APPENDIX

B Curl, Divergence, and Gradient in Cylindrical and Spherical Coordinate Systems

In Sections 3.1, 3.4, and 6.1, we introduced the curl, divergence, and gradient, respectively, and derived the expressions for them in the Cartesian coordinate system. In this appendix, we shall derive the corresponding expressions in the cylindrical and spherical coordinate systems. Considering first the cylindrical coordinate system, we recall from Appendix A that the infinitesimal box defined by the three orthogonal surfaces intersecting at point $P(r, \theta, \phi)$ and the three orthogonal surfaces intersecting at point $Q(r + dr, \phi + d\phi, z + dz)$ is as shown in Figure B.1.

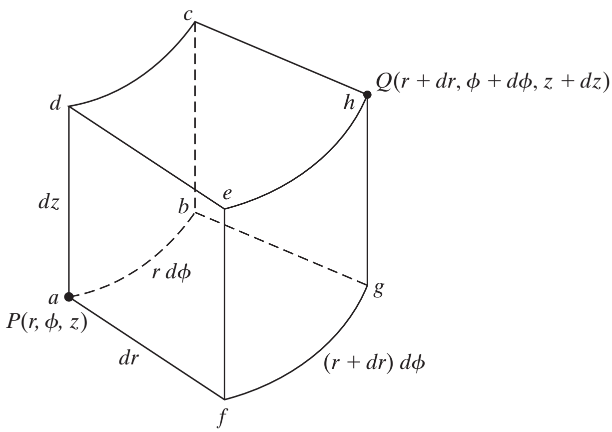

FIGURE B.1

Infinitesimal box formed by incrementing the coordinates in the cylindrical coordinate system.

From the basic definition of the curl of a vector introduced in Section 3.3 and given by

$$\nabla \times \mathbf{A} = \lim_{\Delta S \to 0} \left[\frac{\oint_C \mathbf{A} \cdot d\mathbf{l}}{\Delta S} \right]_{\max} \mathbf{a}_n \tag{B.1}$$

we find the components of $\nabla \times \mathbf{A}$ as follows with the aid of Figure B.1:

$$\begin{aligned}
(\nabla \times \mathbf{A})_r &= \lim_{\substack{d\phi \to 0 \\ dz \to 0}} \frac{\oint_{abcda} \mathbf{A} \cdot d\mathbf{l}}{\text{area } abcd} \\
&= \lim_{\substack{d\phi \to 0 \\ dz \to 0}} \frac{\left\{ \begin{array}{l} [A_\phi]_{(r,\, z)}\, r\, d\phi + [A_z]_{(r,\, \phi + d\phi)}\, dz \\ -\, [A_\phi]_{(r,\, z+dz)} r\, d\phi - [A_z]_{(r,\, \phi)}\, dz \end{array} \right\}}{r\, d\phi\, dz} \\
&= \lim_{d\phi \to 0} \frac{[A_z]_{(r,\, \phi + d\phi)} - [A_z]_{(r,\, \phi)}}{r\, d\phi} + \lim_{dz \to 0} \frac{[A_\phi]_{(r,\, z)} - [A_\phi]_{(r,\, z+dz)}}{dz} \\
&= \frac{1}{r} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}
\end{aligned} \tag{B.2a}$$

$$\begin{aligned}
(\nabla \times \mathbf{A})_\phi &= \lim_{\substack{dz \to 0 \\ dr \to 0}} \frac{\oint_{adefa} \mathbf{A} \cdot d\mathbf{l}}{\text{area } adef} \\
&= \lim_{\substack{dz \to 0 \\ dr \to 0}} \frac{\left\{ \begin{array}{l} [A_z]_{(r,\, \phi)}\, dz + [A_r]_{(\phi,\, z+dz)}\, dr \\ -\, [A_z]_{(r+dr,\, \phi)}\, dz - [A_r]_{(\phi,\, z)}\, dr \end{array} \right\}}{dr\, dz} \\
&= \lim_{dz \to 0} \frac{[A_r]_{(\phi,\, z+dz)} - [A_r]_{(\phi,\, z)}}{dz} + \lim_{dr \to 0} \frac{[A_z]_{(r,\, \phi)} - [A_z]_{(r+dr,\, \phi)}}{dr} \\
&= \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}
\end{aligned} \tag{B.2b}$$

