SM Consider the thermodynamic equilibrium of the ionization reaction

$$H \rightleftharpoons e^- + p^+$$

between a gas of neutral hydrogen atoms and unbound protons and electrons.

Suppose that in a volume V there are N_{p^+} unbound protons (charge $+e$), N_H neutral hydrogen atoms, and N_{e^-} unbound electrons. The gas has no net charge, so $N_{p^+} = N_{e^-}$. The binding energy of the hydrogen atom is $\Delta = 13.6$ eV. Let N be the total number of protons present, $x = N_{p^+}/N$ the fraction of free protons, and $N_H = (1-x)N$ the number of un-ionized hydrogen atoms. Treat all three gases as ideal. The micro-states of the hydrogen gas have energies $E_H = -\Delta + |\mathbf{p}|^2/2m_H$. The protons and electrons have micro-state energies $E = |\mathbf{p}|^2/2m$ where $m = m_{p^+}$ or m_{e^-} respectively.

a) From the appropriate N-particle partition functions, compute the chemical potentials μ_{e^-}, μ_H, and μ_{p^+}.

b) When the reaction $H \rightleftharpoons e^- + p^+$ is in equilibrium there will be a relation between μ_{e^-}, μ_H and μ_{p^+}. Write down (no derivation required) this relation.

c) From parts (a) and (b) deduce that in equilibrium

$$\frac{x^2}{1-x} = \kappa(T, V, N) e^{-\Delta/k_B T},$$

where $\kappa(T, V, N)$ is an expression involving $\hbar$, $k_B T$, V, N, m_{e^-}, m_{p^+} and m_H that you should obtain.

d) Does x increase or decrease as the density N/V increases? (If you did not answer the earlier parts, you can still get credit for this part provided you give a physical justification for your answer.)

Useful :
$$\frac{1}{N!}\left[\frac{V}{\hbar^3}\int \frac{d^3p}{(2\pi)^3} \exp\left\{-\frac{p^2}{2mk_BT}\right\}\right]^N = \frac{1}{N!}\left[V\frac{[2\pi m k_B T]^{3/2}}{h^3}\right]^N.$$