CHAPTER

# 7 Transmission-Line Analysis

In the previous chapter, we introduced the transmission line and the transmission-line equations. The transmission-line equations enable us to discuss the wave propagation phenomena along an arrangement of two parallel conductors having uniform cross section in terms of circuit quantities instead of in terms of field quantities. This chapter is devoted to the analysis of lossless transmission-line systems first in frequency domain, that is, for sinusoidal steady state, and then in time domain, that is, for arbitrary variation with time.

In the frequency domain, we shall study the *standing wave* phenomenon by considering the short-circuited line. From the frequency dependence of the input impedance of the short-circuited line, we shall learn that the condition for the quasistatic approximation for the input behavior of physical structures is that the physical length of the structure must be a small fraction of the wavelength. We shall study reflection and transmission at the junction between two lines in cascade and introduce the Smith® Chart, a useful graphical aid in the solution of transmission-line problems.

In the time domain, we shall begin with a line terminated by a resistive load and learn the *bounce diagram* technique of studying the transient bouncing of waves back and forth on the line for a constant voltage source as well as for a pulse voltage source. We shall apply the bounce diagram technique for an initially charged line. Finally, we shall introduce the *load-line* technique of analysis of a line terminated by a nonlinear element and apply it for the analysis of interconnections between logic gates.

## A. FREQUENCY DOMAIN

In Chapter 6, we introduced transmission lines, and learned that the voltage and current on a line are governed by the transmission-line equations

$$\frac{\partial V}{\partial z} = -\mathscr{L}\frac{\partial I}{\partial t} \tag{7.1a}$$

Smith® Chart is a registered trademark of Analog Instrument Co., P.O. Box 950, New Providence, NJ 07974, USA.

$$\frac{\partial I}{\partial z} = -\mathscr{G}V - \mathscr{C}\frac{\partial V}{\partial t} \tag{7.1b}$$

For the sinusoidally time-varying case, the corresponding differential equations for the phasor voltage $\bar{V}$ and phasor current $\bar{I}$ are given by

$$\frac{\partial \bar{V}}{\partial z} = -j\omega\mathscr{L}\bar{I} \tag{7.2a}$$

$$\frac{\partial \bar{I}}{\partial z} = -\mathscr{G}\bar{V} - j\omega\mathscr{C}\bar{V} = -(\mathscr{G} + j\omega\mathscr{C})\bar{V} \tag{7.2b}$$

Combining (7.2a) and (7.2b) by eliminating $\bar{I}$, we obtain the wave equation for $\bar{V}$ as

$$\begin{aligned}\frac{\partial^2 \bar{V}}{\partial z^2} &= -j\omega\mathscr{L}\frac{\partial \bar{I}}{\partial z} = j\omega\mathscr{L}(\mathscr{G} + j\omega\mathscr{C})\bar{V} \\ &= \bar{\gamma}^2\bar{V} \end{aligned} \tag{7.3}$$

where

$$\bar{\gamma} = \sqrt{j\omega\mathscr{L}(\mathscr{G} + j\omega\mathscr{C})} \tag{7.4}$$

is the propagation constant associated with the wave propagation on the line. The solution for $\bar{V}$ is given by

$$\bar{V}(z) = \bar{A}e^{-\bar{\gamma}z} + \bar{B}e^{\bar{\gamma}z} \tag{7.5}$$

where $\bar{A}$ and $\bar{B}$ are arbitrary constants to be determined from the boundary conditions. The corresponding solution for $\bar{I}$ is then given by

$$\begin{aligned}\bar{I}(z) &= -\frac{1}{j\omega\mathscr{L}}\frac{\partial \bar{V}}{\partial z} = -\frac{1}{j\omega\mathscr{L}}(-\bar{\gamma}\bar{A}e^{-\bar{\gamma}z} + \bar{\gamma}\bar{B}e^{\bar{\gamma}z}) \\ &= \sqrt{\frac{\mathscr{G} + j\omega\mathscr{C}}{j\omega\mathscr{L}}}(\bar{A}e^{-\bar{\gamma}z} - \bar{B}e^{\bar{\gamma}z}) \\ &= \frac{1}{\bar{Z}_0}(\bar{A}e^{-\bar{\gamma}z} - \bar{B}e^{\bar{\gamma}z}) \end{aligned} \tag{7.6}$$

where

$$\bar{Z}_0 = \sqrt{\frac{j\omega\mathscr{L}}{\mathscr{G} + j\omega\mathscr{C}}} \tag{7.7}$$

is known as the *characteristic impedance* of the transmission line.

The solutions for the line voltage and line current given by (7.5) and (7.6), respectively, represent the superposition of (+) and (−) waves, that is, waves propagating in the positive $z$- and negative $z$-directions, respectively. They are completely analogous to the solutions for the electric and magnetic fields in the medium between the conductors of the line. In fact, the propagation constant given by (7.4) is the same as the propagation constant $\sqrt{j\omega\mu(\sigma + j\omega\epsilon)}$, as it should be. The characteristic impedance of the line is analogous to (but not equal to) the intrinsic impedance of the material medium between the conductors of the line.

For a *lossless line*, that is, for a line consisting of a perfect dielectric medium between the conductors, $\mathcal{G} = 0$, and

$$\bar{\gamma} = \alpha + j\beta = \sqrt{j\omega\mathcal{L} \cdot j\omega\mathcal{C}} = j\omega\sqrt{\mathcal{L}\mathcal{C}} \tag{7.8}$$

Thus, the attenuation constant $\alpha$ is equal to zero, which is to be expected, and the phase constant $\beta$ is equal to $\omega\sqrt{\mathcal{L}\mathcal{C}}$. We can then write the solutions for $\bar{V}$ and $\bar{I}$ as

$$\bar{V}(z) = \bar{A}e^{-j\beta z} + \bar{B}e^{j\beta z} \tag{7.9a}$$

$$\bar{I}(z) = \frac{1}{Z_0}(\bar{A}e^{-j\beta z} - \bar{B}e^{j\beta z}) \tag{7.9b}$$

where

$$Z_0 = \sqrt{\frac{\mathcal{L}}{\mathcal{C}}} \tag{7.10}$$

is purely real and independent of frequency. Note also that

$$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mathcal{L}\mathcal{C}}} = \frac{1}{\sqrt{\mu\epsilon}} \tag{7.11}$$

as it should be, and independent of frequency.

Thus, provided that $\mathcal{L}$ and $\mathcal{C}$ are independent of frequency, which is the case if $\mu$ and $\epsilon$ are independent of frequency and the transmission line is uniform, that is, its dimensions remain constant transverse to the direction of propagation of the waves, the lossless line is characterized by no dispersion, a phenomenon discussed in Section 8.3. We shall be concerned with such lines only in this book.

## 7.1 SHORT-CIRCUITED LINE AND FREQUENCY BEHAVIOR

Let us now consider a lossless line short-circuited at the far end $z = 0$, as shown in Figure 7.1(a), in which the double-ruled lines represent the conductors of the transmission line. Note that the line is characterized by $Z_0$ and $\beta$, which is equivalent to specifying $\mathcal{L}$, $\mathcal{C}$, and $\omega$. In actuality, the arrangement may consist, for example, of a perfectly conducting rectangular sheet joining the two conductors of a parallel-plate line as in Figure 7.1(b) or a perfectly conducting ring-shaped sheet joining the two conductors of a coaxial cable as in Figure 7.1(c). We shall assume that the line is driven by a voltage generator of frequency $\omega$ at the left end $z = -l$ so that waves are set up on the line. The short circuit at $z = 0$ requires that the tangential electric field on the surface of the conductor comprising the short circuit be zero. Since the voltage between the conductors of the line is proportional to this electric field, which is transverse to them, it follows that the voltage across the short circuit has to be zero. Thus, we have

$$\bar{V}(0) = 0 \tag{7.12}$$

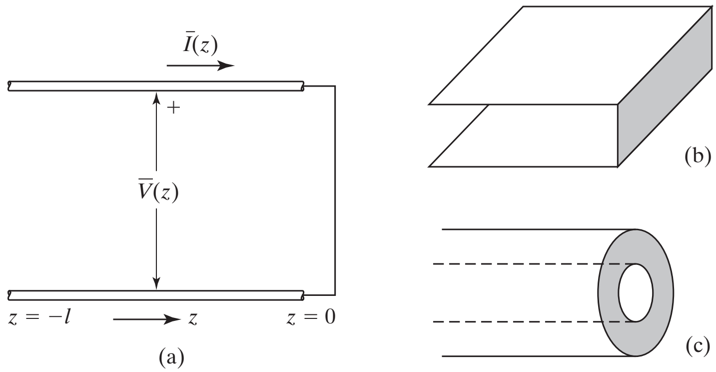



FIGURE 7.1

Transmission line short-circuited at the far end.

Applying the boundary condition given by (7.12) to the general solution for $\bar{V}$ given by (7.9a), we have

$$\bar{V}(0) = \bar{A}e^{-j\beta(0)} + \bar{B}e^{j\beta(0)} = 0$$

or

$$\bar{B} = -\bar{A} \tag{7.13}$$

Thus, we find that the short circuit gives rise to a $(-)$ or reflected wave whose voltage is exactly the negative of the $(+)$ or incident wave voltage, at the short circuit. Substituting this result in (7.9a) and (7.9b), we get the particular solutions for the complex voltage and current on the short-circuited line to be

$$\bar{V}(z) = \bar{A}e^{-j\beta z} - \bar{A}e^{j\beta z} = -2j\bar{A}\sin\beta z \tag{7.14a}$$

$$\bar{I}(z) = \frac{1}{Z_0}(\bar{A}e^{-j\beta z} + \bar{A}e^{j\beta z}) = \frac{2\bar{A}}{Z_0}\cos\beta z \tag{7.14b}$$

The real voltage and current are then given by

$$\begin{aligned} V(z,t) &= \mathrm{Re}[\bar{V}(z)e^{j\omega t}] = \mathrm{Re}(2e^{-j\pi/2}\,Ae^{j\theta}\sin\beta z\, e^{j\omega t}) \\ &= 2A\sin\beta z\sin(\omega t + \theta) \end{aligned} \tag{7.15a}$$

$$\begin{aligned} I(z,t) &= \mathrm{Re}[\bar{I}(z)e^{j\omega t}] = \mathrm{Re}\left[\frac{2}{Z_0}Ae^{j\theta}\cos\beta z\, e^{j\omega t}\right] \\ &= \frac{2A}{Z_0}\cos\beta z\cos(\omega t + \theta) \end{aligned} \tag{7.15b}$$

where we have replaced $\bar{A}$ by $Ae^{j\theta}$ and $-j$ by $e^{-j\pi/2}$. The instantaneous power flow down the line is given by

$$\begin{aligned} P(z, t) &= V(z, t)I(z, t) \\ &= \frac{4A^2}{Z_0} \sin \beta z \cos \beta z \sin (\omega t + \theta) \cos (\omega t + \theta) \\ &= \frac{A^2}{Z_0} \sin 2\beta z \sin 2(\omega t + \theta) \end{aligned} \tag{7.15c}$$

These results for the voltage, current, and power flow on the short-circuited line given by (7.15a), (7.15b), and (7.15c), respectively, are illustrated in Figure 7.2, which shows the variation of each of these quantities with distance from the short circuit for several values of time. The numbers 1, 2, 3, . . . , 9 beside the curves in Figure 7.2 represent the order of the curves corresponding to values of $(\omega t + \theta)$ equal to 0, $\pi/4$, $\pi/2, \ldots, 2\pi$. It can be seen that the phenomenon is one in which the voltage, current, and power flow oscillate sinusoidally with time with different amplitudes at different locations on the line, unlike in the case of traveling waves, in which a given point on the waveform progresses in distance with time. These waves are therefore known as *standing waves*. In particular, they represent *complete standing waves*, in view of the zero amplitudes of the voltage, current, and power flow at certain locations on the line, as shown by Figure 7.2.

The line voltage amplitude is zero for values of $z$ given by $\sin \beta z = 0$ or $\beta z = -m\pi$, $m = 1, 2, 3, \ldots$, or $z = -m\lambda/2$, $m = 1, 2, 3, \ldots$, that is, at multiples of $\lambda/2$ from the short circuit. The line current amplitude is zero for values of $z$ given by $\cos \beta z = 0$ or $\beta z = -(2m + 1)\pi/2$, $m = 0, 1, 2, 3, \ldots$, or $z = -(2m + 1)\lambda/4$, $m = 0, 1, 2, 3, \ldots$, that is, at odd multiples of $\lambda/4$ from the short circuit. The power flow amplitude is zero for values of $z$ given by $\sin 2\beta z = 0$ or $\beta z = -m\pi/2, m = 1, 2, 3, \ldots$, or $z = -m\lambda/4$, $m = 1, 2, 3, \ldots$, that is, at multiples of $\lambda/4$ from the short circuit. Proceeding further, we find that the time-average power flow down the line, that is, power flow averaged over one period of the source voltage, is

$$\begin{aligned} \langle P \rangle &= \frac{1}{T}\int_{t=0}^{T} P(z, t)\, dt = \frac{\omega}{2\pi}\int_{t=0}^{2\pi/\omega} P(z, t)\, dt \\ &= \frac{\omega}{2\pi}\frac{A^2}{Z_0} \sin 2\beta z \int_{t=0}^{2\pi/\omega} \sin 2(\omega t + \theta)\, dt = 0 \end{aligned}$$

Thus, the time average power flow down the line is zero at all points on the line. This is characteristic of complete standing waves.

From (7.14a) and (7.14b) or (7.15a) and (7.15b), or from Figures 7.2(a) and 7.2(b), we find that the amplitudes of the sinusoidal time-variations of the line voltage and line current as functions of distance along the line are

$$|\bar{V}(z)| = 2A|\sin \beta z| = 2A\left|\sin \frac{2\pi}{\lambda} z\right| \tag{7.16a}$$

$$|\bar{I}(z)| = \frac{2A}{Z_0}|\cos \beta z| = \frac{2A}{Z_0}\left|\cos \frac{2\pi}{\lambda} z\right| \tag{7.16b}$$

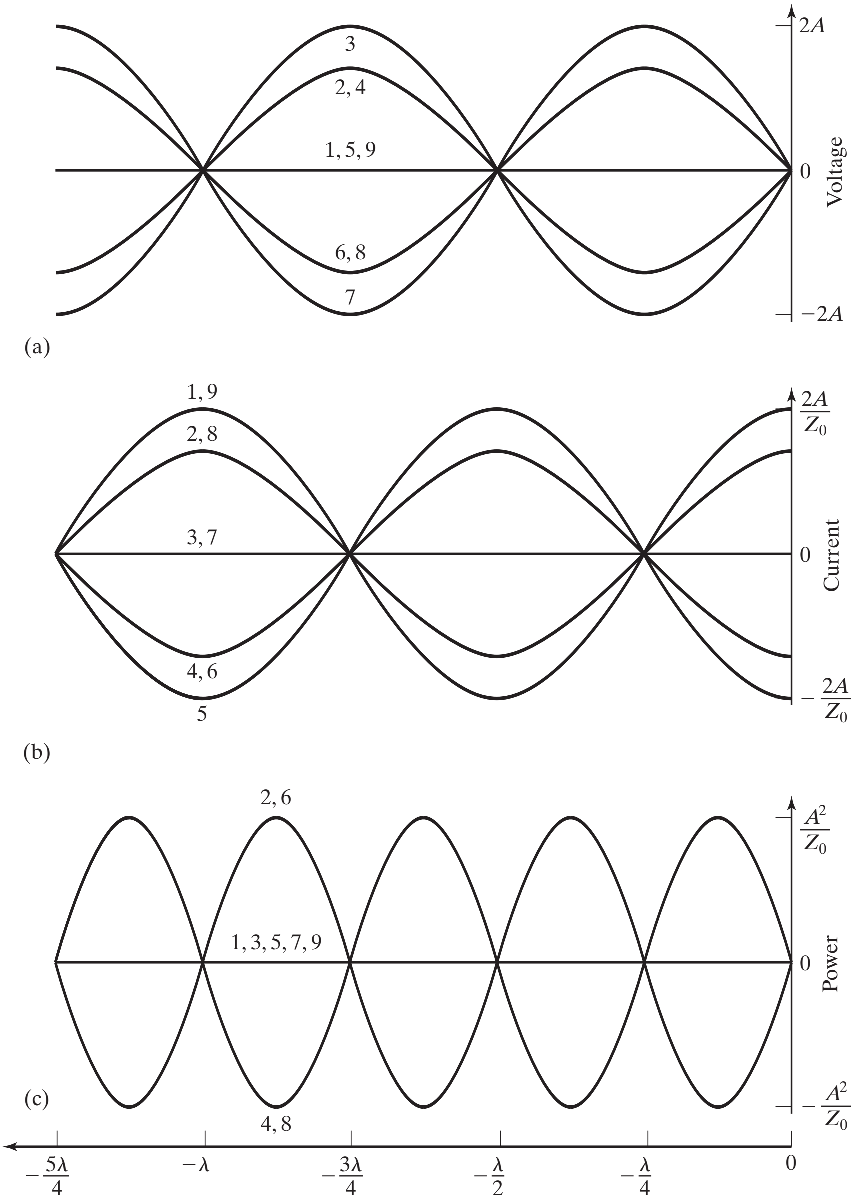



FIGURE 7.2

Time variations of voltage, current, and power flow associated with standing waves on a short-circuited transmission line.

Sketches of these quantities versus $z$ are shown in Figure 7.3. These are known as the *standing wave patterns*. They are the patterns of line voltage and line current one would obtain by connecting an a.c. voltmeter between the conductors of the line and an a.c. ammeter in series with one of the conductors of the line and observing their readings at various points along the line. Alternatively, one can sample the electric and magnetic fields by means of probes.

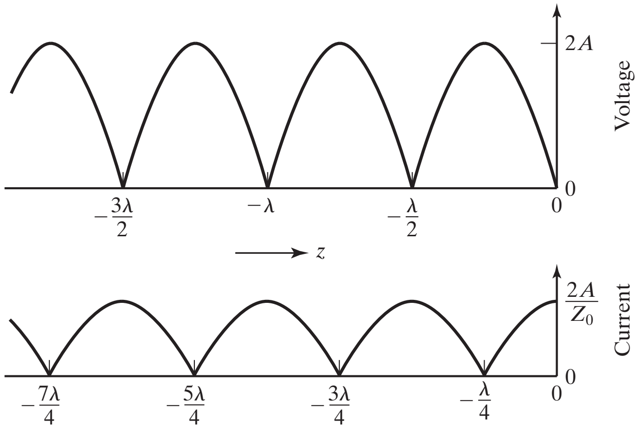



FIGURE 7.3

Standing wave patterns for voltage and current on a short-circuited line.

Returning now to the solutions for $\bar{V}(z)$ and $\bar{I}(z)$ given by (7.14a) and (7.14b), respectively, we can find the input impedance of the short-circuited line of length $l$ by taking the ratio of the complex line voltage to the complex line current at the input $z = -l$. Thus,

$$\begin{aligned}\bar{Z}_{\text{in}} &= \frac{\bar{V}(-l)}{\bar{I}(-l)} = \frac{-2j\bar{A}\sin\beta(-l)}{\dfrac{2\bar{A}}{Z_0}\cos\beta(-l)} \\ &= jZ_0\tan\beta l = jZ_0\tan\frac{2\pi}{\lambda}l \\ &= jZ_0\tan\frac{2\pi f}{v_p}l \end{aligned} \qquad (7.17)$$

We note from (7.17) that the input impedance of the short-circuited line is purely reactive. As the frequency is varied from a low value upward, the input reactance changes from inductive to capacitive and back to inductive, and so on, as illustrated in Figure 7.4. The input reactance is zero for values of frequency equal to multiples of $v_p/2l$. These are the frequencies for which $l$ is equal to multiples of $\lambda/2$ so that the line voltage is zero at the input and hence the input sees a short circuit. The input reactance is infinity for values of frequency equal to odd multiples of $v_p/4l$. These are the frequencies for which $l$ is equal to odd multiples of $\lambda/4$ so that the line current is zero at the input and hence the input sees an open circuit.

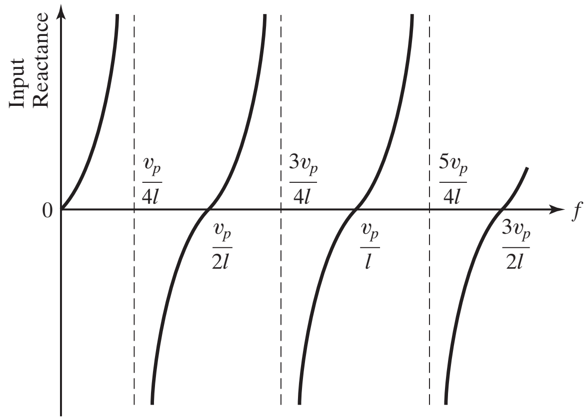



FIGURE 7.4

Variation of the input reactance of a short-circuited transmission line with frequency.

## Example 7.1

From the foregoing discussion of the input reactance of the short-circuited line, we note that as the frequency of the generator is varied continuously upward, the current drawn from it undergoes alternatively maxima and minima corresponding to zero input reactance and infinite input reactance conditions, respectively. This behavior can be utilized for determining the location of a short circuit in the line.

