APPENDIX

A

Cylindrical and Spherical Coordinate Systems

In Section 1.2, we learned that the Cartesian coordinate system is defined by a set of three mutually orthogonal surfaces, all of which are planes. The cylindrical and spherical coordinate systems also involve sets of three mutually orthogonal surfaces. For the cylindrical coordinate system, the three surfaces are a cylinder and two planes, as shown in Figure A.1(a). One of these planes is the same as the z = constant plane in the Cartesian coordinate system. The second plane contains the z-axis and makes an angle ϕ with a reference plane, conveniently chosen to be the xz-plane of the Cartesian coordinate system. This plane is therefore defined by ϕ = constant. The cylindrical surface has the z-axis as its axis. Since the radial distance r from the z-axis to points on the cylindrical surface is a constant, this surface is defined by r = constant. Thus, the three orthogonal surfaces defining the cylindrical coordinates of a point are r = constant, ϕ = constant, and z = constant. Only two of these coordinates (r and z) are distances; the third coordinate (ϕ) is an angle. We note that the entire space is spanned by varying r from 0 to ∞, ϕ from 0 to 2π, and z from $-\infty$ to ∞.

The origin is given by $r = 0, \phi = 0$, and $z = 0$. Any other point in space is given by the intersection of three mutually orthogonal surfaces obtained by incrementing the coordinates by appropriate amounts. For example, the intersection of the three surfaces $r = 2, \phi = \pi/4$, and $z = 3$ defines the point $A(2, \pi/4, 3)$, as shown in Figure A.1(a). These three orthogonal surfaces define three curves that are mutually perpendicular. Two of these are straight lines and the third is a circle. We draw unit vectors, $\mathbf{a}_r$, $\mathbf{a}_\phi$, and $\mathbf{a}_z$ tangential to these curves at the point A and directed toward increasing values of r, ϕ, and z, respectively. These three unit vectors form a set of mutually orthogonal unit vectors in terms of which vectors drawn at A can be described. In a similar manner, we can draw unit vectors at any other point in the cylindrical coordinate system, as shown, for example, for point $B(1, 3\pi/4, 5)$ in Figure A.1(a). It can now be seen that the unit vectors $\mathbf{a}_r$ and $\mathbf{a}_\phi$ at point B are not parallel to the corresponding unit vectors at point A. Thus, unlike in the Cartesian coordinate system, the unit vectors $\mathbf{a}_r$ and $\mathbf{a}_\phi$ in the cylindrical coordinate system do not have the same directions everywhere, that is, they are not uniform. Only the unit vector $\mathbf{a}_z$, which is the same as in the Cartesian coordinate system, is uniform. Finally, we note that for the choice of ϕ as in Figure A.l(a),

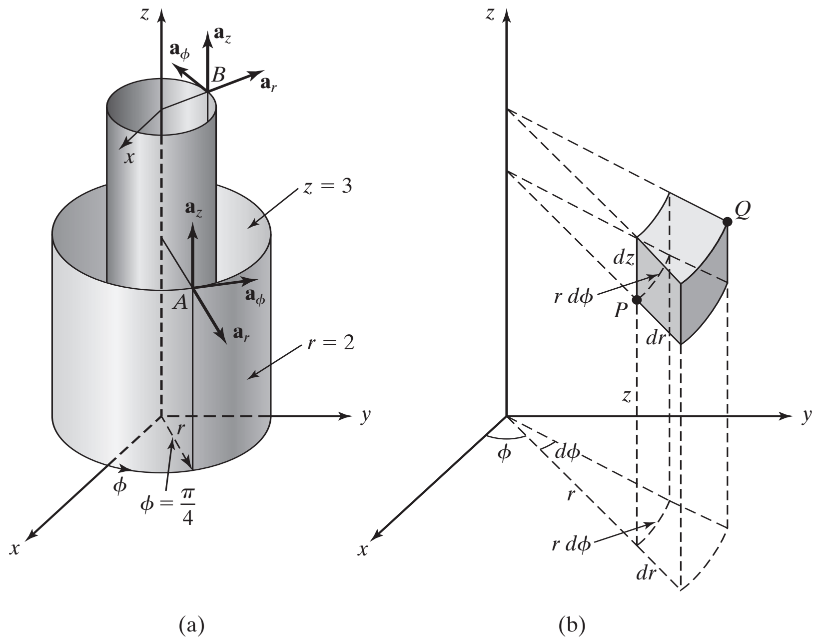

FIGURE A.1

Cylindrical coordinate system. (a) Orthogonal surfaces and unit vectors. (b) Differential volume formed by incrementing the coordinates.

that is, increasing from the positive x-axis toward the positive y-axis, the coordinate system is right-handed, that is, $\mathbf{a}_r \times \mathbf{a}_\phi = \mathbf{a}_z$.

