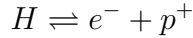


**SM** Consider the thermodynamic equilibrium of the ionization reaction



between a gas of neutral hydrogen atoms and unbound protons and electrons.

Suppose that in a volume  $V$  there are  $N_{p^+}$  unbound protons (charge  $+e$ ),  $N_H$  neutral hydrogen atoms, and  $N_{e^-}$  unbound electrons. The gas has no net charge, so  $N_{p^+} = N_{e^-}$ . The binding energy of the hydrogen atom is  $\Delta = 13.6$  eV. Let  $N$  be the total number of protons present,  $x = N_{p^+}/N$  the fraction of free protons, and  $N_H = (1 - x)N$  the number of un-ionized hydrogen atoms. Treat all three gases as ideal. The micro-states of the hydrogen gas have energies  $E_H = -\Delta + |\mathbf{p}|^2/2m_H$ . The protons and electrons have micro-state energies  $E = |\mathbf{p}|^2/2m$  where  $m = m_{p^+}$  or  $m_{e^-}$  respectively.

- From the appropriate  $N$ -particle partition functions, compute the chemical potentials  $\mu_{e^-}$ ,  $\mu_H$ , and  $\mu_{p^+}$ .
- When the reaction  $H \rightleftharpoons e^- + p^+$  is in equilibrium there will be a relation between  $\mu_{e^-}$ ,  $\mu_H$  and  $\mu_{p^+}$ . Write down (no derivation required) this relation.
- From parts (a) and (b) deduce that in equilibrium

$$\frac{x^2}{1 - x} = \kappa(T, V, N)e^{-\Delta/k_B T},$$

where  $\kappa(T, V, N)$  is an expression involving  $\hbar$ ,  $k_B T$ ,  $V$ ,  $N$ ,  $m_{e^-}$ ,  $m_{p^+}$  and  $m_H$  that you should obtain.

- Does  $x$  increase or decrease as the density  $N/V$  increases? (If you did not answer the earlier parts, you can still get credit for this part provided you give a physical justification for your answer.)

**Useful :** 
$$\frac{1}{N!} \left[ \frac{V}{\hbar^3} \int \frac{d^3 p}{(2\pi)^3} \exp \left\{ -\frac{p^2}{2mk_B T} \right\} \right]^N = \frac{1}{N!} \left[ V \frac{[2\pi mk_B T]^{3/2}}{h^3} \right]^N .$$