Since the difference between a pair of consecutive frequencies for which the input reactance values are zero and infinity is $v_p/4l$, as can be seen from Figure 7.4, it follows that the difference between successive frequencies for which the currents drawn from the generator are maxima and minima is $v_p/4l$. As a numerical example, if for an air dielectric line it is found that as the frequency is varied from 50 MHz upward, the current reaches a minimum for 50.01 MHz and then a maximum for 50.04 MHz, then the distance $l$ of the short circuit from the generator is given by

$$\frac{v_p}{4l} = (50.04 - 50.01) \times 10^6 = 0.03 \times 10^6 = 3 \times 10^4$$

Since $v_p = 3 \times 10^8$ m/s, it follows that

$$l = \frac{3 \times 10^8}{4 \times 3 \times 10^4} = 2500 \text{ m} = 2.5 \text{ km}$$

## Example 7.2

We found that the input impedance of a short-circuited line of length $l$ is given by

$$\bar{Z}_{\text{in}} = jZ_0 \tan \beta l$$

Let us investigate the low-frequency behavior of this input impedance.

First, we note that for any arbitrary value of $\beta l$,

$$\tan \beta l = \beta l + \frac{1}{3}(\beta l)^3 + \frac{2}{15}(\beta l)^5 + \cdots$$

For $\beta l \ll 1$, that is, $\dfrac{2\pi}{\lambda} l \ll 1$ or $l \ll \dfrac{\lambda}{2\pi}$ or $f \ll \dfrac{v_p}{2\pi l}$,

$$\tan \beta l \approx \beta l$$

$$\bar{Z}_{\text{in}} \approx jZ_0 \beta l = j\sqrt{\frac{\mathscr{L}}{\mathscr{C}}}\,\omega\sqrt{\mathscr{L}\mathscr{C}}\,l = j\omega\mathscr{L}l$$

Thus, for frequencies $f \ll v_p/2\pi l$, the short-circuited line as seen from its input behaves essentially like a single inductor of value $\mathscr{L}l$, the total inductance of the line, as shown in Figure 7.5(a).

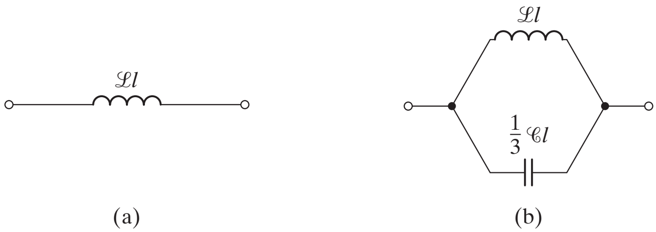



FIGURE 7.5

Equivalent circuits for the input behavior of a short-circuited transmission line.

Proceeding further, we observe that if the frequency is slightly beyond the range for which the above approximation is valid, then

$$\tan \beta l \approx \beta l + \frac{1}{3}(\beta l)^3$$

$$\begin{aligned}
\bar{Z}_{\text{in}} &\approx jZ_0\left(\beta l + \frac{1}{3}\beta^3 l^3\right) \\
&= j\sqrt{\frac{\mathscr{L}}{\mathscr{C}}}\left(\omega\sqrt{\mathscr{L}\mathscr{C}}\,l + \frac{1}{3}\,\omega^3\mathscr{L}^{3/2}\mathscr{C}^{3/2}l^3\right) \\
&= j\omega\mathscr{L}l\left(1 + \frac{1}{3}\,\omega^2\mathscr{L}\mathscr{C}l^2\right) \\
\bar{Y}_{\text{in}} &= \frac{1}{\bar{Z}_{\text{in}}} = \frac{1}{j\omega\mathscr{L}l}\left(1 + \frac{1}{3}\,\omega^2\mathscr{L}\mathscr{C}l^2\right)^{-1} \\
&\approx \frac{1}{j\omega\mathscr{L}l}\left(1 - \frac{1}{3}\,\omega^2\mathscr{L}\mathscr{C}l^2\right) \\
&= \frac{1}{j\omega\mathscr{L}l} + j\frac{1}{3}\omega\mathscr{C}l
\end{aligned}$$

Thus, for frequencies somewhat above those for which the approximation $f \ll v_p/2\pi l$ is valid, the short-circuited line as seen from its input behaves like an inductor of value $\mathscr{L}l$ in parallel with a capacitance of value $\dfrac{1}{3}\mathscr{C}l$, as shown in Figure 7.5(b).

These findings illustrate that a physical structure that can be considered as an inductor at low frequencies $f \ll v_p/2\pi l$ no longer behaves like an inductor, if the frequency is increased beyond that range. In fact, it has a *stray* capacitance associated with it. As the frequency is still increased, the equivalent circuit becomes further complicated. With reference to the question posed in Section 6.5 as to the limit on the frequency beyond which the quasistatic approximation for the input behavior of a physical structure is not valid, it can now be seen that the condition $\beta l \ll 1$ dictates the range of validity for the quasistatic approximation. In terms of the frequency $f$ of the source, this condition means that $f \ll v_p/2\pi l$, or in terms of the period $T = 1/f$, it means that $T \gg 2\pi(l/v_p)$. Thus, quasistatic fields are low-frequency approximations of time-varying fields that are complete solutions to Maxwell's equations, which represent wave propagation phenomena and can be approximated to the quasistatic character only when the times of interest are much greater than the propagation time, $l/v_p$, corresponding to the length of the structure. In terms of space variations of the fields at a fixed time, the wavelength $\lambda(= 2\pi/\beta)$, must be such that $l \ll \lambda/2\pi$; thus, the physical length of the structure must be a small fraction of the wavelength. In terms of the line voltage and current amplitudes, what this means is that over the length of the structure, these amplitudes are fractional portions of the first one-quarter sinusoidal variations at the $z = 0$ end in Figure 7.3, with the boundary conditions at the two ends of the structure always satisfied. Thus, because of the sin $\beta z$ dependence of $V$ on $z$, the line voltage amplitude varies linearly with $z$, whereas because of the cos $\beta z$ dependence of $I$ on $z$, the line current amplitude is essentially a constant. These are exactly the nature of the variations of the zero-order electric field and the first-order magnetic field, as discussed under magnetoquasistatic fields in Example 6.7.

## 7.2 TRANSMISSION-LINE DISCONTINUITY

Let us now consider the case of two transmission lines, 1 and 2, having different characteristic impedances $Z_{01}$ and $Z_{02}$, respectively, and phase constants $\beta_1$ and $\beta_2$, respectively, connected in cascade and driven by a generator at the left end of line 1, as shown in Figure 7.6(a). Physically, the arrangement may, for example, consist of two parallel-plate lines or two coaxial cables of different dielectrics in cascade, as shown in Figures 7.6(b) and 7.6(c), respectively. In view of the discontinuity at the junction $z = 0$

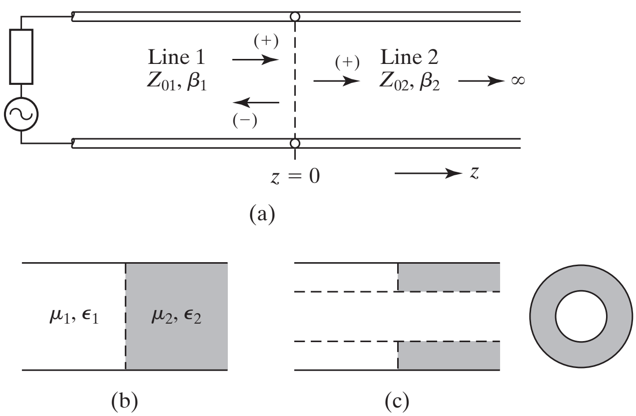



FIGURE 7.6

Two transmission lines connected in cascade.

between the two lines, the incident (+) wave on the junction sets up a reflected (−) wave in line 1 and a transmitted (+) wave in line 2. We shall assume that line 2 is infinitely long so that there is no (−) wave in that line.

We can now write the solutions for the complex voltage and complex current in line 1 as

$$\bar{V}_1(z) = \bar{V}_1^+ e^{-j\beta_1 z} + \bar{V}_1^- e^{j\beta_1 z} \tag{7.18a}$$

$$\begin{aligned} \bar{I}_1(z) &= \bar{I}_1^+ e^{-j\beta_1 z} + \bar{I}_1^- e^{j\beta_1 z} \\ &= \frac{1}{Z_{01}}(\bar{V}_1^+ e^{-j\beta_1 z} - \bar{V}_1^- e^{j\beta_1 z}) \end{aligned} \tag{7.18b}$$

where $\bar{V}_1^+, \bar{V}_1^-, \bar{I}_1^+$, and $\bar{I}_1^-$ are the (+) and (−) wave voltages and currents at $z = 0-$ in line 1, that is, just to the left of the junction. The solutions for the complex voltage and complex current in line 2 are

$$\bar{V}_2(z) = \bar{V}_2^+ e^{-j\beta_2 z} \tag{7.19a}$$

$$\bar{I}_2(z) = \bar{I}_2^+ e^{-j\beta_2 z} = \frac{1}{Z_{02}}\bar{V}_2^+ e^{-j\beta_2 z} \tag{7.19b}$$

where $\bar{V}_2^+$ and $\bar{I}_2^+$ are the (+) wave voltage and current at $z = 0+$ in line 2, that is, just to the right of the junction.

At the junction, the boundary conditions require that the components of **E** and **H** tangential to the dielectric interface be continuous, as shown, for example, for the parallel-plate arrangement in Figure 7.7(a). These are, in fact, the only components present, since the transmission line fields are entirely transverse to the direction of propagation. Now, since the line voltage and current are related to these electric and magnetic fields, respectively, it then follows that the line voltage and line current be

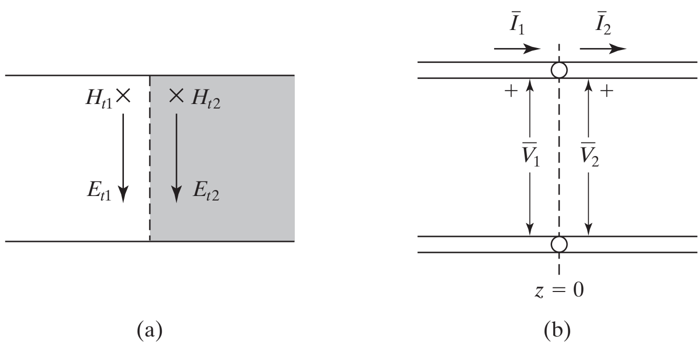



FIGURE 7.7

Application of boundary conditions at the junction between two transmission lines.

continuous at the junction, as shown in Figure 7.7(b). Thus, we obtain the boundary conditions at the junction in terms of line voltage and line current as

$$[\bar{V}_1]_{z=0-} = [\bar{V}_2]_{z=0+} \tag{7.20a}$$

$$[\bar{I}_1]_{z=0-} = [\bar{I}_2]_{z=0+} \tag{7.20b}$$

Applying these boundary conditions to the solutions given by (7.18a) and (7.18b), we obtain

$$\bar{V}_1^+ + \bar{V}_1^- = \bar{V}_2^+ \tag{7.21a}$$

$$\frac{1}{Z_{01}}(\bar{V}_1^+ - \bar{V}_1^-) = \frac{1}{Z_{02}}\bar{V}_2^+ \tag{7.21b}$$

Eliminating $\bar{V}_2^+$ from (7.21a) and (7.21b), we get

$$\bar{V}_1^+\left(\frac{1}{Z_{02}} - \frac{1}{Z_{01}}\right) + \bar{V}_1^-\left(\frac{1}{Z_{02}} + \frac{1}{Z_{01}}\right) = 0$$

or

$$\bar{V}_1^- = \bar{V}_1^+\frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}} \tag{7.22}$$

We now define the voltage reflection coefficient at the junction, $\Gamma_V$, as the ratio of the reflected wave voltage ($\bar{V}_1^-$) at the junction to the incident wave voltage ($\bar{V}_1^+$) at the junction. Thus,

$$\Gamma_V = \frac{\bar{V}_1^-}{\bar{V}_1^+} = \frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}} \tag{7.23}$$

The current reflection coefficient at the junction, $\Gamma_I$, which is the ratio of the reflected wave current ($\bar{I}_1^-$) at the junction to the incident wave current ($\bar{I}_1^+$) at the junction is then given by

$$\Gamma_I = \frac{\bar{I}_1^-}{\bar{I}_1^+} = \frac{-\bar{V}_1^-/Z_{01}}{\bar{V}_1^+/Z_{01}} = -\frac{\bar{V}_1^-}{\bar{V}_1^+} = -\Gamma_V \tag{7.24}$$

We also define the voltage transmission coefficient at the junction, $\tau_V$, as the ratio of the transmitted wave voltage ($\bar{V}_2^+$) at the junction to the incident wave voltage ($\bar{V}_1^+$) at the junction.Thus,

$$\tau_V = \frac{\bar{V}_2^+}{\bar{V}_1^+} = \frac{\bar{V}_1^+ + \bar{V}_1^-}{\bar{V}_1^+} = 1 + \frac{\bar{V}_1^-}{\bar{V}_1^+} = 1 + \Gamma_V \tag{7.25}$$

The current transmission coefficient at the junction, $\tau_I$, which is the ratio of the transmitted wave current ($\bar{I}_2^+$) at the junction to the incident wave current ($\bar{I}_1^+$) at the junction, is given by

$$\tau_I = \frac{\bar{I}_2^+}{\bar{I}_1^+} = \frac{\bar{I}_1^+ + \bar{I}_1^-}{\bar{I}_1^+} = 1 + \frac{\bar{I}_1^-}{\bar{I}_1^+} = 1 - \Gamma_V \tag{7.26}$$

We note that for $Z_{02} = Z_{01}, \Gamma_V = 0, \Gamma_I = 0, \tau_V = 1$, and $\tau_I = 1$. Thus, the incident wave is entirely transmitted, as we may expect since there is no discontinuity at the junction.

---

### Example 7.3

Let us consider the junction of two lines having characteristic impedances $Z_{01} = 50\ \Omega$ and $Z_{02} = 75\ \Omega$, as shown in Figure 7.8, and compute the various quantities.

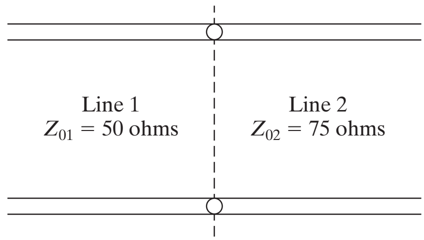



FIGURE 7.8

For the computation of several quantities pertinent to reflection and transmission at the junction between two transmission lines.

From (7.23)–(7.26), we have

$$\Gamma_V = \frac{75 - 50}{75 + 50} = \frac{25}{125} = \frac{1}{5}; \qquad \bar{V}_1^- = \frac{1}{5}\bar{V}_1^+$$

$$\Gamma_I = -\Gamma_V = -\frac{1}{5}; \qquad \bar{I}_1^- = -\frac{1}{5}\bar{I}_1^+$$

$$\tau_V = 1 + \Gamma_V = 1 + \frac{1}{5} = \frac{6}{5}; \qquad \bar{V}_2^+ = \frac{6}{5}\bar{V}_1^+$$

$$\tau_I = 1 - \Gamma_V = 1 - \frac{1}{5} = \frac{4}{5}; \qquad \bar{I}_2^+ = \frac{4}{5}\bar{I}_1^+$$

The fact that the transmitted wave voltage is greater than the incident wave voltage should not be of concern, since it is the power balance that must be satisfied at the junction. We can verify this by noting that if the incident power on the junction is $P_i$, then

$$\text{reflected power, } P_r = \Gamma_V \Gamma_I P_i = -\frac{1}{25}P_i$$

$$\text{transmitted power, } P_t = \tau_V \tau_I P_i = \frac{24}{25}P_i$$

Recognizing that the minus sign for $P_r$ signifies power flow in the negative $z$-direction, we find that power balance is indeed satisfied at the junction.

---

Returning now to the solutions for the voltage and current in line 1 given by (7.18a) and (7.18b), respectively, we obtain, by replacing $\bar{V}_1^-$ by $\Gamma_V \bar{V}_1^+$,

$$\begin{aligned} \bar{V}_1(z) &= \bar{V}_1^+ e^{-j\beta_1 z} + \Gamma_V \bar{V}_1^+ e^{j\beta_1 z} \\ &= \bar{V}_1^+ e^{-j\beta_1 z}(1 + \Gamma_V e^{j2\beta_1 z}) \end{aligned} \tag{7.27a}$$

$$\bar{I}_1(z) = \frac{1}{Z_{01}}(\bar{V}_1^+ e^{-j\beta_1 z} - \Gamma_V \bar{V}_1^+ e^{j\beta_1 z})$$

$$= \frac{\bar{V}_1^+}{Z_{01}} e^{-j\beta_1 z}(1 - \Gamma_V e^{j2\beta_1 z}) \tag{7.27b}$$

The amplitudes of the sinusoidal time-variations of the line voltage and line current as functions of distance along the line are then given by

$$\begin{aligned} |\bar{V}_1(z)| &= |\bar{V}_1^+||e^{-j\beta_1 z}||1 + \Gamma_V e^{j2\beta_1 z}| \\ &= |\bar{V}_1^+||1 + \Gamma_V \cos 2\beta_1 z + j\Gamma_V \sin 2\beta_1 z| \\ &= |\bar{V}_1^+|\sqrt{1 + \Gamma_V^2 + 2\Gamma_V \cos 2\beta_1 z} \end{aligned} \tag{7.28a}$$

$$\begin{aligned} |\bar{I}_1(z)| &= \frac{|\bar{V}_1^+|}{Z_{01}}|e^{-j\beta_1 z}||1 - \Gamma_V e^{j2\beta_1 z}| \\ &= \frac{|\bar{V}_1^+|}{Z_{01}}|1 - \Gamma_V \cos 2\beta_1 z - j\Gamma_V \sin 2\beta_1 z| \\ &= \frac{|\bar{V}_1^+|}{Z_{01}}\sqrt{1 + \Gamma_V^2 - 2\Gamma_V \cos 2\beta_1 z} \end{aligned} \tag{7.28b}$$

From (7.28a) and (7.28b), we note the following:

1. The line voltage amplitude undergoes alternate maxima and minima equal to $|\bar{V}_1^+|(1 + |\Gamma_V|)$ and $|\bar{V}_1^+|(1 - |\Gamma_V|)$, respectively. The line voltage amplitude at $z = 0$ is a maximum on minimum depending on whether $\Gamma_V$ is positive or negative. The distance between a voltage maximum and the adjacent voltage minimum is $\pi/2\beta_1$ or $\lambda_1/4$.
2. The line current amplitude undergoes alternate maxima and minima equal to $(\bar{V}_1^+/Z_{01})(1 + |\Gamma_V|)$ and $(\bar{V}_1^+/Z_{01})(1 - |\Gamma_V|)$, respectively. The line current amplitude at $z = 0$ is a minimum or maximum depending on whether $\Gamma_V$ is positive or negative. The distance between a current maximum and the adjacent current minimum is $\pi/2\beta_1$ or $\lambda_1/4$.

Knowing these properties of the line voltage and current amplitudes, we now sketch the voltage and current standing wave patterns, as shown in Figure 7.9, assuming $\Gamma_V > 0$. Since these standing wave patterns do not contain perfect nulls, as in the case of the short-circuited line of Section 7.1, these are said to correspond to *partial standing waves.*

We now define a quantity known as the *standing wave ratio* (SWR) as the ratio of the maximum voltage, $V_{max}$, to the minimum voltage, $V_{min}$, of the standing wave pattern. Thus, we find that

$$\text{SWR} = \frac{V_{max}}{V_{min}} = \frac{|\bar{V}_1^+|(1 + |\Gamma_V|)}{|\bar{V}_1^+|(1 - |\Gamma_V|)} = \frac{1 + |\Gamma_V|}{1 - |\Gamma_V|} \tag{7.29}$$

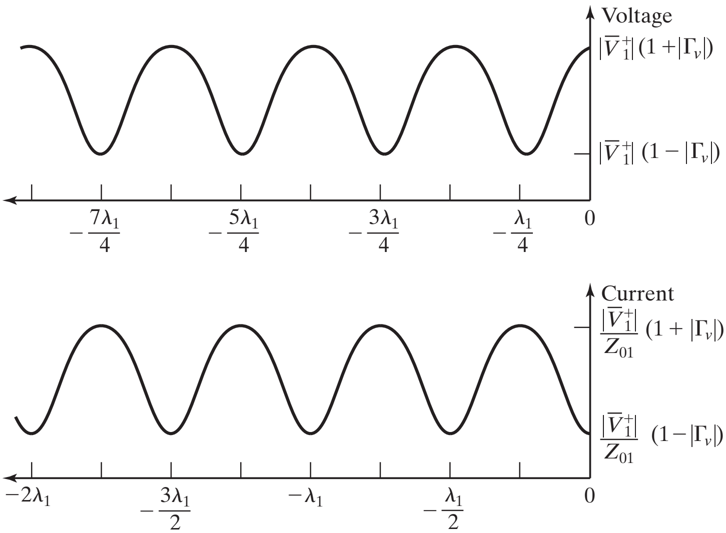



FIGURE 7.9

Standing wave patterns for voltage and current on a transmission line terminated by another transmission line.

The SWR is an important parameter in transmission-line matching. It is an indicator of the degree of the existence of standing waves on the line. We shall, however, not pursue the topic here any further. Finally, we note that for the case of Example 7.3, the SWR in line 1 is $(1+\frac{1}{5})/(1-\frac{1}{5})$, or 1.5. The SWR in line 2 is, of course, equal to 1 since there is no reflected wave in that line.

## 7.3 THE SMITH CHART

In the previous section, we studied reflection and transmission at the junction of two transmission lines, shown in Figure 7.10. In this section, we shall introduce the Smith Chart, which is a useful graphical aid in the solution of transmission-line and many other problems.