To obtain expressions for the differential lengths, surfaces, and volume in the cylindrical coordinate system, we now consider two points $P(r, \phi, z)$ and $Q(r + dr, \phi + d\phi, z + dz)$, where Q is obtained by incrementing infinitesimally each coordinate from its value at P, as shown in Figure A.1(b). The three orthogonal surfaces intersecting at P and the three orthogonal surfaces intersecting at Q define a box which can be considered to be rectangular, since dr, $d\phi$, and dz are infinitesimally small. The three differential length elements, forming the contiguous sides of this box are $dr\,\mathbf{a}_r$, $r\,d\phi\,\mathbf{a}_\phi$, and $dz\,\mathbf{a}_z$. The differential length vector $d\mathbf{l}$ from P to Q is thus given by

$$d\mathbf{l} = dr\,\mathbf{a}_r + r\,d\phi\,\mathbf{a}_\phi + dz\,\mathbf{a}_z \tag{A.1}$$

The differential surfaces formed by pairs of the differential length elements are

$$\pm dS\,\mathbf{a}_z = \pm(dr)(r\,d\phi)\mathbf{a}_z = \pm dr\,\mathbf{a}_r \times r\,d\phi\,\mathbf{a}_\phi \tag{A.2a}$$

$$\pm dS\,\mathbf{a}_r = \pm(r\,d\phi)(dz)\mathbf{a}_r = \pm r\,d\phi\,\mathbf{a}_\phi \times dz\,\mathbf{a}_z \tag{A.2b}$$

$$\pm dS\,\mathbf{a}_\phi = \pm(dz)(dr)\mathbf{a}_\phi = \pm dz\,\mathbf{a}_z \times dr\,\mathbf{a}_r \tag{A.2c}$$

Finally, the differential volume dv formed by the three differential lengths is simply the volume of the box, that is,

$$dv = (dr)(r\,d\phi)(dz) = r\,dr\,d\phi\,dz \tag{A.3}$$

For the spherical coordinate system, the three mutually orthogonal surfaces are a sphere, a cone, and a plane, as shown in Figure A.2(a). The plane is the same as the $\phi =$ constant plane in the cylindrical coordinate system. The sphere has the origin as its center. Since the radial distance r from the origin to points on the spherical surface is a constant, this surface is defined by $r =$ constant. The spherical coordinate r should not be confused with the cylindrical coordinate r. When these two coordinates appear in the same expression, we shall use the subscripts c and s to distinguish between cylindrical and spherical. The cone has its vertex at the origin and its surface is symmetrical about the z-axis. Since the angle θ is the angle that the conical surface makes with the z-axis, this surface is defined by $\theta =$ constant. Thus, the three orthogonal surfaces defining the spherical coordinates of a point are $r =$ constant, $\theta =$ constant, and $\phi =$ constant. Only one of these coordinates (r) is distance; the other two coordinates (θ and ϕ) are angles. We note that the entire space is spanned by varying r from 0 to ∞, θ from 0 to π, and ϕ from 0 to 2π.