$$\begin{aligned}
(\nabla \times \mathbf{A})_z &= \lim_{\substack{dr \to 0 \\ d\phi \to 0}} \frac{\oint_{afgba} \mathbf{A} \cdot d\mathbf{l}}{\text{area } afgb} \\
&= \lim_{\substack{dr \to 0 \\ d\phi \to 0}} \frac{\left\{ \begin{array}{l} [A_r]_{(\phi,\, z)}\, dr + [A_\phi]_{(r+dr,\, z)}\, (r + dr)\, d\phi \\ -\, [A_r]_{(\phi + d\phi,\, z)}\, dr - [A_\phi]_{(r,\, z)} r\, d\phi \end{array} \right\}}{r\, dr\, d\phi} \\
&= \lim_{dr \to 0} \frac{[rA_\phi]_{(r+dr,\, z)} - [rA_\phi]_{(r,\, z)}}{r\, dr} + \lim_{d\phi \to 0} \frac{[A_r]_{(\phi,\, z)} - [A_r]_{(\phi + d\phi,\, z)}}{r\, d\phi} \\
&= \frac{1}{r} \frac{\partial}{\partial r} (rA_\phi) - \frac{1}{r} \frac{\partial A_r}{\partial \phi}
\end{aligned} \tag{B.2c}$$

Combining (B.2a), (B.2b), and (B.2c), we obtain the expression for the curl of a vector in cylindrical coordinates as

$$\nabla \times \mathbf{A} = \left[\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right]\mathbf{a}_r + \left[\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right]\mathbf{a}_\phi + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial \phi}\right]\mathbf{a}_z$$

$$= \begin{vmatrix} \dfrac{\mathbf{a}_r}{r} & \mathbf{a}_\phi & \dfrac{\mathbf{a}_z}{r} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix} \tag{B.3}$$

To find the expression for the divergence, we make use of the basic definition of the divergence of a vector, introduced in Section 3.6 and given by

$$\nabla \cdot \mathbf{A} = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta v} \tag{B.4}$$

Evaluating the right side of (B.4) for the box of Figure B.1, we obtain

$$\nabla \cdot \mathbf{A} = \lim_{\substack{dr \to 0 \\ d\phi \to 0 \\ dz \to 0}} \frac{\left\{\begin{matrix}[A_r]_{r+dr}(r + dr)\, d\phi\, dz - [A_r]_r r\, d\phi\, dz + [A_\phi]_{\phi+d\phi}\, dr\, dz \\ - [A_\phi]_\phi\, dr\, dz + [A_z]_{z+dz} r\, dr\, d\phi - [A_z]_z r\, dr\, d\phi\end{matrix}\right\}}{r\, dr\, d\phi\, dz}$$

$$= \lim_{dr \to 0} \frac{[rA_r]_{r+dr} - [rA_r]_r}{r\, dr} + \lim_{d\phi \to 0} \frac{[A_\phi]_{\phi+d\phi} - [A_\phi]_\phi}{r\, d\phi}$$

$$+ \lim_{dz \to 0} \frac{[A_z]_{z+dz} - [A_z]_z}{dz}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \tag{B.5}$$

