CHAPTER

6 Statics, Quasistatics, and Transmission Lines

In the preceding chapters, we learned that the phenomenon of wave propagation is based upon the interaction between the time-varying or dynamic electric and magnetic fields. In this chapter, we shall use the thread of statics-quasistatics-waves to bring out the frequency behavior of physical structures. Static fields are studied by setting the time derivatives in Maxwell's equations equal to zero. We will introduce the *lumped* circuit elements familiar in circuit theory, through the different classifications of static fields. For a nonzero frequency, the fields are dynamic. The exact solutions are solutions to the complete Maxwell's equations for time-varying fields. However, a class of fields, known as quasistatic fields, can be studied as low-frequency extensions of static fields. They are approximations to the exact solutions. We will learn that for quasistatic fields, the circuit equivalent for the input behavior of a physical structure is essentially same as the lumped circuit equivalent for the corresponding static case. As the frequency is increased beyond the quasistatic approximation, the lumped circuit equivalent is no longer valid and the *distributed* circuit equivalent comes into play, leading to the transmission line.

We begin the chapter with electric potential, a scalar that is related to the static electric field intensity through a vector operation known as the *gradient.* We shall introduce the gradient and the electric potential and then consider two important differential equations involving the potential, known as *Poisson's equation* and *Laplace's equation.* Beginning with static field involving the solution of the Laplace's equation, we shall then embark on the study based on the thread of statics-quasistatics-waves.

6.1 GRADIENT AND ELECTRIC POTENTIAL

For static fields, $\partial/\partial t = 0$, and Maxwell's curl equations given for time-varying fields by

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{6.1}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{6.2}$$

reduce to

$$\nabla \times \mathbf{E} = 0 \tag{6.3}$$

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{6.4}$$

respectively. Equation (6.3) states that the curl of the static electric field is equal to zero. If the curl of a vector is zero, then that vector can be expressed as the *gradient* of a scalar, since the curl of the gradient of a scalar is identically equal to zero. The gradient of a scalar, say Φ, denoted $\nabla\Phi$ (del Φ) is given in Cartesian coordinates by

$$\begin{aligned}\nabla\Phi &= \left(\mathbf{a}_x \frac{\partial}{\partial x} + \mathbf{a}_y \frac{\partial}{\partial y} + \mathbf{a}_z \frac{\partial}{\partial z}\right)\Phi \\ &= \frac{\partial\Phi}{\partial x}\mathbf{a}_x + \frac{\partial\Phi}{\partial y}\mathbf{a}_y + \frac{\partial\Phi}{\partial z}\mathbf{a}_z \end{aligned} \tag{6.5}$$

The curl of $\nabla\Phi$ is then given by

$$\begin{aligned}\nabla \times \nabla\Phi &= \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (\nabla\Phi)_x & (\nabla\Phi)_y & (\nabla\Phi)_z \end{vmatrix} \\ &= \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial\Phi}{\partial x} & \dfrac{\partial\Phi}{\partial y} & \dfrac{\partial\Phi}{\partial z} \end{vmatrix} \\ &= 0 \end{aligned} \tag{6.6}$$

To discuss the physical interpretation of the gradient, we note that

$$\begin{aligned}\nabla\Phi \cdot d\mathbf{l} &= \left(\frac{\partial\Phi}{\partial x}\mathbf{a}_x + \frac{\partial\Phi}{\partial y}\mathbf{a}_y + \frac{\partial\Phi}{\partial z}\mathbf{a}_z\right) \cdot (dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z) \\ &= \frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{\partial y}dy + \frac{\partial\Phi}{\partial z}dz \\ &= d\Phi \end{aligned} \tag{6.7}$$

Let us consider a surface on which Φ is equal to a constant, say Φ_0, and a point P on that surface, as shown in Figure 6.1(a). If we now consider another point Q_1 on the same surface and an infinitesimal distance away from P, $d\Phi$ between these two points is zero since Φ is constant on the surface. Thus, for the vector $d\mathbf{l}_1$ drawn from P to Q_1, $[\nabla\Phi]_P \cdot d\mathbf{l}_1 = 0$ and hence $[\nabla\Phi]_P$ is perpendicular to $d\mathbf{l}_1$. Since this is true for all points $Q_1, Q_2, Q_3, \ldots$ on the constant Φ surface, it follows that $[\nabla\Phi]_P$ must be normal to all possible infinitesimal displacement vectors $d\mathbf{l}_1, d\mathbf{l}_2, d\mathbf{l}_3, \ldots$ drawn at P and hence

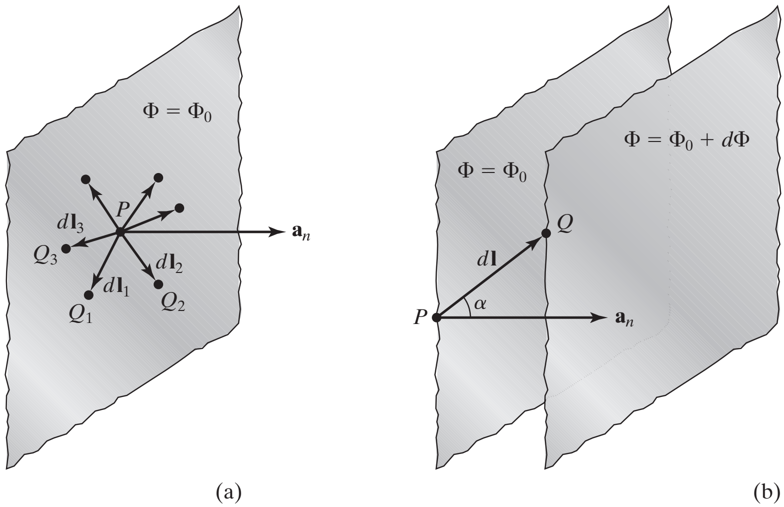

FIGURE 6.1

For discussing the physical interpretation of the gradient of a scalar function.

is normal to the surface. Denoting $\mathbf{a}_n$ to be the unit normal vector to the surface at P, we then have

$$[\nabla\Phi]_P = |\nabla\Phi|_P\,\mathbf{a}_n \tag{6.8}$$

Let us now consider two surfaces on which Φ is constant, having values Φ_0 and $\Phi_0 + d\Phi$, as shown in Figure 6.1(b). Let P and Q be points on the $\Phi = \Phi_0$ and $\Phi = \Phi_0 + d\Phi$ surfaces, respectively, and $d\mathbf{l}$ be the vector drawn from P to Q. Then from (6.7) and (6.8),

$$\begin{aligned} d\Phi &= [\nabla\Phi]_P \cdot d\mathbf{l} \\ &= |\nabla\Phi|_P\,\mathbf{a}_n \cdot d\mathbf{l} \\ &= |\nabla\Phi|_P\,dl\cos\alpha \end{aligned} \tag{6.9}$$

where α is the angle between $\mathbf{a}_n$ at P and $d\mathbf{l}$. Thus,

$$|\nabla\Phi|_P = \frac{d\Phi}{dl\cos\alpha} \tag{6.10}$$

Since $dl\cos\alpha$ is the distance between the two surfaces along $\mathbf{a}_n$ and hence is the shortest distance between them, it follows that $|\nabla\Phi|_P$ is the maximum rate of increase of Φ at the point P. Thus, the gradient of a scalar function Φ at a point is a vector having magnitude equal to the maximum rate of increase of Φ at that point and is directed along the direction of the maximum rate of increase, which is normal to the constant Φ surface passing through that point. This concept of the gradient of a scalar function is often utilized to find a unit vector normal to a given surface. We shall illustrate this by means of an example.

Example 6.1

Let us find the unit vector normal to the surface $y = x^2$ at the point $(2, 4, 1)$ by using the concept of the gradient of a scalar.

Writing the equation for the surface as

$$x^2 - y = 0$$

we note that the scalar function that is constant on the surface is given by

$$\Phi(x, y, z) = x^2 - y$$

The gradient of the scalar function is then given by

$$\begin{aligned} \nabla\Phi &= \nabla(x^2 - y) \\ &= \frac{\partial(x^2 - y)}{\partial x}\mathbf{a}_x + \frac{\partial(x^2 - y)}{\partial y}\mathbf{a}_y + \frac{\partial(x^2 - y)}{\partial z}\mathbf{a}_z \\ &= 2x\mathbf{a}_x - \mathbf{a}_y \end{aligned}$$

The value of the gradient at the point $(2, 4, 1)$ is $2(2)\mathbf{a}_x - \mathbf{a}_y = 4\mathbf{a}_x - \mathbf{a}_y$. Thus, the required unit vector is

$$\mathbf{a}_n = \pm\frac{4\mathbf{a}_x - \mathbf{a}_y}{|4\mathbf{a}_x - \mathbf{a}_y|} = \pm\left(\frac{4}{\sqrt{17}}\mathbf{a}_x - \frac{1}{\sqrt{17}}\mathbf{a}_y\right)$$

Returning to Maxwell's curl equation for the static electric field given by (6.3), we can now express **E** as the gradient of a scalar function, say, Φ. The question then arises as to what this scalar function is. To obtain the answer, let us consider a region of static electric field. Then we can draw a set of surfaces orthogonal everywhere to the field lines, as shown in Figure 6.2. These surfaces correspond to the constant Φ

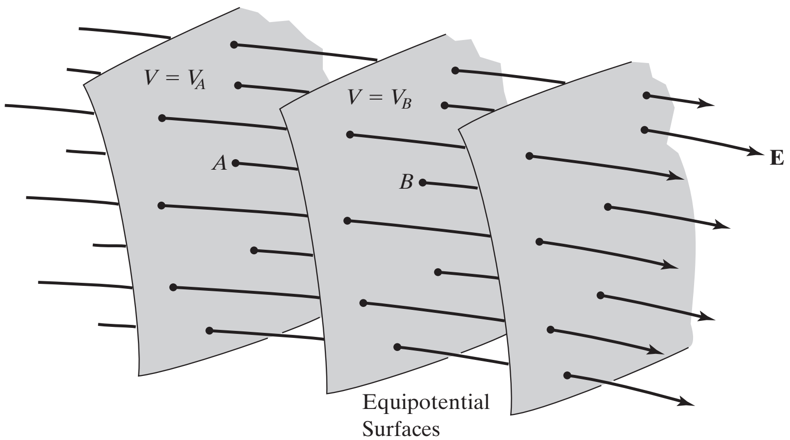

FIGURE 6.2

A set of equipotential surfaces in a region of static electric field.

surfaces. Since on any such surface $\mathbf{E} \cdot d\mathbf{l} = 0$, no work is involved in the movement of a test charge from one point to another on the surface. Such surfaces are known as the *equipotential surfaces.* Since they are orthogonal to the field lines, they may physically be occupied by conductors without affecting the field distribution.

Movement of a test charge from a point, say A, on one equipotential surface to a point, say B, on another equipotential surface involves an amount of work per unit charge equal to $\int_A^B \mathbf{E} \cdot d\mathbf{l}$ to be done by the field. This quantity is known as the *electric potential difference* between the points A and B and is denoted by the symbol $[V]_A^B$. It has the units of volts. There is a potential drop from A to B if work is done by the field and a potential rise if work is done against the field by an external agent. The situation is similar to that in the earth's gravitational field for which there is a potential drop associated with the movement of a mass from a point of higher elevation to a point of lower elevation and a potential rise for just the opposite case.

It is convenient to define an *electric potential* associated with each point. The potential at point A, denoted V_A, is simply the potential difference between point A and a reference point, say O. It is the amount of work per unit charge done by the field in connection with the movement of a test charge from A to O, or the amount of work per unit charge done against the field by an external agent in moving the test charge from O to A. Thus,

$$V_A = \int_A^O \mathbf{E} \cdot d\mathbf{l} = -\int_O^A \mathbf{E} \cdot d\mathbf{l} \tag{6.11}$$

and

$$\begin{aligned} [V]_A^B &= \int_A^B \mathbf{E} \cdot d\mathbf{l} = \int_A^O \mathbf{E} \cdot d\mathbf{l} + \int_O^B \mathbf{E} \cdot d\mathbf{l} \\ &= \int_A^O \mathbf{E} \cdot d\mathbf{l} - \int_B^O \mathbf{E} \cdot d\mathbf{l} \\ &= V_A - V_B \end{aligned} \tag{6.12}$$

If we now consider points A and B to be separated by infinitesimal length $d\mathbf{l}$ from A to B, then the incremental potential drop from A to B is $\mathbf{E}_A \cdot d\mathbf{l}$, or the incremental potential rise dV along the length $d\mathbf{l}$ is given by

$$dV = -\mathbf{E}_A \cdot d\mathbf{l} \tag{6.13}$$

Writing

$$dV = [\nabla V]_A \cdot d\mathbf{l} \tag{6.14}$$

in accordance with (6.7), we then have

$$[\nabla V]_A \cdot d\mathbf{l} = -\mathbf{E}_A \cdot d\mathbf{l} \tag{6.15}$$

Since (6.15) is true at any point A in the static electric field, it follows that

$$\mathbf{E} = -\nabla V \tag{6.16}$$

Thus, we have obtained the result that the static electric field is the negative of the gradient of the electric potential.

Before proceeding further, we note that the potential difference we have defined here has the same meaning as the voltage between two points, defined in Section 2.1. We, however, recall that the voltage between two points A and B in a time-varying field is in general dependent on the path followed from A to B to evaluate $\int_A^B \mathbf{E} \cdot d\mathbf{l}$, since, according to Faraday's law,

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{S} \tag{6.17}$$

is not in general equal to zero. On the other hand, the potential difference (or voltage) between two points A and B in a static electric field is independent of the path followed from A to B to evaluate $\int_A^B \mathbf{E} \cdot d\mathbf{l}$, since, for static fields, $\partial/\partial t = 0$, and (6.17) reduces to

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0 \tag{6.18}$$

Thus, the potential difference between two points in a static electric field has a unique value. Fields for which the line integral around a closed path is zero are known as *conservative* fields. The static electric field is a conservative field. The earth's gravitational field is another example of a conservative field, since the work done in moving a mass around a closed path is equal to zero.

Returning now to the discussion of electric potential, let us consider the electric field of a point charge and investigate the electric potential due to the point charge. To do this, we recall from Section 1.5 that the electric field intensity due to a point charge Q is directed radially away from the point charge and its magnitude is $Q/4\pi\epsilon_0 R^2$, where R is the radial distance from the point charge. Since the equipotential surfaces are everywhere orthogonal to the field lines, it then follows that they are spherical surfaces centered at the point charge, as shown by the cross-sectional view in Figure 6.3. If we now consider two equipotential surfaces of radii R and $R + dR$, the potential drop from the surface of radius R to the surface of radius $R + dR$ is $(Q/4\pi\epsilon_0 R^2)\, dR$, or, the incremental potential rise dV is given by

$$\begin{aligned} dV &= -\frac{Q}{4\pi\epsilon_0 R^2}\, dR \\ &= d\left(\frac{Q}{4\pi\epsilon_0 R} + C\right) \end{aligned} \tag{6.19}$$

where C is a constant. Thus,

$$V(R) = \frac{Q}{4\pi\epsilon_0 R} + C \tag{6.20}$$

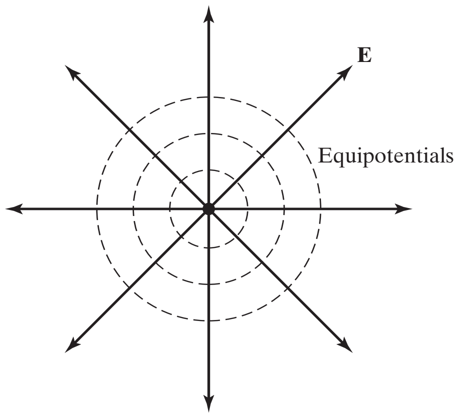

FIGURE 6.3

Cross-sectional view of equipotential surfaces and electric field lines for a point charge.

