by making a scaling transformation of the z-coordinate operator, with new (scaled) coordinate $\hat{z}' = \hat{z}\lambda$. In terms of the coordinate operators $\hat{x}$, $\hat{y}$, and $\hat{z}'$, the potential is again spherically symmetric.

(c) By using the fact that a canonical transformation preserves the canonical commutation relations, determine the scaled momentum $\hat{p}'_z$ that is canonically conjugate to $\hat{z}'$.

(d) Write the hamiltonian $\hat{H}$ of part (b) in terms of the scaled coordinates and momenta, $\hat{\mathbf{r}}' = (\hat{x}, \hat{y}, \hat{z}')$ and $\hat{\mathbf{p}}' = (\hat{p}_x, \hat{p}_y, \hat{p}'_z)$.

(e) Consider a well only slightly distorted from spherical, so that $\lambda - 1 \equiv \epsilon \ll 1$. In terms of the scaled coordinates and momenta one can write $\hat{H}$ in the form of a sum of the hamiltonian for the spherical system plus a small perturbation:

$$\hat{H} = \hat{H}_0 + \epsilon \hat{H}_1 + \mathcal{O}(\epsilon^2),$$
$$\hat{H}_0 = \frac{1}{2m}|\hat{\mathbf{p}}'|^2 + V(\hat{\mathbf{r}}').$$

Give an explicit expression for the first-order perturbation $\hat{H}_1$ in terms of $\hat{\mathbf{r}}'$ and $\hat{\mathbf{p}}'$.

(f) Find the ground state energy of the particle in the distorted well in terms of m and a, to first order in ϵ, and explain the sign of the first order term.