To obtain the expression for the gradient of a scalar, we recall from Appendix A that in cylindrical coordinates,

$$d\mathbf{l} = dr\, \mathbf{a}_r + r\, d\phi\, \mathbf{a}_\phi + dz\, \mathbf{a}_z \tag{B.6}$$

and hence

$$d\Phi = \frac{\partial \Phi}{\partial r} dr + \frac{\partial \Phi}{\partial \phi} d\phi + \frac{\partial \Phi}{\partial z} dz$$

$$= \left(\frac{\partial \Phi}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial \Phi}{\partial \phi}\mathbf{a}_\phi + \frac{\partial \Phi}{\partial z}\mathbf{a}_z\right) \cdot (dr\, \mathbf{a}_r + r\, d\phi\, \mathbf{a}_\phi + dz\, \mathbf{a}_z)$$

$$= \nabla \Phi \cdot d\mathbf{l} \tag{B.7}$$

Thus,

$$\nabla\Phi = \frac{\partial\Phi}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\phi}\mathbf{a}_\phi + \frac{\partial\Phi}{\partial z}\mathbf{a}_z \tag{B.8}$$

Turning now to the spherical coordinate system, we recall from Appendix A that the infinitesimal box defined by the three orthogonal surfaces intersecting at $P(r, \theta, \phi)$ and the three orthogonal surfaces intersecting at $Q(r + dr, \theta + d\theta, \phi + d\phi)$ is as shown in Figure B.2. From the basic definition of the curl of a vector given by (B.1), we then find the components of $\nabla \times \mathbf{A}$ as follows with the aid of Figure B.2:

$$\begin{aligned}
(\nabla \times \mathbf{A})_r &= \lim_{\substack{d\theta\to 0\\ d\phi\to 0}} \frac{\oint_{abcda} \mathbf{A}\cdot d\mathbf{l}}{\text{area } abcd} \\
&= \lim_{\substack{d\theta\to 0\\ d\phi\to 0}} \frac{\left\{\begin{array}{l}[A_\theta]_{(r,\phi)} r\,d\theta + [A_\phi]_{(r,\theta+d\theta)} r\sin(\theta + d\theta)\,d\phi \\ -[A_\theta]_{(r,\phi+d\phi)} r\,d\theta - [A_\phi]_{(r,\theta)} r\sin\theta\,d\phi\end{array}\right\}}{r^2\sin\theta\,d\theta\,d\phi} \\
&= \lim_{d\theta\to 0} \frac{[A_\phi\sin\theta]_{(r,\theta+d\theta)} - [A_\phi\sin\theta]_{(r,\theta)}}{r\sin\theta\,d\theta} \\
&\quad + \lim_{d\phi\to 0} \frac{[A_\theta]_{(r,\phi)} - [A_\theta]_{(r,\phi+d\phi)}}{r\sin\theta\,d\phi} \\
&= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\phi\sin\theta) - \frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial\phi}
\end{aligned} \tag{B.9a}$$

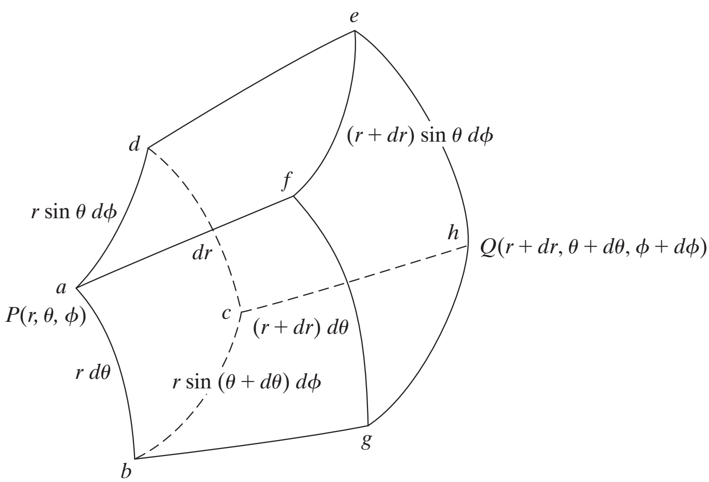

FIGURE B.2

Infinitesimal box formed by incrementing the coordinates in the spherical coordinate system.