We can conveniently set C equal to zero by noting that it is equal to $V(\infty)$ and by choosing $R = \infty$ for the reference point. Thus, we obtain the electric potential due to a point charge Q to be

$$V = \frac{Q}{4\pi\epsilon_0 R} \tag{6.21}$$

We note that the potential drops off inversely with the radial distance away from the point charge. Equation (6.21) is often the starting point for the computation of the potential field due to static charge distributions and the subsequent determination of the electric field by using (6.16).

6.2 POISSON'S AND LAPLACE'S EQUATIONS

In the previous section, we learned that for the static electric field, $\nabla \times \mathbf{E}$ is equal to zero, and hence

$$\mathbf{E} = -\nabla V$$

Substituting this result into Maxwell's divergence equation for $\mathbf{D}$, and assuming ϵ to be uniform, we obtain

$$\begin{aligned}\nabla \cdot \mathbf{D} &= \nabla \cdot \epsilon \mathbf{E} = \epsilon \nabla \cdot \mathbf{E} \\ &= \epsilon \nabla \cdot (-\nabla V) = \rho\end{aligned}$$

or

$$\nabla \cdot \nabla V = -\frac{\rho}{\epsilon}$$

The quantity $\nabla \cdot \nabla V$ is known as the *Laplacian* of V, denoted $\nabla^2 V$ (del squared V). Thus, we have

$$\nabla^2 V = -\frac{\rho}{\epsilon} \tag{6.22}$$

This equation is known as the *Poisson's equation.* It governs the relationship between the volume charge density ρ in a region and the potential in that region. In Cartesian coordinates,

$$\begin{aligned}\nabla^2 V &= \nabla \cdot \nabla V \\ &= \left(\mathbf{a}_x \frac{\partial}{\partial x} + \mathbf{a}_y \frac{\partial}{\partial y} + \mathbf{a}_z \frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z\right) \\ &= \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \qquad (6.23)\end{aligned}$$

and Poisson's equation becomes

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho}{\epsilon} \qquad (6.24)$$

For the one-dimensional case in which V varies with x only, $\partial^2 V/\partial y^2$ and $\partial^2 V/\partial z^2$ are both equal to zero, and (6.24) reduces to

$$\frac{\partial^2 V}{\partial x^2} = \frac{d^2 V}{dx^2} = -\frac{\rho}{\epsilon} \qquad (6.25)$$

We shall illustrate the application of (6.25) by means of an example.

Example 6.2

Let us consider the space charge layer in a *p-n* junction semiconductor with zero bias, as shown in Figure 6.4(a), in which the region $x < 0$ is doped *p*-type and the region $x > 0$ is doped *n*-type. To review briefly the formation of the space charge layer, we note that since the density of the holes on the *p* side is larger than that on the *n* side, there is a tendency for the holes to diffuse to the *n* side and recombine with the electrons. Similarly, there is a tendency for the electrons on the *n* side to diffuse to the *p* side and recombine with the holes. The diffusion of holes leaves behind negatively charged acceptor atoms, and the diffusion of electrons leaves behind positively charged donor atoms. Since these acceptor and donor atoms are immobile, a space charge layer, also known as the *depletion layer*, is formed in the region of the junction, with negative charges on the *p* side and positive charges on the *n* side. This space charge gives rise to an electric field directed from the *n* side of the junction to the *p* side so that it opposes diffusion of the mobile carriers across the junction, thereby resulting in an equilibrium. For simplicity, let us consider an abrupt junction, that is, a junction in which the impurity concentration is constant on either side of the junction. Let N_A and N_D be the acceptor and donor ion concentrations, respectively, and d_p and d_n be the widths in the *p* and *n* regions, respectively, of the depletion layer. The space charge density ρ is then given by

$$\rho = \begin{cases} -|e|N_A & \text{for } -d_p < x < 0 \\ |e|N_D & \text{for } 0 < x < d_n \end{cases} \qquad (6.26)$$

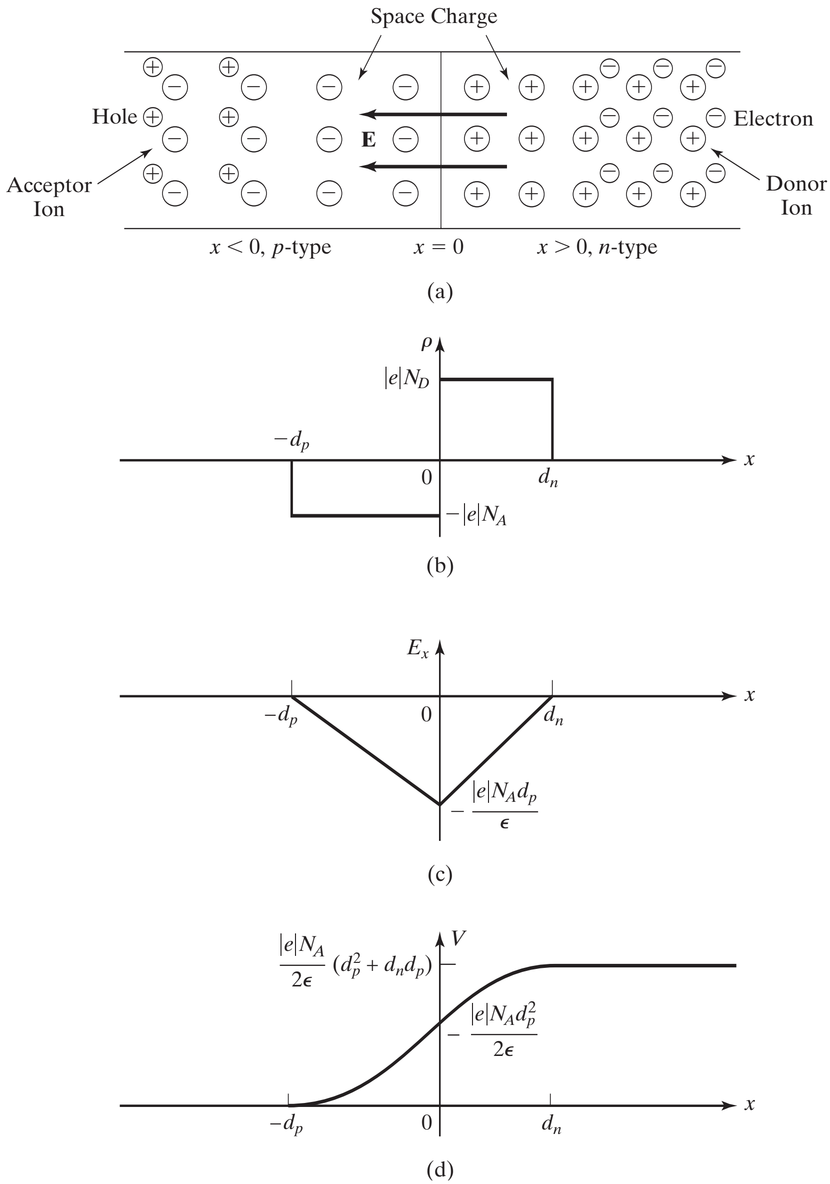

FIGURE 6.4

For illustrating the application of Poisson's equation for the determination of the potential distribution for a *p-n* junction semiconductor.

as shown in Figure 6.4(b), where $|e|$ is the magnitude of the electronic charge. Since the semiconductor is electrically neutral, the total acceptor charge must be equal to the total donor charge; that is,

$$|e|N_A d_p = |e|N_D d_n \tag{6.27}$$

We wish to find the potential distribution in the depletion layer and the depletion layer width in terms of the potential difference across the depletion layer and the acceptor and donor ion concentrations.

Substituting (6.26) into (6.25), we obtain the equation governing the potential distribution to be

$$\frac{d^2V}{dx^2} = \begin{cases} \dfrac{|e|N_A}{\epsilon} & \text{for } -d_p < x < 0 \\ -\dfrac{|e|N_D}{\epsilon} & \text{for } 0 < x < d_n \end{cases} \tag{6.28}$$

To solve (6.28) for V, we integrate it once and obtain

$$\frac{dV}{dx} = \begin{cases} \dfrac{|e|N_A}{\epsilon}x + C_1 & \text{for } -d_p < x < 0 \\ -\dfrac{|e|N_D}{\epsilon}x + C_2 & \text{for } 0 < x < d_n \end{cases}$$

where C_1 and C_2 are constants of integration. To evaluate C_1 and C_2, we note that since $\mathbf{E} = -\nabla V = -(\partial V/\partial x)\mathbf{a}_x$, $\partial V/\partial x$ is simply equal to $-E_x$. Since the electric field lines begin on the positive charges and end on the negative charges, and in view of (6.27), the field and, hence, $\partial V/\partial x$ must vanish at $x = -d_p$ and $x = d_n$, giving us

$$\frac{dV}{dx} = \begin{cases} \dfrac{|e|N_A}{\epsilon}(x + d_p) & \text{for } -d_p < x < 0 \\ -\dfrac{|e|N_D}{\epsilon}(x - d_n) & \text{for } 0 < x < d_n \end{cases} \tag{6.29}$$

The field intensity, that is, $-dV/dx$, may now be sketched as a function of x, as shown in Figure 6.4(c).

Proceeding further, we integrate (6.29) and obtain

$$V = \begin{cases} \dfrac{|e|N_A}{2\epsilon}(x + d_p)^2 + C_3 & \text{for } -d_p < x < 0 \\ -\dfrac{|e|N_D}{2\epsilon}(x - d_n)^2 + C_4 & \text{for } 0 < x < d_n \end{cases}$$

where C_3 and C_4 are constants of integration. To evaluate C_3 and C_4, we first set the potential at $x = -d_p$ arbitrarily equal to zero to obtain C_3 equal to zero. Then we make use of the condition that the potential be continuous at $x = 0$, since the discontinuity in dV/dx at $x = 0$ is finite, to obtain

$$\frac{|e|N_A}{2\epsilon}d_p^2 = -\frac{|e|N_D}{2\epsilon}d_n^2 + C_4$$

or

$$C_4 = \frac{|e|}{2\epsilon}(N_A d_p^2 + N_D d_n^2)$$

Substituting this value for C_4 and setting C_3 equal to zero in the expression for V, we get the required solution

$$V = \begin{cases} \dfrac{|e|N_A}{2\epsilon}(x + d_p)^2 & \text{for } -d_p < x < 0 \\ -\dfrac{|e|N_D}{2\epsilon}(x^2 - 2xd_n) + \dfrac{|e|N_A}{2\epsilon}d_p^2 & \text{for } 0 < x < d_n \end{cases} \tag{6.30}$$

The variation of potential with x as given by (6.30) is shown in Figure 6.4(d).

We can proceed further and find the width $d = d_p + d_n$ of the depletion layer by setting $V(d_n)$ equal to the contact potential, V_0, that is, the potential difference across the depletion layer resulting from the electric field in the layer. Thus,

$$\begin{aligned} V_0 = V(d_n) &= \frac{|e|N_D}{2\epsilon}d_n^2 + \frac{|e|N_A}{2\epsilon}d_p^2 \\ &= \frac{|e|}{2\epsilon}\frac{N_D(N_A + N_D)}{N_A + N_D}d_n^2 + \frac{|e|}{2\epsilon}\frac{N_A(N_A + N_D)}{N_A + N_D}d_p^2 \\ &= \frac{|e|}{2\epsilon}\frac{N_A N_D}{N_A + N_D}(d_n^2 + d_p^2 + 2d_n d_p) \\ &= \frac{|e|}{2\epsilon}\frac{N_A N_D}{N_A + N_D}d^2 \end{aligned}$$

where we have made use of (6.27). Finally, we obtain the result that

$$d = \sqrt{\frac{2\epsilon V_0}{|e|}\left(\frac{1}{N_A} + \frac{1}{N_D}\right)}$$

which tells us that the depletion layer width is smaller, the heavier the doping is. This property is used in tunnel diodes to achieve layer widths on the order of 10^{-6} cm by heavy doping as compared to widths on the order of 10^{-4} cm in ordinary *p-n* junctions.

We have just illustrated an example of the application of Poisson's equation involving the solution for the potential distribution for a given charge distribution. Poisson's equation is even more useful for the solution of problems in which the charge distribution is the quantity to be determined, given the functional dependence of the charge density on the potential. We shall, however, not pursue this topic any further.

If the charge density in a region is zero, then Poisson's equation reduces to

$$\nabla^2 V = 0 \tag{6.31}$$

This equation is known as *Laplace's equation.* It governs the behavior of the potential in a charge-free region. In Cartesian coordinates, it is given by

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \tag{6.32}$$

The problems for which Laplace's equation is applicable consist of finding the potential distribution in the region between two conductors given the charge distribution on the surfaces of the conductors or the potentials of the conductors or a combination of the two. The procedure involves the solving of Laplace's equation subject to the boundary conditions on the surfaces of the conductors. We shall do this in the following section.

6.3 STATIC FIELDS AND CIRCUIT ELEMENTS

In the previous two sections, we considered static fields with reference to electric field alone. In this section, we shall expand the treatment to all types of static fields, for the purpose of introducing circuit elements. Thus, for static fields, $\partial/\partial t = 0$. Maxwell's equations in integral form and the law of conservation of charge become

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0 \tag{6.33a}$$

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S} \tag{6.33b}$$

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho \, dv \tag{6.33c}$$

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0 \tag{6.33d}$$

$$\oint_S \mathbf{J} \cdot d\mathbf{S} = 0 \tag{6.33e}$$

whereas Maxwell's equations in differential form and the continuity equation reduce to

$$\nabla \times \mathbf{E} = 0 \tag{6.34a}$$

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{6.34b}$$

$$\nabla \cdot \mathbf{D} = \rho \tag{6.34c}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{6.34d}$$

$$\nabla \cdot \mathbf{J} = 0 \tag{6.34e}$$

Immediately, one can see that, unless **J** includes a component due to conduction current, the equations involving the electric field are completely independent of those involving the magnetic field. Thus, the fields can be subdivided into *static electric fields*, or *electrostatic fields*, governed by (6.33a) and (6.33c), or (6.34a) and (6.34c), and *static magnetic fields*, or *magnetostatic fields*, governed by (6.33b) and (6.33d), or (6.34b) and (6.34d). The source of a static electric field is ρ, whereas the source of a static magnetic field is **J**. One can also see from (6.33e) or (6.34e) that no relationship exists between **J** and ρ. If **J** includes a component due to conduction current, then, since $\mathbf{J}_c = \sigma\mathbf{E}$, a coupling between the electric and magnetic fields exists for that part of the total field associated with $\mathbf{J}_c$. However, the coupling is only one way, since the right side of (6.33a) or (6.34a) is still zero. The field is then referred to as *electromagnetostatic field*. It can also be seen, then, that for consistency, the right sides of (6.33c) and (6.34c) must be zero, since the right sides of (6.33e) and (6.34e) are zero. We shall now consider each of the three types of static fields separately and discuss some fundamental aspects.