First we define the line impedance $\bar{Z}(z)$ at a given value of $z$ on the line as the ratio of the complex line voltage to the complex line current at that value of $z$, that is,

$$\bar{Z}(z) = \frac{\bar{V}(z)}{\bar{I}(z)} \tag{7.30}$$

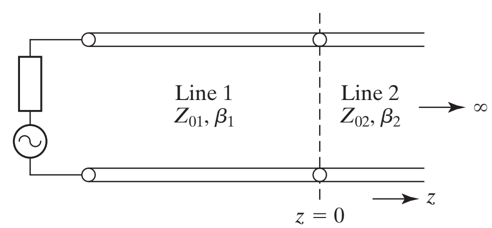



FIGURE 7.10

A transmission line terminated by another infinitely long transmission line.

From the solutions for the line voltage and line current on line 2 given by (7.19a) and (7.19b), respectively, the line impedance in line 2 is given by

$$\bar{Z}_2(z) = \frac{\bar{V}_2(z)}{\bar{I}_2(z)} = Z_{02}$$

Thus, the line impedance at all points on line 2 is simply equal to the characteristic impedance of that line. This is because the line is infinitely long and hence there is only a (+) wave on the line. From the solutions for the line voltage and line current in line 1 given by (7.18a) and (7.18b), respectively, the line impedance for that line is given by

$$\begin{aligned}\bar{Z}_1(z) &= \frac{\bar{V}_1(z)}{\bar{I}_1(z)} = Z_{01}\frac{\bar{V}_1^+ e^{-j\beta_1 z} + \bar{V}_1^- e^{j\beta_1 z}}{\bar{V}_1^+ e^{-j\beta_1 z} - \bar{V}_1^+ e^{j\beta_1 z}} \\ &= Z_{01}\frac{1 + \bar{\Gamma}_V(z)}{1 - \bar{\Gamma}_V(z)} \qquad (7.31)\end{aligned}$$

where

$$\bar{\Gamma}_V(z) = \frac{\bar{V}_1^- e^{j\beta_1 z}}{\bar{V}_1^+ e^{-j\beta_1 z}} = \bar{\Gamma}_V(0)e^{j2\beta_1 z} \qquad (7.32)$$

$$\bar{\Gamma}_V(0) = \frac{\bar{V}_1^-}{\bar{V}_1^+} = \frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}} \qquad (7.33)$$

The quantity $\bar{\Gamma}_V(0)$ is the voltage reflection coefficient at the junction $z = 0$, and $\bar{\Gamma}_V(z)$ is the voltage reflection coefficient at any value of $z$.

To compute the line impedance at a particular value of $z$, we first compute $\bar{\Gamma}_V(0)$ from a knowledge of $Z_{02}$, which is the terminating impedance to line 1. We then compute $\bar{\Gamma}_V(z) = \bar{\Gamma}_V(0)e^{j2\beta_1 z}$, which is a complex number having the same magnitude as that of $\bar{\Gamma}_V(0)$ but a phase angle equal to $2\beta_1 z$ plus the phase angle of $\bar{\Gamma}_V(0)$. The computed value of $\bar{\Gamma}_V(z)$ is then substituted in (7.31) to find $\bar{Z}_1(z)$. All of this complex algebra is eliminated through the use of the Smith Chart.

The Smith Chart is a mapping of the values of normalized line impedance onto the reflection coefficient ($\bar{\Gamma}_V$) plane. The normalized line impedance $\bar{Z}_n(z)$ is the ratio of the line impedance to the characteristic impedance of the line. From (7.31), and omitting the subscript 1 for the sake of generality, we have

$$\bar{Z}_n(z) = \frac{\bar{Z}(z)}{Z_0} = \frac{1 + \bar{\Gamma}_V(z)}{1 - \bar{\Gamma}_V(z)} \qquad (7.34)$$

Conversely,

$$\bar{\Gamma}_V(z) = \frac{\bar{Z}_n(z) - 1}{\bar{Z}_n(z) + 1} \qquad (7.35)$$

Writing $\bar{Z}_n = r + jx$ and substituting into (10.35), we find that

$$|\bar{\Gamma}_V| = \left|\frac{r + jx - 1}{r + jx + 1}\right| = \frac{\sqrt{(r-1)^2 + x^2}}{\sqrt{(r+1)^2 + x^2}} \leq 1 \qquad \text{for } r \geq 0$$

Thus, we note that all passive values of normalized line impedances, that is, points in the right half of the complex $\bar{Z}_n$-plane shown in Figure 7.11(a) are mapped onto the region within the circle of radius unity in the complex $\bar{\Gamma}_V$-plane shown in Figure 7.11(b).

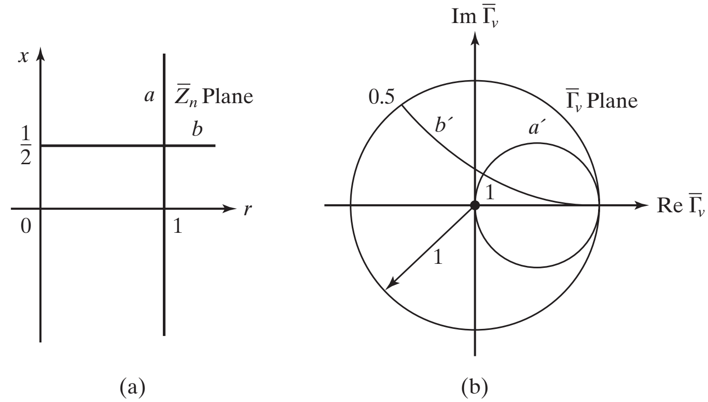



FIGURE 7.11

For illustrating the development of the Smith Chart.

We can now assign values for $\bar{Z}_n$, compute the corresponding values of $\bar{\Gamma}_V$, and plot them on the $\bar{\Gamma}_V$-plane but indicating the values of $\bar{Z}_n$ instead of the values of $\bar{\Gamma}_V$. To do this in a systematic manner, we consider contours in the $\bar{Z}_n$-plane corresponding to constant values of $r$, as shown for example by the line marked $a$ for $r = 1$, and corresponding to constant values of $x$, as shown for example by the line marked $b$ for $x = \frac{1}{2}$ in Figure 7.11(a).

By considering several points along line $a$, computing the corresponding values of $\bar{\Gamma}_V$, plotting them on the $\bar{\Gamma}_V$-plane, and joining them, we obtain the contour marked $a'$ in Figure 7.11(b). Although it can be shown analytically that this contour is a circle of radius $\frac{1}{2}$ and centered at $(1/2, 0)$, it is a simple task to write a computer program to perform this operation, including the plotting. Similarly, by considering several points along line $b$ and following the same procedure, we obtain the contour marked $b'$ in Figure 7.11(b). Again, it can be shown analytically that this contour is a portion of a circle of radius 2 and centered at $(1, 2)$. We can now identify the points on contour $a'$ as corresponding to $r = 1$ by placing the number 1 beside it and the points on contour $b'$ as corresponding to $x = \frac{1}{2}$ by placing the number 0.5 beside it. The point of intersection of contours $a'$ and $b'$ then corresponds to $\bar{Z}_n = 1 + j0.5$.

When the procedure discussed above is applied to many lines of constant $r$ and constant $x$ covering the entire right half of the $\bar{Z}_n$-plane, we obtain the Smith Chart. In a commercially available form shown in Figure 7.12, the Smith Chart contains contours of constant $r$ and constant $x$ at appropriate increments of $r$ and $x$ in the range $0 < r < \infty$ and $-\infty < x < \infty$ so that interpolation between the contours can be carried out to a good degree of accuracy.

Let us now consider the transmission line system shown in Figure 7.13, which is the same as that in Figure 7.10 except that a reactive element having susceptance (reciprocal of reactance) $B$ is connected in parallel with line 1 at a distance $l$ from the junction.

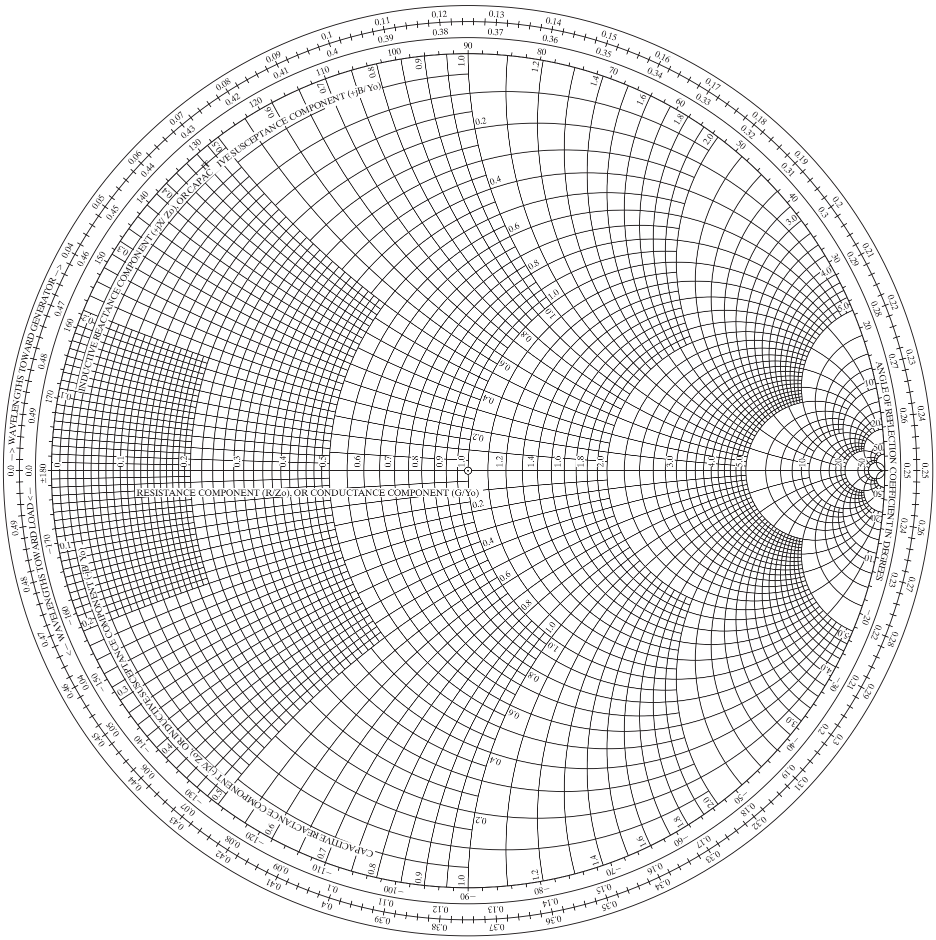



FIGURE 7.12

A commercially available form of the Smith Chart (reproduced with the courtesy of Analog Instrument Co., P.O. Box 950, New Providence, NJ 07974, USA).

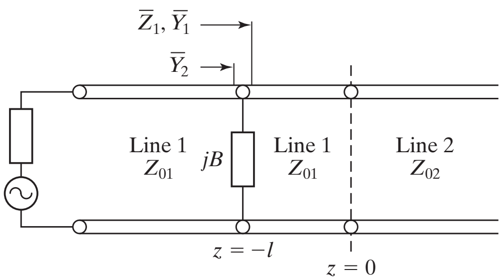



FIGURE 7.13

A transmission-line system for illustrating the computation of several quantities by using the Smith Chart.

Let us assume $Z_{01} = 150\ \Omega$, $Z_{02} = 50\ \Omega$, $B = -0.003$ S, and $l = 0.375\lambda_1$, where $\lambda_1$ is the wavelength in line 1 corresponding to the source frequency, and find the following quantities by using the Smith Chart, as shown in Figure 7.14:

**1.** $\bar{Z}_1$, line impedance just to the right of $jB$: First we note that since line 2 is infinitely long, the load for line 1 is simply 50 Ω. Normalizing this with respect to the characteristic impedance of line 1, we obtain the normalized load impedance for line 1 to be

$$\bar{Z}_n(0) = \frac{50}{150} = \frac{1}{3}$$

Locating this on the Smith Chart at point $A$ in Figure 7.14 amounts to computing the reflection coefficient at the junction, that is, $\bar{\Gamma}_V(0)$. Now the reflection coefficient at $z = -l = -0.375\lambda_1$, being equal to $\bar{\Gamma}_V(0)e^{-j2\beta_1 l} = \bar{\Gamma}_V(0)e^{-j1.5\pi}$, can be located on the Smith Chart by moving $A$ such that the magnitude remains constant but the phase angle decreases by $1.5\pi$. This is equivalent to moving it on a circle with its center at the center of the Smith Chart and in the clockwise direction by $1.5\pi$ or 270° so that point $B$ is reached. Actually, it is not necessary to compute this angle, since the Smith Chart contains a distance scale in terms of $\lambda$ along its periphery for movement from load toward generator and vice versa, based on a complete revolution for one-half wavelength. The normalized impedance at point $B$ can now be read off the chart and multiplied by the characteristic impedance of the line to obtain the required impedance value. Thus,

$$\bar{Z}_1 = (0.6 - j0.8)150 = (90 - j120)\ \Omega.$$

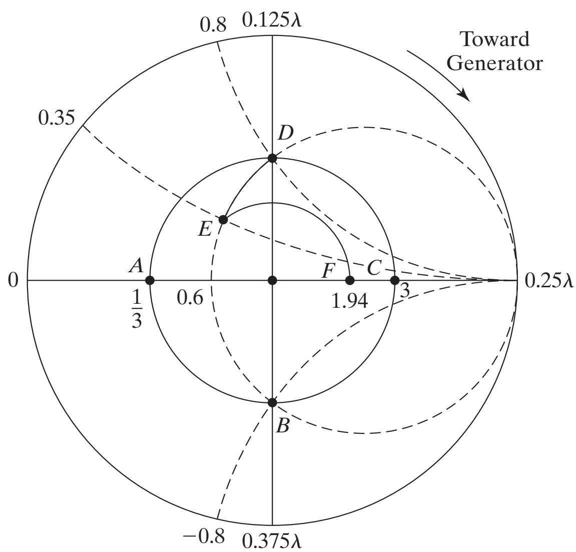



FIGURE 7.14

For illustrating the use of the Smith Chart in the computation of several quantities for the transmission-line system of Figure 7.13.

**2.** SWR on line 1 to the right of $jB$: From (7.29)

$$\text{SWR} = \frac{1 + |\Gamma_V|}{1 - |\Gamma_V|} = \frac{1 + |\bar{\Gamma}_V|e^{j0}}{1 - |\bar{\Gamma}_V|e^{j0}} \tag{7.36}$$

Comparing the right side of (7.36) with the expression for $\bar{Z}_n$ given by (7.34), we note that it is simply equal to $\bar{Z}_n$ corresponding to phase angle of $\bar{\Gamma}_V$ equal to zero. Thus, to find the SWR, we locate the point on the Smith Chart having the same $|\bar{\Gamma}_V|$ as that for $z = 0$, but having a phase angle equal to zero, that is, the point $C$ in Figure 7.14, and then read off the normalized resistance value at that point. Here, it is equal to 3 and hence the required SWR is equal to 3. In fact, the circle passing through $C$ and having its center at the center of the Smith Chart is known as the *constant SWR* (=3) *circle*, since for any normalized load impedance to line 1 lying on that circle, the SWR is the same (and equal to 3).

**3.** $\bar{Y}_1$, line admittance just to the right of $jB$: To find this, we note that the normalized line admittance $\bar{Y}_n$ at any value of $z$, that is, the line admittance normalized with respect to the line characteristic admittance $Y_0$ (reciprocal of $Z_0$) is given by

$$\begin{aligned}
\bar{Y}_n(z) &= \frac{\bar{Y}(z)}{Y_0} = \frac{Z_0}{\bar{Z}(z)} = \frac{1}{\bar{Z}_n(z)} \\
&= \frac{1 - \bar{\Gamma}_V(z)}{1 + \bar{\Gamma}_V(z)} = \frac{1 + \bar{\Gamma}_V(z)e^{\pm j\pi}}{1 - \bar{\Gamma}_V(z)e^{\pm j\pi}} \\
&= \frac{1 + \bar{\Gamma}_V(z)e^{\pm j2\beta\lambda/4}}{1 - \bar{\Gamma}_V(z)e^{\pm j2\beta\lambda/4}} = \frac{1 + \bar{\Gamma}_V(z \pm \lambda/4)}{1 - \bar{\Gamma}_V(z \pm \lambda/4)} \\
&= \bar{Z}_n\left(z \pm \frac{\lambda}{4}\right)
\end{aligned} \tag{7.37}$$

Thus, $\bar{Y}_n$ at a given value of $z$ is equal to $\bar{Z}_n$ at a value of $z$ located $\lambda/4$ from it. On the Smith Chart, this corresponds to the point on the constant SWR circle passing through $B$ and diametrically opposite to it, that is, the point $D$. Thus,

$$\bar{Y}_{n1} = 0.6 + j0.8$$

and

$$\begin{aligned}
\bar{Y}_1 &= Y_{01}\bar{Y}_{n1} = \frac{1}{150}(0.6 + j0.8) \\
&= (0.004 + j0.0053)\ \text{S}
\end{aligned}$$

In fact, the Smith Chart can be used as an admittance chart instead of as an impedance chart, that is, by knowing the line admittance at one point on the line, the line admittance at another point on the line can be found by proceeding in the same manner as for impedances. As an example, to find $\bar{Y}_1$, we can first find the normalized line admittance at $z = 0$ by locating the point $C$ diametrically opposite to point $A$ on the constant SWR circle. Then we find $\bar{Y}_{n1}$ by simply going on the constant SWR circle by the distance $l(= 0.375\lambda_1)$ toward the generator. This leads to point $D$, thereby giving us the same result for $\bar{Y}_1$ as found above.

**4.** SWR on line 1 to the left of $jB$: To find this, we first locate the normalized line admittance just to the left of $jB$, which then determines the constant SWR circle corresponding to the portion of line 1 to the left of $jB$. Thus, noting that $\bar{Y}_2 = \bar{Y}_1 + jB$, or $\bar{Y}_{n2} = \bar{Y}_{n1} + jB/Y_{01}$, and hence

$$\mathrm{Re}[\bar{Y}_{n2}] = \mathrm{Re}[\bar{Y}_{n1}] \tag{7.38a}$$

$$\mathrm{Im}[\bar{Y}_{n2}] = \mathrm{Im}[\bar{Y}_{n1}] + \frac{B}{Y_{01}} \tag{7.38b}$$

we start at point $D$ and go along the constant real part (conductance) circle to reach point $E$ for which the imaginary part differs from the imaginary part at $D$ by the amount $B/Y_{01}$, that is, $-0.003/(1/150)$, or $-0.45$. We then draw the constant SWR circle passing through $E$ and then read off the required SWR value at point $F$. This value is equal to 1.94.

The steps outlined above in part 4 can be applied is reverse to determine the location and the value of the susceptance required to achieve an SWR of unity to the left of it, that is, a condition of no standing waves. This procedure is known as transmission-line *matching*. It is important from the point of view of eliminating or minimizing certain undesirable effects of standing waves in electromagnetic energy transmission.

To illustrate the solution to the matching problem, we first recognize that an SWR of unity is represented by the center point of the Smith Chart. Hence, matching is achieved if $\bar{Y}_{n2}$ falls at the center of the Smith Chart. Now since the difference between $\bar{Y}_{n1}$ and $\bar{Y}_{n2}$ is only in the imaginary part as indicated by (7.38a) and (7.38b), $\bar{Y}_{n1}$ must lie on the constant conductance circle passing through the center of the Smith Chart (this circle is known as the *unit conductance circle*, since it corresponds to normalized real part equal to unity). $\bar{Y}_{n1}$ must also lie on the constant SWR circle corresponding to the portion of the line to the right of $jB$. Hence, it is given by the point(s) of intersection of this constant SWR circle and the unit conductance circle. There are two such points, $G$ and $H$, as shown in Figure 7.15, in which the points $A$ and $C$ are repeated from Figure 7.14. There are thus two solutions to the matching problem. If we choose $G$ to correspond to $\bar{Y}_{n1}$, then, since the distance from $C$ to $G$ is $(0.333 - 0.250)\lambda_1$, or $0.083\lambda_1$, $jB$ must be located at $z = -0.083\lambda_1$. To find the value of $jB$, we note that the normalized susceptance value corresponding to $G$ is $-1.16$ and hence $B/Y_{01} = 1.16$, or $jB = j1.16\, Y_{01} = j0.00773$ S. If, however, we choose the point $H$ to correspond to $\bar{Y}_{n1}$, then we find in a similar manner that $jB$ must be located at $z = (0.250 + 0.167)\lambda_1$ or $0.417\lambda_1$ and its value must be $-j0.00773$ S.

The reactive element $jB$ used to achieve the matching is commonly realized by means of a short-circuited section of line, known as a *stub*. This is based on the fact that the input impedance of a short-circuited line is purely reactive, as shown in Section 7.1.

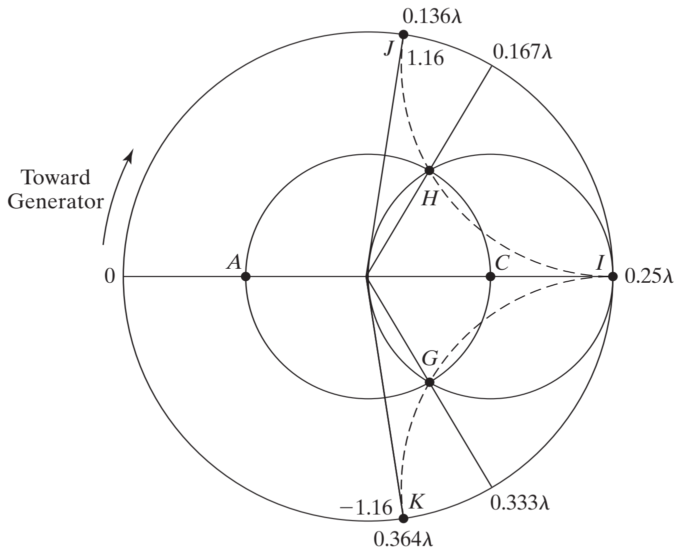



FIGURE 7.15

Solution of transmission-line matching problem by using the Smith Chart.