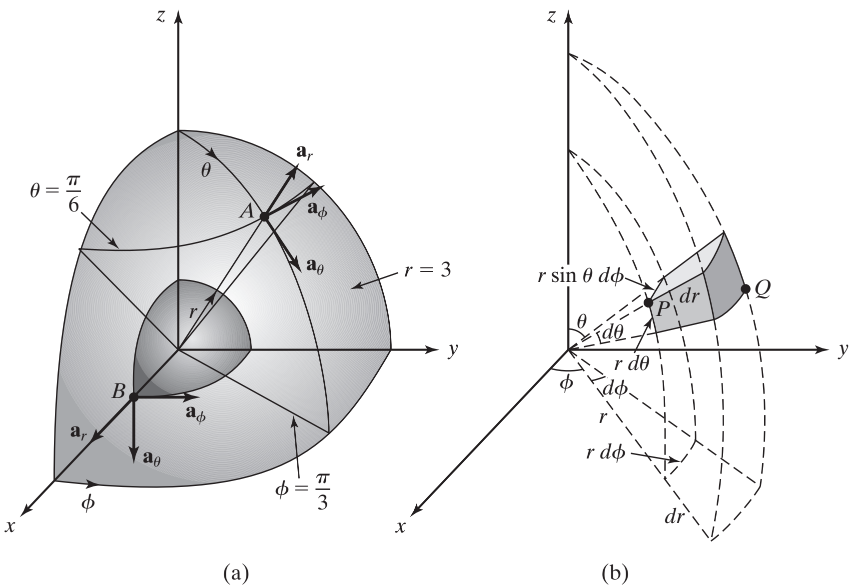

FIGURE A.2

Spherical coordinate system. (a) Orthogonal surfaces and unit vectors. (b) Differential volume formed by incrementing the coordinates.

The origin is given by $r = 0, \theta = 0$, and $\phi = 0$. Any other point in space is given by the intersection of three mutually orthogonal surfaces obtained by incrementing the coordinates by appropriate amounts. For example, the intersection of the three surfaces $r = 3, \theta = \pi/6$, and $\phi = \pi/3$ defines the point $A(3, \pi/6, \pi/3)$ as shown in Figure A.2(a). These three orthogonal surfaces define three curves that are mutually perpendicular. One of these is a straight line and the other two are circles. We draw unit vectors $\mathbf{a}_r$, $\mathbf{a}_\theta$, and $\mathbf{a}_\phi$ tangential to these curves at point A and directed toward increasing values of r, θ, and ϕ, respectively. These three unit vectors form a set of mutually orthogonal unit vectors in terms of which vectors drawn at A can be described. In a similar manner, we can draw unit vectors at any other point in the spherical coordinate system, as shown, for example, for point $B(1, \pi/2, 0)$ in Figure A.2(a). It can now be seen that these unit vectors at point B are not parallel to the corresponding unit vectors at point A. Thus, in the spherical coordinate system, all three unit vectors $\mathbf{a}_r$, $\mathbf{a}_\theta$, and $\mathbf{a}_\phi$ do not have the same directions everywhere, that is, they are not uniform. Finally, we note that for the choice of θ as in Figure A.2(a), that is, increasing from the positive z-axis toward the xy-plane, the coordinate system is right-handed, that is, $\mathbf{a}_r \times \mathbf{a}_\theta = \mathbf{a}_\phi$.

To obtain expressions for the differential lengths, surfaces, and volume in the spherical coordinate system, we now consider two points $P(r, \theta, \phi)$ and $Q(r + dr, \theta + d\theta, \phi + d\phi)$, where Q is obtained by incrementing infinitesimally each coordinate from its value at P, as shown in Figure A.2(b). The three orthogonal surfaces intersecting at P and the three orthogonal surfaces intersecting at Q define a box that can be considered to be rectangular, since dr, $d\theta$, and $d\phi$ are infinitesimally small. The three differential length elements forming the contiguous sides of this box are $dr\,\mathbf{a}_r$, $r\,d\theta\,\mathbf{a}_\theta$, and $r \sin\theta\,d\phi\,\mathbf{a}_\phi$. The differential length vector $d\mathbf{l}$ from P to Q is thus given by

$$d\mathbf{l} = dr\,\mathbf{a}_r + r\,d\theta\,\mathbf{a}_\theta + r\sin\theta\,d\phi\,\mathbf{a}_\phi \tag{A.4}$$