Electrostatic Fields and Capacitance

The equations of interest are (6.33a) and (6.33c), or (6.34a) and (6.34c). The first of each pair of these equations simply tells us that the electrostatic field is a conservative field, and the second of each pair of these equations enables us, in principle, to determine the electrostatic field for a given charge distribution. Alternatively, the Poisson's equation, equation (6.22), can be used to find the electric scalar potential, V, from which the electrostatic field can be determined by using (6.16).

In a charge-free region, the Poisson's equation reduces to the Laplace's equation, (6.31). The field is then due to charges outside the region, such as surface charge on conductors bounding the region. The situation is then one of solving a boundary value problem, as we shall illustrate by means of an example.

Example 6.3

Figure 6.5(a) is that of a parallel-plate arrangement in which two parallel, perfectly conducting plates ($\sigma = \infty, \mathbf{E} = 0$) of dimensions w along the y-direction and l along the z-direction lie in the $x = 0$ and $x = d$ planes. The region between the plates is a perfect dielectric ($\sigma = 0$) of material parameters ϵ and μ. The thickness of the plates is shown exaggerated for convenience in illustration. A potential difference of V_0 is maintained between the plates by connecting a direct voltage source at the end $z = -l$. If fringing of the field due to the finite dimensions of the structure normal to the x-direction is neglected, or, if it is assumed that the structure is part of one which is infinite in extent normal to the x-direction, then the problem can be treated as one-dimensional with x as the variable, and (6.31) reduces to

$$\frac{d^2V}{dx^2} = 0 \tag{6.35}$$

We wish to carry out the electrostatic field analysis for this arrangement.

The solution for the potential in the charge-free region between the plates is given by

$$V(x) = \frac{V_0}{d}(d - x) \tag{6.36}$$

which satisfies (6.35), as well as the boundary conditions of $V = 0$ at $x = d$ and $V = V_0$ at $x = 0$. The electric field intensity between the plates is then given by

$$\mathbf{E} = -\nabla V = \frac{V_0}{d}\mathbf{a}_x \tag{6.37}$$

as depicted in the cross-sectional view in Figure 6.5(b), and resulting in displacement flux density

$$\mathbf{D} = \frac{\epsilon V_0}{d}\mathbf{a}_x \tag{6.38}$$

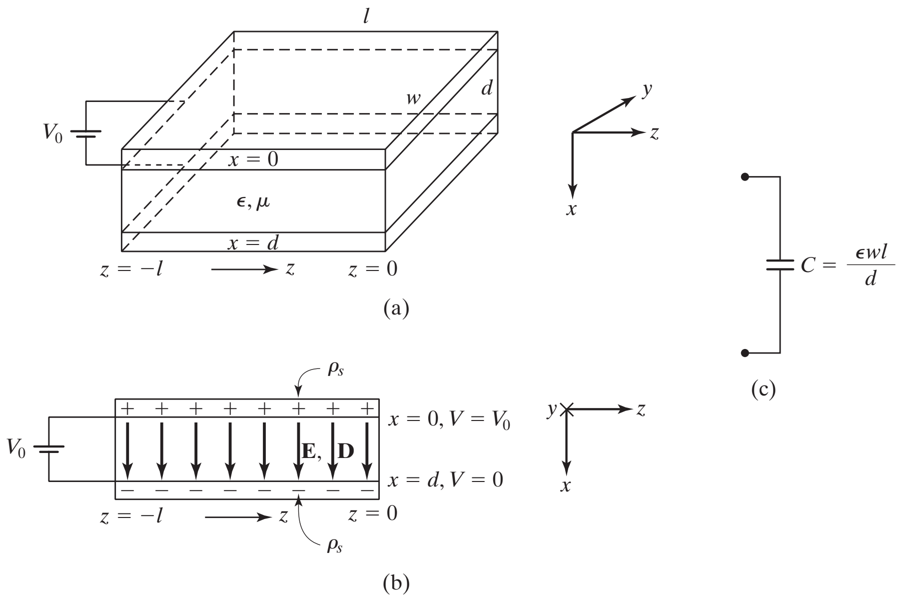

FIGURE 6.5

Electrostatic field analysis for a parallel-plate arrangement.

Then, using the boundary condition for the normal component of **D** given by (5.94c), we obtain the magnitude of the charge on either plate to be

$$Q = \left(\frac{\epsilon V_0}{d}\right)(wl) = \frac{\epsilon wl}{d} V_0 \tag{6.39}$$

We can now find the familiar circuit parameter, the capacitance, C, of the parallel-plate arrangement, which is defined as the ratio of the magnitude of the charge on either plate to the potential difference V_0. Thus,

$$C = \frac{Q}{V_0} = \frac{\epsilon wl}{d} \tag{6.40}$$

Note that the units of C are the units of ϵ times meter, that is, farads. The phenomenon associated with the arrangement is that energy is stored in the capacitor in the form of electric field energy between the plates, as given by

$$\begin{aligned} W_e &= \left(\frac{1}{2}\epsilon E_x^2\right)(wld) \\ &= \frac{1}{2}\left(\frac{\epsilon wl}{d}\right)V_0^2 \\ &= \frac{1}{2}CV_0^2 \end{aligned} \tag{6.41}$$

the familiar expression for energy stored in a capacitor.

Magnetostatic Fields and Inductance

The equations of interest are (6.33b) and (6.33d), or (6.34b) and (6.34d). The second of each pair of these equations simply tells us that the magnetostatic field is solenoidal, which as we know holds for any magnetic field, and the first of each pair of these equations enables us, in principle, to determine the magnetostatic field for a given current distribution.

In a current-free region, $\mathbf{J} = 0$. The field is then due to currents outside the region, such as surface currents on conductors bounding the region. The situation is then one of solving a boundary value problem as in the case of (6.31). However, since the boundary condition (5.94b) relates the magnetic field directly to the surface current density, it is straightforward and more convenient to determine the magnetic field directly by using (6.34b) and (6.34d). We shall illustrate by means of an example.

Example 6.4

Figure 6.6(a) is that of the parallel-plate arrangement of Figure 6.5(a) with the plates connected by another conductor at the end $z = 0$ and driven by a source of direct current I_0 at the end $z = -l$. If fringing of the field due to the finite dimensions of the structure normal to the x-direction is neglected, or, if it is assumed that the structure is part of one which is infinite in extent normal to the x-direction, then the problem can be treated as one-dimensional with x as the variable and we can write the current density on the plates to be

$$\mathbf{J}_S = \begin{cases} (I_0/w)\,\mathbf{a}_z & \text{on the plate } x = 0 \\ (I_0/w)\,\mathbf{a}_x & \text{on the plate } z = 0 \\ -(I_0/w)\,\mathbf{a}_z & \text{on the plate } x = d \end{cases} \tag{6.42}$$

We wish to carry out the magnetostatic field analysis for this arrangement.

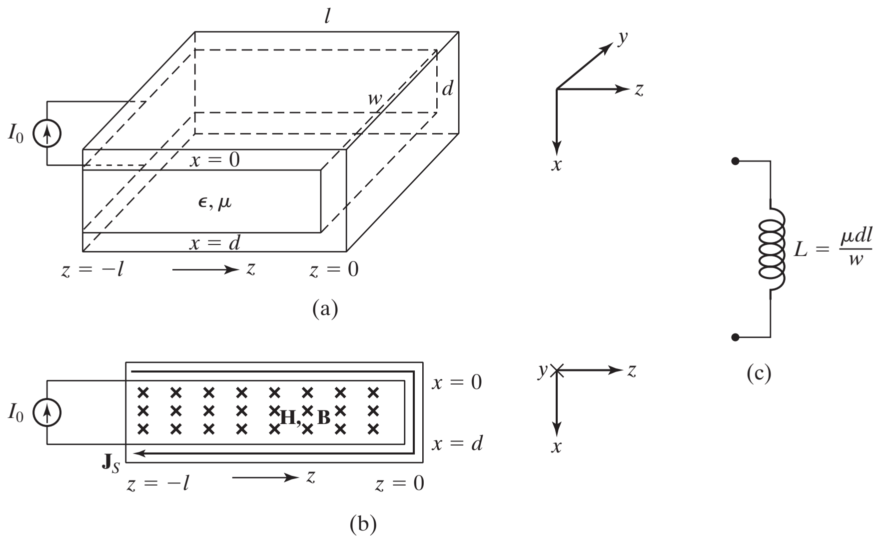

FIGURE 6.6

Magnetostatic field analysis for a parallel-plate arrangement.

In the current-free region between the plates, (6.34b) reduces to

$$\begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & 0 & 0 \\ H_x & H_y & H_z \end{vmatrix} = 0 \tag{6.43}$$

and (6.34d) reduces to

$$\frac{\partial B_x}{\partial x} = 0 \tag{6.44}$$

so that each component of the field, if it exists, has to be uniform. This automatically forces H_x and H_z to be zero, since nonzero value of these components do not satisfy the boundary conditions (5.94b) and (5.94d) on the plates, keeping in mind that the field is entirely in the region between the conductors. Thus, as depicted in the cross-sectional view in Figure 6.6(b),

$$\mathbf{H} = \frac{I_0}{w}\mathbf{a}_y \tag{6.45}$$

which satisfies the boundary condition (5.94b) on all three plates, and results in magnetic flux density

$$\mathbf{B} = \frac{\mu I_0}{w}\mathbf{a}_y \tag{6.46}$$

The magnetic flux, ψ, linking the current I_0, is then given by

$$\psi = \left(\frac{\mu I_0}{w}\right)(dl) = \left(\frac{\mu dl}{w}\right)I_0 \tag{6.47}$$

We can now find the familiar circuit parameter, the inductance, L, of the parallel-plate arrangement, which is defined as the ratio of the magnetic flux linking the current to the current. Thus,

$$L = \frac{\psi}{I_0} = \frac{\mu dl}{w} \tag{6.48}$$

Note that the units of L are the units of μ times meter, that is, henrys. The phenomenon associated with the arrangement is that energy is stored in the inductor in the form of magnetic field energy between the plates, as given by

$$\begin{aligned} W_m &= \left(\frac{1}{2}\mu H^2\right)(wld) \\ &= \frac{1}{2}\left(\frac{\mu dl}{w}\right)I_0^2 \\ &= \frac{1}{2}LI_0^2 \end{aligned} \tag{6.49}$$

the familiar expression for energy stored in an inductor.

Electromagnetostatic Fields and Conductance

The equations of interest are

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0 \tag{6.50a}$$

$$\oint_C \mathrm{H} \cdot d\mathbf{l} = \oint_S \mathbf{J}_c \cdot d\mathbf{S} = \sigma \oint_S \mathbf{E} \cdot d\mathbf{S} \tag{6.50b}$$

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = 0 \tag{6.50c}$$

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0 \tag{6.50d}$$

or, in differential form,

$$\nabla \times \mathbf{E} = 0 \tag{6.51a}$$

$$\nabla \times \mathbf{H} = \mathbf{J}_c = \sigma\mathbf{E} \tag{6.51b}$$

$$\nabla \cdot \mathbf{D} = 0 \tag{6.51c}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{6.51d}$$

From (6.51a) and (6.51c), we note that Laplace's equation for the electrostatic potential, (6.31), is satisfied, so that, for a given problem, the electric field can be found in the same manner as in the case of the example of Figure 6.6. The magnetic field is then found by using (6.51b), and making sure that (6.51d) is also satisfied. We shall illustrate by means of an example.

Example 6.5

Figure 6.7(a) is that of the parallel-plate arrangement of Figure 6.5(a) but with an imperfect dielectric material of parameters σ, ϵ, and μ, between the plates. We wish to carry out the electromagnetostatic field analysis of the arrangement.

The electric field between the plates is the same as that given by (6.37), that is,

$$\mathbf{E} = \frac{V_0}{d}\mathbf{a}_x \tag{6.52}$$

resulting in a conduction current of density

$$\mathbf{J}_c = \frac{\sigma V_0}{d}\mathbf{a}_x \tag{6.53}$$

from the top plate to the bottom plate, as depicted in the cross-sectional view of Figure 6.7(b). Since $\partial\rho/\partial t = 0$ at the boundaries between the plates and the slab, continuity of current is satisfied by the flow of surface current on the plates. At the input $z = -l$, this surface current, which is the current drawn from the source, must be equal to the total current flowing from the top to the bottom plate. It is given by

$$I_c = \left(\frac{\sigma V_0}{d}\right)(wl) = \frac{\sigma wl}{d}V_0 \tag{6.54}$$

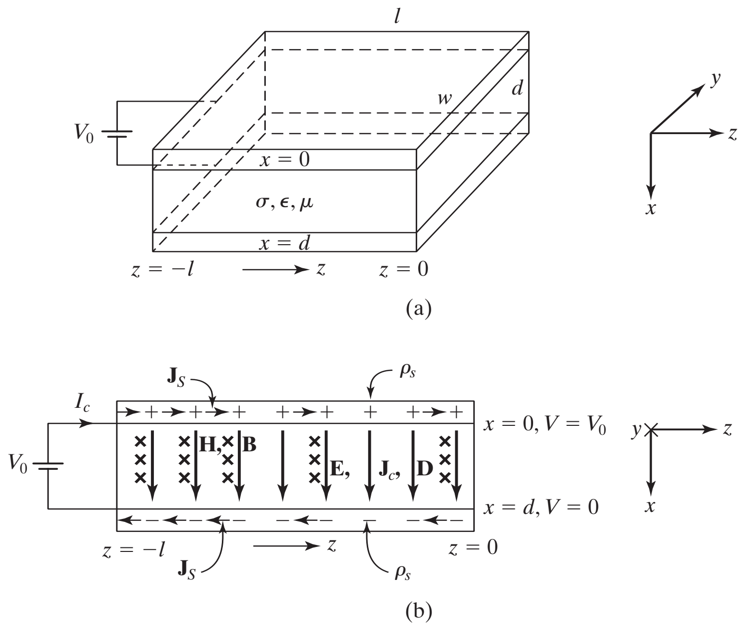

FIGURE 6.7

Electromagnetostatic field analysis for a parallel-plate arrangement.

We can now find the familiar circuit parameter, the conductance, G, of the parallel-plate arrangement, which is defined as the ratio of the current drawn from the source to the source voltage V_0. Thus,

$$G = \frac{I_c}{V_0} = \frac{\sigma w l}{d} \tag{6.55}$$

Note that the units of G are the units of σ times meter, that is, siemens (S). The reciprocal quantity, R, the resistance of the parallel-plate arrangement, is given by

$$R = \frac{V_0}{I_c} = \frac{d}{\sigma w l} \tag{6.56}$$

The unit of R is ohms. The phenomenon associated with the arrangement is that power is dissipated in the material between the plates, as given by

$$\begin{aligned} P_d &= (\sigma E^2)(wld) \\ &= \left(\frac{\sigma w l}{d}\right) V_0^2 \\ &= G V_0^2 \\ &= \frac{V_0^2}{R} \end{aligned} \tag{6.57}$$

the familiar expression for power dissipated in a resistor.

Proceeding further, we find the magnetic field between the plates by using (6.51b), and noting that the geometry of the situation requires a y-component of $\mathbf{H}$, dependent on z, to satisfy the equation. Thus,

$$\mathbf{H} = H_y(z)\,\mathbf{a}_y \tag{6.58a}$$

$$\frac{\partial H_y}{\partial z} = -\frac{\sigma V_0}{d} \tag{6.58b}$$

$$\mathbf{H} = -\frac{\sigma V_0}{d}\,z\,\mathbf{a}_y \tag{6.58c}$$

where the constant of integration is set to zero, since the boundary condition at $z = 0$ requires H_y to be zero for z equal to zero. Note that the magnetic field is directed in the positive y-direction (since z is negative) and increases linearly from $z = 0$ to $z = -l$, as depicted in Figure 6.7(b). It also satisfies the boundary condition at $z = -l$ by being consistent with the current drawn from the source to be $w[H_y]_{z=-l} = (\sigma V_0/d)(wl) = I_c$.