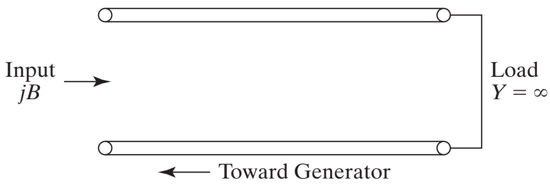



FIGURE 7.16

A short-circuited stub.

The length of the stub for a required input susceptance can be found by considering the short circuit as the load, as shown in Figure 7.16, and using the Smith Chart. The admittance corresponding to a short circuit is infinity, and hence the load admittance normalized with respect to the characteristic admittance of the stub is also equal to infinity. This is located on the Smith Chart at point $I$ in Figure 7.15. We then go along the constant SWR circle passing through $I$ (the outermost circle) toward the generator (input) until we reach the point corresponding to the required input susceptance of the stub normalized with respect to the characteristic admittance of the stub. Assuming the characteristic impedance of the stub to be the same as that of the line, this quantity is here equal to $j1.16$ or $-j1.16$, depending on whether point $G$ or point $H$ is chosen for the location of the stub. This leads us to point $J$ or point $K$, and hence the stub length is $(0.25 + 0.136)\lambda_1$, or $0.386\lambda_1$, for $jB = j1.16$, and $(0.364 - 0.25)\lambda_1$, or $0.114\lambda_1$, for $jB = -j1.16$. The arrangement of the stub corresponding to the solution for which the stub location is at $z = -0.083\lambda_1$, and the stub length is $0.386\lambda_1$, is shown in Figure 7.17.

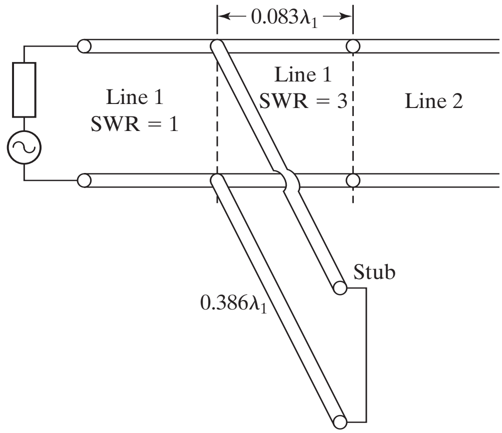



FIGURE 7.17

A solution to the matching problem for the transmission-line system of Figure 7.10.

## B. TIME DOMAIN

For a lossless line, the transmission-line equations (6.86a) and (6.86b) or (7.1a) and (7.1b) reduce to

$$\frac{\partial V}{\partial z} = -\mathcal{L}\frac{\partial I}{\partial t} \tag{7.39a}$$

$$\frac{\partial I}{\partial z} = -\mathcal{C}\frac{\partial V}{\partial t} \tag{7.39b}$$

In time domain, the solutions are given by

$$V(z, t) = Af(t - z\sqrt{\mathcal{L}\mathcal{C}}) + Bg(t + z\sqrt{\mathcal{L}\mathcal{C}}) \tag{7.40a}$$

$$I(z, t) = \frac{1}{\sqrt{\mathcal{L}/\mathcal{C}}}[Af(t - z\sqrt{\mathcal{L}\mathcal{C}}) - Bg(t + z\sqrt{\mathcal{L}\mathcal{C}})] \tag{7.40b}$$

which can be verified by substituting them into (7.39a) and (7.39b). These solutions represent voltage and current traveling waves propagating with velocity

$$v_p = \frac{1}{\sqrt{\mathcal{L}\mathcal{C}}} \tag{7.41}$$

in view of the arguments $(t \mp z\sqrt{\mathcal{L}\mathcal{C}})$ for the functions $f$ and $g$, and characteristic impedance

$$Z_0 = \sqrt{\frac{\mathcal{L}}{\mathcal{C}}} \tag{7.42}$$

They can also be inferred from the fact that $v_p$ and $Z_0$ are independent of frequency.

We now rewrite (7.40a) and (7.40b) as

$$V(z, t) = V^+\left(t - \frac{z}{v_p}\right) + V^-\left(t + \frac{z}{v_p}\right) \tag{7.43a}$$

$$I(z, t) = \frac{1}{Z_0}\left[V^+\left(t - \frac{z}{v_p}\right) - V^-\left(t + \frac{z}{v_p}\right)\right] \tag{7.43b}$$

or, more concisely,

$$V = V^+ + V^- \tag{7.44a}$$

$$I = \frac{1}{Z_0}(V^+ - V^-) \tag{7.44b}$$

with the understanding that $V^+$ is a function of $(t - z/v_p)$ and $V^-$ is a function of $(t + z/v_p)$. In terms of $(+)$ and $(-)$ wave currents, the solution for the current may also be written as

$$I = I^+ + I^- \tag{7.45}$$

Comparing (7.44b) and (7.45), we see that

$$I^+ = \frac{V^+}{Z_0} \tag{7.46a}$$

$$I^- = -\frac{V^-}{Z_0} \tag{7.46b}$$

The minus sign in (7.46b) can be understood if we recognize that in writing (7.44a) and (7.45), we follow the notation that both $V^+$ and $V^-$ have the same polarities with one conductor (say, $a$) positive with respect to the other conductor (say, $b$) and that both $I^+$ and $I^-$ flow in the positive $z$-direction along conductor $a$ and return in the negative $z$-direction along conductor $b$, as shown in Figure 7.18. The power flow associated with either wave, as given by the product of the corresponding voltage and current, is then directed in the positive $z$-direction, as shown in Figure 7.18. Thus,

$$P^+ = V^+I^+ = V^+\left(\frac{V^+}{Z_0}\right) = \frac{(V^+)^2}{Z_0} \tag{7.47a}$$

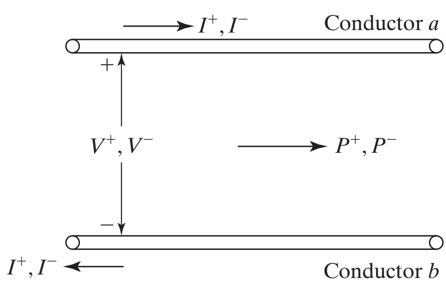



FIGURE 7.18

Polarities for voltages and currents associated with $(+)$ and $(-)$ waves.

Since $(V^+)^2$ is always positive, regardless of whether $V^+$ is numerically positive or negative, (7.47a) indicates that the (+) wave power does actually flow in the positive $z$-direction, as it should. On the other hand,

$$P^- = V^- I^- = V^-\left(-\frac{V^-}{Z_0}\right) = -\frac{(V^-)^2}{Z_0} \tag{7.47b}$$

Since $(V^-)^2$ is always positive, regardless of whether $V^-$ is numerically positive or negative, the minus sign in (7.47b) indicates that $P^-$ is negative, and, hence, the (−) wave power actually flows in the negative $z$-direction, as it should.

## 7.4 LINE TERMINATED BY RESISTIVE LOAD

Let us now consider a line of length $l$ terminated by a load resistance $R_L$ and driven by a constant voltage source $V_0$ in series with internal resistance $R_g$, as shown in Figure 7.19. Note again that the conductors of the transmission line are represented by double-ruled lines, whereas the connections to the conductors are single-ruled lines, to be treated as lumped circuits. We assume that no voltage and current exist on the line for $t < 0$ and the switch $S$ is closed at $t = 0$. We wish to discuss the transient wave phenomena on the line for $t > 0$. The characteristic impedance of the line and the velocity of propagation are $Z_0$ and $v_p$, respectively.

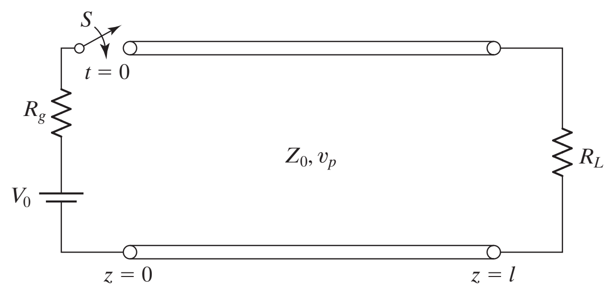



FIGURE 7.19

Transmission line terminated by a load resistance $R_L$ and driven by a constant voltage source in series with an internal resistance $R_g$.

When the switch $S$ is closed at $t = 0$, a (+) wave originates at $z = 0$ and travels toward the load. Let the voltage and current of this (+) wave be $V^+$ and $I^+$, respectively. Then we have the situation at $z = 0$, as shown in Figure 7.20(a). Note that the load resistor does not come into play here since the phenomenon is one of wave propagation; hence, until the (+) wave goes to the load, sets up a reflection, and the reflected wave arrives back at the source, the source does not even know of the existence of $R_L$. This is a fundamental distinction between ordinary (lumped-) circuit theory and transmission-line (distributed-circuit) theory. In ordinary circuit theory, no time delay is involved; the effect of a transient in one part of the circuit is felt in all branches of the circuit instantaneously. In a transmission-line system, the effect of a transient at one location on the line is felt at a different location on the line only after an interval of time that the wave takes to travel from the first location to the second. Returning now to the circuit

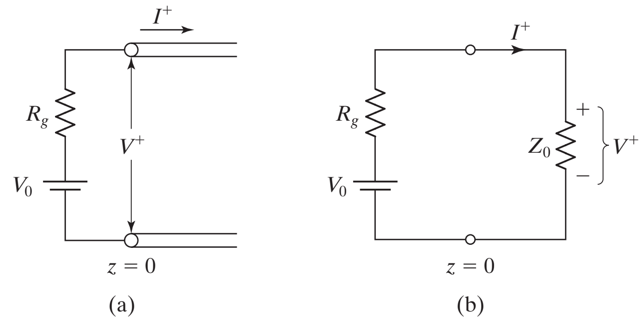



FIGURE 7.20

(a) For obtaining the (+) wave voltage and current at $z = 0$ for the line of Figure 7.19. (b) Equivalent circuit for (a).

in Figure 7.20(a), the various quantities must satisfy the boundary condition, that is, Kirchhoff's voltage law around the loop. Thus, we have

$$V_0 - I^+R_g - V^+ = 0 \tag{7.48a}$$

We, however, know from (6.31a) that $I^+ = V^+/Z_0$. Hence, we get

$$V_0 - \frac{V^+}{Z_0}R_g - V^+ = 0 \tag{7.48b}$$

or

$$V^+ = V_0\frac{Z_0}{R_g + Z_0} \tag{7.49a}$$

$$I^+ = \frac{V^+}{Z_0} = \frac{V_0}{R_g + Z_0} \tag{7.49b}$$

Thus, we note that the situation in Figure 7.20(a) is equivalent to the circuit shown in Figure 7.20(b); that is, the voltage source views a resistance equal to the characteristic impedance of the line, across $z = 0$. This is to be expected, since only a (+) wave exists at $z = 0$ and the ratio of the voltage to current in the (+) wave is equal to $Z_0$.

The (+) wave travels toward the load and reaches the termination at $t = l/v_p$. It does not, however, satisfy the boundary condition there, since this condition requires the voltage across the load resistance to be equal to the current through it times its value, $R_L$. But the voltage-to-current ratio in the (+) wave is equal to $Z_0$. To resolve this inconsistency, there is only one possibility, which is the setting up of a (−) wave, or a reflected wave. Let the voltage and current in this reflected wave be $V^-$ and $I^-$, respectively. Then the total voltage across $R_L$ is $V^+ + V^-$, and the total current through it is $I^+ + I^-$, as shown in Figure 7.21(a). To satisfy the boundary condition, we have

$$V^+ - V^- = R_L(I^+ + I^-) \tag{7.50a}$$

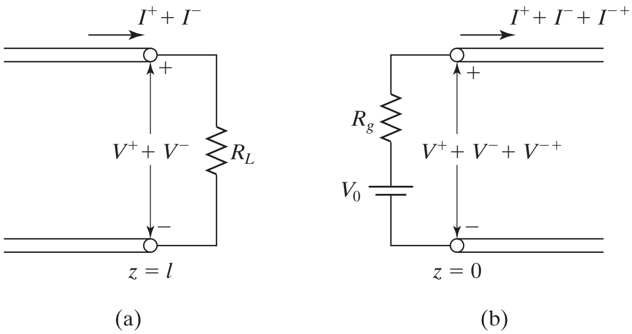



FIGURE 7.21

For obtaining the voltages and currents associated with (a) the (−) wave and (b) the (−+) wave, for the line of Figure 7.19.

But from (7.46a) and (7.46b), we know that $I^+ = V^+/Z_0$ and $I^- = -V^-/Z_0$, respectively. Hence,

$$V^+ - V^- = R_L\left(\frac{V^+}{Z_0} - \frac{V^-}{Z_0}\right) \tag{7.50b}$$

or

$$V^- = V^+\frac{R_L - Z_0}{R_L + Z_0} \tag{7.51}$$

We now denote the *voltage reflection coefficient*, that is, the ratio of the reflected voltage to the incident voltage, by the symbol $\Gamma$ (previously $\Gamma_V$). Thus,

$$\Gamma = \frac{V^-}{V^+} = \frac{R_L - Z_0}{R_L + Z_0} \tag{7.52}$$

We then note that the *current reflection coefficient* is

$$\frac{I^-}{I^+} = \frac{-V^-/Z_0}{V^+/Z_0} = -\frac{V^-}{V^+} = -\Gamma \tag{7.53}$$

Now, returning to the reflected wave, we observe that this wave travels back toward the source and that it reaches there at $t = 2l/v_p$. Since the boundary condition at $z = 0$, which was satisfied by the original (+) wave alone, is then violated, a reflection of the reflection, or a re-reflection, will be set up and it travels toward the load. Let us assume the voltage and current in this re-reflected wave, which is a (+) wave, to be $V^{-+}$ and $I^{-+}$, respectively, with the superscripts denoting that the (+) wave is a consequence of the (−) wave. Then the total line voltage and the line current at $z = 0$ are $V^+ + V^- + V^{-+}$ and $I^+ + I^- + I^{-+}$, respectively, as shown in Figure 7.21(b). To satisfy the boundary condition, we have

$$V^+ + V^- + V^{-+} = V_0 - R_g(I^+ + I^- + I^{-+}) \tag{7.54a}$$

But we know that $I^+ = V^+/Z_0$, $I^- = -V^-/Z_0$, and $I^{-+} = V^{-+}/Z_0$. Hence,

$$V^+ + V^- + V^{-+} = V_0 - \frac{R_g}{Z_0}(V^+ - V^- + V^{-+}) \tag{7.54b}$$

Furthermore, substituting for $V^+$ from (7.49a), simplifying, and rearranging, we get

$$V^{-+}\left(1 + \frac{R_g}{Z_0}\right) = V^-\left(\frac{R_g}{Z_0} - 1\right)$$

or

$$V^{-+} = V^- \frac{R_g - Z_0}{R_g + Z_0} \tag{7.55}$$

Comparing (7.55) with (7.51), we note that the reflected wave views the source with internal resistance as the internal resistance alone; that is, the voltage source is equivalent to a short circuit insofar as the $(-)$ wave is concerned. A moment's thought will reveal that superposition is at work here. The effect of the voltage source is taken into account by the constant outflow of the original $(+)$ wave from the source. Hence, for the reflection of the reflection, that is, for the $(-+)$ wave, we need only consider the internal resistance $R_g$. Thus, the voltage reflection coefficient formula (7.52) is a general formula and will be used repeatedly. In view of its importance, a brief discussion of the values of $\Gamma$ for some special cases is in order, as follows:

1. $R_L = 0$, or short-circuited line.

$$\Gamma = \frac{0 - Z_0}{0 + Z_0} = -1$$

   The reflected voltage is exactly the negative of the incident voltage, thereby keeping the voltage across $R_L$ (short circuit) always zero.

2. $R_L = \infty$, or open-circuited line.

$$\Gamma = \frac{\infty - Z_0}{\infty + Z_0} = 1$$

   and the current reflection coefficient $= -\Gamma = -1$. Thus, the reflected current is exactly the negative of the incident current, thereby keeping the current through $R_L$ (open circuit) always zero.

3. $R_L = Z_0$, or line terminated by its characteristic impedance.

$$\Gamma = \frac{Z_0 - Z_0}{Z_0 + Z_0} = 0$$

   This corresponds to no reflection, which is to be expected since $R_L(= Z_0)$ is consistent with the voltage-to-current ratio in the $(+)$ wave alone, and, hence, there is no violation of boundary condition and no need for the setting up of a reflected wave. Thus, a line terminated by its characteristic impedance is equivalent to an infinitely long line insofar as the source is concerned.

Returning to the discussion of the re-reflected wave, we note that this wave reaches the load at time $t = 3l/v_p$ and sets up another reflected wave. This process of bouncing back and forth of waves goes on indefinitely until the steady state is reached. To keep track of this transient phenomenon, we make use of the *bounce-diagram* technique. Some other names given for this diagram are *reflection diagram* and *space-time diagram*. We shall introduce the bounce diagram through a numerical example.

## Example 7.4

Let us consider the system shown in Figure 7.22. Note that we have introduced a new quantity $T$, which is the one-way travel time along the line from $z = 0$ to $z = l$; that is, instead of specifying two quantities $l$ and $v_p$, we specify $T(= l/v_p)$. Using the bounce-diagram technique, we wish to obtain and plot line voltage and current versus $t$ for fixed values $z$ and line voltage and current versus $z$ for fixed values $t$.

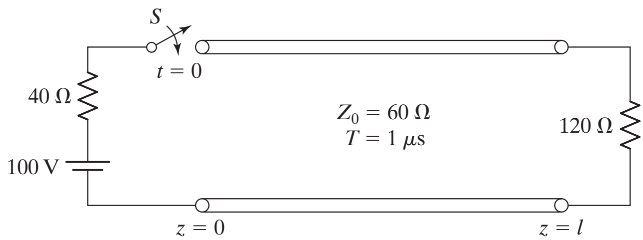



FIGURE 7.22

Transmission-line system for illustrating the bounce-diagram technique of keeping track of the transient phenomenon.

Before we construct the bounce diagram, we need to compute the following quantities:

$$\text{Voltage carried by the initial (+) wave} = 100\,\frac{60}{40 + 60} = 60\text{ V}$$

$$\text{Current carried by the initial (+) wave} = \frac{60}{60} = 1\text{ A}$$

$$\text{Voltage reflection coefficient at load, } \Gamma_R = \frac{120-60}{120 + 60} = \frac{1}{3}$$

$$\text{Voltage reflection coefficient at source, } \Gamma_S = \frac{40 - 60}{40 + 60} = -\frac{1}{5}$$

The bounce diagram is essentially a two-dimensional representation of the transient waves bouncing back and forth on the line. Separate bounce diagrams are drawn for voltage and current, as shown in Figure 7.23(a) and (b), respectively. Position ($z$) on the line is represented horizontally and the time ($t$) vertically. Reflection coefficient values for the two ends are shown at the top of the diagrams for quick reference. Note that current reflection coefficients are $-\Gamma_R = -\frac{1}{3}$ and $-\Gamma_S = \frac{1}{5}$, respectively, at the load and at the source. Crisscross lines are drawn as shown in the figures to indicate the progress of the wave as a function of both $z$ and $t$, with the numerical value for each leg of travel shown beside the line corresponding to that leg and approximately at the center of the line. The arrows indicate the directions of travel. Thus, for example, the first line on the voltage bounce diagram indicates that the initial (+) wave of 60 V takes a time of 1 $\mu$s to reach the load end of the line. It sets up a reflected wave of 20 V, which travels back to the source, reaching there at a time of 2 $\mu$s, which then gives rise to a (+) wave of voltage $-4$ V, and so on, with the process continuing indefinitely.

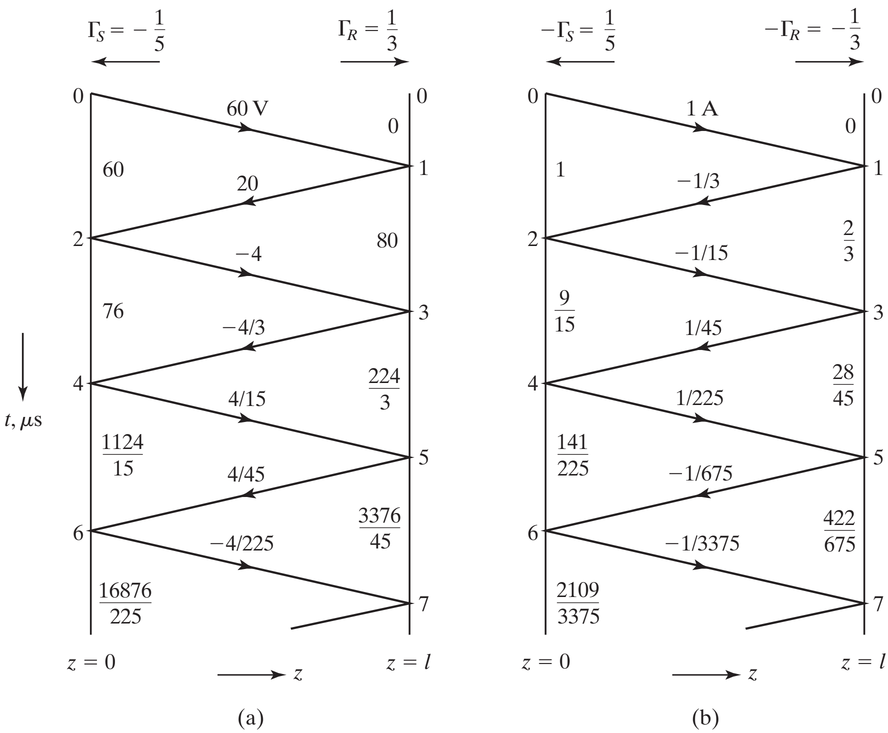



FIGURE 7.23

(a) Voltage and (b) current bounce diagrams, depicting the bouncing back and forth of the transient waves for the system of Figure 7.22.

