ASM. A quantum harmonic oscillator has Hamiltonian

$$\hat{H} = \frac{1}{2m}(\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2) + \frac{1}{2}m\omega^2(\hat{x}^2 + \hat{y}^2 + \hat{z}^2)$$

a) In this part of the question label the states by quantum numbers n_x, n_y, n_z, that are related to motion in the x, y, z directions. Use this labelling to write down a formula for the energy eigenvalues E_{n_x,n_y,n_z}, and find an expression for the degeneracy of each energy level (*i.e.* the number of states with a given allowed energy).

b) Compute the partition function for the quantum oscillator, and hence the average energy and entropy of the system at temperature T.

c) As this system is rotationally invariant, we can uniquely label the energy eigenstates by the total angular momentum ℓ, the eigenvalue m_ℓ of the z component of angular momentum, and a principal quantum number k that takes the values $k = 0, 1, 2, 3, \ldots$. In terms of these quantities the energy eigenvalues are

$$E_{k,\ell,m_\ell} = \hbar\omega(2k + \ell + 3/2).$$

Recompute the partition function using this energy formula and show that it equals your partition function from part (b). For a system at temperature T, what is the average value of the orbital angular momentum ℓ? Show that for large T the average value of ℓ rises linearly with T.

Possibly useful formulæ:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad \sum_{n=0}^{\infty} n x^n = \frac{x}{(1-x)^2} \qquad \sum_{n=0}^{\infty} n^2 x^n = \frac{x(1+x)}{(1-x)^3}$$

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty}(n+1)x^n \qquad \frac{1}{(1-x)^3} = \sum_{n=0}^{\infty}\frac{1}{2}(n+1)(n+2)x^n$$