Because of the existence of the magnetic field, the arrangement is characterized by an inductance, which can be found either by using the flux linkage concept or by the energy method. To use the flux linkage concept, we recognize that a differential amount of magnetic flux $d\psi' = \mu H_y d(dz')$ between z equal to $(z' - dz')$ and z equal to z', where $-l < z' < 0$, links only that part of the current that flows from the top plate to the bottom plate between $z = z'$ and $z = 0$, thereby giving a value of $(-z'/l)$ for the fraction, N, of the total current linked. Thus, the inductance, familiarly known as the internal inductance, denoted L_i, since it is due to magnetic field internal to the current distribution, as compared to that in (6.48) for which the magnetic field is external to the current distribution, is given by

$$\begin{aligned} L_i &= \frac{1}{I_c}\int_{z'=-1}^{0} N\,d\psi' \\ &= \frac{1}{3}\frac{\mu dl}{w} \end{aligned} \tag{6.59}$$

or, 1/3 times the inductance of the structure if $\sigma = 0$ and the plates are joined at $z = 0$, as in Figure 6.6(b).

Alternatively, if the energy method is used by computing the energy stored in the magnetic field and setting it equal to $\frac{1}{2}L_i I_c^2$, then we have

$$\begin{aligned} L_i &= \frac{1}{I_c^2}(dw)\int_{z=-l}^{0} \mu H_y^2\,dz \\ &= \frac{1}{3}\frac{\mu dl}{w} \end{aligned} \tag{6.60}$$

same as in (6.59).

Finally, recognizing that there is energy storage associated with the electric field between the plates, we note that the arrangement has also associated with it a capacitance C, equal to $\epsilon wl/d$. Thus, all three properties of conductance, capacitance, and inductance are associated with the structure. Since for $\sigma = 0$ the situation reduces to that of Figure 6.5, we can represent the arrangement of Figure 6.7 to be equivalent to the circuit shown in Figure 6.8. Note that the capacitor is charged to the voltage V_0 and the current through it is zero (open circuit condition).

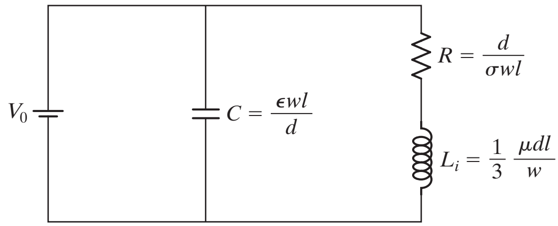

FIGURE 6.8

Circuit equivalent for the arrangement of Figure 6.7.

The voltage across the inductor is zero (short circuit condition) and the current through it is V_0/R. Thus, the current drawn from the voltage source is V_0/R and the voltage source views a single resistor R, as far as the current drawn from it is concerned.

6.4 LOW-FREQUENCY BEHAVIOR VIA QUASISTATICS

In the preceding section, we introduced circuit elements via static fields. A class of dynamic fields for which certain features can be analyzed as though the fields were static are known as *quasistatic fields*. In terms of behavior in the frequency domain, they are low-frequency extensions of static fields present in a physical structure, when the frequency of the source driving the structure is zero, or low-frequency approximations of time-varying fields in the structure that are complete solutions to Maxwell's equations. In this section, we consider the approach of low-frequency extensions of static fields. Thus, for a given structure, we begin with a time-varying field having the same spatial characteristics as that of the static field solution for the structure, and obtain field solutions containing terms up to and including the first power (which is the lowest power) in ω for their amplitudes. Depending on whether the predominant static field is electric or magnetic, quasistatic fields are called *electroquasistatic fields* or *magnetoquasistatic fields*. We shall now consider these separately.

Electroquasistatic Fields

For electroquasistatic fields, we begin with the electric field having the spatial dependence of the static field solution for the given arrangement. We shall illustrate by means of an example.

Example 6.6

Figure 6.9 shows the cross-sectional view of the arrangement of Figure 6.5(a) excited by a sinusoidally time-varying voltage source $V_g(t) = V_0 \cos \omega t$ instead of a direct voltage source. We wish to carry out the electroquasistatic field analysis for the arrangement.

From (6.37), we write

$$\mathbf{E}_0 = \frac{V_0}{d} \cos \omega t \; \mathbf{a}_x \tag{6.61}$$

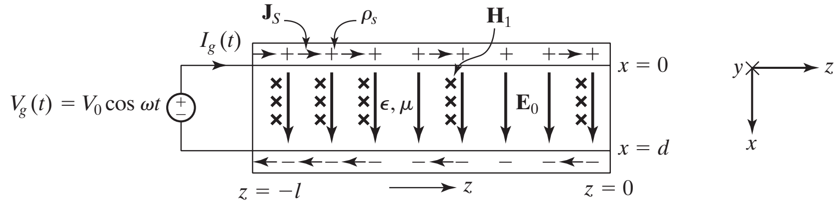

FIGURE 6.9

Electroquasistatic field analysis for the parallel-plate structure of Figure 6.5.

where the subscript 0 denotes that the amplitude of the field is of the zeroth power in ω. This results in a magnetic field in accordance with Maxwell's equation for the curl of **H**, given by (3.28). Thus, noting that $\mathbf{J} = 0$ in view of the perfect dielectric medium, we have for the geometry of the arrangement,

$$\frac{\partial H_{y1}}{\partial z} = -\frac{\partial D_{x0}}{\partial t} = \frac{\omega \epsilon V_0}{d} \sin \omega t$$

$$\mathbf{H}_1 = \frac{\omega \epsilon V_0 z}{d} \sin \omega t \, \mathbf{a}_y \tag{6.62}$$

where we have also satisfied the boundary condition at $z = 0$ by choosing the constant of integration such that $[H_{y1}]_{z=0}$ is zero, and the subscript 1 denotes that the amplitude of the field is of the first power in ω. Note that the amplitude of H_{y1}varies linearly with z, from zero at $z = 0$ to a maximum at $z = -l$.

We stop the solution here, because continuing the process by substituting (6.62) into Maxwell's curl equation for **E**, (3.17), to obtain the resulting electric field will yield a term having amplitude proportional to the second power in ω. This simply means that the fields given as a pair by (6.61) and (6.62) do not satisfy (3.17), and hence are not complete solutions to Maxwell's equations. They are the quasistatic fields. The complete solutions are obtained by solving Maxwell's equations simultaneously and subject to the boundary conditions for the given problem.

Proceeding further, we obtain the current drawn from the voltage source to be

$$\begin{aligned} I_g(t) &= w[H_{y1}]_{z=-l} \\ &= -\omega\left(\frac{\epsilon w l}{d}\right) V_0 \sin \omega t \\ &= C \frac{dV_g(t)}{dt} \end{aligned} \tag{6.63a}$$

or,

$$\bar{I}_g = j\omega C \bar{V}_g \tag{6.63b}$$

where $C = (\epsilon wl/d)$ is the capacitance of the arrangement obtained from static field considerations. Thus, the input admittance of the structure is $j\omega C$, such that its low frequency input behavior is essentially that of a single capacitor of value same as that found from static field analysis of the structure. Indeed, from considerations of power flow, using Poynting's theorem, we obtain the power flowing into the structure to be

$$\begin{aligned} P_{\text{in}} &= wd[E_{x0}H_{y1}]_{z=0} \\ &= -\left(\frac{\epsilon wl}{d}\right)\omega V_0^2 \sin \omega t \cos \omega t \\ &= \frac{d}{dt}\left(\frac{1}{2}CV_g^2\right) \end{aligned} \tag{6.64}$$

which is consistent with the electric energy stored in the structure for the static case, as given by (6.41).

Magnetoquasistatic Fields

For magnetoquasistatic fields, we begin with the magnetic field having the spatial dependence of the static field solution for the given arrangement. We shall illustrate by means of an example.

Example 6.7

Figure 6.10 shows the cross-sectional view of the arrangement of Figure 6.6(a), excited by a sinusoidally time-varying current source $I_g(t) = I_0 \cos \omega t$ instead of a direct current source. We wish to carry out the magnetoquasistatic field analysis for the arrangement.

From (6.45) we write

$$\mathbf{H}_0 = \frac{I_0}{w} \cos \omega t \, \mathbf{a}_y \tag{6.65}$$

where the subscript 0 again denotes that the amplitude of the field is of the zeroth power in ω. This results in an electric field in accordance with Maxwell's curl equation for $\mathbf{E}$, given by (3.17). Thus, we have for the geometry of the arrangement,

$$\begin{aligned} \frac{\partial E_{x1}}{\partial z} &= -\frac{\partial B_{y0}}{\partial t} = \frac{\omega \mu I_0}{w} \sin \omega t \\ \mathbf{E}_1 &= \frac{\omega \mu I_0 z}{w} \sin \omega t \, \mathbf{a}_x \end{aligned} \tag{6.66}$$

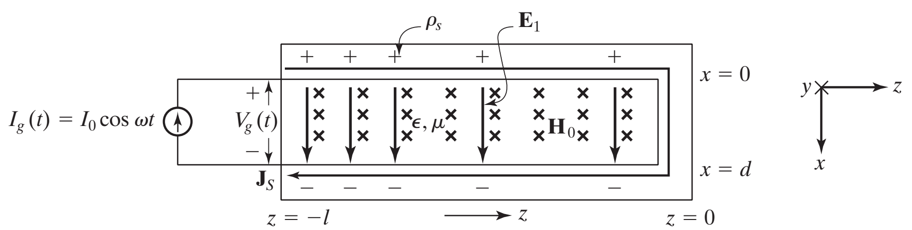

FIGURE 6.10

Magnetoquasistatic field analysis for the parallel-plate structure of Figure 6.6.

where we have also satisfied the boundary condition at $z = 0$ by choosing the constant of integration such that $[E_{x1}]_{z=0}$ is equal to zero, and again the subscript 1 denotes that the amplitude of the field is of the first power in ω. Note that the amplitude of E_{x1} varies linearly with z, from zero at $z = 0$ to a maximum at $z = -l$.

As in the case of electroquasistatic fields, we stop the process here, because continuing it by substituting (6.66) into Maxwell's curl equation for **H**, (3.28), to obtain the resulting magnetic field will yield a term having amplitude proportional to the second power in ω. This simply means that the fields given as a pair by (6.65) and (6.66) do not satisfy (3.28), and hence are not complete solutions to Maxwell's equations. They are the quasistatic fields. The complete solutions are obtained by solving Maxwell's equations simultaneously and subject to the boundary conditions for the given problem.

Proceeding further, we obtain the voltage across the current source to be

$$\begin{aligned} V_g(t) &= d[E_{x1}]_{z=-l} \\ &= -\omega\left(\frac{\mu dl}{w}\right)I_0 \sin \omega t \\ &= L\frac{dI_g(t)}{dt} \end{aligned} \tag{6.67a}$$

or

$$\bar{V}_g = j\omega L\bar{I}_g \tag{6.67b}$$

where $L = (\mu dl/w)$ is the inductance of the arrangement obtained from static field considerations. Thus, the input impedance of the structure is $j\omega L$, such that its low frequency input behavior is essentially that of a single inductor of value same as that found from static field analysis of the structure. Indeed, from considerations of power flow, using Poynting's theorem, we obtain the power flowing into the structure to be

$$\begin{aligned} P_{\text{in}} &= wd[E_{x1}H_{y0}]_{z=-l} \\ &= -\left(\frac{\mu dl}{w}\right)\omega I_0^2 \sin \omega t \cos \omega t \\ &= \frac{d}{dt}\left(\frac{1}{2}LI_g^2\right) \end{aligned} \tag{6.68}$$

which is consistent with the magnetic energy stored in the structure for the static case, as given by (6.49).

Quasistatic Fields in a Conductor

If the dielectric slab in an arrangement is conductive, then both electric and magnetic fields exist in the static case, because of the conduction current, as discussed under electromagnetostatic fields in Section 6.3. Furthermore, the electric field of amplitude proportional to the first power in ω contributes to the creation of magnetic field of amplitude proportional to the first power in ω, in addition to that from electric field of amplitude proportional to the zeroth power in ω. We shall illustrate by means of an example.

Example 6.8

Let us consider that the dielectric slab in the arrangement of Figure 6.9 is conductive, as shown in Figure 6.11(a), and carry out the quasistatic field analysis for the arrangement.

Using the results from the static field analysis from the arrangement of Figure 6.7, we have for the arrangement of Figure 6.11(a),

$$\mathbf{E}_0 = \frac{V_0}{d}\cos\omega t\,\mathbf{a}_x \tag{6.69}$$

$$\mathbf{J}_{c0} = \sigma\mathbf{E}_0 = \frac{\sigma V_0}{d}\cos\omega t\,\mathbf{a}_x \tag{6.70}$$

$$\mathbf{H}_0 = -\frac{\sigma V_0 z}{d}\cos\omega t\,\mathbf{a}_y \tag{6.71}$$

as depicted in the figure. Also, the variations with z of the amplitudes of E_{x0} and H_{y0} are shown in Figure 6.11(b).

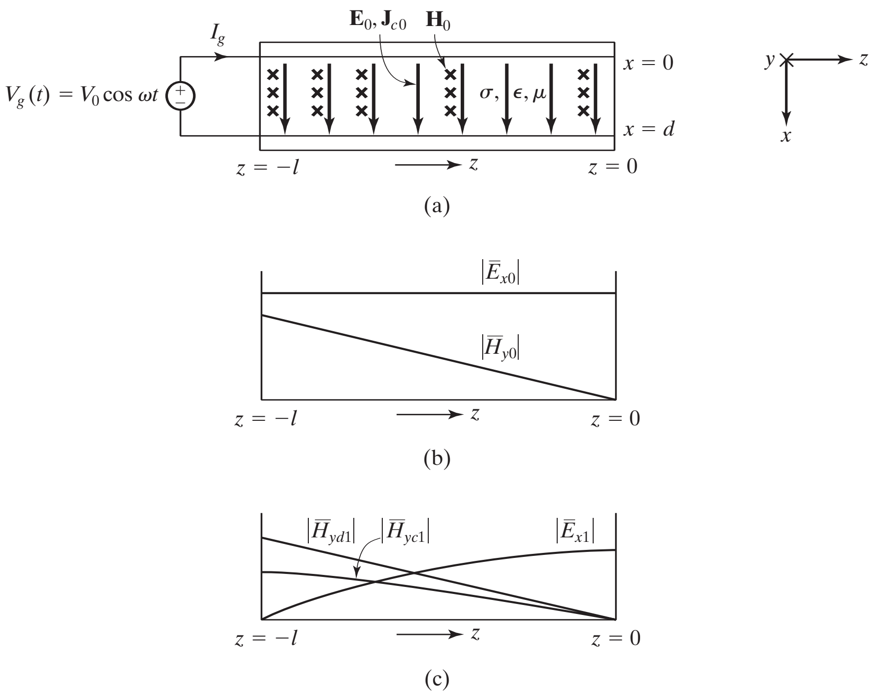

FIGURE 6.11

(a) Zero-order fields for the parallel-plate structure of Figure 6.7. (b) Variations of amplitudes of the zero-order fields along the structure. (c) Variations of amplitdes of the first-order fields along the structure.