Now, to sketch the line voltage and/or current versus time at any value of $z$, we note that since the voltage source is a constant voltage source, each individual wave voltage and current, once the wave is set up at that value of $z$, continues to exist there forever. Thus, at any particular time, the voltage (or current) at that value of $z$ is a superposition of all the voltages (or currents) corresponding to the crisscross lines preceding that value of time. These values are marked on the bounce diagrams for $z = 0$ and $z = l$. Noting that each value corresponds to the 2-$\mu$s time interval between adjacent crisscross lines, we now sketch the time variations of line voltage and current at $z = 0$ and $z = l$, as shown in Figures 7.24(a) and (b), respectively. Similarly, by observing that the numbers written along the time axis for $z = 0$ are actually valid for any pair of $z$ and $t$ within the triangle ($\triangleright$) inside which they lie and that the numbers written along the time axis for $z = l$ are actually valid for any pair of $z$ and $t$ within the triangle ($\triangleleft$) inside which they lie, we can draw the sketches of line voltage and current versus time for any other value of $z$. This is done for $z = l/2$ in Figure 7.24(c).

It can be seen from the sketches of Figure 7.24 that as time progresses, the line voltage and current tend to converge to certain values, which we can expect to be the steady-state values. In the steady state, the situation consists of a single (+) wave, which is actually a superposition of the infinite number of transient (+) waves, and a single (−) wave, which is actually a superposition of the infinite number of transient (−) waves. Denoting the steady-state (+) wave voltage and

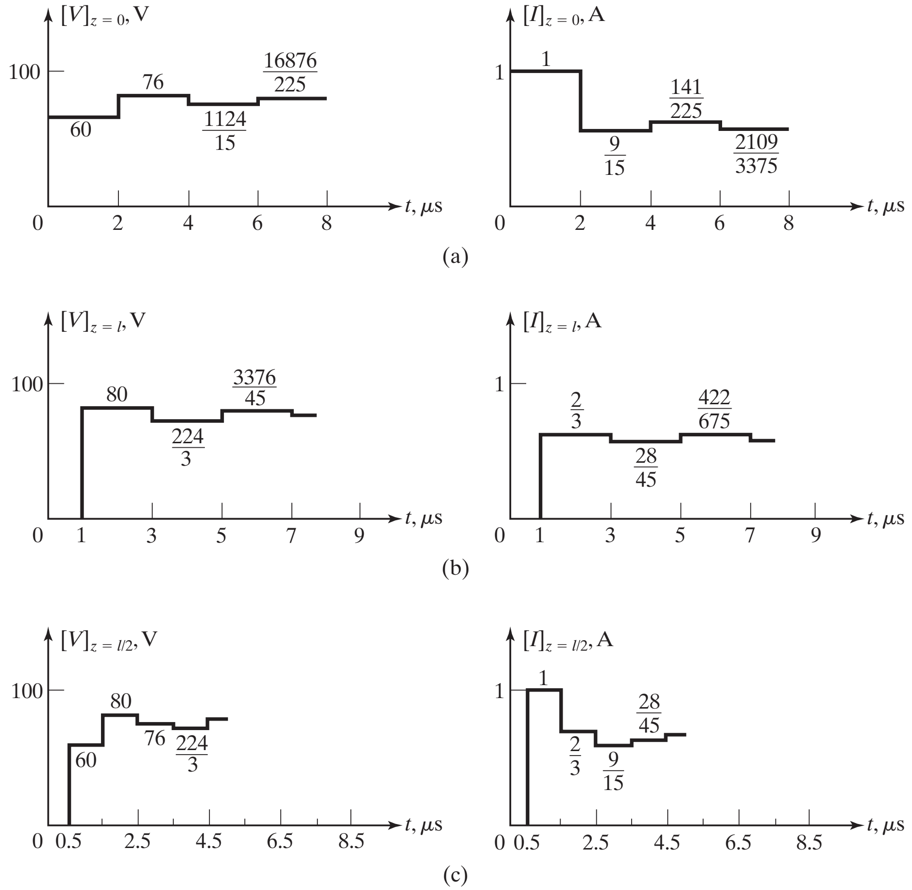



FIGURE 7.24

Time variations of line voltage and line current at (a) $z = 0$, (b) $z = l$, and (c) $z = l/2$ for the system of Figure 7.22.

current to be $V_{SS}^{+}$ and $I_{SS}^{+}$, respectively, and the steady-state (−) wave voltage and current to be $V_{SS}^{-}$ and $I_{SS}^{-}$, respectively, we obtain from the bounce diagrams

$$V_{SS}^{+} = 60 - 4 + \frac{4}{15} - \cdots = 60\left(1 - \frac{1}{15} + \frac{1}{15^2} - \cdots\right) = 56.25 \text{ V}$$

$$I_{SS}^{+} = 1 - \frac{1}{15} + \frac{1}{225} - \cdots = 1 - \frac{1}{15} + \frac{1}{15^2} - \cdots = 0.9375 \text{ A}$$

$$V_{SS}^{-} = 20 - \frac{4}{3} + \frac{4}{45} - \cdots = 20\left(1 - \frac{1}{15} + \frac{1}{15^2} - \cdots\right) = 18.75 \text{ V}$$

$$I_{SS}^{-} = -\frac{1}{3} + \frac{1}{45} - \frac{1}{675} + \cdots = -\frac{1}{3}\left(1 - \frac{1}{15} + \frac{1}{15^2} - \cdots\right) = -0.3125 \text{ A}$$

Note that $I_{SS}^+ = V_{SS}^+/Z_0$ and $I_{SS}^- = -V_{SS}^-/Z_0$, as they should be. The steady-state line voltage and current can now be obtained to be

$$V_{SS} = V_{SS}^+ + V_{SS}^- = 75 \text{ V}$$

$$I_{SS} = I_{SS}^+ + I_{SS}^- = 0.625 \text{ A}$$

These are the same as the voltage across $R_L$ and current through $R_L$ if the source and its internal resistance were connected directly to $R_L$, as shown in Figure 7.25. This is to be expected since the series inductors and shunt capacitors of the distributed equivalent circuit behave like short circuits and open circuits, respectively, for the constant voltage source in the steady state.

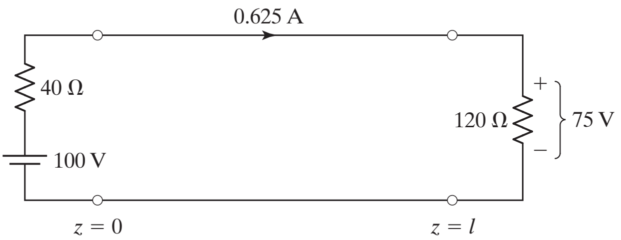



FIGURE 7.25

Steady-state equivalent for the system of Figure 7.22.

Sketches of line voltage and current as functions of distance ($z$) along the line for any particular time can also be drawn from considerations similar to those employed for the sketches of Figure 7.24. For example, suppose we wish to draw the sketch of line voltage versus $z$ for $t = 2.5\ \mu$s. Then we note from the voltage bounce diagram that for $t = 2.5\ \mu$s, the line voltage is 76 V from $z = 0$ to $z = l/2$ and 80 V from $z = l/2$ to $z = l$. This is shown in Figure 7.26(a). Similarly, Figure 7.26(b) shows the variation of line current versus $z$ for $t = 1\frac{1}{3}\ \mu$s.

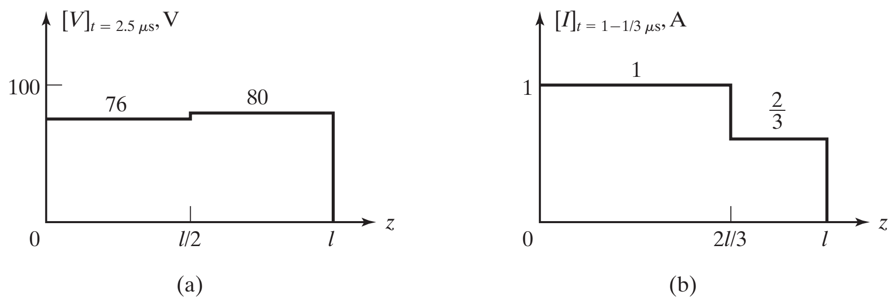



FIGURE 7.26

Variations with $z$ of (a) line voltage for $t = 2.5\ \mu$s and (b) line current for $t = 1\frac{1}{3}\ \mu$s, for the system of Figure 7.22.

In Example 7.4, we introduced the bounce-diagram technique for a constant-voltage source. The technique can also be applied if the voltage source is a pulse. In the case of a rectangular pulse, this can be done by representing the pulse as the superposition of two step functions, as shown in Figure 7.27, and superimposing the bounce diagrams for the two sources one on another. In doing so, we should note that the bounce diagram for one source begins at a value of time greater than zero. Alternatively, the time variation for each wave can be drawn alongside the time axes beginning at the time of start of the wave. These can then be used to plot the required sketches. An example is in order, to illustrate this technique, which can also be used for a pulse of arbitrary shape.

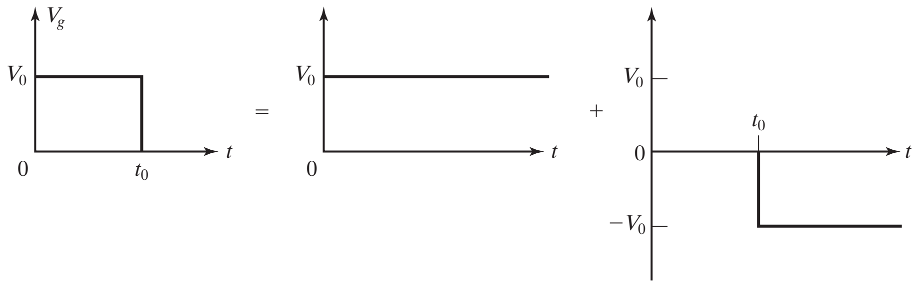



FIGURE 7.27

Representation of a rectangular pulse as the superposition of two step functions.

## Example 7.5

Let us assume that the voltage source in the system of Figure 7.22 is a 100-V rectangular pulse extending from $t = 0$ to $t = 1\ \mu$s and extend the bounce-diagram technique.

Considering, for example, the voltage bounce diagram, we reproduce in Figure 7.28 part of the voltage bounce diagram of Figure 7.23(a) and draw the time variations of the individual pulses alongside the time axes, as shown in the figure. Note that voltage axes are chosen such that positive values are to the left at the left end ($z = 0$) of the diagram, but to the right at the right end ($z = l$) of the diagram.

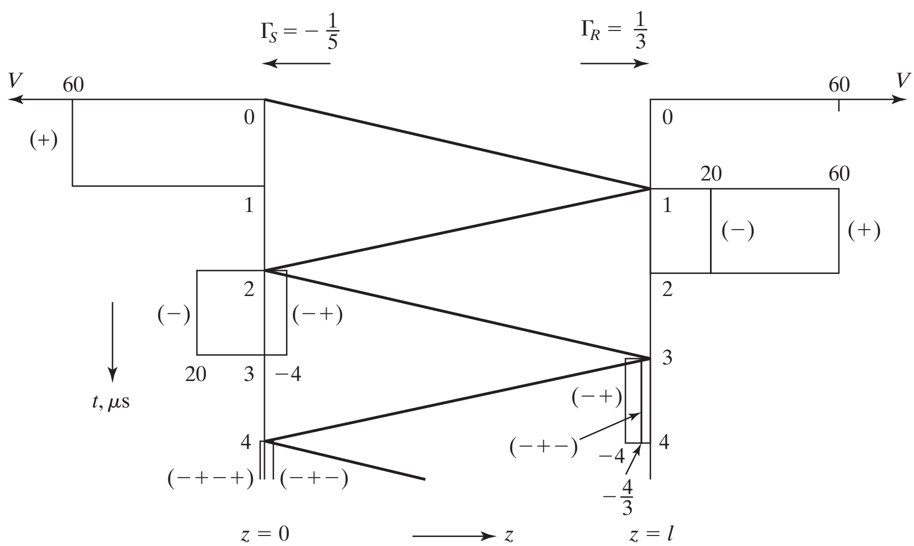



FIGURE 7.28

Voltage bounce diagram for the system of Figure 7.22 except that the voltage source is a rectangular pulse of 1-$\mu$s duration from $t = 0$ to $t = 1\ \mu$s.

From the voltage bounce diagram, sketches of line voltage versus time at $z = 0$ and $z = l$ can be drawn, as shown in Figures 7.29(a) and (b), respectively. To draw the sketch of line voltage versus time for any other value of $z$, we note that as time progresses, the (+) wave pulses slide down the crisscross lines from left to right, whereas the (−) wave pulses slide down from right to

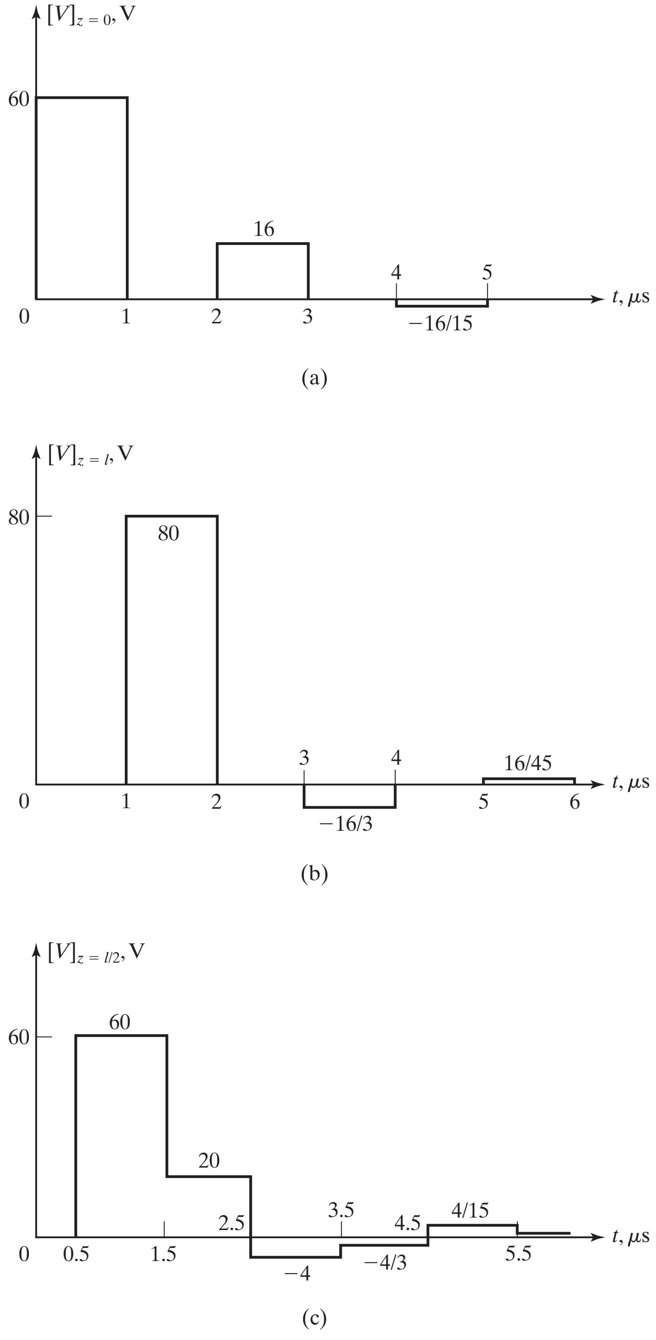



FIGURE 7.29

Time variations of line voltage at (a) $z = 0$, (b) $z = l$, and (c) $z = l/2$ for the system of Figure 7.22, except that the voltage source is a rectangular pulse of 1-$\mu$s duration from $t = 0$ to $t = 1$ $\mu$s.

left. Thus, to draw the sketch for $z = l/2$, we displace the time plots of the $(+)$ waves at $z = 0$ and of the $(-)$ waves at $z = l$ forward in time by 0.5 $\mu$s, that is, delay them by 0.5 $\mu$s, and add them to obtain the plot shown in Figure 7.29(c).

Sketches of line voltage versus distance ($z$) along the line for fixed values of time can also be drawn from the bounce diagram, based on the phenomenon of the individual pulses sliding down the crisscross lines. Thus, if we wish to sketch $V(z)$ for $t = 2.25$ $\mu$s, then we take the portion from $t = 2.25$ $\mu$s back to $t = 2.25 - 1 = 1.25$ $\mu$s (since the one-way travel time on the line is 1 $\mu$s) of all the $(+)$ wave pulses at $z = 0$ and lay them on the line from $z = 0$ to $z = l$, and we take the portion from $t = 2.25$ $\mu$s back to $t = 2.25 - 1 = 1.25$ $\mu$s of all the $(-)$ wave pulses at $z = l$ and lay them on the line from $z = l$ back to $z = 0$. In this case, we have only one $(+)$ wave pulse, that of the $(-+)$ wave, and only one $(-)$ wave pulse, that of the $(-)$ wave, as shown in Figures 7.30(a) and (b). The line voltage is then the superposition of these two waveforms, as shown in Figure 7.30(c).

Similar considerations apply for the current bounce diagram and plots of line current versus $t$ for fixed values of $z$ and line current versus $z$ for fixed values of $t$.

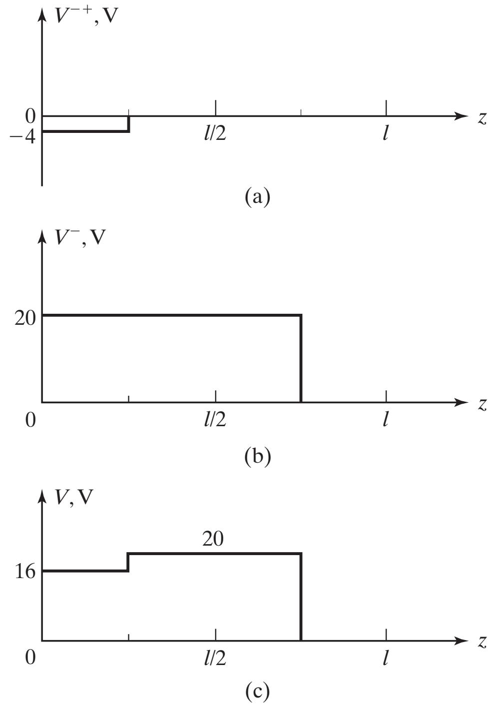



FIGURE 7.30

Variations with $z$ of (a) the $(-+)$ wave voltage, (b) the $(-)$ wave voltage, and (c) the total line voltage, at $t = 2.25$ $\mu$s for the system of Figure 7.22, except that the voltage source is a rectangular pulse of 1-$\mu$s duration from $t = 0$ to $t = 1$ $\mu$s.

## 7.5 LINES WITH INITIAL CONDITIONS

Thus far, we have considered lines with quiescent initial conditions, that is, with no initial voltages and currents on them. As a prelude to the discussion of analysis of interconnections between logic gates, we shall now consider lines with nonzero initial conditions. We discuss first the general case of arbitrary initial voltage and current distributions by decomposing them into (+) and (−) wave voltages and currents. To do this, we consider the example shown in Figure 7.31, in which a line open-circuited at both ends is charged initially, say, at $t = 0$, to the voltage and current distributions shown in the figure.

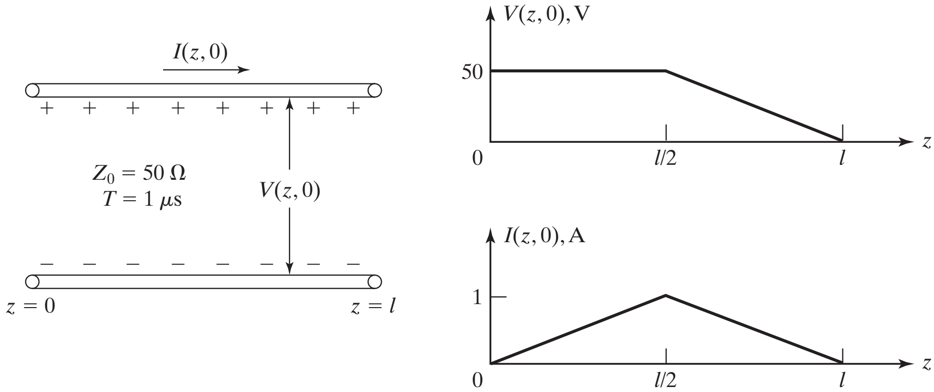



FIGURE 7.31

Line open-circuited at both ends and initially charged to the voltage and current distributions $V(z, 0)$ and $I(z, 0)$, respectively.

Writing the line voltage and current distributions as sums of (+) and (−) wave voltages and currents, we have

$$V^+(z, 0) + V^-(z, 0) = V(z, 0) \tag{7.56a}$$

$$I^+(z, 0) + I^-(z, 0) = I(z, 0) \tag{7.56b}$$

But we know that $I^+ = V^+/Z_0$ and $I^- = -V^-/Z_0$. Substituting these into (7.56b) and multiplying by $Z_0$, we get

$$V^+(z, 0) - V^-(z, 0) = Z_0 I(z, 0) \tag{7.57}$$

Solving (7.56a) and (7.57), we obtain

$$V^+(z, 0) = \tfrac{1}{2}[V(z, 0) + Z_0 I(z, 0)] \tag{7.58a}$$

$$V^-(z, 0) = \tfrac{1}{2}[V(z, 0) - Z_0 I(z, 0)] \tag{7.58b}$$

Thus, for the distributions $V(z, 0)$ and $I(z, 0)$ given in Figure 7.31, we obtain the distributions of $V^+(z, 0)$ and $V^-(z, 0)$, as shown by Figure 7.32(a), and hence of $I^+(z, 0)$ and $I^-(z, 0)$, as shown by Figure 7.32(b).

