SMB: Consider neutral hydrogen gas maintained at a fixed density n_0. At finite temperature some of the hydrogen atoms will be ionized through the process

$$H \rightleftharpoons e + p.$$

In thermal equilibrium the density n_H of un-ionized hydrogen atoms, the density n_p of protons, and the density n_e of electrons are related by *Saha's equation*

$$\frac{n_e n_p}{n_h} = \chi, \quad \text{where} \quad \chi = \left(\frac{m_e k_B T}{2\pi\hbar^2}\right)^{3/2} \exp\{-R/k_B T\}.$$

Here m_e is the electron mass, k_B is Boltzmann's constant and $R = 13.6$ eV is the Rydberg constant equal to the ionization energy of hydrogen.

a) Write down two conservation laws that relate n_0, n_e, n_p and n_H, and hold for all temperatures.

b) Use the conservation laws and the equilibrium equation to derive a quadratic equation whose solution gives n_p/n_0 as a function of the dimensionless quantity χ/n_0

Now consider hydrogen gas with total density $n_0 = 3 \times 10^{18}$ m^{-3}.

c) What fraction of these hydrogen atoms are ionized on the surface of the sun (photosphere) where the temperature $T \sim 5800$ K.

d) What fraction of these hydrogen atoms are **not** ionized in the corona outside the sun where the temperature $T \sim 10^6$ K.

e) Does n_p/n_0 go up or down as the density n_0 is decreased? Why?

Hint: The answers to (c) and (d) are sufficiently small that they can be found to sufficient accuracy without making use of the quadratic equation from (b).

Useful: (i) $n_Q \equiv \left(\frac{m_e k_B T}{2\pi\hbar^2}\right)^{3/2} = 2.5 \times 10^{27}$ m^{-3} at $T = 10,000$ K; (ii) $k_B T =$ 13.6 eV at $T = 158,000$ K.