CM79 1196B

This question concerns a simple model for the formation of rings around a planet. The system we consider is a central planet of mass M around which a cloud of small particles of identical mass $m << M$ revolve. There is no particular symmetry in the cloud initially; however, the total angular momentum of the cloud around the planet is non-zero. Assume that the particles do not interact with each other except for collisions that obey the following simplified collision rule:

$$\mathbf{v}_i' - \mathbf{v}_j' = -e_r(\mathbf{v}_i - \mathbf{v}_j),$$

where e_r (the coefficient of restitution) obeys $0 < e_r < 1$, $\mathbf{v}_i$ is the velocity of the i-th particle, and primes denote the velocities after collision. The collisions are instantaneous and do not change the masses of the colliding particles.

(a) Consider two colliding particles with velocities before collision $\mathbf{v}_1$ and $\mathbf{v}_2$. Write down the loss of kinetic energy ΔK in terms of m, e_r and the velocities before the collision.

(b) Suppose that the particles in (a) have angular momentum $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ around the central planet. Demonstrate that upon collision, $|\boldsymbol{L}_1 - \boldsymbol{L}_2|$ diminishes.

(c) The Laplace-Runge-Lenz vector of a particle is defined by

$$\varepsilon = \frac{\mathbf{v} \times \boldsymbol{L}}{Gm(M+m)} - \frac{\mathbf{r}}{\mathrm{r}},$$

where $\boldsymbol{L}$ is the particle angular momentum, $\mathbf{v}$ the particle velocity, $\mathbf{r}$ the particle position whose origin is at the central planet and G the gravitational constant. Demonstrate that upon collision the magnitude of the difference between the Laplace-Runge-Lenz vectors of the colliding particles diminishes.

(d) Given that $\boldsymbol{L}$ and ε uniquely determine the Kepler orbit of the particle, and that the Laplace-Runge-Lenz vector is invariant when there are no collisions, use your results from (b) and (c) to argue that eventually stable rings are formed around the planet.