Suppose that we wish to find the voltage and current distributions at some later value of time, say, $t = 0.5\ \mu s$. Then, we note that as the (+) and (−) waves propagate

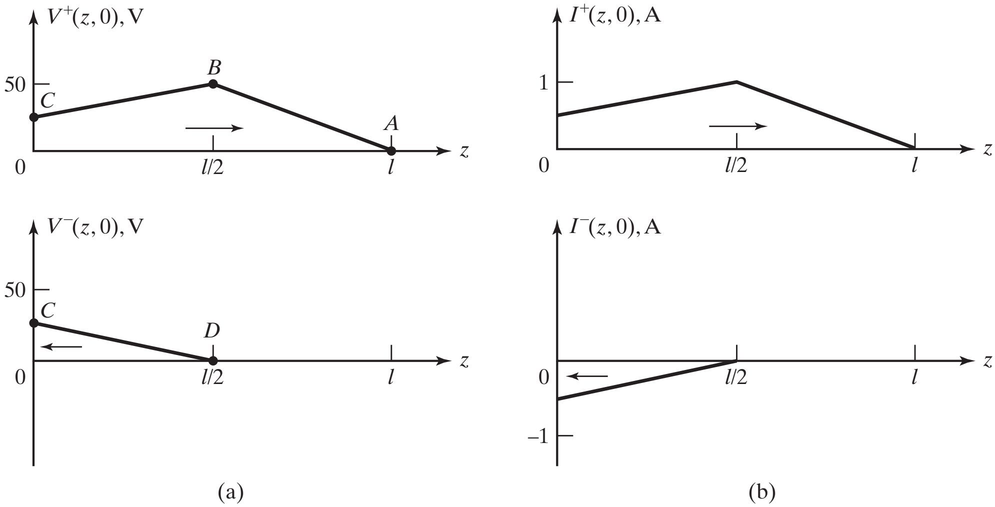



FIGURE 7.32

Distributions of (a) voltage and (b) current in the $(+)$ and $(-)$ waves obtained by decomposing the voltage and current distributions of Figure 7.31.

and impinge on the open circuits at $z = l$ and $z = 0$, respectively, they produce the $(-)$ and $(+)$ waves, respectively, consistent with a voltage reflection coefficient of 1 and current reflection coefficient of $-1$ at both ends. Hence, at $t = 0.5\ \mu$s, the $(+)$ and $(-)$ wave voltage and current distributions and their sum distributions are as shown in Figure 7.33, in which the points $A$, $B$, $C$, and $D$ correspond to the points $A$, $B$, $C$, and $D$, respectively, in Figure 7.32. Proceeding in this manner, one can obtain the voltage and current distributions for any value of time.

Suppose that we connect a resistor of value $Z_0$ at the end $z = l$ at $t = 0$ instead of keeping it open-circuited. Then the reflection coefficient at that end becomes zero thereafter, and the $(+)$ wave, as it impinges on the resistor, gets absorbed in it instead of producing the $(-)$ wave. The line therefore completely discharges into the resistor by the time $t = 1.5\ \mu$s, with the resulting time variation of voltage across $R_L$, as shown in Figure 7.34, where the points $A$, $B$, $C$, and $D$ correspond to the points $A$, $B$, $C$, and $D$, respectively, in Figure 7.32.

For a line with uniform initial voltage and current distributions, the analysis can be performed in the same manner as for arbitrary initial voltage and current distributions. Alternatively, and more conveniently, the analysis can be carried out with the aid of superposition and bounce diagrams. The basis behind this method lies in the fact that the uniform distribution corresponds to a situation in which the line voltage and current remain constant with time at all points on the line until a change is made at some point on the line. The boundary condition is then violated at that point, and a transient wave of constant voltage and current is set up, to be superimposed on the initial distribution. We shall illustrate this technique of analysis by means of an example.

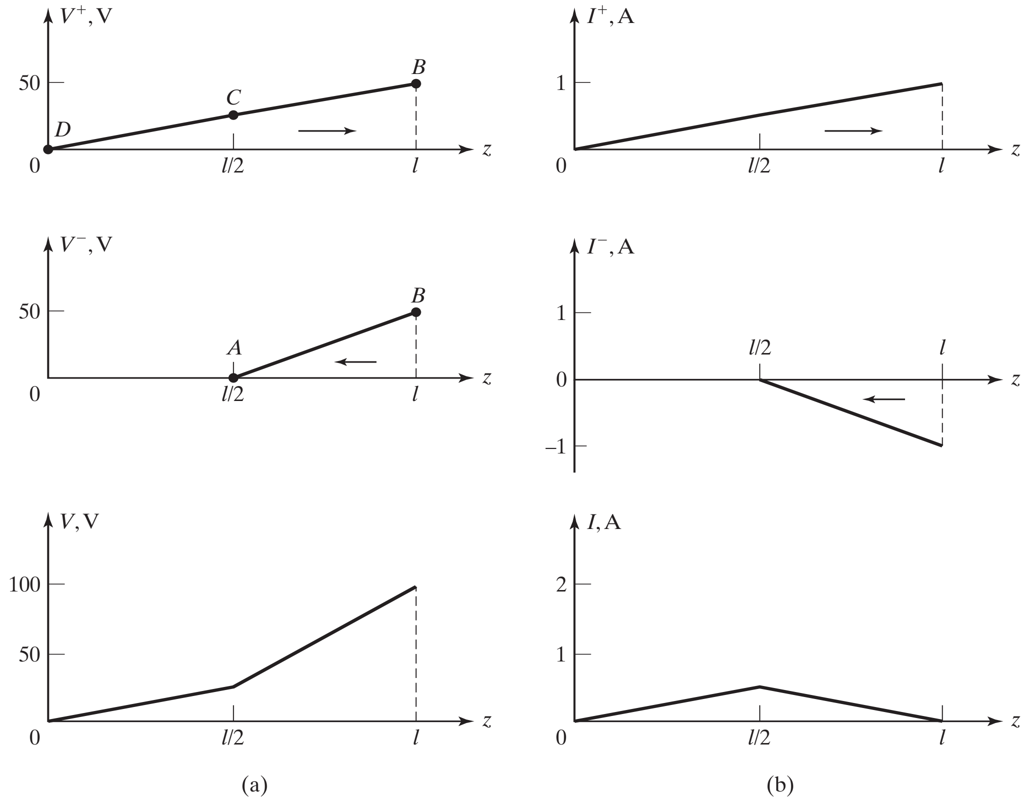



FIGURE 7.33

Distributions of (a) voltage and (b) current in the (+) and (−) waves and their sum for $t = 0.5\ \mu s$ for the initially charged line of Figure 7.31.

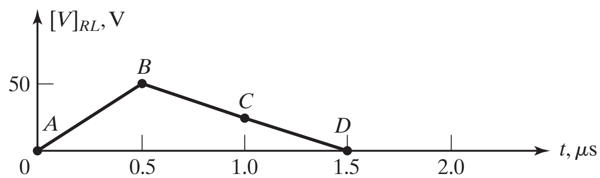



FIGURE 7.34

Voltage across $R_L(= Z_0 = 50\ \Omega)$ resulting from connecting it at $t = 0$ to the end $z = l$ of the line of Figure 7.31.

## Example 7.6

Let us consider a line of $Z_0 = 50\ \Omega$ and $T = 1\ \mu s$ initially charged to uniform voltage $V_0 = 100$ V and zero current. A resistor $R_L = 150\ \Omega$ is connected at $t = 0$ to the end $z = 0$ of the line, as shown in Figure 7.35(a). We wish to obtain the time variation of the voltage across $R_L$ for $t > 0$.

Since the change is made at $z = 0$ by connecting $R_L$ to the line, a (+) wave originates at $z = 0$, so that the total line voltage at that point is $V_0 + V^+$ and the total line current

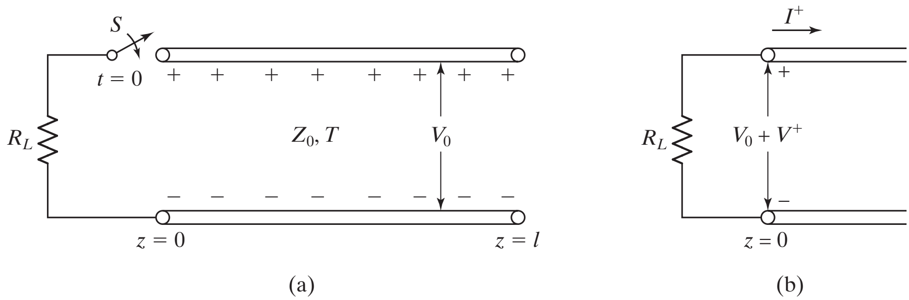



FIGURE 7.35

(a) Transmission line charged initially to uniform voltage $V_0$. (b) For obtaining the voltage and current associated with the transient (+) wave resulting from the closure of the switch in (a).

is $0 + I^+$, or $I^+$, as shown in Figure 7.35(b). To satisfy the boundary condition at $z = 0$, we then write

$$V_0 + V^+ = -R_L I^+ \tag{7.59}$$

But we know that $I^+ = V^+/Z_0$. Hence, we have

$$V_0 + V^+ = -\frac{R_L}{Z_0}V^+ \tag{7.60}$$

or

$$V^+ = -V_0\frac{Z_0}{R_L + Z_0} \tag{7.61a}$$

$$I^+ = -V_0\frac{1}{R_L + Z_0} \tag{7.61b}$$

For $V_0 = 100$ V, $Z_0 = 50\ \Omega$, and $R_L = 150\ \Omega$, we obtain $V^+ = -25$ V and $I^+ = -0.5$ A.

We may now draw the voltage and current bounce diagrams, as shown in Figure 7.36. We note that in these bounce diagrams, the initial conditions are accounted for by the horizontal lines drawn at the top, with the numerical values of voltage and current indicated on them. Sketches of line voltage and current versus $z$ for fixed values of $t$ can be drawn from these bounce diagrams in the usual manner. Sketches of line voltage and current versus $t$ for any fixed value of $z$ also can be drawn from the bounce diagrams in the usual manner. Of particular interest is the voltage across $R_L$, which illustrates how the line discharges into the resistor. The time variation of this voltage is shown in Figure 7.37.

It is also instructive to check the energy balance, that is, to verify that the energy dissipated in the 150-$\Omega$ resistor for $t > 0$ is indeed equal to the energy stored in the line at $t = 0-$, since the line is lossless. To do this, we note that, in general, energy is stored in both electric and magnetic fields in the line, with energy densities $\frac{1}{2}\mathscr{C}V^2$ and $\frac{1}{2}\mathscr{L}I^2$, respectively. Thus, for a line charged uniformly to voltage $V_0$ and current $I_0$, the total electric and magnetic stored energies are given by

$$\begin{aligned} W_e &= \frac{1}{2}\mathscr{C}V_0^2 l = \frac{1}{2}\mathscr{C}V_0^2 v_p T \\ &= \frac{1}{2}\mathscr{C}V_0^2\frac{1}{\sqrt{\mathscr{L}\mathscr{C}}}T = \frac{1}{2}\frac{V_0^2}{Z_0}T \end{aligned} \tag{7.62a}$$

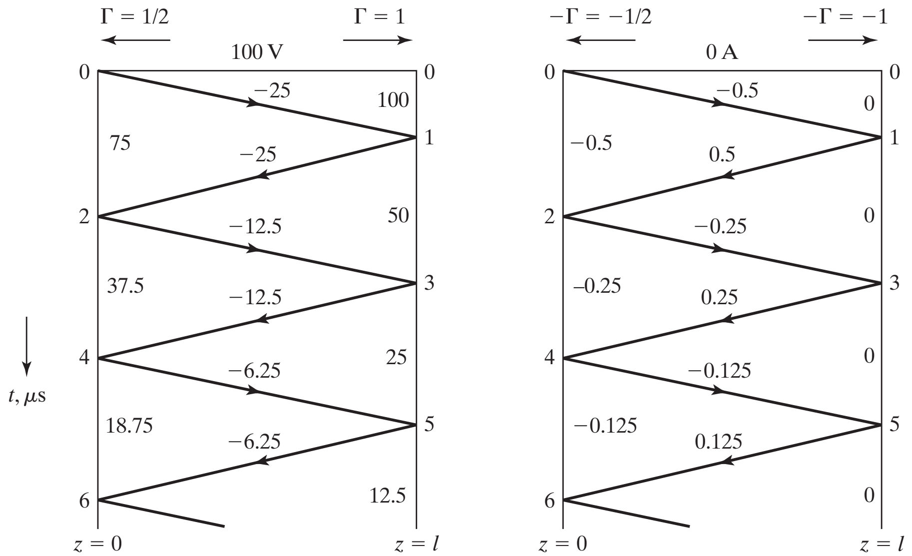



FIGURE 7.36

Voltage and current bounce diagrams depicting the transient phenomenon for $t > 0$ for the line of Figure 7.35(a), for $V_0 = 100$ V, $Z_0 = 50\ \Omega$, $R_L = 150\ \Omega$, and $T = 1\ \mu$s.

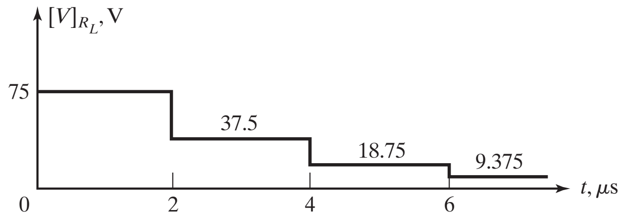



FIGURE 7.37

Time variation of voltage across $R_L$ for $t > 0$ in Figure 7.35(a) for $V_0 = 100$ V, $Z_0 = 50\ \Omega$, $R_L = 150\ \Omega$, and $T = 1\ \mu$s.

$$\begin{aligned} W_m &= \frac{1}{2}\mathcal{L}I_0^2 l = \frac{1}{2}\mathcal{L}I_0^2 v_p T \\ &= \frac{1}{2}\mathcal{L}I_0^2 \frac{1}{\sqrt{\mathcal{L}\mathcal{C}}}T = \frac{1}{2}I_0^2 Z_0 T \end{aligned} \tag{7.62b}$$

Since, for the example under consideration, $V_0 = 100$ V, $I_0 = 0$, and $T = 1\ \mu$s, $W_e = 10^{-4}$ J and $W_m = 0$. Thus, the total initial stored energy in the line is $10^{-4}$ J. Now, denoting the power dissipated in the resistor to be $P_d$, we obtain the energy dissipated in the resistor to be

$$\begin{aligned} W_d &= \int_{t=0}^{\infty} P_d\, dt \\ &= \int_0^{2\times10^{-6}} \frac{75^2}{150}\,dt + \int_{2\times10^{-6}}^{4\times10^{-6}} \frac{37.5^2}{150}\,dt + \int_{4\times10^{-6}}^{6\times10^{-6}} \frac{18.75^2}{150}\,dt + \cdots \\ &= \frac{2\times10^{-6}}{150}\times 75^2\left(1+\frac{1}{4}+\frac{1}{16}+\cdots\right) = 10^{-4}\ \text{J} \end{aligned}$$

which is exactly the same as the initial stored energy in the line, thereby satisfying the energy balance.

## 7.6 INTERCONNECTIONS BETWEEN LOGIC GATES

Thus far, we have been concerned with time-domain analysis for lines with terminations and discontinuities made up of linear circuit elements. Logic gates present nonlinear resistive terminations to the interconnecting transmission lines in digital circuits. The analysis is then made convenient by a graphical technique known as the *load-line* technique. We shall first introduce this technique by means of an example.

### Example 7.7

Let us consider the transmission-line system shown in Figure 7.38, in which the line is terminated by a passive nonlinear element having the indicated *V-I* relationship. We wish to obtain the time variations of the voltages $V_S$ and $V_L$ at the source and load ends, respectively, following the closure of the switch $S$ at $t = 0$, using the load-line technique.

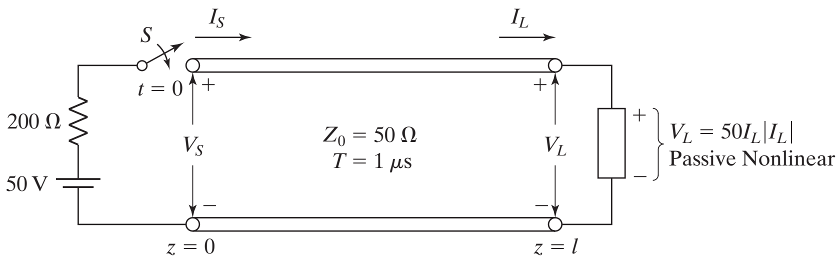



FIGURE 7.38

Line terminated by a passive nonlinear element and driven by a constant-voltage source in series with internal resistance.

With reference to the notation shown in Figure 7.38, we can write the following equations pertinent to $t = 0+$ at $z = 0$:

$$50 = 200 I_S + V_S \tag{7.63a}$$

$$V_S = V^+$$

$$I_S = I^+ = \frac{V^+}{Z_0} = \frac{V_S}{50} \tag{7.63b}$$

where $V^+$ and $I^+$ are the voltage and current, respectively, of the $(+)$ wave set up immediately after closure of the switch. The two equations (7.63a) and (7.63b) can be solved graphically by constructing the straight lines representing them, as shown in Figure 7.39, and obtaining the point of intersection $A$, which gives the values of $V_S$ and $I_S$. Note in particular that (7.63b) is a straight line of slope $1/50$ and passing through the origin.

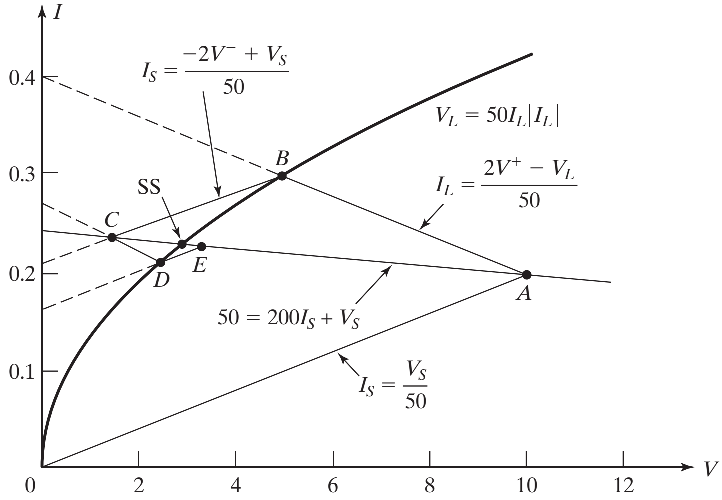



FIGURE 7.39

Graphical solution for obtaining time variations of $V_S$ and $V_L$ for $t > 0$ in the transmission-line system of Figure 7.38.

When the $(+)$ wave reaches the load end $z = l$ at $t = T$, a $(-)$ wave is set up. We can then write the following equations pertinent to $t = T+$ at $z = l$:

$$V_L = 50I_L|I_L| \tag{7.64a}$$

$$\begin{aligned} V_L &= V^+ + V^- \\ I_L &= I^+ + I^- = \frac{V^+ - V^-}{Z_0} \\ &= \frac{V^+ - (V_L - V^+)}{50} = \frac{2V^+ - V_L}{50} \end{aligned} \tag{7.64b}$$

where $V^-$ and $I^-$ are the $(-)$ wave voltage and current, respectively. The solution for $V_L$ and $I_L$ is then given by the intersection of the nonlinear curve representing (7.64a) and the straight line of slope $-1/50$ corresponding to (7.64b). Noting from (7.64b) that for $V_L = V^+$, $I_L = V^+/50$, we see that the straight line passes through point $A$. Thus, the solution of (7.64a) and (7.64b) is given by point $B$ in Figure 7.39.

When the $(-)$ wave reaches the source end $z = 0$ at $t = 2T$, it sets up a reflection. Denoting this to be the $(-+)$ wave, we can then write the following equations pertinent to $t = 2T+$ at $z = 0$:

$$50 = 200I_S + V_S \tag{7.65a}$$

$$\begin{aligned} V_S &= V^+ + V^- + V^{-+} \\ I_S &= I^+ + I^- + I^{-+} = \frac{V^+ - V^- + V^{-+}}{Z_0} \\ &= \frac{V^+ - V^- + (V_S - V^+ - V^-)}{50} = \frac{-2V^- + V_S}{50} \end{aligned} \tag{7.65b}$$

where $V^{-+}$ and $I^{-+}$ are the $(-+)$ wave voltage and current, respectively. Noting from (7.65a) that for $V_S = V^+ + V^-$, $I_S = (V^+ - V^-)/50$, we see that (7.65b) represents a straight line of slope 1/50 passing through $B$. Thus, the solution of (7.65a) and (7.65b) is given by point $C$ in Figure 7.39.

Continuing in this manner, we observe that the solution consists of obtaining the points of intersection on the source and load $V$-$I$ characteristics by drawing successively straight lines of slope $1/Z_0$ and $-1/Z_0$, beginning at the origin (the initial state) and with each straight line originating at the previous point of intersection, as shown in Figure 7.39. The points $A, C, E, \ldots$, give the voltage and current at the source end for $0 < t < 2T, 2T < t < 4T, 4T < t < 6T, \ldots$, whereas the points $B, D, \ldots$, give the voltage and current at the load end for $T < t < 3T, 3T < t < 5T, \ldots$. Thus, for example, the time variations of $V_S$ and $V_L$ are shown in Figures 7.40(a) and (b), respectively. Finally, it can be seen from Figure 7.39 that the steady-state values of line voltage and current are reached at the point of intersection (denoted SS) of the source and load $V$-$I$ characteristics.

