Consider the one-dimensional Ising model, that is a set of $N+1$ spins which can take values $\sigma_i = \pm 1$. The Hamiltonian is

$$H = -J\sum_{i=1}^{N}\sigma_i\sigma_{i+1} \qquad\qquad J > 0$$

In the following you may assume that $N >> 1$ and drop terms of relative order $N^{-1/2}$ or smaller.

(a) What configurations of the σ_i correspond to the lowest energy level? What is the degeneracy of this level?

(b) What is the number of different states in which just $M(\leq N)$ pairs of nearest-neighbor spins are oppositely oriented? [Hint: The number of ways of assigning N distinguishable objects to two boxes so that M of them are in one box is $N!/(M!(N-M)!) \equiv C_M^N$.]

(c) Given the result of (b), what is the entropy $S(M)$ associated with a particular value of M?

(d) We can define the "constrained" Helmholtz free energy $F(M)$ associated with a particular value of M as $E(M) - TS(M)$. Find the free energy per spin (i.e. $N^{-1}F(M)$) as a function of $m \equiv M/N$.

(e) Using Stirling's approximation $\ell n(N!) \approx N(\ell nN - 1)$, and treating m as a continuous variable, show that the most probable value of m as a function of T is $\overline{m} = (\exp(J/k_BT)+1)^{-1}$. You may assume that the mean value of a variable is equal to its most probable value.

(f) Evaluate the specific heat as a function of T [Hint: It is easier to do this using the energy rather than the free energy.] Why does the specific heat tend to zero for $T\to\infty$?