The magnetic field given by (6.69) gives rise to an electric field having amplitude proportional to the first power in ω, in accordance with Maxwell's curl equation for **E**, (3.17). Thus,

$$\frac{\partial E_{x1}}{\partial z} = -\frac{\partial B_{y0}}{\partial t} = -\frac{\omega\mu\sigma V_0 z}{d}\sin\omega t$$

$$E_{x1} = -\frac{\omega\mu\sigma V_0}{2d}(z^2 - l^2)\sin\omega t \tag{6.72}$$

where we have also made sure that the boundary condition at $z = -l$ is satisfied. This boundary condition requires that E_x be equal to V_g/d at $z = -l$. Since this is satisfied by E_{x0} alone, it follows that E_{x1} must be zero at $z = -l$.

The electric field given by (6.69) and that given by (6.72) together give rise to a magnetic field having terms with amplitudes proportional to the first power in ω, in accordance with Maxwell's curl equation for **H**, (3.28). Thus,

$$\frac{\partial H_{y1}}{\partial z} = -\sigma E_{x1} - \epsilon\frac{\partial E_{x0}}{\partial t}$$

$$= \frac{\omega\mu\sigma^2 V_0}{2d}(z^2 - l^2)\sin\omega t + \frac{\omega\epsilon V_0}{d}\sin\omega t$$

$$H_{y1} = \frac{\omega\mu\sigma^2 V_0(z^3 - 3zl^2)}{6d}\sin\omega t + \frac{\omega\epsilon V_0 z}{d}\sin\omega t \tag{6.73}$$

where we have also made sure that the boundary condition at $z = 0$ is satisfied. This boundary condition requires that H_y be equal to zero at $z = 0$, which means that all of its terms must be zero at $z = 0$. Note that the first term on the right side of (6.73) is the contribution from the conduction current in the material resulting from E_{x1}, and the second term is the contribution from the displacement current resulting from E_{x0}. Denoting these to be H_{yc1} and H_{yd1}, respectively, we show the variations with z of the amplitudes of all the field components having amplitudes proportional to the first power in ω in Figure 6.11(c).

Now, adding up the contributions to each field, we obtain the total electric and magnetic fields up to and including the terms with amplitudes proportional to the first power in ω to be

$$E_x = \frac{V_0}{d}\cos\omega t - \frac{\omega\mu\sigma V_0}{2d}(z^2 - l^2)\sin\omega t \tag{6.74a}$$

$$H_y = -\frac{\sigma V_0 z}{d}\cos\omega t + \frac{\omega\epsilon V_0 z}{d}\sin\omega t + \frac{\omega\mu\sigma^2 V_0(z^3 - 3zl^2)}{6d}\sin\omega t \tag{6.74b}$$

or

$$\bar{E}_x = \frac{\bar{V}_g}{d} + j\omega\frac{\mu\sigma}{2d}(z^2 - l^2)\bar{V}_g \tag{6.75a}$$

$$\bar{H}_y = -\frac{\sigma z}{d}\bar{V}_g - j\omega\frac{\epsilon z}{d}\bar{V}_g - j\omega\frac{\mu\sigma^2(z^3 - 3zl^2)}{6d}\bar{V}_g \tag{6.75b}$$

Finally, the current drawn from the voltage source is given by

$$\bar{I}_g = w[\bar{H}_y]_{z=-l}$$

$$= \left(\frac{\sigma wl}{d} + j\omega\frac{\epsilon wl}{d} - j\omega\frac{\mu\sigma^2 wl^3}{3d}\right)\bar{V}_g \tag{6.76}$$

The input admittance of the structure is given by

$$\begin{aligned}\bar{Y}_{\text{in}} &= \frac{\bar{I}_g}{\bar{V}_g} = j\omega\frac{\epsilon wl}{d} + \frac{\sigma wl}{d}\left(1 - j\omega\frac{\mu\sigma l^2}{3}\right) \\ &\approx j\omega\frac{\epsilon wl}{d} + \frac{1}{\dfrac{d}{\sigma wl}\left(1 + j\omega\dfrac{\mu\sigma l^2}{3}\right)}\end{aligned} \tag{6.77}$$

where we have used the approximation $[1 + j\omega(\mu\sigma l^2/3)]^{-1} \approx [1 - j\omega(\mu\sigma l^2/3)]$. Proceeding further, we have

$$\begin{aligned}\bar{Y}_{\text{in}} &= j\omega\frac{\epsilon wl}{d} + \frac{1}{\dfrac{d}{\sigma wl} + j\omega\dfrac{\mu dl}{3w}} \\ &= j\omega C + \frac{1}{R + j\omega L_i}\end{aligned} \tag{6.78}$$

where $C = \epsilon wl/d$ is the capacitance of the structure if the material is a perfect dielectric, $R = d/\sigma wl$ is the resistance of the structure, and $L_i = \mu dl/3w$ is the internal inductance of the structure, all computed from static field analysis of the structure.

The equivalent circuit corresponding to (6.78) consists of capacitance C in parallel with the series combination of resistance R and internal inductance L_i, the same as in Figure 6.8. Thus, the low-frequency input behavior of the structure is essentially the same as that of the equivalent circuit of Figure 6.8, with the understanding that its input admittance must also be approximated to first-order terms. Note that for $\sigma = 0$, the input admittance of the structure is purely capacitive. For nonzero σ, a critical value of σ equal to $\sqrt{3\epsilon/\mu l^2}$ exists for which the input admittance is purely conductive. For values of σ smaller than the critical value, the input admittance is complex and capacitive, and for values of σ larger than the critical value, the input admittance is complex and inductive.

6.5 THE DISTRIBUTED CIRCUIT CONCEPT AND THE PARALLEL-PLATE TRANSMISSION LINE

In the preceding section, we have seen that, from the circuit point of view, the parallel-plate structure of Figure 6.5 behaves like a capacitor for the static case and the capacitive character is essentially retained for its input behavior for sinusoidally time-varying excitation at frequencies low enough to be within the range of validity of the quasistatic approximation. Likewise, we have seen that, from a circuit point of view, the parallel-plate structure of Figure 6.6 behaves like an inductor for the static case and the inductive character is essentially retained for its input behavior for sinusoidally time-varying excitation at frequencies low enough to be within the range of validity of the quasistatic approximation. For both structures, at an arbitrarily high enough frequency, the input behavior can be obtained only by obtaining complete (wave) solutions to Maxwell's equations, subject to the appropriate boundary conditions.

Two questions to ask at this point are (1) whether there is a circuit equivalent for the structure itself, independent of the termination, that is representative of the phenomenon taking place along the structure and valid at any arbitrary frequency, to the extent that the material parameters themselves are independent of frequency, and (2) what the limit on frequency is beyond which the quasistatic approximation is not valid. The answer to the first question is, yes, under a certain condition, giving rise to the concept of the *distributed circuit*, which we shall develop in this section by considering the parallel-plate structure, to be then known as the *parallel-plate transmission line.* The condition is that the waves propagating along the structure be the so-called *transverse electromagnetic* or TEM waves, meaning that the directions of the electric and magnetic fields are entirely traverse to the direction of propagation of the waves. The answer to the second question is that for the quasistatic approximation to hold, the length of the physical structure along the direction of propagation of the waves must be very small compared to the wavelength corresponding to the frequency of the source, in the dielectric region between the plates. While this can be obtained by extending the solution for the quasistatic case beyond the terms of the first power in ω by successive solution of Maxwell's equations (as in Section 4.3) and finding the condition under which the term of the first power in ω is predominant, it is more straightforward to obtain the exact solution by resorting to simultaneous solution of Maxwell's equations and finding the condition for which it approximates to the quasistatic solution. We shall do this in Section 7.1 by considering the structure of Figure 6.10 as a short-circuited transmission line and finding its input impedance.

Now, to develop and discuss the concept of the distributed circuit, we consider the parallel-plate arrangement of Figure 6.7(a) excited by a sinusoidally time-varying source of arbitrary frequency, as shown in Figure 6.12(a). Then, for an exact solution, the equations to be solved are

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\mu\frac{\partial \mathbf{H}}{\partial t} \tag{6.79a}$$

$$\nabla \times \mathbf{H} = \mathbf{J}_c + \frac{\partial \mathbf{D}}{\partial t} = \sigma\mathbf{E} + \epsilon\frac{\partial \mathbf{E}}{\partial t} \tag{6.79b}$$

For the geometry of the arrangement, neglecting fringing of the fields at the edges or assuming that the structure is part of a much larger-sized configuration, $\mathbf{E} = E_x(z, t)\mathbf{a}_x$ and $\mathbf{H} = H_y(z, t)\mathbf{a}_y$, so that (6.79a) and (6.79b) simplify to

$$\frac{\partial E_x}{\partial z} = -\mu\frac{\partial H_y}{\partial t} \tag{6.80a}$$

$$\frac{\partial H_y}{\partial z} = -\sigma E_x - \epsilon\frac{\partial E_x}{\partial t} \tag{6.80b}$$

The situation is one of uniform plane electromagnetic waves propagating in the z-direction as though the conductors are not present, being guided by them, since all the boundary conditions are satisfied. We then have the simple case of a *parallel-plate transmission line.* Now, since E_z and H_z are zero in a given constant-z plane, that is, a plane *transverse* to the direction of propagation of the wave, as shown in Figure 6.12(b), we can uniquely define a voltage between the plates in terms of the electric field intensity in that plane, and a current crossing that plane in one direction on the top plate and in the opposite direction on the bottom plate in terms of the magnetic

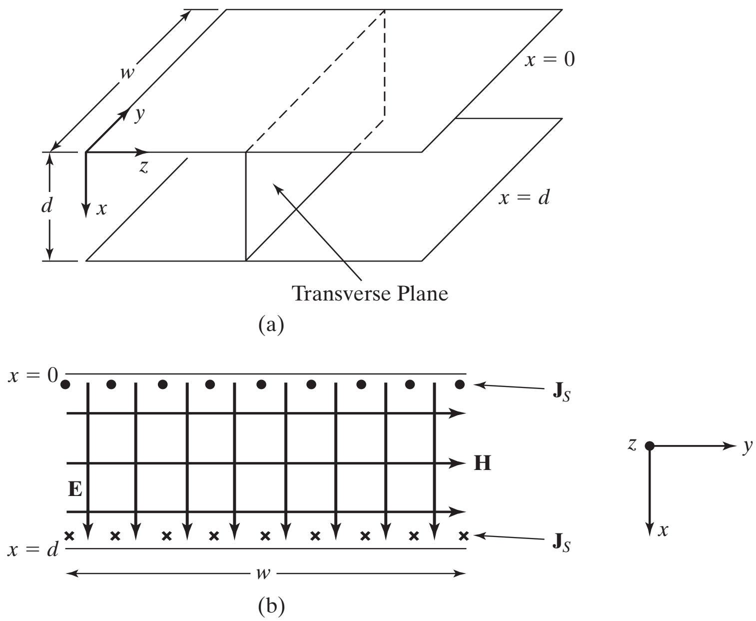

FIGURE 6.12

(a) Parallel-plate transmission line. (b) A transverse plane of the parallel-plate transmission line.

field intensity in that plane. These are given by

$$V(z,t) = \int_{x=0}^{d} E_x(z,t)\,dx = E_x(z,t)\int_{x=0}^{d} dx = dE_x(z,t) \tag{6.81a}$$

$$\begin{aligned} I(z,t) &= \int_{y=0}^{w} J_S(z,t)\,dy = \int_{y=0}^{w} H_y(z,t)\,dy = H_y(z,t)\int_{y=0}^{w} dy \\ &= wH_y(z,t) \end{aligned} \tag{6.81b}$$

Proceeding further, we can find the power flow down the line by evaluating the surface integral of the Poynting vector over a given transverse plane. Thus,

$$\begin{aligned} P(z,t) &= \int_{\text{transverse plane}} (\mathbf{E}\times\mathbf{H})\cdot d\mathbf{S} \\ &= \int_{x=0}^{d}\int_{y=0}^{w} E_x(z,t)H_y(z,t)\,\mathbf{a}_z\cdot dx\,dy\,\mathbf{a}_z \\ &= \int_{x=0}^{d}\int_{y=0}^{w} \frac{V(z,t)}{d}\,\frac{I(z,t)}{w}\,dx\,dy \\ &= V(z,t)I(z,t) \end{aligned} \tag{6.82}$$

which is the familiar relationship employed in circuit theory.

From (6.81a) and (6.81b), we have

$$E_x = \frac{V}{d} \tag{6.83a}$$

$$H_y = \frac{I}{w} \tag{6.83b}$$

Substituting for E_x and H_y in (6.80a) and (6.80b) from (6.83a) and (6.83b), respectively, we now obtain two differential equations for voltage and current along the line as

$$\frac{\partial}{\partial z}\left(\frac{V}{d}\right) = -\mu\frac{\partial}{\partial t}\left(\frac{I}{w}\right) \tag{6.84a}$$

$$\frac{\partial}{\partial z}\left(\frac{I}{w}\right) = -\sigma\left(\frac{V}{d}\right) - \epsilon\frac{\partial}{\partial t}\left(\frac{V}{d}\right) \tag{6.84b}$$

or

$$\frac{\partial V}{\partial z} = -\left(\frac{\mu d}{w}\right)\frac{\partial I}{\partial t} \tag{6.85a}$$

$$\frac{\partial I}{\partial z} = -\left(\frac{\sigma w}{d}\right)V - \left(\frac{\epsilon w}{d}\right)\frac{\partial V}{\partial t} \tag{6.85b}$$

We now recognize the quantities in parentheses in (6.85a) and (6.85b) to be the circuit parameters L, G, and C, divided by the length l of the structure in the z-direction. Thus, these are the inductance per unit length, capacitance per unit length, and conductance per unit length, of the line, denoted to be $\mathscr{L}$, $\mathscr{G}$, and $\mathscr{C}$, respectively, and we can write the equations in terms of these parameters as

$$\frac{\partial V}{\partial z} = -\mathscr{L}\frac{\partial I}{\partial t} \tag{6.86a}$$

$$\frac{\partial I}{\partial z} = -\mathscr{G}V - \mathscr{C}\frac{\partial V}{\partial t} \tag{6.86b}$$

where

$$\mathscr{L} = \frac{\mu d}{w} \tag{6.87a}$$

$$\mathscr{C} = \frac{\epsilon w}{d} \tag{6.87b}$$

$$\mathscr{G} = \frac{\sigma w}{d} \tag{6.87c}$$

We note that $\mathscr{L}$, $\mathscr{C}$, and $\mathscr{G}$ are purely dependent on the dimensions of the line and

$$\mathscr{L}\mathscr{C} = \mu\epsilon \tag{6.88a}$$

$$\frac{\mathscr{G}}{\mathscr{C}} = \frac{\sigma}{\epsilon} \tag{6.88b}$$

Equations (6.86a) and (6.86b) are known as the *transmission line equations.* They characterize the wave propagation along the line in terms of the circuit quantities instead of in terms of the field quantities. It should, however, not be forgotten that the actual phenomenon is one of electromagnetic waves guided by the conductors of the line.