$$(\nabla \times \mathbf{A})_\theta = \lim_{\substack{d\phi \to 0 \\ dr \to 0}} \frac{\oint_{adefa} \mathbf{A} \cdot d\mathbf{l}}{\text{area } adef}$$

$$= \lim_{\substack{d\phi \to 0 \\ dr \to 0}} \frac{\left\{ \begin{array}{l} [A_\phi]_{(r,\theta)} r \sin\theta \, d\phi + [A_r]_{(\theta,\phi+d\phi)} \, dr \\ - [A_\phi]_{(r+dr,\theta)} (r + dr) \sin\theta \, d\phi - [A_r]_{(\theta,\phi)} \, dr \end{array} \right\}}{r \sin\theta \, dr \, d\phi}$$

$$= \lim_{d\phi \to 0} \frac{[A_r]_{(\theta,\phi+d\phi)} - [A_r]_{(\theta,\phi)}}{r \sin\theta \, d\phi}$$

$$+ \lim_{dr \to 0} \frac{[rA_\phi]_{(r,\theta)} - [rA_\phi]_{(r+dr,\theta)}}{r \, dr}$$

$$= \frac{1}{r \sin\theta} \frac{\partial A_r}{\partial \phi} - \frac{1}{r} \frac{\partial}{\partial r}(rA_\phi) \qquad \text{(B.9b)}$$

$$(\nabla \times \mathbf{A})_\phi = \lim_{\substack{dr \to 0 \\ d\theta \to 0}} \frac{\oint_{afgba} \mathbf{A} \cdot d\mathbf{l}}{\text{area } afgb}$$

$$= \lim_{\substack{dr \to 0 \\ d\theta \to 0}} \frac{\left\{ \begin{array}{l} [A_r]_{(\theta,\phi)} \, dr + [A_\theta]_{(r+dr,\phi)} (r + dr) \, d\theta \\ - [A_r]_{(\theta+d\theta,\phi)} \, dr - [A_\theta]_{(r,\phi)} r \, d\theta \end{array} \right\}}{r \, dr \, d\theta}$$

$$= \lim_{dr \to 0} \frac{[rA_\theta]_{(r+dr,\phi)} - [rA_\theta]_{(r,\phi)}}{r \, dr}$$

$$+ \lim_{d\theta \to 0} \frac{[A_r]_{(\theta,\phi)} \, dr - [A_r]_{(\theta+d\theta,\phi)} \, dr}{r \, d\theta}$$

$$= \frac{1}{r} \frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r} \frac{\partial A_r}{\partial \theta} \qquad \text{(B.9c)}$$

Combining (B.9a), (B.9b), and (B.9c), we obtain the expression for the curl of a vector in spherical coordinates as

$$\nabla \times \mathbf{A} = \frac{1}{r \sin\theta} \left[\frac{\partial}{\partial \theta}(A_\phi \sin\theta) - \frac{\partial A_\theta}{\partial \phi} \right] \mathbf{a}_r$$

$$+ \frac{1}{r} \left[\frac{1}{\sin\theta} \frac{\partial A_r}{\partial \phi} - \frac{\partial}{\partial r}(rA_\phi) \right] \mathbf{a}_\theta + \frac{1}{r} \left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta} \right] \mathbf{a}_\phi$$

$$= \begin{vmatrix} \dfrac{\mathbf{a}_r}{r^2 \sin\theta} & \dfrac{\mathbf{a}_\theta}{r \sin\theta} & \dfrac{\mathbf{a}_\phi}{r} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ A_r & rA_\theta & r \sin\theta \, A_\phi \end{vmatrix} \qquad \text{(B.10)}$$