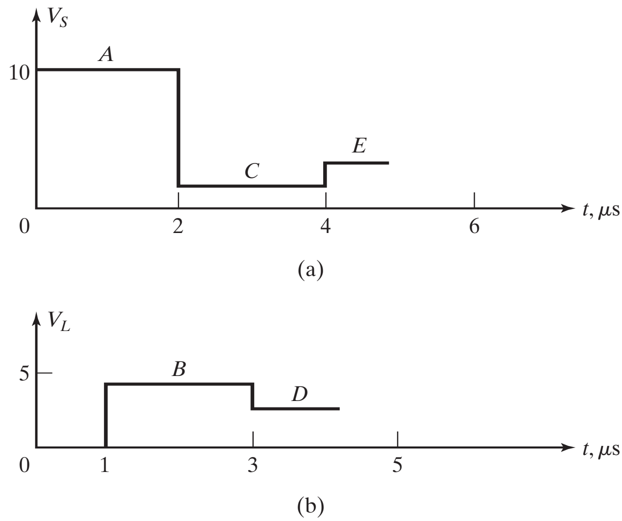



FIGURE 7.40

Time variations of (a) $V_S$ and (b) $V_L$, for the transmission-line system of Figure 7.38. The voltage levels $A$, $B$, $C$, ... correspond to those in Figure 7.39.

Now, going back to Example 7.6, the behavior of the system for the uniformly charged line can be analyzed by using the load-line technique, as an alternative to the solution using the bounce-diagram technique. Thus, noting that the terminal voltage-current characteristics at the ends $z = 0$ and $z = l$ of the system in Figure 7.35 are given by $V = -IR_L = -150I$ and $I = 0$, respectively, and that the characteristic impedance of the line is 50 Ω, we can carry out the load-line construction, as shown in Figure 7.41, beginning at the point $A$ (100 V, 0 A), and drawing alternately straight lines of slope 1/50 and −1/50 to obtain the points of intersection $B, C, D, \ldots$. The points $B, D, F, \ldots$ give the line voltage and current values at the end $z = 0$ for intervals of 2 μs beginning at $t = 0$ μs, 2 μs, 4 μs, ..., whereas the points $C, E, \ldots$ give the line voltage and current values at the end $z = l$ for intervals of 2 μs beginning at $t = 1$ μs, 3 μs, .... For example, the time variation of the line voltage at $z = 0$ provided by the load-line construction is the same as in Figure 7.37.

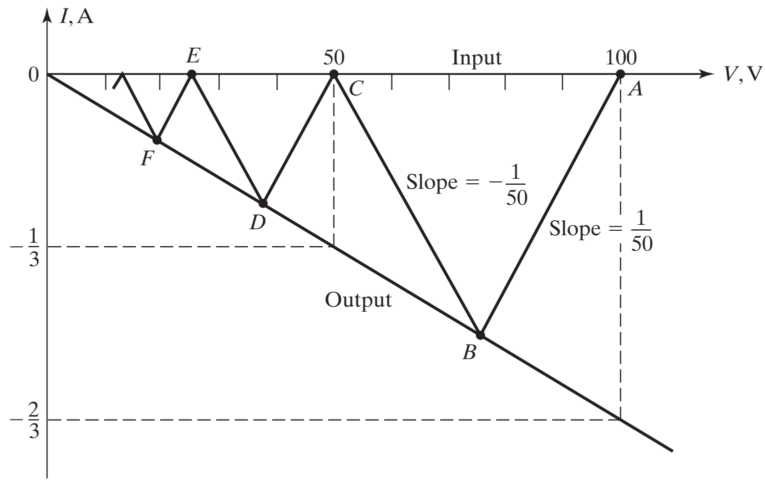
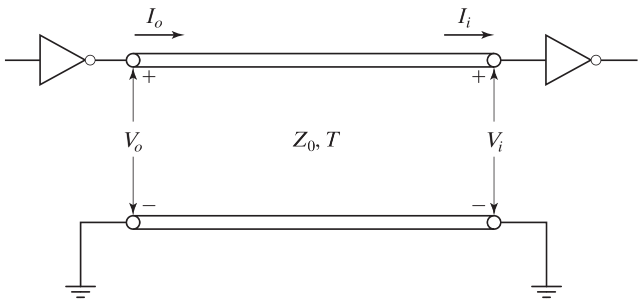



FIGURE 7.41

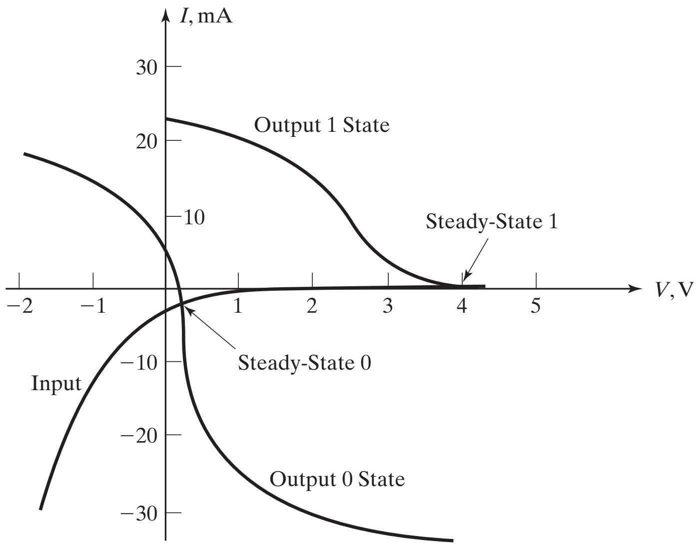
Load-line construction for the analysis of the system of Figure 7.35(a).

We shall now apply the procedure for the use of the load-line technique for a line with uniform initial distribution, just illustrated, to the analysis of the system in Figure 7.42(a) in which two transistor-transistor logic (TTL) inverters are interconnected by using a transmission line of characteristic impedance $Z_0$ and one-way travel time $T$. As the name inverter implies, the gate has an output that is the inverse of the input. Thus, if the input is in the HIGH (logic 1) range, the output will be in the LOW (logic 0) range, and vice versa. Typical $V$-$I$ characteristics for a TTL inverter are shown in Figure 7.42(b). As shown in this figure, when the system is in the steady state with the output of the first inverter in the 0 state, the voltage and current along the line are given by the intersection of the output 0 characteristic and the input characteristic; when the system is in the steady state with the output of the first inverter in the 1 state, the voltage and current along the line are given by the intersection of the output 1 characteristic and the input characteristic. Thus, the line is charged to 0.2 V for the steady-state 0 condition and to 4 V for the steady-state 1 condition. We wish to study the transient phenomena corresponding to the transition when the output of the first gate switches from the 0 to the 1 state, and vice versa, assuming $Z_0$ of the line to be 30 Ω.

Considering first the transition from the 0 state to the 1 state, and following the line of argument in Example 7.7, we carry out the construction shown in Figure 7.43(a). This construction consists of beginning at the point corresponding to the steady-state 0 (the initial state) and drawing a straight line of slope $1/30$ to intersect with the output 1 characteristic at point $A$, then drawing from point $A$ a straight line of slope $-1/30$ to intersect the input characteristic at point $B$, and so on. From this construction, the variation of the voltage $V_i$ at the input of the second gate can be sketched as shown in Figure 7.43(b), in which the voltage levels correspond to the




(a)



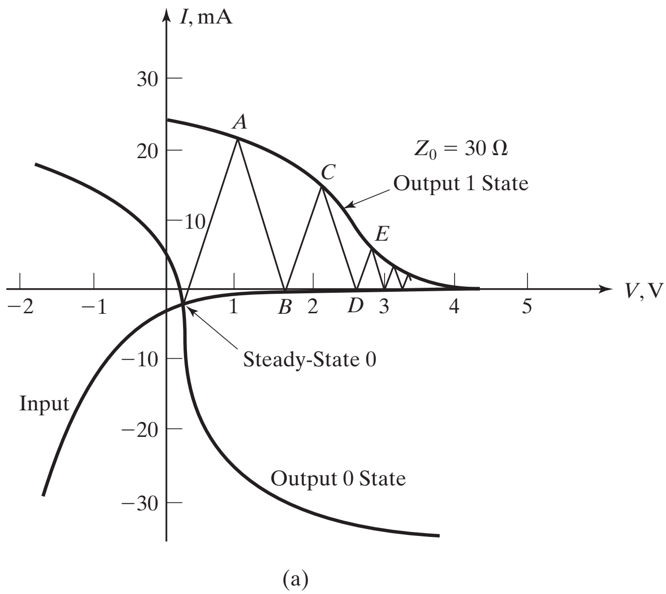

(b)

FIGURE 7.42

(a) Transmission-line interconnection between two logic gates. (b) Typical $V$-$I$ characteristics for the logic gates.

points 0, $B$, $D$, ..., in Figure 7.43(a). The effect of the transients on the performance of the system may now be seen by noting from Figure 7.43(b) that depending on the value of the minimum gate voltage that will reliably be recognized as logic 1, a time delay in excess of $T$ may be involved in the transition from 0 to 1. Thus, if this minimum voltage is 2 V, the interconnecting line will result in an extra time delay of $2T$ for the input of the second gate to switch from 0 to 1, since $V_i$ does not exceed 2 V until $t = 3T+$.

Considering next the transition from the 1 state to the 0 state, we carry out the construction shown in Figure 7.44(a), with the crisscross lines beginning at the point



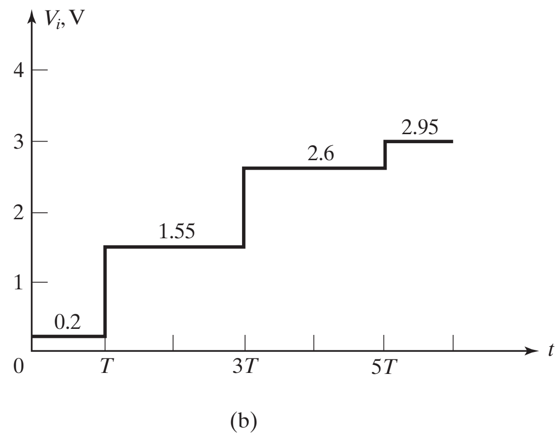




FIGURE 7.43

(a) Construction based on the load-line technique for analysis of the 0-to-1 transition for the system of Figure 7.42(a). (b) Plot of $V_i$ versus $t$ obtained from the construction in (a).

corresponding to the steady-state 1. From this construction, we obtain the plot of $V_i$ versus $t$, as shown in Figure 7.44(b), in which the voltage levels correspond to the points $1, B, D, \ldots$, in Figure 7.44(a). If we assume a maximum gate input voltage that can be readily recognized as logic 0 to be 1 V, it can once again be seen that an extra time delay of $2T$ is involved in the switching of the input of the second gate from 1 to 0, since $V_i$ does not drop below 1 V until $t = 3T+$.

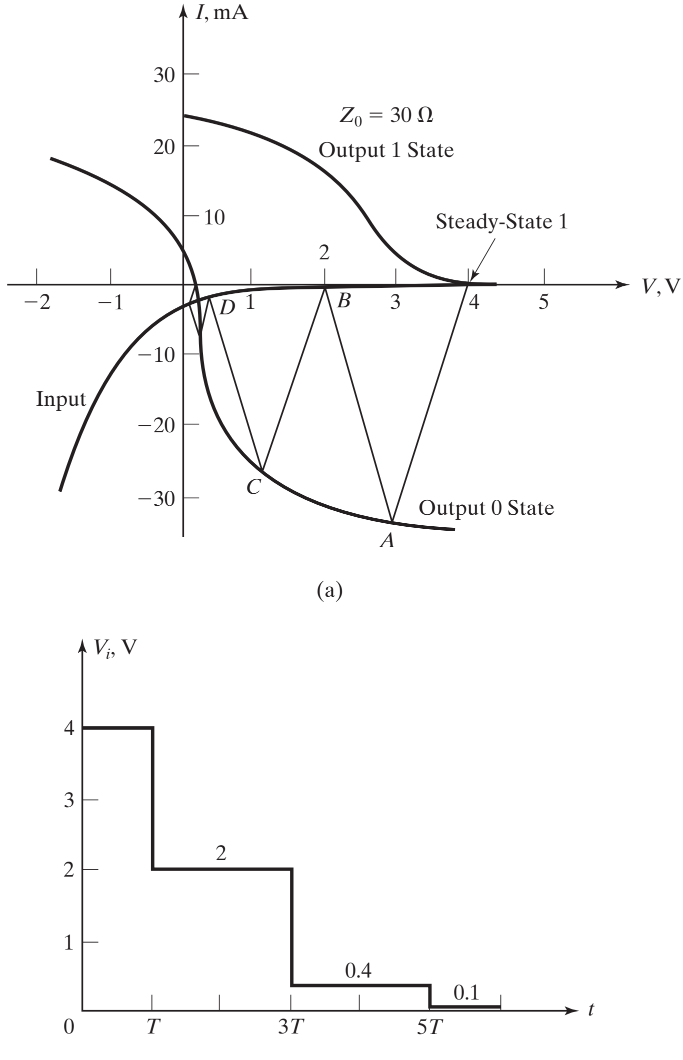



FIGURE 7.44

(a) Construction based on the load-line technique for analysis of the 1-to-0 transition for the system of Figure 7.42(a). (b) Plot of $V_i$ versus $t$ obtained from the construction in (a).

## SUMMARY

In this chapter, we first studied frequency domain analysis of transmission lines. The general solutions to the transmission-line equations, expressed in phasor form, that is,

$$\frac{\partial \bar{V}}{\partial z} = -j\omega\mathscr{L}\bar{I} \tag{7.66a}$$

$$\frac{\partial \bar{I}}{\partial z} = -\mathscr{G}\bar{V} - j\omega\mathscr{C}\bar{V} \tag{7.66b}$$

are given by

$$\bar{V}(z) = \bar{A}e^{-\bar{\gamma}z} + \bar{B}e^{\bar{\gamma}z} \tag{7.67a}$$

$$\bar{I}(z) = \frac{1}{\bar{Z}_0}(\bar{A}e^{-\bar{\gamma}z} - \bar{B}e^{\bar{\gamma}z}) \tag{7.67b}$$

where

$$\bar{\gamma} = \sqrt{j\omega\mathscr{L}(\mathscr{G} + j\omega\mathscr{C})} \qquad [=\sqrt{j\omega\mu(\sigma + j\omega\epsilon)}]$$

$$\bar{Z}_0 = \sqrt{\frac{j\omega\mathscr{L}}{\mathscr{G} + j\omega\mathscr{C}}} \qquad \left[\neq\sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}\right]$$

are the propagation constant and the characteristic impedance, respectively, of the line. For a lossless line ($\mathscr{G} = 0$), these reduce to

$$\bar{\gamma} = j\beta = j\omega\sqrt{\mathscr{L}\mathscr{C}} \qquad (=j\omega\sqrt{\mu\epsilon})$$

$$\bar{Z}_0 = Z_0 = \sqrt{\frac{\mathscr{L}}{\mathscr{C}}} \qquad (\neq\sqrt{\mu/\epsilon})$$

so that for a lossless line,

$$\bar{V}(z) = \bar{A}e^{-j\beta z} + \bar{B}e^{j\beta z} \tag{7.68a}$$

$$\bar{I}(z) = \frac{1}{Z_0}(\bar{A}e^{-j\beta z} - \bar{B}e^{j\beta z}) \tag{7.68b}$$

The solutions given by (7.67a) and (7.67b) or (7.68a) and (7.68b) represent the superposition of (+) and (−) waves propagating in the medium between the conductors of the line, expressed in terms of the line voltage and current instead of in terms of the electric and magnetic fields.

By applying these general solutions to the case of a lossless line short circuited at the far end and obtaining the particular solutions for that case, we discussed the standing wave phenomenon and the standing wave patterns resulting from the complete reflection of waves by the short circuit. We also examined the frequency behavior of the input impedance of a short-circuited line of length $l$, given by

$$\bar{Z}_{\text{in}} = jZ_0 \tan \beta l$$

and (a) illustrated its application in a technique for the location of short circuit in a line, and (b) learned that for a circuit element to behave as assumed by conventional (lumped) circuit theory, its dimensions must be a small fraction of the wavelength corresponding to the frequency of operation.

Next, we studied reflection and transmission of waves at a junction between two lossless lines. By applying them to the general solutions for the line voltage and current on either side of the junction, we deduced the ratio of the reflected wave voltage to the incident wave voltage, that is, the voltage reflection coefficient, to be

$$\Gamma_V = \frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}}$$

where $Z_{01}$ is the characteristic impedance of the line from which the wave is incident and $Z_{02}$ is the characteristic impedance of the line on which the wave is incident. The ratio of the transmitted wave voltage to the incident wave voltage, that is, the voltage transmission coefficient, is given by

$$\tau_V = 1 + \Gamma_V$$

The current reflection and transmission coefficients are given by

$$\Gamma_I = -\Gamma_V$$
$$\tau_I = 1 - \Gamma_V$$

We discussed the standing wave pattern resulting from the partial reflection of the wave at the junction and defined a quantity known as the standing wave ratio (SWR), which is a measure of the reflection phenomenon. In terms of $\Gamma_V$, it is given by

$$\text{SWR} = \frac{1 + |\Gamma_V|}{1 - |\Gamma_V|}$$

We then introduced the Smith Chart, which is a graphical aid in the solution of transmission-line problems. After first discussing the basis behind the construction of the Smith Chart, we illustrated its use by considering a transmission-line system and computing several quantities of interest. We concluded the section on Smith Chart with the solution of a transmission-line matching problem.

We devoted the remainder of the chapter to time-domain analysis of transmission lines. For a lossless line, the transmission-line equations in time domain are given by

$$\frac{\partial V}{\partial z} = -\mathcal{L}\frac{\partial I}{\partial t} \tag{7.69a}$$

$$\frac{\partial I}{\partial z} = -\mathcal{C}\frac{\partial V}{\partial t} \tag{7.69b}$$

The solutions to these equations are

$$V(z, t) = Af\left(t - \frac{z}{v_p}\right) + Bg\left(t + \frac{z}{v_p}\right) \tag{7.70a}$$

$$I(z, t) = \frac{1}{Z_0}\left[Af\left(t - \frac{z}{v_p}\right) - Bg\left(t + \frac{z}{v_p}\right)\right] \tag{7.70b}$$

where $Z_0 = \sqrt{\mathcal{L}/\mathcal{C}}$ is the characteristic impedance of the line, and $v_p = 1/\sqrt{\mathcal{L}\mathcal{C}}$ is the velocity of propagation on the line.

We then discussed time-domain analysis of a transmission line terminated by a load resistance $R_L$ and excited by a constant voltage source $V_0$ in series with internal resistance $R_g$. Writing the general solutions (7.70a) and (7.70b) concisely in the manner

$$V = V^+ + V^-$$
$$I = I^+ + I^-$$

where

$$I^+ = \frac{V^+}{Z_0}$$

$$I^- = -\frac{V^-}{Z_0}$$

we found that the situation consists of the bouncing back and forth of transient (+) and (−) waves between the two ends of the line. The initial (+) wave voltage is $V^+Z_0/(R_g + Z_0)$. All other waves are governed by the reflection coefficients at the two ends of the line, given for the voltage by

$$\Gamma_R = \frac{R_L - Z_0}{R_L + Z_0}$$

and

$$\Gamma_S = \frac{R_g - Z_0}{R_g + Z_0}$$

for the load and source ends, respectively. In the steady state, the situation is the superposition of all the transient waves, equivalent to the sum of a single (+) wave and a single (−) wave. We discussed the bounce-diagram technique of keeping track of the transient phenomenon and extended it to a pulse voltage source.

As a prelude to the consideration of interconnections between logic gates, we discussed time-domain analysis of lines with nonzero initial conditions. For the general case, the initial voltage and current distributions $V(z, 0)$ and $I(z, 0)$ are decomposed into (+) and (−) wave voltages and currents as given by

$$V^+(z, 0) = \frac{1}{2}[V(z, 0) + Z_0 I(z, 0)]$$

$$V^-(z, 0) = \frac{1}{2}[V(z, 0) - Z_0 I(z, 0)]$$

$$I^+(z, 0) = \frac{1}{Z_0}V^+(z, 0)$$

$$I^-(z, 0) = -\frac{1}{Z_0}V^-(z, 0)$$

The voltage and current distributions for $t > 0$ are then obtained by keeping track of the bouncing of these waves at the two ends of the line. For the special case of uniform distribution, the analysis can be performed more conveniently by considering the situation as one in which a transient wave is superimposed on the initial distribution and using the bounce-diagram technique. We then introduced the load-line technique of time-domain analysis, and applied it to the analysis of transmission-line interconnection between logic gates.

## REVIEW QUESTIONS

**7.1.** Discuss the solutions for the transmission-line equations in frequency domain.

**7.2.** Discuss the propagation constant and characteristic impedance associated with wave propagation on transmission lines.

**7.3.** What is the boundary condition to be satisfied at a short circuit on a line?

**7.4.** For an open-circuited line, what would be the boundary condition to be satisfied at the open circuit?

**7.5.** What is a standing wave? How do complete standing waves arise? Discuss their characteristics and give an example in mechanics.

**7.6.** What is a standing wave pattern? Discuss the voltage and current standing wave patterns for the short-circuited line.

**7.7.** What would be the voltage and current standing wave patterns for an open-circuited line?

**7.8.** Discuss the variation with frequency of the input reactance of a short-circuited line and its application in the determination of the location of a short circuit.

**7.9.** Can you suggest an alternative procedure to that described in Example 7.1 to locate a short circuit in a transmission line?