The differential surfaces formed by pairs of the differential length elements are

$$\pm dS\,\mathbf{a}_\phi = \pm(dr)(r\,d\theta)\mathbf{a}_\phi = \pm dr\,\mathbf{a}_r \times r\,d\theta\,\mathbf{a}_\theta \tag{A.5a}$$

$$\pm dS\,\mathbf{a}_r = \pm(r\,d\theta)(r\sin\theta\,d\phi)\mathbf{a}_r = \pm r\,d\theta\,\mathbf{a}_\theta \times r\sin\theta\,d\phi\,\mathbf{a}_\phi \tag{A.5b}$$

$$\pm dS\,\mathbf{a}_\theta = \pm(r\sin\theta\,d\phi)(dr)\mathbf{a}_\theta = \pm r\sin\theta\,d\phi\,\mathbf{a}_\phi \times dr\,\mathbf{a}_r \tag{A.5c}$$

Finally, the differential volume dv formed by the three differential lengths is simply the volume of the box, that is,

$$dv = (dr)(r\,d\theta)(r\sin\theta\,d\phi) = r^2\sin\theta\,dr\,d\theta\,d\phi \tag{A.6}$$

In the study of electromagnetics it is sometimes useful to be able to convert the coordinates of a point and vectors drawn at a point from one coordinate system to another, particularly from the Cartesian system to the cylindrical system and vice versa, and from the Cartesian system to the spherical system and vice versa. To derive first the relationships for the conversion of the coordinates, let us consider Figure A.3(a), which illustrates the geometry pertinent to the coordinates of a point P in the three different coordinate systems. Thus, from simple geometrical considerations, we have

$$x = r_c\cos\phi \qquad y = r_c\sin\phi \qquad z = z \tag{A.7}$$

$$x = r_s\sin\theta\cos\phi \qquad y = r_s\sin\theta\sin\phi \qquad z = r_s\cos\theta \tag{A.8}$$

Conversely, we have

$$r_c = \sqrt{x^2 + y^2} \qquad \phi = \tan^{-1}\frac{y}{x} \qquad z = z \tag{A.9}$$

$$r_s = \sqrt{x^2 + y^2 + z^2} \qquad \theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z} \qquad \phi = \tan^{-1}\frac{y}{x} \tag{A.10}$$

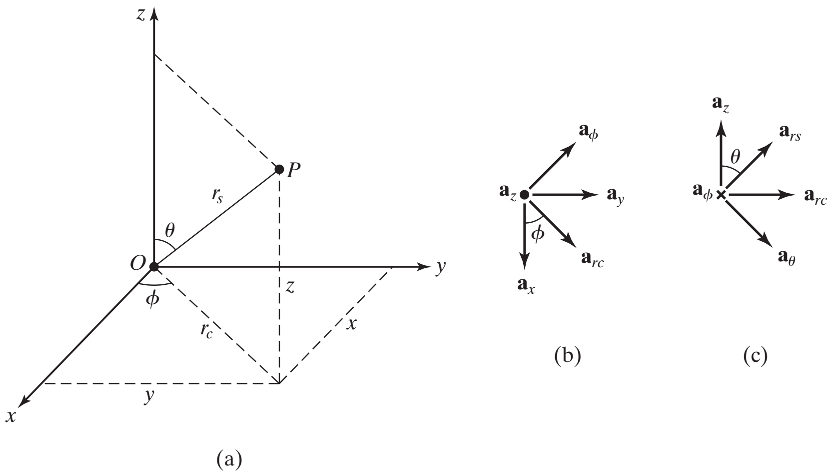

FIGURE A.3

(a) For conversion of coordinates of a point from one coordinate system to another.
(b) and (c) For expressing unit vectors in cylindrical and spherical coordinate systems, respectively, in terms of unit vectors in the Cartesian coordinate system.

Relationships (A.7) and (A.9) correspond to conversion from cylindrical coordinates to Cartesian coordinates and vice versa. Relationships (A.8) and (A.10) correspond to conversion from spherical coordinates to Cartesian coordinates and vice versa.

