

CMB A surprisingly accurate approximation for the motion of a mass m orbiting a black hole of mass M can be obtained by using ordinary **non-relativistic Newtonian mechanics**, but slightly modifying the usual $1/r$ Keplerian potential to

$$U(r) = -\frac{GmM}{(r - r_g)}.$$

Here G is the gravitational constant and r_g is the radius of the black-hole event horizon. Orbits with $r < r_g$ are inside the black hole and so unphysical.

- a) By means of a Lagrangian or otherwise, obtain, for a general potential $U(r)$ and orbital angular momentum l , an equation for the radius $r(t)$ in the form

$$m\ddot{r} + \frac{\partial}{\partial r}W_{\text{eff}}(r, l) = 0.$$

You must give an explicit equation for W_{eff} .

- b) Use the potential $U(r)$ given above to find the value of l that will allow a circular orbit of radius r_0 around the black hole.
- c) Explain how you would use some property of W_{eff} to determine whether the circular orbit you found in part (b) is stable or unstable when the particle is given a small kick that does not alter its orbital angular momentum.
- d) Exploit your result from part (b) to show that for the potential $U(r)$ *no* circular orbit in the range $r_g < r_0 < 3r_g$ is stable.