A single, one-dimensional, harmonic oscillator has natural frequency ω and, correspondingly, its energy eigenstates are non-degenerate and have energies $\epsilon_n = \hbar\omega(n + \frac{1}{2})$ (with $n = 0, 1, 2, \ldots$).

a) Calculate the canonical partition function Z and the Helmholtz free energy F_1, each as a function of the temperature T and ω. Express F_1 in the form

$$F_1 = k_B T\, Y(\hbar\omega/2k_B T),$$

and state the required form of the function Y.

b) Determine the internal energy E, as a function of T and ω. Derive the leading behavior at low temperatures (i.e., for $T \ll \hbar\omega/k_B$) and at high temperatures (i.e., for $T \gg \hbar\omega/k_B$), and give a brief physical explanation of your result for E in each of these regimes. Sketch E as a function of T for all physical values of T.

Now consider a model of a crystal consisting of N of these harmonic oscillators. The coordinates of the oscillators are supposed to represent the displacements of the atoms of the crystal from their equilibrium positions. *However, you are to neglect any coupling between the oscillators.*

c) Write down the total Helmholtz free energy F_N governing the crystal, in terms of F_1.

d) Suppose now that the natural frequency ω is no longer a constant, but instead depends on V/N (i.e., the volume per atom in the crystal), via

$$\omega = \Omega_0 - \Omega_1 \ln\left(V/Na^3\right),$$

where Ω_0 and Ω_1 are constant frequencies and a is a constant length. Determine the isothermal compressibility of the crystal κ_T, defined as

$$\kappa_T \equiv -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T,$$

where p is the pressure, expressing your answer in terms of T, Ω_0, Ω_1, N, V and a.

e) The volume of the crystal is increased, quasi-statically, from V_1 to V_2, while T is held constant. Show that the heat transferred to the crystal is given by the formula

$$T\int_{V_1}^{V_2} dV \left(\frac{\partial p}{\partial T}\right)_V,$$

but do not attempt to perform the integration.