SM A theorist hypothesises that there is a type of weakly interacting particle whose Hamiltonian is given by

$$H = \gamma(p_x^2 + p_y^2 + p_z^2)^\alpha,$$

where α, γ are real and positive. She wonders if this strange energy-momentum relation will change the usual ideal gas law $PV = Nk_BT$ relating the pressure P of a gas of N particles confined in a box of volume V to its temperature T. She asks you for help.

a) Write down the one-particle partition function $Z_1 = \int_0^\infty \rho(E)e^{-E/k_BT}dE$ for one such particle confined in a macroscopic volume V. Here $\rho(E) = dn/dE$ is the one-particle density of states. Perform the integral and express your answer in terms of $V, \alpha, \gamma, \hbar, k_BT$, and the Euler Gamma function (see below for definition).

b) Write down, in terms of Z_1, the partition function Z_N for a gas of N such identical weakly-interacting particles in the volume V. You may assume that T is large enough that Boltzmann statistics are applicable.

c) Using a thermodynamic relation and your Z_N from part (b), show that the hypothetical gas obeys an ideal gas law of the form

$$PV = \zeta N k_B T,$$

where ζ is a number that you should find.

d) As a check on your calculation, use the elementary kinetic theory of gases to compute the pressure exerted by the gas due to the particles colliding elastically with the side of the box that confines the particles to the region $x < 0$. Remember that only particles with positive $v_x \equiv \partial H/\partial p_x$ will bounce off the wall! Do you get the same answer as in part (c)? If not why not?

Useful: The Euler Gamma function is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt,$$

and obeys

$$x\Gamma(x) = \Gamma(x+1).$$