It is customary to represent a transmission line by means of its circuit equivalent, derived from the transmission-line equations (6.86a) and (6.86b). To do this, let us consider a section of infinitesimal length Δz along the line between z and $z + \Delta z$. From (6.86a), we then have

$$\lim_{\Delta z \to 0} \frac{V(z + \Delta z, t) - V(z, t)}{\Delta z} = -\mathscr{L}\frac{\partial I(z, t)}{\partial t}$$

or, for $\Delta z \to 0$,

$$V(z + \Delta z, t) - V(z, t) = -\mathscr{L}\,\Delta z \frac{\partial I(z, t)}{\partial t} \tag{6.89a}$$

This equation can be represented by the circuit equivalent shown in Figure 6.13(a), since it satisfies Kirchhoff's voltage law written around the loop *abcda*. Similarly, from (6.86b), we have

$$\lim_{\Delta z \to 0} \frac{I(z + \Delta z, t) - I(z, t)}{\Delta z} = \lim_{\Delta z \to 0}\left[-\mathscr{G}V(z + \Delta z, t) - \mathscr{C}\frac{\partial V(z + \Delta z, t)}{\partial t}\right]$$

or, for $\Delta z \to 0$,

$$I(z + \Delta z, t) - I(z, t) = -\mathscr{G}\,\Delta z\, V(z + \Delta z, t) - \mathscr{C}\,\Delta z \frac{\partial V(z + \Delta z, t)}{\partial t} \tag{6.89b}$$

This equation can be represented by the circuit equivalent shown in Figure 6.13(b), since it satisfies Kirchhoff's current law written for node *c*. Combining the two equations, we then obtain the equivalent circuit shown in Figure 6.13(c) for a section Δz of the line. It

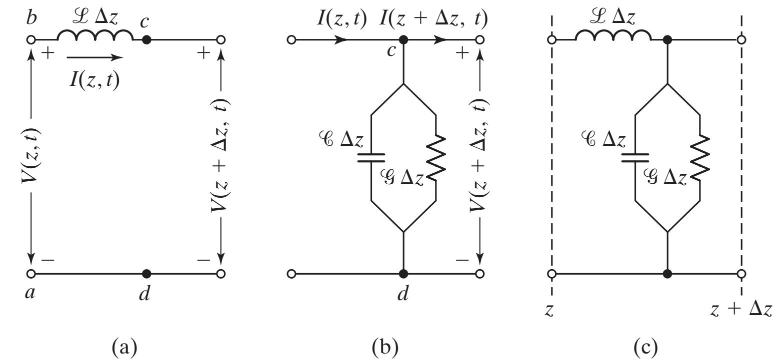

FIGURE 6.13

Development of circuit equivalent for an infinitesimal length Δz of a transmission line.

then follows that the circuit representation for a portion of length l of the line consists of an infinite number of such sections in cascade, as shown in Figure 6.14. Such a circuit is known as a *distributed circuit* as opposed to the *lumped circuits* that are familiar in circuit theory. The distributed circuit notion arises from the fact that the inductance, capacitance, and conductance are distributed uniformly and overlappingly along the line.

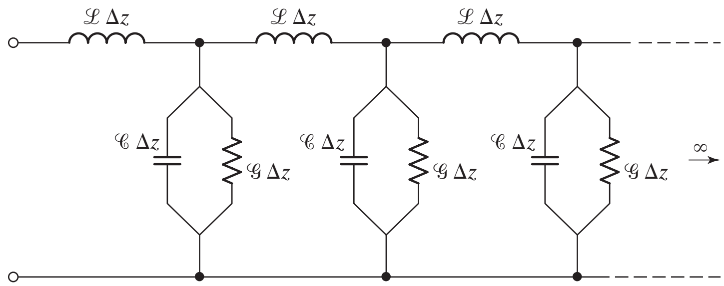

FIGURE 6.14

Distributed circuit representation of a transmission line.

A more physical interpretation of the distributed circuit concept follows from energy considerations. We know that the uniform plane wave propagation between the conductors of the line is characterized by energy storage in the electric and magnetic fields and power dissipation due to the conduction current flow. If we consider a section Δz of the line, the energy stored in the electric field in this section is given by

$$\begin{aligned} W_e &= \frac{1}{2}\epsilon E_x^2(\text{volume}) = \frac{1}{2}\epsilon E_x^2(dw\,\Delta z) \\ &= \frac{1}{2}\frac{\epsilon w}{d}(E_x d)^2\,\Delta z = \frac{1}{2}\mathcal{C}\,\Delta z\,V^2 \end{aligned} \tag{6.90a}$$

The energy stored in the magnetic field in that section is given by

$$\begin{aligned} W_m &= \frac{1}{2}\mu H_y^2\,(\text{volume}) = \frac{1}{2}\mu H_y^2(dw\,\Delta z) \\ &= \frac{1}{2}\frac{\mu d}{w}(H_y w)^2\,\Delta z = \frac{1}{2}\mathcal{L}\,\Delta z\,I^2 \end{aligned} \tag{6.90b}$$

The power dissipated due to conduction current flow in that section is given by

$$\begin{aligned} P_d &= \sigma E_x^2(\text{volume}) = \sigma E_x^2(dw\,\Delta z) \\ &= \frac{\sigma w}{d}(E_x d)^2\,\Delta z = \mathcal{G}\,\Delta z\,V^2 \end{aligned} \tag{6.90c}$$

Thus, $\mathcal{L}$, $\mathcal{C}$, and $\mathcal{G}$ are elements associated with energy storage in the magnetic field, energy storage in the electric field, and power dissipation due to the conduction current flow in the dielectric, respectively, for a given infinitesimal section of the line. Since these phenomena occur continuously and since they overlap, the inductance, capacitance, and conductance must be distributed uniformly and overlappingly along the line. In actual practice, the conductors of the transmission line are imperfect, resulting in slight penetration of the fields into the conductors, in accordance with the skin effect phenomenon. This gives rise to power dissipation and magnetic field energy storage in the conductors, which are taken into account by including a resistance and additional inductance in the series branch of the transmission-line equivalent circuit.

6.6 TRANSMISSION LINE WITH AN ARBITRARY CROSS SECTION

In the previous section, we considered the parallel-plate transmission line made up of perfectly conducting sheets lying in the planes $x = 0$ and $x = \mathrm{d}$ so that the boundary conditions of zero tangential component of the electric field and zero normal component of the magnetic field are satisfied by the uniform plane wave characterized by the fields

$$\mathbf{E} = E_x(z, t)\mathbf{a}_x$$
$$\mathbf{H} = H_y(z, t)\mathbf{a}_y$$

thereby leading to the situation in which the uniform plane wave is guided by the conductors of the transmission line. In the general case, however, the conductors of the transmission line have arbitrary cross sections and the fields consist of both x- and y-components and are dependent on x- and y-coordinates in addition to the z-coordinate. Thus, the fields between the conductors are given by

$$\mathbf{E} = E_x(x, y, z, t)\mathbf{a}_x + E_y(x, y, z, t)\mathbf{a}_y$$
$$\mathbf{H} = H_x(x, y, z, t)\mathbf{a}_x + H_y(x, y, z, t)\mathbf{a}_y$$

These fields are no longer uniform in x and y but are directed entirely transverse to the direction of propagation, that is, the z-axis, which is the axis of the transmission line. Hence, they are known as *transverse electromagnetic waves*, or *TEM waves.* The uniform plane waves are simply a special case of the transverse electromagnetic waves.

To extend the computation of the transmission line parameters $\mathcal{L}$, $\mathcal{C}$, and $\mathcal{G}$ to the general case, let us consider a transmission line made up of parallel, perfect conductors of arbitrary cross sections, as shown by the cross-sectional view in Figure 6.15(a). Let us assume that the inner conductor is positive with respect to the outer conductor and that the current flows along the positive z-direction (into the page) on the inner conductor and along the negative z-direction (out of the page) on the outer conductor. We can then draw a *field map*, that is, a graphical sketch of the direction lines of the fields between the conductors, from the following considerations: (a) The electric field lines must originate on the inner conductor and be normal to it and must terminate on the outer conductor and be normal to it, since the tangential component of the electric field on a perfect conductor surface must be zero. (b) The magnetic field lines must be everywhere perpendicular to the electric field lines; although this can be shown by a rigorous mathematical proof, it is intuitively obvious, since, first, the magnetic field lines must be tangential near the conductor surfaces and, second, at any arbitrary point the fields correspond to those of a locally uniform plane wave. Thus, suppose that we

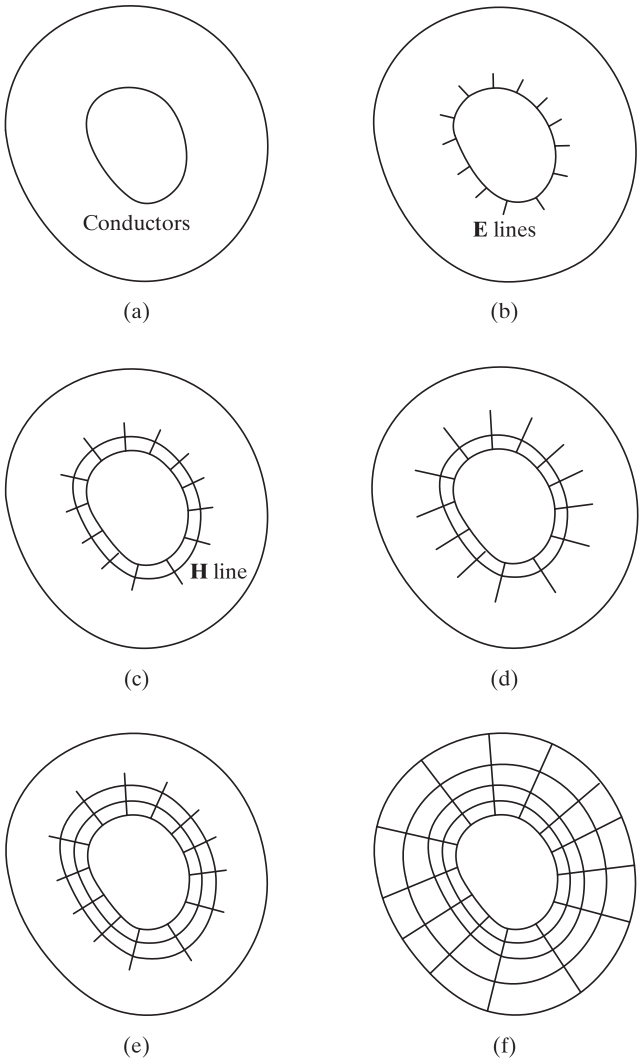

FIGURE 6.15

Construction of a transmission line field map consisting of curvilinear rectangles.

start with the inner conductor and draw several lines normal to it at several points on the surface, as shown in Figure 6.15(b). We can then draw a curved line displaced from the conductor surface and such that it is perpendicular everywhere to the electric field lines of Figure 6.15(b), as shown in Figure 6.15(c). This contour represents a magnetic field line and forms the basis for further extension of the electric field lines, as shown in Figure 6.15(d). A second magnetic field line can then be drawn so that it is everywhere perpendicular to the extended electric field lines, as shown in Figure 6.15(e). This procedure is continued until the entire cross section between the conductors is filled with two sets of orthogonal contours, as shown in Figure 6.15(f), thereby resulting in a field map made up of curvilinear rectangles.

By drawing the field lines with very small spacings, we can make the rectangles so small that each of them can be considered to be the cross section of a parallel-plate line. In fact, by choosing the spacings appropriately, we can even make them a set of squares. If we now replace the magnetic field lines by perfect conductors, since it does not violate any boundary condition, it can be seen that the arrangement can be viewed as the parallel combination, in the angular direction, of m number of series combinations of n number of parallel-plate lines in the radial direction, where m is the number of squares in the angular direction, that is, along a magnetic field line, and n is the number of squares in the radial direction, that is, along an electric field line. We can then find simple expressions for $\mathcal{L}$, $\mathcal{C}$, and $\mathcal{G}$ of the line in the following manner.

Let us for simplicity consider the field map of Figure 6.16, consisting of eight segments $1, 2, \ldots, 8$ in the angular direction and two segments a and b in the radial direction. The arrangement is then a parallel combination, in the angular direction, of eight series combinations of two lines in the radial direction, each having a curvilinear rectangular cross section. Let $I_1, I_2, \ldots, I_8$ be the currents associated with the segments $1, 2, \ldots, 8$, respectively, and let ψ_a and ψ_b be the magnetic fluxes per unit length in the z-direction associated with the segments a and b, respectively. Then the inductance per

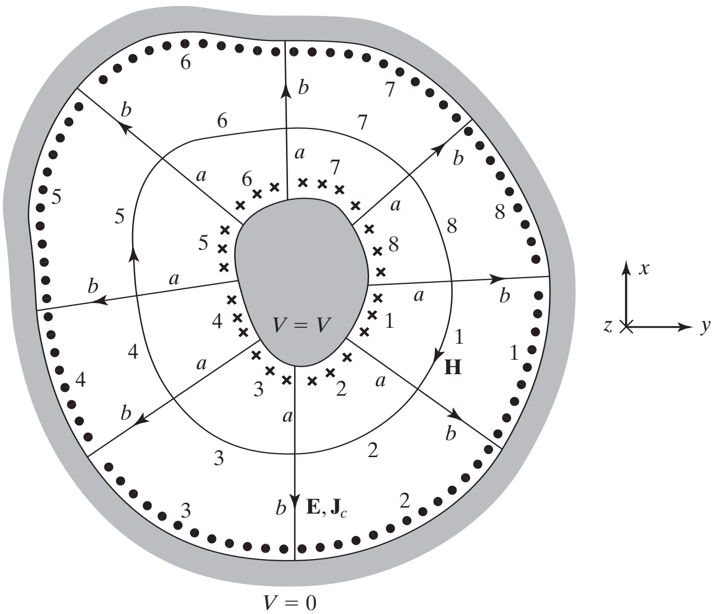

FIGURE 6.16

For deriving the expressions for the transmission-line parameters from the field map.

unit length of the transmission line is given by

$$\begin{aligned}\mathscr{L} &= \frac{\psi}{I} = \frac{\psi_a + \psi_b}{I_1 + I_2 + \cdots + I_8} \\ &= \frac{1}{\dfrac{I_1}{\psi_a} + \dfrac{I_2}{\psi_a} + \cdots \dfrac{I_8}{\psi_a}} + \frac{1}{\dfrac{I_1}{\psi_b} + \dfrac{I_2}{\psi_b} \cdots + \dfrac{I_8}{\psi_b}} \\ &= \frac{1}{\dfrac{1}{\mathscr{L}_{1a}} + \dfrac{1}{\mathscr{L}_{2a}} + \cdots + \dfrac{1}{\mathscr{L}_{8a}}} + \frac{1}{\dfrac{1}{\mathscr{L}_{1b}} + \dfrac{1}{\mathscr{L}_{2b}} + \cdots + \dfrac{1}{\mathscr{L}_{8b}}}\end{aligned} \tag{6.91a}$$

Let $Q_1, Q_2, \ldots, Q_8$ be the charges per unit length in the z-direction associated with the segments $1, 2, \ldots, 8$, respectively, and let V_a and V_b be the voltages associated with the segments a and b, respectively. Then the capacitance per unit length of the transmission line is given by

$$\begin{aligned}\mathscr{C} &= \frac{Q}{V} = \frac{Q_1 + Q_2 + \cdots + Q_8}{V_a + V_b} \\ &= \frac{1}{\dfrac{V_a}{Q_1} + \dfrac{V_b}{Q_1}} + \frac{1}{\dfrac{V_a}{Q_2} + \dfrac{V_b}{Q_2}} + \cdots + \frac{1}{\dfrac{V_a}{Q_8} + \dfrac{V_b}{Q_8}} \\ &= \frac{1}{\dfrac{1}{\mathscr{C}_{1a}} + \dfrac{1}{\mathscr{C}_{1b}}} + \frac{1}{\dfrac{1}{\mathscr{C}_{2a}} + \dfrac{1}{\mathscr{C}_{2b}}} + \cdots + \frac{1}{\dfrac{1}{\mathscr{C}_{8a}} + \dfrac{1}{\mathscr{C}_{8b}}}\end{aligned} \tag{6.91b}$$