**7.10.** Discuss the condition for the validity of the quasistatic approximation for the input behavior of a physical structure.

**7.11.** Discuss the input behavior of a short-circuited line for frequencies slightly beyond those for which the quasistatic approximation is valid.

**7.12.** What are the boundary conditions for the voltage and current at the junction between two transmission lines?

**7.13.** What is the voltage reflection coefficient at the junction between two transmission lines? How are the current reflection coefficient and the voltage and current transmission coefficients related to the voltage reflection coefficient?

**7.14.** What is the voltage reflection coefficient at the short circuit for a short-circuited line?

**7.15.** Can the transmitted wave current at the junction between two transmission lines be greater than the incident wave current? Explain.

**7.16.** What is a partial standing wave? Discuss the standing wave patterns corresponding to partial standing waves.

**7.17.** Define standing wave ratio (SWR). What are the standing wave ratios for (a) an infinitely long line, (b) a short-circuited line, (c) an open-circuited line, and (d) a line terminated by its characteristic impedance?

**7.18.** Define line impedance. What is its value for an infinitely long line?

**7.19.** What is the basis behind the construction of the Smith Chart? How does the Smith Chart simplify the solution of transmission-line problems?

**7.20.** Briefly discuss the mapping of the normalized line impedances from the complex $\bar{Z}_n$-plane onto the Smith Chart.

**7.21.** Why is a circle with its center at the center of the Smith Chart known as a constant SWR circle? Where on the circle is the corresponding SWR value marked?

**7.22.** Using the Smith Chart, how do you find the normalized line admittance at a point on the line given the normalized line impedance at that point?

**7.23.** Briefly discuss the solution of the transmission-line matching problem.

**7.24.** How is the length of a short-circuited stub for a required input susceptance determined by using the Smith Chart?

**7.25.** Discuss the general solutions for the line voltage and current in time-domain and the notation associated with their representation in concise form.

**7.26.** What is the fundamental distinction between the occurrence of the response in one branch of a lumped circuit to the application of an excitation in a different branch of the circuit and the occurrence of the response at one location on a transmission line to the application of an excitation at a different location on the line?

**7.27.** Describe the phenomenon of the bouncing back and forth of transient waves on a transmission line excited by a constant voltage source in series with internal resistance and terminated by a resistance.

**7.28.** Discuss the values of the voltage reflection coefficient for some special cases.

**7.29.** What is the steady-state equivalent of a line excited by a constant voltage source? What is the actual situation in the steady state?

**7.30.** Discuss the bounce-diagram technique of keeping track of the bouncing back and forth of the transient waves on a transmission line for a constant voltage source.

**7.31.** Discuss the bounce-diagram technique of keeping track of the bouncing back and forth of the transient waves on a transmission line for a pulse voltage source.

**7.32.** Discuss the determination of the voltage and current distributions on an initially charged line for any given time from the knowledge of the initial voltage and current distributions.

**7.33.** Discuss with the aid of an example the discharging of an initially charged line into a resistor.

**7.34.** Discuss the bounce-diagram technique of transient analysis of a line with uniform initial voltage and current distributions.

**7.35.** Discuss the load-line technique of obtaining the time variations of the voltages and currents at the source and load ends of a line from a knowledge of the terminal *V-I* characteristics.

**7.36.** Discuss the analysis of transmission-line interconnection between two logic gates.

## PROBLEMS

**7.1.** For a transmission line of arbitrary cross section and with the medium between the conductors characterized by $\sigma = 10^{-16}$ S/m, $\epsilon = 2.5\epsilon_0$, and $\mu = \mu_0$, it is known that $\mathscr{C} = 10^{-10}$ F/m. (a) Find $\mathscr{L}$ and $\mathscr{G}$. (b) Find $\bar{Z}_0$ for $f = 10^6$ Hz.

**7.2.** For the coaxial cable of Example 6.9 employing air dielectric, find the ratio of the outer to the inner radii for which the characteristic impedance of the cable is 75 Ω.

**7.3.** Using the general solutions for the complex line voltage and current on a lossless line given by (7.9a) and (7.9b), respectively, obtain the particular solutions for the complex voltage and current on an open-circuited line. Then find the input impedance of an open-circuited line of length *l*.

**7.4.** Solve Example 7.1 by considering the standing wave patterns between the short circuit and the generator for the two frequencies of interest and by deducing the number of wavelengths at one of the two frequencies.

**7.5.** For an air dielectric short-circuited line of characteristic impedance 50 Ω, find the minimum values of the length for which its input impedance is equivalent to that of (a) an inductor of value $0.25 \times 10^{-6}$ H at 100 MHz and (b) a capacitor of value $10^{-10}$ F at 100 MHz.

**7.6.** A transmission line of length 2 m having a nonmagnetic ($\mu = \mu_0$) perfect dielectric is short-circuited at the far end. A variable-frequency generator is connected at its input and the current drawn is monitored. It is found that the current reaches a maximum for $f = 500$ MHz and then a minimum for $f = 525$ MHz. Find the permittivity of the dielectric.

**7.7.** A voltage generator is connected to the input of a lossless line short-circuited at the far end. The frequency of the generator is varied and the line voltage and line current at the input terminals are monitored. It is found that the voltage reaches a maximum value of 10 V at 405 MHz and the current reaches a maximum value of 0.2 A at 410 MHz. (a) Find the characteristic impedance of the line. (b) Find the voltage and current values at 407 MHz.

**7.8.** Assuming that the criterion $f \ll v_p/2\pi l$ is satisfied for frequencies less than $0.1 v_p/2\pi l$, compute the maximum length of an air dielectric short-circuited line for which the input impedance is approximately that of an inductor of value equal to the total inductance of the line for $f = 100$ MHz.

**7.9.** A lossless transmission line of length 2 m and having $\mathcal{L} = 0.5\mu_0$ and $\mathcal{C} = 18\epsilon_0$ is short circuited at the far end. (a) Find the phase velocity, $v_p$. (b)Find the wavelength, the length of the line in terms of the number of wavelengths, and the input impedance of the line for each of the following frequencies: 100 Hz; 100 MHz; and 12.5 MHz.

**7.10.** Repeat Example 7.3 with the values of $Z_{01}$ and $Z_{02}$ interchanged.

**7.11.** In the transmission-line system shown in Figure 7.45, a power $P_i$ is incident on the junction from line 1. Find (a) the power reflected into line 1, (b) the power transmitted into line 2, and (c) the power transmitted into line 3.

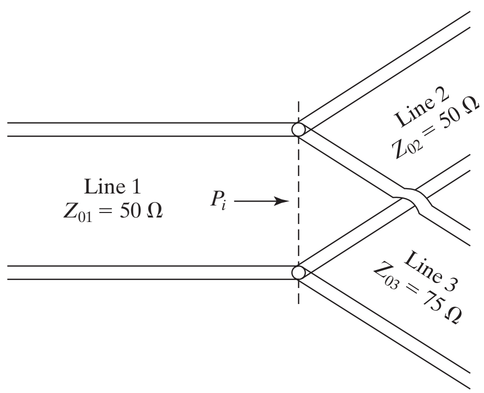



FIGURE 7.45

For Problem 7.11.

**7.12.** Show that the voltage minima of the standing wave pattern of Figure 7.9 are sharper than the voltage maxima by computing the voltage amplitude halfway between the locations of voltage maxima and minima.

**7.13.** A line assumed to be infinitely long and of unknown characteristic impedance is connected to a line of characteristic impedance 50 Ω on which standing wave measurements are made. It is found that the standing wave ratio is 3 and that two consecutive voltage minima exist at 15 cm and 25 cm from the junction of the two lines. Find the unknown characteristic impedance.

**7.14.** A line assumed to be infinitely long and of unknown characteristic impedance when connected to a line of characteristic impedance 50 Ω produces a standing wave ratio of value 2 in the 50-Ω line. The same line when connected to a line of characteristic impedance 150 Ω produces a standing wave ratio of value 1.5 in the 150-Ω line. Find the unknown characteristic impedance.

**7.15.** Compute values of $\bar{\Gamma}_V$ corresponding to several points along line $a$ in Figure 7.11(a) and show that the contour $a'$ in Figure 7.11(b) is a circle of radius $\frac{1}{2}$ and centered at $(1/2, 0)$.

**7.16.** Compute values of $\bar{\Gamma}_V$ corresponding to several points along line $b$ in Figure 7.11(a) and show that the contour $b'$ in Figure 7.11(b) is a portion of a circle of radius 2 and centered at $(1, 2)$.

**7.17.** For the transmission-line system of Figure 7.13, and for the values of $Z_{01}$, $Z_{02}$, and $l$ specified in the text, find the value of $B$ that minimizes the SWR to the left of $jB$. What is the minimum value of SWR?

**7.18.** In Figure 7.13, assume $Z_{01} = 300\ \Omega$, $Z_{02} = 75\ \Omega$, $B = 0.002$ S, and $l = 0.145\lambda_1$, and find (a) $\bar{Z}_1$, (b) SWR on line 1 to the right of $jB$, (c) $\bar{Y}_1$, and (d) SWR on line 1 to the left of $jB$.

**7.19.** A transmission line of characteristic impedance 50 Ω is terminated by a load impedance of $(73 + j0)\ \Omega$. Find the location and the length of a short-circuited stub of characteristic impedance 50 Ω for achieving a match between the line and the load.

**7.20.** Show that (7.40a) and (7.40b) satisfy the transmission-line equations (7.39a) and (7.39b).

**7.21.** In the system shown in Figure 7.46, assume that $V_g$ is a constant voltage source of 100 V and the switch $S$ is closed at $t = 0$. Find and sketch: (a) the line voltage versus $z$ for $t = 0.2\ \mu$s; (b) the line current versus $z$ for $t = 0.4\ \mu$s; (c) the line voltage versus $t$ for $z = 30$ m; and (d) the line current versus $t$ for $z = -40$ m.

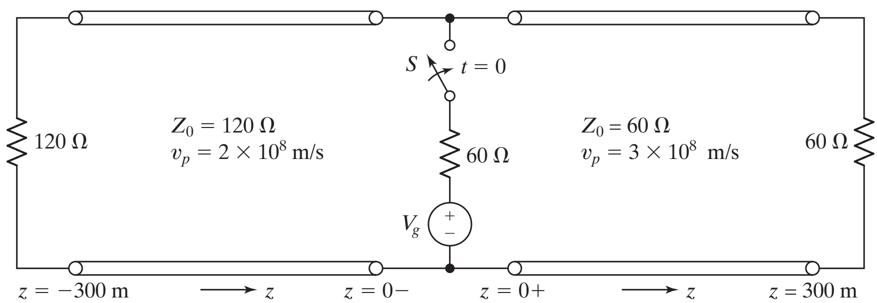



FIGURE 7.46

For Problem 7.21.

**7.22.** In the system shown in Figure 7.47(a), the switch $S$ is closed at $t = 0$. The line voltage variations with time at $z = 0$ and $z = l$ for the first 5 $\mu$s are observed to be as shown in Figure 7.47(b) and (c), respectively. Find the values of $V_0$, $R_g$, $R_L$, and $T$.

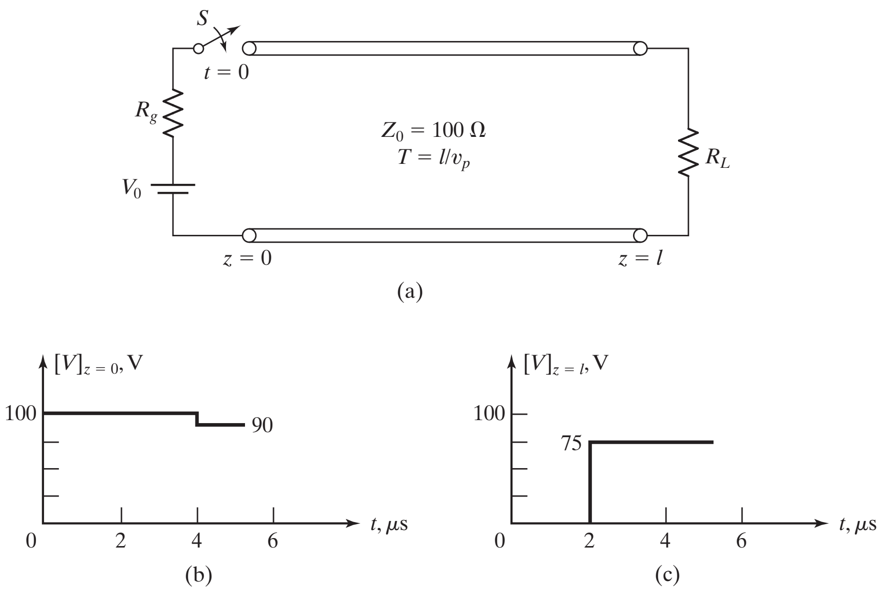



FIGURE 7.47

For Problem 7.22.

**7.23.** The system shown in Figure 7.48 is in steady state. Find (a) the line voltage and current, (b) the (+) wave voltage and current, and (c) the (−) wave voltage and current.

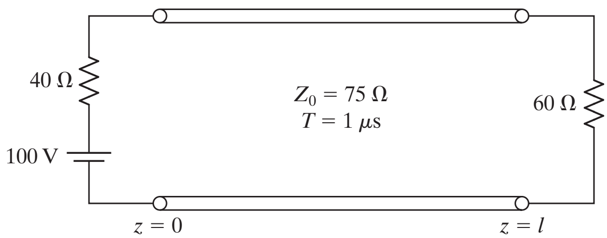



FIGURE 7.48

For Problem 7.23.

**7.24.** In the system shown in Figure 7.49, the switch $S$ is closed at $t = 0$. Assume $V_g(t)$ to be a direct voltage of 90 V and draw the voltage and current bounce diagrams. From these bounce diagrams, sketch: (a) the line voltage and line current versus $t$ (up to $t = 7.25\ \mu$s) at $z = 0$, $z = l$, and $z = l/2$; and (b) the line voltage and line current versus $z$ for $t = 1.2\ \mu$s and $t = 3.5\ \mu$s.

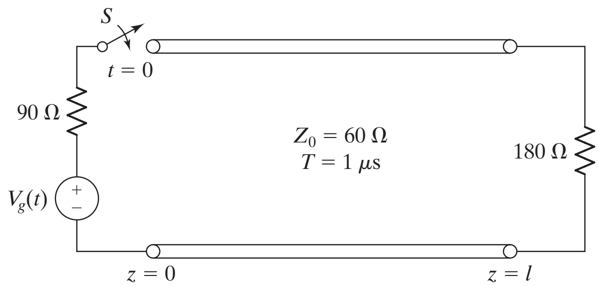



FIGURE 7.49

For Problem 7.24.

**7.25.** Repeat Problem 7.21 assuming $V_g$ to be a triangular pulse, as shown in Figure 7.50.

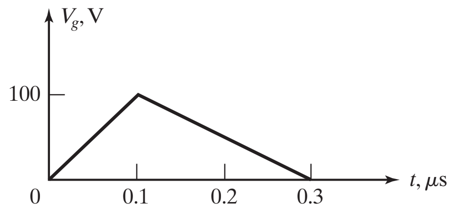



FIGURE 7.50

For Problem 7.25.

**7.26.** For the system of Problem 7.24, assume that the voltage source is of 0.3-$\mu$s duration instead of being of infinite duration. Find and sketch the line voltage and line current versus $z$ for $t = 1.2\ \mu$s and $t = 3.5\ \mu$s.

**7.27.** In the system shown in Figure 7.51, the switch $S$ is closed at $t = 0$. Find and sketch: (a) the line voltage versus $z$ for $t = 2\frac{1}{2}\ \mu$s; (b) the line current versus $z$ for $t = 2\frac{1}{2}\ \mu$s; and (c) the line voltage at $z = l$ versus $t$ up to $t = 4\ \mu$s.

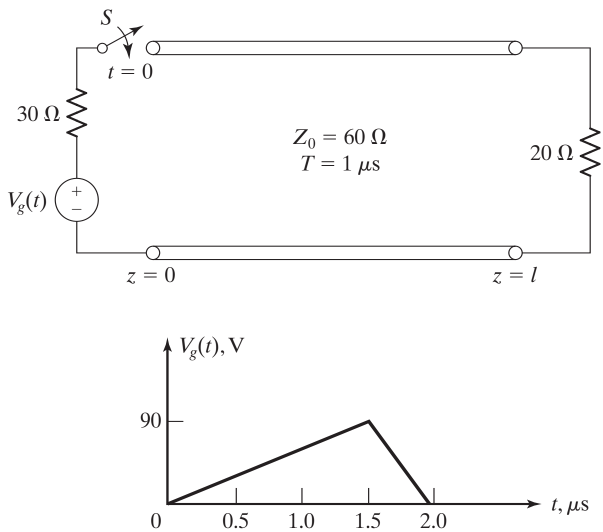



FIGURE 7.51

For Problem 7.27.

**7.28.** In the system shown in Figure 7.52, the switch $S$ is closed at $t = 0$. Draw the voltage and current-bounce diagrams and sketch (a) the line voltage and line current versus $t$ for $z = 0$ and $z = l$ and (b) the line voltage and line current versus $z$ for $t = 2, 9/4, 5/2, 11/4$, and 3 $\mu$s. Note that the period of the source voltage is 2 $\mu$s, which is equal to the two-way travel time on the line.

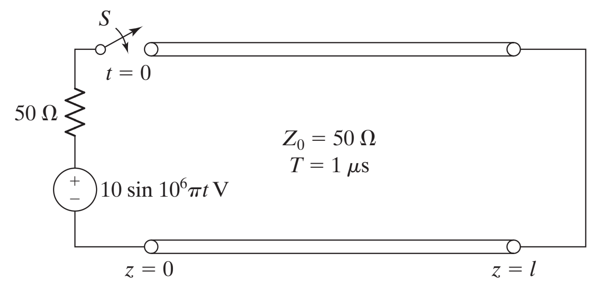



FIGURE 7.52

For Problem 7.28.

**7.29.** In the system shown in Figure 7.53, a passive nonlinear element having the indicated volt-ampere characteristic is connected to an initially charged line at $t = 0$. Find the voltage across the nonlinear element immediately after closure of the switch.

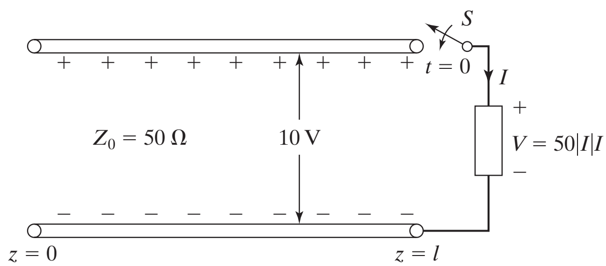



FIGURE 7.53

For Problem 7.29.

**7.30.** In the system shown in Figure 7.54, steady-state conditions are established with the switch $S$ closed. At $t = 0$, the switch is opened. (a) Find the sketch the voltage across the 150-$\Omega$ resistor for $t \geq 0$, with the aid of a bounce diagram. (b) Show that the total energy dissipated in the 150-$\Omega$ resistor after opening the switch is exactly the same as the energy stored in the line before opening the switch.

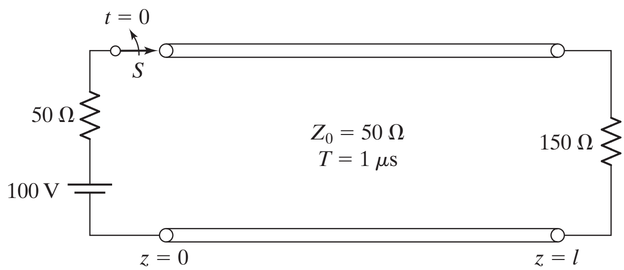



FIGURE 7.54

For Problem 7.30.

**7.31.** In the system shown in Figure 7.55, steady-state conditions are established with the switch $S$ closed. At $t = 0$, the switch is opened. (a) Sketch the voltage and current along the system for $t = 0-$. (b) Find the total energy stored in the lines for $t = 0-$. (c) Find and sketch the voltages across the two resistors for $t > 0$. (d) From your sketches of part (c), find the total energy dissipated in the resistors for $t > 0$.

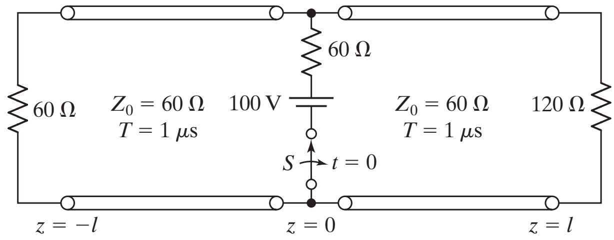



FIGURE 7.55

For Problem 7.31.

**7.32.** For the system of Problem 7.24, use the load-line technique to obtain and plot line voltage and line current versus $t$ (up to $t = 5.25\ \mu s$) at $z = 0$ and $z = l$. Also obtain the steady-state values of line voltage and current from the load-line construction.

**7.33.** For the system of Problem 7.29, use the load-line technique to obtain and plot line voltage versus $t$ from $t = 0$ up to $t = 7l/v_p$ at $z = 0$ and $z = l$.

**7.34.** For the example of interconnection between logic gates of Figure 7.42(a), repeat the load-line constructions for $Z_0 = 50\ \Omega$ and draw graphs of $V_i$ versus $t$ for both 0-to-1 and 1-to-0 transitions.

**7.35.** For the example of interconnection between logic gates of Figure 7.42(a), find (a) the minimum value of $Z_0$ such that for the transition form 0 to 1, the voltage $V_i$ reaches 2V at $t = T+$ and (b) the minimum value of $Z_0$ such that for the transition from 1 to 0, the voltage $V_i$ reaches 1 V at $t = T+$.