To find the expression for the divergence, we make use of the basic definition of the divergence of a vector given by (B.4) and by evaluating its right side for the box of Figure B.2, we obtain

$$\nabla \cdot \mathbf{A} = \lim_{\substack{dr\to 0\\ d\theta\to 0\\ d\phi\to 0}} \frac{\left\{\begin{array}{l}[A_r]_{r+dr}(r+dr)^2 \sin\theta\, d\theta\, d\phi - [A_r]_r r^2 \sin\theta\, d\theta\, d\phi \\ +\, [A_\theta]_{\theta+d\theta} r \sin(\theta+d\theta)\, dr\, d\phi - [A_\theta]_\theta r\sin\theta\, dr\, d\phi \\ +\, [A_\phi]_{\phi+d\phi} r\, dr\, d\theta - [A_\phi]_\phi r\, dr\, d\theta\end{array}\right\}}{r^2 \sin\theta\, dr\, d\theta\, d\phi}$$

$$= \lim_{dr\to 0} \frac{[r^2A_r]_{r+dr} - [r^2A_r]_r}{r^2\, dr} + \lim_{d\theta\to 0} \frac{[A_\theta \sin\theta]_{\theta+d\theta} - [A_\theta\sin\theta]_\theta}{r\sin\theta\, d\theta}$$

$$+ \lim_{d\phi\to 0} \frac{[A_\phi]_{\phi+d\phi} - [A_\phi]_\phi}{r\sin\theta\, d\phi}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi} \tag{B.11}$$

To obtain the expression for the gradient of a scalar, we recall from Appendix A that in spherical coordinates,

$$d\mathbf{l} = dr\,\mathbf{a}_r + r\,d\theta\,\mathbf{a}_\theta + r\sin\theta\,d\phi\,\mathbf{a}_\phi \tag{B.12}$$

and hence

$$\begin{aligned} d\Phi &= \frac{\partial\Phi}{\partial r}dr + \frac{\partial\Phi}{\partial\theta}d\theta + \frac{\partial\Phi}{\partial\phi}d\phi \\ &= \left(\frac{\partial\Phi}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\mathbf{a}_\phi\right)\cdot(dr\,\mathbf{a}_r + r\,d\theta\,\mathbf{a}_\theta + r\sin\theta\,d\phi\,\mathbf{a}_\phi) \\ &= \nabla\Phi\cdot d\mathbf{l} \end{aligned} \tag{B.13}$$

Thus,

$$\nabla\Phi = \frac{\partial\Phi}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\mathbf{a}_\phi \tag{B.14}$$

REVIEW QUESTIONS

B.1. Briefly discuss the basic definition of the curl of a vector.

B.2. Justify the application of the basic definition of the curl of a vector to determine separately the individual components of the curl.

B.3. How would you generalize the interpretations for the components of the curl of a vector in terms of the lateral derivatives involving the components of the vector to hold in cylindrical and spherical coordinate systems?

B.4. Briefly discuss the basic definition of the divergence of a vector.

B.5. How would you generalize the interpretation for the divergence of a vector in terms of the longitudinal derivatives involving the components of the vector to hold in cylindrical and spherical coordinate systems?

B.6. Provide general interpretation for the components of the gradient of a scalar.

PROBLEMS

B.1. Find the curl and the divergence for each of the following vectors in cylindrical coordinates: (a) $r \cos \phi \, \mathbf{a}_r - r \sin \phi \, \mathbf{a}_\phi$; (b) $\frac{1}{r} \mathbf{a}_r$; (c) $\frac{1}{r} \mathbf{a}_\phi$.

B.2. Find the gradient for each of the following scalar functions in cylindrical coordinates: (a) $\frac{1}{r} \cos \phi$; (b) $r \sin \phi$.

B.3. Find the expansion for the Laplacian, that is, the divergence of the gradient, of a scalar in cylindrical coordinates.

B.4. Find the curl and the divergence for each of the following vectors in spherical coordinates: (a) $r^2 \mathbf{a}_r + r \sin \theta \, \mathbf{a}_\theta$; (b) $\frac{e^{-r}}{r} \mathbf{a}_\theta$; (c) $\frac{1}{r^2} \mathbf{a}_r$.

B.5. Find the gradient for each of the following scalar functions in spherical coordinates: (a) $\frac{\sin \theta}{r}$; (b) $r \cos \theta$.

B.6. Find the expansion for the Laplacian, that is, the divergence of the gradient, of a scalar in spherical coordinates.