Let $I_{c1}, I_{c2}, \ldots, I_{c8}$ be the conduction currents per unit length in the z-direction associated with the segments $1, 2, \ldots, 8$, respectively. Then the conductance per unit length of the transmission line is given by

$$\begin{aligned}\mathscr{G} &= \frac{I_c}{V} = \frac{I_{c1} + I_{c2} + \cdots + I_{c8}}{V_a + V_b} \\ &= \frac{1}{\dfrac{V_a}{I_{c1}} + \dfrac{V_b}{I_{c1}}} + \frac{1}{\dfrac{V_a}{I_{c2}} + \dfrac{V_b}{I_{c2}}} + \cdots + \frac{1}{\dfrac{V_a}{I_{c8}} + \dfrac{V_b}{I_{c8}}} \\ &= \frac{1}{\dfrac{1}{\mathscr{G}_{1a}} + \dfrac{1}{\mathscr{G}_{1b}}} + \frac{1}{\dfrac{1}{\mathscr{G}_{2a}} + \dfrac{1}{\mathscr{G}_{2b}}} + \cdots + \frac{1}{\dfrac{1}{\mathscr{G}_{8a}} + \dfrac{1}{\mathscr{G}_{8b}}}\end{aligned} \tag{6.91c}$$

Generalizing the expressions (6.91a), (6.91b), and (6.91c) to m segments in the angular direction and n segments in the radial direction, we obtain

$$\mathscr{L} = \sum_{j=1}^{n} \frac{1}{\displaystyle\sum_{i=1}^{m} \frac{1}{\mathscr{L}_{ij}}} \tag{6.92a}$$

$$\mathscr{C} = \sum_{i=1}^{m} \frac{1}{\displaystyle\sum_{j=1}^{n} \frac{1}{\mathscr{C}_{ij}}} \tag{6.92b}$$

$$\mathcal{G} = \sum_{i=1}^{m} \frac{1}{\displaystyle\sum_{j=1}^{n} \frac{1}{\mathcal{G}_{ij}}} \tag{6.92c}$$

where $\mathcal{L}_{ij}$, $\mathcal{C}_{ij}$, and $\mathcal{G}_{ij}$ are the inductance, capacitance, and conductance per unit length corresponding to the rectangle *ij*. If the map consists of curvilinear squares, then $\mathcal{L}_{ij}$, $\mathcal{C}_{ij}$, and $\mathcal{G}_{ij}$ are equal to μ, ϵ, and σ, respectively, according to (6.87a), (6.87b), and (6.87c), respectively, since the width w of the plates is equal to the spacing d of the plates for each square. Thus, we obtain simple expressions for $\mathcal{L}$, $\mathcal{C}$, and $\mathcal{G}$ as given by

$$\mathcal{L} = \mu \frac{n}{m} \tag{6.93a}$$

$$\mathcal{C} = \epsilon \frac{m}{n} \tag{6.93b}$$

$$\mathcal{G} = \sigma \frac{m}{n} \tag{6.93c}$$

The computation of $\mathcal{L}$, $\mathcal{C}$, and $\mathcal{G}$ then consists of sketching a field map consisting of curvilinear squares, counting the number of squares in each direction, and substituting these values in (6.93a), (6.93b), and (6.93c). Note that once again

$$\mathcal{L}\mathcal{C} = \mu\epsilon \tag{6.94a}$$

$$\frac{\mathcal{G}}{\mathcal{C}} = \frac{\sigma}{\epsilon} \tag{6.94b}$$

We shall now consider an example of the application of the curvilinear squares technique.

Example 6.9

The coaxial cable is a transmission line made up of parallel, coaxial, cylindrical conductors. Let the radius of the inner conductor be a and that of the outer conductor be b. We wish to find expressions for $\mathcal{L}$, $\mathcal{C}$, and $\mathcal{G}$ of the coaxial cable by using the curvilinear squares technique.

Figure 6.17 shows the cross-sectional view of the coaxial cable and the field map. In view of the symmetry associated with the conductor configuration, the construction of the field map is simplified in this case. The electric field lines are radial lines from one conductor to the other, and the magnetic field lines are circles concentric with the conductors, as shown in the figure. Let the number of curvilinear squares in the angular direction be m. Then to find the number of curvilinear squares in the radial direction, we note that the angle subtended at the center of the conductors by adjacent pairs of electric field lines is equal to $2\pi/m$. Hence, at any arbitrary radius r between the two conductors, the side of the curvilinear square is equal to $r(2\pi/m)$. The number of squares in an infinitesimal distance dr in the radial direction is then equal to $\dfrac{dr}{r(2\pi/m)}$, or $\dfrac{m}{2\pi}\dfrac{dr}{r}$. The total number of squares in the radial direction from the inner to the outer conductor is given by

$$n = \int_{r=a}^{b} \frac{m}{2\pi}\frac{dr}{r} = \frac{m}{2\pi} \ln \frac{b}{a}$$

The required expressions for $\mathcal{L}$, $\mathcal{C}$, and $\mathcal{G}$ are then given by

$$\mathcal{L} = \mu \frac{n}{m} = \frac{\mu}{2\pi} \ln \frac{b}{a} \tag{6.95a}$$

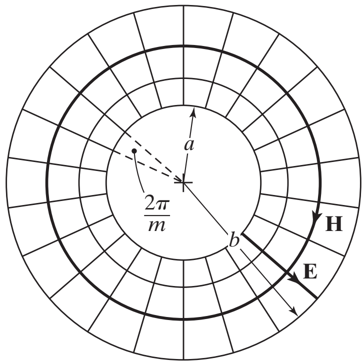

FIGURE 6.17

Field map consisting of curvilinear squares for a coaxial cable.

$$\mathscr{C} = \epsilon \frac{m}{n} = \frac{2\pi\epsilon}{\ln (b/a)} \tag{6.95b}$$

$$\mathscr{G} = \sigma \frac{m}{n} = \frac{2\pi\sigma}{\ln (b/a)} \tag{6.95c}$$

These expressions are exact. We have been able to obtain exact expressions in this case because of the geometry involved. When the geometry is not so simple, we can only obtain approximate values for $\mathscr{L}$, $\mathscr{C}$, and $\mathscr{G}$.

We have just discussed an example of the determination of the transmission-line parameters $\mathscr{L}$, $\mathscr{C}$, and $\mathscr{G}$ for a coaxial cable. There are other configurations having different cross sections for which one can obtain the parameters either by the curvilinear squares technique or by other analytical or experimental techniques. The parameters for some cases for which exact expressions are available are listed in Table 6.1, along with those for the parallel-plate line and coaxial cable.

TABLE 6.1 Conductance, Capacitance, and Inductance per Unit Length for Some Structures Consisting of Infinitely Long Conductors Having the Cross Sections Shown in Figure 6.18

Description	Capacitance per unit length, $\mathscr{C}$	Conductance per unit length, $\mathscr{G}$	Inductance per unit length, $\mathscr{L}$
Parallel-plane conductors, Figure 6.18(a)	$\epsilon \frac{w}{d}$	$\sigma \frac{w}{d}$	$\mu \frac{d}{w}$
Coaxial cylindrical conductors, Figure 6.18(b)	$\frac{2\pi\epsilon}{\ln (b/a)}$	$\frac{2\pi\sigma}{\ln (b/a)}$	$\frac{\mu}{2\pi} \ln \frac{b}{a}$
Parallel cylindrical wires, Figure 6.18(c)	$\frac{\pi\epsilon}{\cosh^{-1} (d/a)}$	$\frac{\pi\sigma}{\cosh^{-1} (d/a)}$	$\frac{\mu}{\pi} \cosh^{-1} \frac{d}{a}$
Eccentric inner conductor, Figure 6.18(d)	$\frac{2\pi\epsilon}{\cosh^{-1}\left(\frac{a^2 + b^2 - d^2}{2ab}\right)}$	$\frac{2\pi\sigma}{\cosh^{-1}\left(\frac{a^2 + b^2 - d^2}{2ab}\right)}$	$\frac{\mu}{2\pi} \cosh^{-1}\left(\frac{a^2 + b^2 + d^2}{2ab}\right)$
Shielded parallel cylindrical wires, Figure 6.18(e)	$\frac{\pi\epsilon}{\ln \frac{d(b^2 - d^2/4)}{a(b^2 + d^2/4)}}$	$\frac{\pi\sigma}{\ln \frac{d(b^2 - d^2/4)}{a(b^2 + d^2/4)}}$	$\frac{\mu}{\pi} \ln \frac{d(b^2 - d^2/4)}{a(b^2 + d^2/4)}$

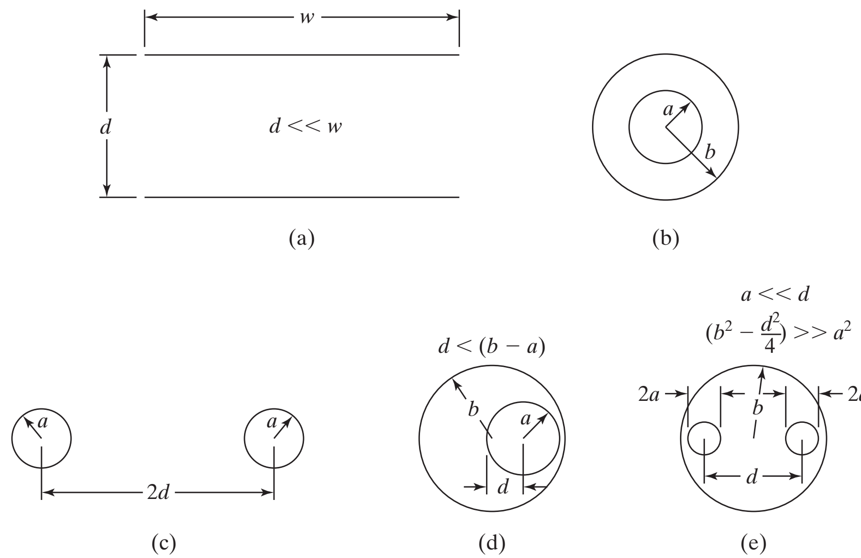

FIGURE 6.18

Cross sections of some common configurations of parallel, infinitely long conductors.

SUMMARY

In this chapter, we first introduced the electric potential from the fact that for the static case,

$$\nabla \times \mathbf{E} = 0 \tag{6.96}$$

and, since the curl of the gradient of a scalar function is identically zero, **E** can be expressed as the gradient of a scalar function. The gradient of a scalar function Φ is given in Cartesian coordinates by

$$\nabla\Phi = \frac{\partial\Phi}{\partial x}\,\mathbf{a}_x + \frac{\partial\Phi}{\partial y}\,\mathbf{a}_y + \frac{\partial\Phi}{\partial z}\,\mathbf{a}_z$$

The magnitude of $\nabla\Phi$ at a given point is the maximum rate of increase of Φ at that point, and its direction is the direction in which the maximum rate of increase occurs, that is, normal to the constant Φ surface passing through that point.

From considerations of work associated with the movement of a test charge in the static electric field, we found that for the case of the static electric field, the scalar function is $-V$, so that

$$\mathbf{E} = -\nabla V \tag{6.97}$$

where V is the electric potential. The electric potential V_A at a point A is the amount of work per unit charge done by the field in the movement of a test charge from the point A to a reference point O. It is the potential difference between A and O. Thus,

$$V_A = [V]_A^O = \int_A^O \mathbf{E}\cdot d\mathbf{l} = -\int_O^A \mathbf{E}\cdot d\mathbf{l}$$

The potential difference between two points has the same physical meaning as the voltage between the two points. The voltage in a time-varying field is, however, not a unique quantity, since it depends on the path employed for evaluating it, whereas the potential difference in a static field, being independent of the path, has a unique value.

We considered the potential field of a point charge and found that for the point charge

$$V = \frac{Q}{4\pi\epsilon R}$$

where R is the radial distance away from the point charge. The equipotential surfaces for the point charge are thus spherical surfaces centered at the point charge.

Substituting (6.97) into Maxwell's divergence equation for $\mathbf{D}$, we derived the Poisson's equation

$$\nabla^2 V = -\frac{\rho}{\epsilon} \tag{6.98}$$

which states that the Laplacian of the electric potential at a point is equal to $-1/\epsilon$ times the volume charge density at that point. In Cartesian coordinates,

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

For the one-dimensional case in which the charge density is a function of x only, (6.98) reduces to

$$\frac{\partial^2 V}{\partial x^2} = \frac{\partial^2 V}{\partial x^2} = -\frac{\rho}{\epsilon}$$

We illustrated the solution of this equation by considering the example of a p–n junction diode.

If $\rho = 0$, Poisson's equation reduces to Laplace's equation

$$\nabla^2 V = 0 \tag{6.99}$$

This equation is applicable for a charge-free dielectric region as well as for a conducting medium.

To introduce circuit elements, we next began with Maxwell's equations in differential form and the continuity equation for static fields, given by

$$\nabla \times \mathbf{E} = 0 \tag{6.100a}$$
$$\nabla \times \mathbf{H} = \mathbf{J} \tag{6.100b}$$
$$\nabla \cdot \mathbf{D} = \rho \tag{6.100c}$$
$$\nabla \cdot \mathbf{B} = 0 \tag{6.100d}$$
$$\nabla \cdot \mathbf{J} = 0 \tag{6.100e}$$

and considered three cases of static fields: (a) electrostatic fields, (b) magnetostatic fields, and (c) electromagnetostatic fields. From these three cases, we introduced the circuit elements, capacitance (C), inductance (L) and conductance (G), respectively, by considering a parallel-plate arrangement.

We then turned to the quasistatic extension of the static field solution as a means of obtaining the low-frequency behavior of a physical structure. The quasistatic field approach involves starting with a time-varying field having the same spatial characteristics as the static field in the physical structure and then obtaining field solutions containing terms up to and including the first power in frequency by using Maxwell's curl equations for time-varying fields. We applied this approach for the same three cases as for the static fields, and found that the input behavior of the structure remains essentially the same as for the corresponding static case.

The quasistatic approximation holds for frequencies for which the wavelength corresponding to the frequency of the source is large compared to the length of the structure along the direction of propagation of the waves, which is to be derived in Section 7.1. Beyond the range of validity of the quasistatic approximation, the input behavior can be obtained only by obtaining complete solutions to Maxwell's equations, subject to the boundary conditions, which led us to the concept of the distributed circuit and the parallel-plate structure becoming a parallel-plate transmission line. We derived the *transmission-line equations*,

$$\frac{\partial V}{\partial z} = -\mathcal{L}\frac{\partial I}{\partial t} \tag{6.101a}$$

$$\frac{\partial I}{\partial z} = -\mathcal{G}V - \mathcal{C}\frac{\partial V}{\partial t} \tag{6.101b}$$

These equations are applicable to all transmission lines, characterized by transverse electromagnetic wave propagation. They govern the wave propagation along the line in terms of circuit quantities instead of in terms of field quantities.

