SM7all02B

A long tube with length L and cross-sectional area A contains N non-interacting bosons each of mass m (N is a macroscopic number). The tube stands vertically in a uniform gravitational field with acceleration g and it is kept at a constant temperature T. Assume that g is large enough to have an observable effect. Take the potential energy at the bottom (height $h = 0$) of the tube to be zero. The tube is very long so that $mgL \gg kT$ (k is the Boltzmann constant) and the area A is large enough so the energy spacing between adjacent quantized levels is much smaller than kT.

(a) You may consider the system as a vertical stack of many layers. Each layer (centered at height h) is a three-dimensional subsystem (with area A and thickness dL) and it is in thermal and diffusive contact with neighboring layers. Calculate the density of states per unit volume $g(E)$ as a function of energy E for a free particle in the subsystem.

(b) Find an integral relation between the density $n(h)$ at height h and the chemical potential μ.

(c) In the high temperature limit, the bosons behave like classical particles (ideal gas). Show that $n(h)$ is proportional to $e^{-mgh/kT}$ and use the fact

$$N = A \int_0^L n(h)dh$$

to find the proportionality constant in terms of (N/A) (the two-dimensional (2D) particle density), T, and other constants.

(d) Now consider the quantum regime. At low temperatures, we expect Bose-Einstein condensation to occur. Find the number of bosons in the ground state (N_0) as a function of T.

(e) Find the chemical potential in terms of N_0, T and other constants near the Bose-Einstein transition.

(f) Find the Bose-Einstein transition temperature T_E in terms (N/A) and other constants. Evaluate T_E for the case $\frac{(2m)^{1/2}}{\hbar^3 g} = 10^{22} eV^{-5/2}$ Å^{-2} and a 2D density $(N/A) = 10^{-12}$ Å^{-2}.

Note: The following integrals may be needed.

$$\int \frac{dx}{ae^x - 1} = \ln\left(\frac{ae^x - 1}{a - 1}\right) - x.$$

$$\int_0^\infty x^{1/2} \ln\left(\frac{e^x}{e^x - 1}\right) dx = \frac{1}{2}\sqrt{\pi} \sum_{n=0}^{\infty} (1+n)^{-5/2} = 1.189.$$