QMB: Suppose a quantum Hamiltonian $H(\lambda)$ depends on some real-valued parameter λ, and for each value of λ the state $|\Psi_\lambda\rangle$ is a non-degenerate energy eigenstate of $H(\lambda)$:

$$H(\lambda)|\Psi_\lambda\rangle = E(\lambda)|\Psi_\lambda\rangle.$$

a) Show that

$$\frac{\partial E(\lambda)}{\partial \lambda} = \frac{\langle \Psi_\lambda | \frac{\partial H(\lambda)}{\partial \lambda} | \Psi_\lambda \rangle}{\langle \Psi_\lambda | \Psi_\lambda \rangle}.$$

b) The formula for part (a) does not hold for degenerate eigenstates. Point out where in your proof you use the fact that the state is non-degenerate, and explain what goes wrong if it *is* degenerate.

A physicist remembers that the ground state of a spinless electron in a hydrogen atom with Hamiltonian

$$H_{\text{hydrogen}} = -\frac{\hbar^2}{2m_e}\nabla^2 - \frac{e^2}{4\pi\epsilon_0 r}$$

is non-degenerate and has energy

$$E_0 = -\frac{m_e e^4}{2(4\pi\epsilon_0)^2\hbar^2}.$$

They also remember that in spherical polar coordinates the ground state wavefunction of H_{hydrogen} has the form

$$\psi_0(r,\theta,\phi) \equiv \langle r,\theta,\phi|\psi_0\rangle = N\exp\{-\alpha r\},$$

but they do *not* remember what the constants α or N are in terms of e^2, $\hbar$, *etc.*

c) Normalize ψ_0 to find N in terms of α, and then compute

$$\left\langle \psi_0 \left| \frac{1}{4\pi\epsilon_0 r} \right| \psi_0 \right\rangle$$

as a function of α. Compare the left and right-hand sides of the formula from part (a) and so find α in terms of the physical constants appearing in the expression for E_0.