Considering next the conversion of vectors from one coordinate system to another, we note that in order to do this, we need to express each of the unit vectors of the first coordinate system in terms of its components along the unit vectors in the second coordinate system. From the definition of the dot product of two vectors, the component of a unit vector along another unit vector, that is, the cosine of the angle between the unit vectors, is simply the dot product of the two unit vectors. Thus, considering the sets of unit vectors in the cylindrical and Cartesian coordinate systems, we have with the aid of Figure A.3(b),

$$\mathbf{a}_{rc}\cdot\mathbf{a}_x = \cos\phi \qquad \mathbf{a}_{rc}\cdot\mathbf{a}_y = \sin\phi \qquad \mathbf{a}_{rc}\cdot\mathbf{a}_z = 0 \tag{A.11a}$$

$$\mathbf{a}_{\phi}\cdot\mathbf{a}_x = -\sin\phi \qquad \mathbf{a}_{\phi}\cdot\mathbf{a}_y = \cos\phi \qquad \mathbf{a}_{\phi}\cdot\mathbf{a}_z = 0 \tag{A.11b}$$

$$\mathbf{a}_{z}\cdot\mathbf{a}_x = 0 \qquad \mathbf{a}_{z}\cdot\mathbf{a}_y = 0 \qquad \mathbf{a}_{z}\cdot\mathbf{a}_z = 1 \tag{A.11c}$$

Similarly, for the sets of unit vectors in the spherical and Cartesian coordinate systems, we obtain with the aid of Figure A.3(c) and Figure A.3(b),

$$\mathbf{a}_{rs}\cdot\mathbf{a}_x = \sin\theta\cos\phi \qquad \mathbf{a}_{rs}\cdot\mathbf{a}_y = \sin\theta\sin\phi \qquad \mathbf{a}_{rs}\cdot\mathbf{a}_z = \cos\theta \tag{A.12a}$$

$$\mathbf{a}_{\theta}\cdot\mathbf{a}_x = \cos\theta\cos\phi \qquad \mathbf{a}_{\theta}\cdot\mathbf{a}_y = \cos\theta\sin\phi \qquad \mathbf{a}_{\theta}\cdot\mathbf{a}_z = -\sin\theta \tag{A.12b}$$

$$\mathbf{a}_{\phi}\cdot\mathbf{a}_x = -\sin\phi \qquad \mathbf{a}_{\phi}\cdot\mathbf{a}_y = \cos\phi \qquad \mathbf{a}_{\phi}\cdot\mathbf{a}_z = 0 \tag{A.12c}$$

We shall now illustrate the use of these relationships by means of an example.

Example A.1

Let us consider the vector $3\mathbf{a}_x + 4\mathbf{a}_y + 5\mathbf{a}_z$ at the point (3, 4, 5) and convert the vector to one in spherical coordinates.

First, from the relationships (A.10), we obtain the spherical coordinates of the point (3, 4, 5) to be

$$r_s = \sqrt{3^2 + 4^2 + 5^2} = 5\sqrt{2}$$
$$\theta = \tan^{-1}\frac{\sqrt{3^2 + 4^2}}{5} = \tan^{-1} 1 = 45^\circ$$
$$\phi = \tan^{-1}\frac{4}{3} = 53.13^\circ$$

Then noting from the relationships (A.12) that at the point under consideration,

$$\begin{aligned}
\mathbf{a}_x &= \sin\theta\cos\phi\,\mathbf{a}_{rs} + \cos\theta\cos\phi\,\mathbf{a}_\theta - \sin\phi\,\mathbf{a}_\phi \\
&= 0.3\sqrt{2}\mathbf{a}_{rs} + 0.3\sqrt{2}\mathbf{a}_\theta - 0.8\mathbf{a}_\phi \\
\mathbf{a}_y &= \sin\theta\sin\phi\,\mathbf{a}_{rs} + \cos\theta\sin\phi\,\mathbf{a}_\theta + \cos\phi\,\mathbf{a}_\phi \\
&= 0.4\sqrt{2}\mathbf{a}_{rs} + 0.4\sqrt{2}\mathbf{a}_\theta + 0.6\mathbf{a}_\phi \\
\mathbf{a}_z &= \cos\theta\,\mathbf{a}_{rs} - \sin\theta\,\mathbf{a}_\theta = 0.5\sqrt{2}\mathbf{a}_{rs} - 0.5\sqrt{2}\mathbf{a}_\theta
\end{aligned}$$

we obtain

$$\begin{aligned}
3\mathbf{a}_x + 4\mathbf{a}_y + 5\mathbf{a}_z &= (0.9\sqrt{2} + 1.6\sqrt{2} + 2.5\sqrt{2})\mathbf{a}_{rs} \\
&\quad + (0.9\sqrt{2} + 1.6\sqrt{2} - 2.5\sqrt{2})\mathbf{a}_\theta + (-2.4 + 2.4)\mathbf{a}_\phi = 5\sqrt{2}\mathbf{a}_{rs}
\end{aligned}$$

