SMSpring95B

An idealized model of a polymer consists of a chain of N links lying in the xy-plane. Each of the links has length a and is free to point along each of the four lattice directions in the plane, as shown in the figure. One end of the chain is fixed to the origin O, and the other end is located at the point $\mathbf{L} = (L_x, L_y)$. Links are allowed to double back or form closed loops, as shown in the figure. The total energy U of the configuration is the sum of link energies: zero for each link lying parallel to the x axis and ϵ (> 0) for each link lying parallel to the y axis. N_x^+, N_x^-, N_y^+ and N_y^- are, respectively, the number of links whose heads point in the positive or negative x or y directions, relative to the previous link.

(a) The exact expression for the number of possible configurations Ω for given N, N_x^+, N_x^-, N_y^+ and N_y^-, is given by

$$\Omega = \frac{N!}{N_x^+!\, N_x^-!\, N_y^+!\, N_y^-!}.$$

Use Stirling's approximation to the factorial function $\left[i.e., \ln M! \approx M\left(\ln M - 1\right)\right]$ to determine the entropy S as a function of N, N_x^+, N_x^-, N_y^+ and N_y^-.

(b) Express N_x^+, N_x^-, N_y^+ and N_y^-, and hence S, in terms of N, L_x, L_y and U.

For the present system the first law of thermodynamics reads

$$dU = T\,dS + \mathbf{I} \cdot d\mathbf{L},$$

where T is the temperature and $\mathbf{I} = (I_x, I_y)$ is the tension applied to the end of the polymer.

(c) Express the components of $\mathbf{I}$ in terms of derivatives of S. Hence, determine $\mathbf{I}$ explicitly in terms of N, L_x, L_y and U. Determine L_y for the situation that $I_y = 0$.

(d) Determine T in terms of N, L_x, L_y and U.

(e) Explain briefly the origin of the formula for Ω given in part (a).