SMB A neutron star is essentially a degenerate Fermi gas. Consider the situation where all the neutrons in the star are <u>ultrarelativistic</u>, *i.e.* they have energy $\epsilon = pc$ where the momentum is $p = \hbar k$ and k is the wavenumber. Consider a large cube of volume V filled with N of these ultrarelativistic neutrons in thermal equilibrium at temperature T.

a) Treat the neutrons as quantum particles in the cube with periodic boundary conditions. The energy density of states $D(\epsilon)$ is defined by replacing the sum over states by an integral according to

$$\sum_{\text{states}} (\ldots) \to \int (\ldots)\, D(\epsilon) d\epsilon,$$

where $(\ldots)$ represents any physical quantity of interest. Show that

$$D(\epsilon) = A\epsilon^{\alpha},$$

where you should determine the quantities A and α. (Do not forget that the neutron has spin $\frac{1}{2}$).

b) Determine the Fermi energy ϵ_F (*i.e.* the chemical potential at temperature $T = 0$) in terms of the neutron number density $n = N/V$ and other physical constants.

c) Write down an integral expression for the average energy density $u = U/V$ of the system at temperature T, but do not attempt to simplify it. Explain why, in the limit $k_{\mathrm{B}}T \ll \epsilon_F$ the expression for $u(T = 0)$ is a good approximation for $u(T)$.

d) Evaluate $u(T = 0)$ exactly, writing your final answer in terms of n and ϵ_F.