This result is to be expected since the given vector has components equal to the coordinates of the point at which it is specified. Hence, its magnitude is equal to the distance of the point from the origin, that is, the spherical coordinate r of the point, and its direction is along the line drawn from the origin to the point, that is, along the unit vector $\mathbf{a}_{rs}$ at that point. In fact, the given vector is a particular case of the vector $x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z = r_s\mathbf{a}_{rs}$, known as the *position vector*, since it is the same as the vector drawn from the origin to the point (x, y, z).

REVIEW QUESTIONS

A.1. What are the three orthogonal surfaces defining the cylindrical coordinate system?

A.2. What are the limits of variation of the cylindrical coordinates?

A.3. Which of the unit vectors in the cylindrical coordinate system are not uniform?

A.4. State whether the vector $3\mathbf{a}_r + 4\mathbf{a}_\phi + 5\mathbf{a}_z$ at the point (1, 0, 2) and the vector $3\mathbf{a}_r + 4\mathbf{a}_\phi + 5\mathbf{a}_z$ at the point $(2, \pi/2, 3)$ are equal or not.

A.5. What are the differential length vectors in cylindrical coordinates?

A.6. What are the three orthogonal surfaces defining the spherical coordinate system?

A.7. What are the limits of variation of the spherical coordinates?

A.8. Which of the unit vectors in the spherical coordinate system are not uniform?

A.9. State if the vector $3\mathbf{a}_r + 4\mathbf{a}_\theta$ at the point $(1, \pi/2, 0)$ and the vector $3\mathbf{a}_r + 4\mathbf{a}_\theta$ at the point $(2, 0, \pi/2)$ are equal or not.

A.10. What are the differential length vectors in spherical coordinates?

A.11. Outline the procedure for converting a vector at a point from one coordinate system to another.

A.12. What is the expression for the position vector in the cylindrical coordinate system?

PROBLEMS

A.1. Express in terms of Cartesian coordinates the vector drawn from the point $P(2, \pi/3, 1)$ to the point $Q(4, 2\pi/3, 2)$ in cylindrical coordinates.

A.2. Express in terms of Cartesian coordinates the vector drawn from the point $P(1, \pi/3, \pi/4)$ to the point $Q(2, 2\pi/3, 3\pi/4)$ in spherical coordinates.

A.3. Determine if the vector $\mathbf{a}_r + \mathbf{a}_\phi + 2\mathbf{a}_z$ at the point $(1, \pi/4, 2)$ and the vector $\sqrt{2}\mathbf{a}_r + 2\mathbf{a}_z$ at the point $(2, \pi/2, 3)$ are equal or not.

A.4. Determine if the vector $3\mathbf{a}_r + \sqrt{3}\mathbf{a}_\theta - 2\mathbf{a}_\phi$ at the point $(2, \pi/3, \pi/6)$ and the vector $\mathbf{a}_r + \sqrt{3}\mathbf{a}_\theta - 2\sqrt{3}\mathbf{a}_\phi$ at the point $(1, \pi/6, \pi/3)$ are equal or not.

A.5. Find the dot and cross products of the unit vector $\mathbf{a}_r$ at the point $(1, 0, 0)$ and the unit vector $\mathbf{a}_\phi$ at the point $(2, \pi/4, 1)$ in cylindrical coordinates.

A.6. Find the dot and cross products of the unit vector $\mathbf{a}_r$ at the point $(1, \pi/4, 0)$ and the unit vector $\mathbf{a}_\theta$ at the point $(2, \pi/2, \pi/2)$ in spherical coordinates.

A.7. Convert the vector $5\mathbf{a}_x + 12\mathbf{a}_y + 6\mathbf{a}_z$ at the point $(5, 12, 4)$ to one in cylindrical coordinates.

A.8. Convert the vector $3\mathbf{a}_x + 4\mathbf{a}_y - 5\mathbf{a}_z$ at the point $(3, 4, 5)$ to one in spherical coordinates.