Precise measurements of the length of a day indicate that the earth's rotation about its axis is slowing down. This is caused by tidal friction—as the moon orbits the earth, the tides rise and fall and convert mechanical energy to heat.

Treat the earth-moon system as isolated and subject to no external forces. Denote the time (t)-dependent orbital angular momentum of the moon around the earth as $L(\mathrm{t})$ and the earth's spin angular momentum about its axis as $S(\mathrm{t})$ and the angular velocity of the earth spinning about its axis as $\omega(\mathrm{t})$. Suppose that S and L are parallel and are both perpendicular to the plane of the earth-moon orbit. Denote the total angular momentum $J = L + S$. Approximate the moon's orbit about the earth as circular with a radius $r(t)$ that changes very slowly with time.

(a) Identify a conserved quantity in this system. Use it and the kinematics of the moon's circular orbit to relate the rate of change of $r(t)$, $\dot{r}(t)/r(t)$, to the rate of change of the angular velocity of the earth spinning about its axis, $\dot{\omega}(t)/\omega(t)$. Your relation will include the ratio of the two angular momenta, $S(t)/L(t)$.

(b) Precise measurements indicate that an earth day is lengthening at a rate of 4.40×10^{-8} sec/day at the present time t_0. Planetary scientists and geologists also estimate the ratio $S(t_0)/L(t_0) \approx 0.342$ at the present time t_0. Find the numerical value of $\dot{r}(t_0)$ and explain its sign. Write $\dot{r}(t_0)$ as a certain number of centimeters per month to get a 'feel' for it. The present radius of the moon's orbit around the earth is 3.86×10^5 km.

Now we want to discover the fate of the moon's orbit around the earth. The mechanical energy E of the system consists of three terms: (1) the kinetic energy of the moon, (2) the gravitational potential energy of the earth-moon system and, (3) the rotational energy of the earth spinning on its axis.

(c) Use the equation of motion of the moon around the earth and the conservation law of part (a) to write E in terms of just one variable, the angular momentum $L(t)$. Find the minimum of $E(L)$ to find the 'final' state of the earth-moon system. Write the result in terms of the angular velocity ω, the angular velocity of the earth spinning around its axis and Ω, the angular velocity of the moon's orbital motion about the earth.

(d) Describe in words the final state of the system found in part (c).