QMA: Consider the quantum mechanics of a particle of mass m in a 3-dimensional isotropic harmonic oscillator potential and governed by the Hamiltonian

$$H_0 = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) + \frac{1}{2}m\omega^2\left(x^2 + y^2 + z^2\right),$$

where ω is a positive constant.

a) Write down (no calculation required) the energies of the three lowest distinct energy levels of H_0 and state their degeneracy.

b) Suppose this system is perturbed by the potential

$$V = \lambda m\omega\, xy, \text{ where } \lambda > 0.$$

Find the physical constraint on λ required for the energy to be bounded from below.

c) Show that a suitable rotation of the axes reduces the problem to that of three uncoupled oscillators and hence write down an *exact* expression for the energy spectrum of $H = H_0 + V$.

d) Assuming $0 < \lambda \ll 1$, use perturbation theory to find the shift in the ground state energy to first order in λ.

e) Take the limit of your answer from part (c) as $\lambda \to 0$ and compare your answer with your perturbation calculation from part (d). Do they agree? If not why not?

Hint: You may use without proof the usual relations between position/momentum and creation/annihilation operators

$$x = \sqrt{\frac{\hbar}{2m\omega}}\left(a_x + a_x^\dagger\right) \text{ and } p_x = i\sqrt{\frac{m\hbar\omega}{2}}\left(-a_x + a_x^\dagger\right),$$

and similarly for y, p_y and z, p_z.