The parameters $\mathcal{L}$, $\mathcal{C}$, and $\mathcal{G}$ in (6.101a) and (6.101b) are the inductance, capacitance, and conductance per unit length of line, which differ from one line to another. For the parellel-plate line having width w of the plates and spacing d between the plates, they are given by

$$\mathcal{L} = \frac{\mu d}{w}$$

$$\mathcal{C} = \frac{\epsilon w}{d}$$

$$\mathcal{G} = \frac{\sigma w}{d}$$

where μ, ϵ, and σ are the material parameters of the medium between the plates and fringing of the fields is neglected. We learned how to compute $\mathcal{L}$, $\mathcal{C}$, and $\mathcal{G}$ for a line of arbitrary cross section by constructing a field map of the transverse electromagnetic wave fields, consisting of curvilinear squares in the cross-sectional plane of the line.

If m is the number of squares tangential to the conductors and n is the number of squares normal to the conductors, then

$$\mathcal{L} = \mu \frac{n}{m}$$

$$\mathcal{C} = \epsilon \frac{m}{n}$$

$$\mathcal{G} = \sigma \frac{m}{n}$$

By applying this technique to the coaxial cable, we found that for a cable of inner radius a and outer radius b,

$$\mathcal{L} = \frac{\mu}{2\pi} \ln \frac{b}{a}$$

$$\mathcal{C} = \frac{2\pi\epsilon}{\ln (b/a)}$$

$$\mathcal{G} = \frac{2\pi\sigma}{\ln (b/a)}$$

REVIEW QUESTIONS

6.1. State Maxwell's curl equations for static fields.

6.2. What is the expansion for the gradient of a scalar in Cartesian coordinates? When can a vector be expressed as the gradient of a scalar?

6.3. Discuss the physical interpretation for the gradient of a scalar function.

6.4. Discuss the application of the gradient concept for the determination of unit vector normal to a surface.

6.5. How would you find the rate of increase of a scalar function along a specified direction by using the gradient concept?

6.6. Define electric potential. What is its relationship to the static electric field intensity?

6.7. Distinguish between voltage, as applied to time-varying fields, and potential difference.

6.8. What is a conservative field? Give two examples of conservative fields.

6.9. Describe the equipotential surfaces for a point charge.

6.10. Discuss the determination of the electric field intensity due to a charge distribution by using the potential concept.

6.11. What is the Laplacian of a scalar? What is its expansion in Cartesian coordinates?

6.12. State Poisson's equation.

6.13. Outline the solution of Poisson's equation for the potential in a region of known charge density varying in one dimension.

6.14. State Laplace's equation. In what regions is it valid?

6.15. State Maxwell's equations for static fields in (a) integral form, and (b) differential form.

6.16. Discuss the classification of static fields with reference to subsets of Maxwell's equations.

6.17. Outline the steps involved in the electrostatic field analysis of a parallel plate structure and the determination of its capacitance.

6.18. Outline the steps involved in the magnetostatic field analysis of a parallel plate structure and the determination of its inductance.

6.19. Outline the steps involved in the electromagnetostatic field analysis of a parallel plate structure and the determination of its circuit equivalent.

6.20. Explain the term *internal inductance.*

6.21. What is meant by the quasistatic extension of the static field in a physical structure?

6.22. Outline the steps involved in the electroquasistatic field analysis of a parallel plate structure and the determination of its input behavior. Compare the input behavior with the electrostatic case.

6.23. Outline the steps involved in the magnetoquasistatic field analysis of a parallel plate structure and the determination of its input behavior. Compare the input behavior with the magnetostatic case.

6.24. Outline the steps involved in the quasistatic field analysis of a parallel plate structure with a conducting slab between the plates and the determination of its input behavior. Compare the input behavior with the electromagnetostatic case.

6.25. Discuss the phenomenon taking place along a parallel-plate structure at any arbitrary frequency and the need for the concept of the *distributed circuit.*

6.26. What is the limit on the frequency beyond which the quasistatic approximation for the input behavior of a physical structure is not valid?

6.27. How is the voltage between the two conductors in a given cross-sectional plane of a parallel-plate transmission line related to the electric field in that plane?

6.28. How is the current flowing on the plates across a given cross-sectional plane of a parallel-plate transmission line related to the magnetic field in that plane?

6.29. What are transmission-line equations? How are they obtained from Maxwell's equations?

6.30. What are the expressions for $\mathscr{L}$, the inductance per unit length, $\mathscr{C}$, the capacitance per unit length, and $\mathscr{G}$, the conductance per unit length, for a parallel-plate transmission line?

6.31. Are the three quantities $\mathscr{L}$, $\mathscr{C}$, and $\mathscr{G}$ independent? If not, how are they dependent on each other?

6.32. Draw the transmission-line equivalent circuit. How is it derived from the transmission-line equations?

6.33. Discuss the concept of the distributed circuit and compare it to a lumped circuit.

6.34. Discuss the physical phenomena associated with each of the elements in the transmission-line equivalent circuit.

6.35. What is a transverse electromagnetic wave?

6.36. What is a field map? Describe the procedure for drawing the field map for a transmission line of arbitrary cross section.

6.37. Draw a rough sketch of the field map for a line made up of two identical parallel cylindrical conductors with their axes separated by four times their radii.

6.38. Describe the procedure for computing the transmission line parameters $\mathscr{L}$, $\mathscr{C}$, and $\mathscr{G}$ from the field map.

6.39. How does a field map consisting of curvilinear squares simplify the computation of the line parameters?

6.40. Discuss the determination of $\mathscr{L}$, $\mathscr{C}$, and $\mathscr{G}$ for a coaxial cable by using the curvilinear squares technique.

PROBLEMS

6.1. Find the gradients of the following scalar functions: (a) $\sqrt{x^2 + y^2 + z^2}$; (b) xyz.

6.2. Determine which of the following vectors can be expressed as the gradient of a scalar function: (a) $y\mathbf{a}_x - x\mathbf{a}_y$; (b) $x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$; (c) $2xy^3z\mathbf{a}_x + 3x^2y^2z\mathbf{a}_y + x^2y^3\mathbf{a}_z$.

6.3. Find the unit vector normal to the plane surface $5x + 2y + 4z = 20$.

6.4. Find the unit vector normal to the surface $x^2 - y^2 = 5$ at the point $(3, 2, 1)$.

6.5. Find the rate of increase of the scalar function x^2y at the point $(1, 2, 1)$ in the direction of the vector $\mathbf{a}_x - \mathbf{a}_y$.

6.6. For the static electric field given by $\mathbf{E} = y\mathbf{a}_x + x\mathbf{a}_y$, find the potential difference between points $A(1, 1, 1)$ and $B(2, 2, 2)$.

6.7. For a point charge Q situated at the point $(1, 2, 0)$, find the potential difference between the point $A(3, 4, 1)$ and the point $B(5, 5, 0)$.

6.8. For the arrangement of a linear electric dipole consisting of point charges Q and $-Q$ at the points $(0, 0, d/2)$ and $(0, 0, -d/2)$, respectively, obtain the expression for the electric potential and hence for the electric field intensity at distances from the dipole large compared to d.

6.9. For a line charge of uniform density 10^{-3} C/m situated along the z-axis between $(0, 0, -1)$ and $(0, 0, 1)$, obtain the series expression for the electric potential at the point $(0, y, 0)$ by dividing the line charge into 100 equal segments and considering the charge in each segment to be a point charge located at the center of the segment. Then find the series expression for the electric field intensity at the point $(0, 1, 0)$.

6.10. Repeat Problem 6.9, assuming the line charge density to be $10^{-3}|z|$ C/m.

6.11. The potential distribution in a simplified model of a vacuum diode consisting of cathode in the plane $x = 0$ and anode in the plane $x = d$ and held at a potential V_0 relative to the cathode is given by

$$V = V_0\left(\frac{x}{d}\right)^{4/3} \qquad \text{for } 0 < x < d$$

(a) Find the space charge density distribution in the region $0 < x < d$.

(b) Find the surface charge densities on the cathode and the anode.

6.12. Show that for the p–n junction diode of Figure 6.4(a), the boundary condition of the continuity of the normal component of displacement flux density at $x = 0$ is automatically satisfied by equation (6.29).

6.13. Assume that the impurity concentration for the p–n junction diode of Figure 6.4(a) is a linear function of distance across the junction. The space charge density distribution is then given by

$$\rho = kx \qquad \text{for } -d/2 < x < d/2$$

where d is the width of the space charge region and k is the proportionality constant. Find the solution for the potential in the space charge region.

6.14. A space-charge density distribution is given by

$$\rho = \begin{cases} \rho_0 \sin x & \text{for } -\pi < x < \pi \\ 0 & \text{otherwise} \end{cases}$$

where ρ_0 is a constant. Find the sketch the potential V versus x for all x. Assume $V = 0$ for $x = 0$.

6.15. The region between the two plates of Figure 6.5 is filled with two perfect dielectric media having permittivities ϵ_1 for $0 < x < t$ and ϵ_2 for $t < x < d$. (a) Find the solutions for the potentials in the two regions $0 < x < t$ and $t < x < d$. (b) Find the potential at the interface $x = t$. (c) Find the capacitance of the arrangement.

6.16. For a dielectric medium of nonuniform permittivity, show that the Poisson's equation is given by

$$\epsilon \nabla^2 V + \nabla\epsilon \cdot \nabla V = -\rho$$

Assume that the region between the two plates of Figure 6.5 is filled with a perfect dielectric of nonuniform permittivity

$$\epsilon = \frac{\epsilon_0}{1 - (x/2d)}$$

Find the solution for the potential between the plates and obtain the expression for the capacitance per unit area of the plates.

6.17. The region between the plates of Figure 6.6 is divided into half in the y-direction. Assume that one half is filled with a material of permeability μ_1 and the other half with a material of permeability μ_2. Find the inductance of the arrangement.

6.18. The region between the two plates of Figure 6.7 is filled with two imperfect dielectric media having conductivities σ_1 for $0 < x < t$ and σ_2 for $t < x < d$. (a) Find the solutions for the potentials in the two regions $0 < x < t$ and $t < x < d$. (b) Find the potential at the interface $x = t$.

6.19. For the structure of Figure 6.9, continue the analysis beyond the quasistatic extension and obtain the input admittance correct to the third power in ω. Determine the equivalent circuit.

6.20. For the structure of Figure 6.10, continue the analysis beyond the quasistatic extension to obtain the input impedance correct to the third power in ω. Determine the equivalent circuit.

6.21. For the structure of Figure 6.10, assume that the medium between the plates is an imperfect dielectric of conductivity σ. (a) Show that the input impedance correct to the first power in ω is the same as if σ were zero. (b) Obtain the input impedance correct to the second power in ω and determine the equivalent circuit.

6.22. Find the condition(s) under which the quasistatic input behavior of the structure of Figure 6.11 is essentially equivalent to (a) a capacitor in parallel with a resistor and (b) a resistor in series with an inductor.

6.23. A parallel-plate transmission line is made up of perfect conductors of width $w = 0.1$ m and lying in the planes $x = 0$ and $x = 0.02$ m. The medium between the conductors is a perfect dielectric of $\mu = \mu_0$. For a uniform plane wave having the electric field

$$\mathbf{E} = 100\pi \cos(2\pi \times 10^6 t - 0.02\pi z)\, \mathbf{a}_x \text{ V/m}$$

propagating between the conductors, find (a) the voltage between the conductors, (b) the current along the conductors, and (c) the power flow along the line.

6.24. A parallel-plate transmission line made up of perfect conductors has $\mathcal{L}$ equal to 10^{-7} H/m. If the medium between the plates is characterized by $\sigma = 10^{-11}$ S/m, $\epsilon = 6\epsilon_0$, and $\mu = \mu_0$, find $\mathcal{C}$ and $\mathcal{G}$ of the line.

6.25. If the conductors of a transmission line are imperfect, then the transmission-line equivalent circuit contains a resistance and additional inductance in the series branch. Assuming that the thickness of the (imperfect) conductors of a parallel-plate line is several skin depths at the frequency of interest, show from considerations of skin effect phenomenon in a good conductor medium that the resistance and inductance per unit length along the conductors are $2/\sigma_c\delta w$ and $2/\omega\sigma_c\delta w$, respectively, where σ_c is the conductivity of the (imperfect) conductors, w is the width, and δ is the skin depth. The factor 2 arises because of two conductors.

6.26. Show that two alternative representations of the circuit equivalent of the transmission-line equations (6.86a) and (6.86b) are as shown in Figures 6.19(a) and (b).

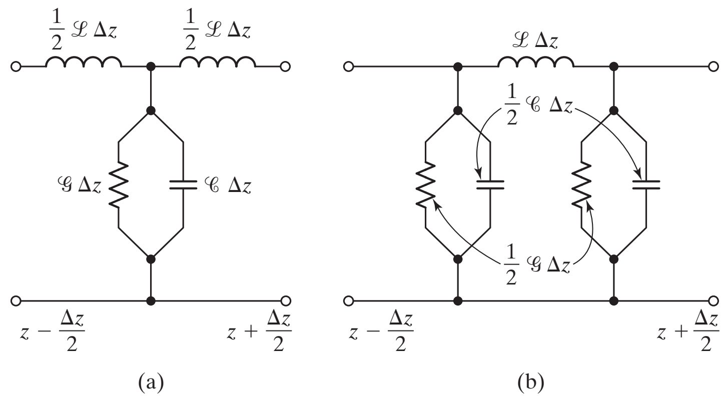

FIGURE 6.19

For Problem 6.26.

6.27. Show that for a transverse electromagnetic wave, the voltage between the conductors and the current along the conductors in a given transverse plane are uniquely defined in terms of the electric and magnetic fields, respectively, in that plane.

6.28. By constructing a field map consisting of curvilinear squares for a coaxial cable having $b/a = 3.5$, obtain the approximate values of the line parameters $\mathcal{L}$, $\mathcal{C}$, and $\mathcal{G}$ in terms of μ, ϵ, and σ of the dielectric. Compare the approximate values with the exact values given by expressions derived in Example 6.9.

6.29. For $d/a = 2$ for the parallel-wire line [see Figure 6.18(c)], construct a field map consisting of curvilinear squares and obtain approximate values for the line parameters $\mathcal{L}$, $\mathcal{C}$, and $\mathcal{G}$. Compare approximate values with the exact values given by the expressions in Table 6.1.

6.30. The shielded strip line, employed in microwave integrated circuits, consists of a center conductor photoetched on the inner faces of two substrates sandwiched between two conductors, as shown by the cross-sectional view in Figure 6.20. For the dimensions shown in the figure, construct a field map consisting of curvilinear squares and compute $\mathcal{L}$ and $\mathcal{C}$, considering the substrate to be a perfect dielectric having $\epsilon = 9\epsilon_0$ and $\mu = \mu_0$. Assume for simplicity that the field is confined to the substrate region.

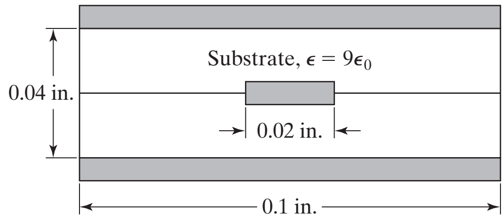

FIGURE 6.20
For Problem 6.30.

6.31. The cross section of an eccentric coaxial cable [see Figure 6.18(d)] consists of an outer circle of radius $a = 5$ cm and an inner circle of radius $b = 2$ cm, with their centers separated by $d = 2$ cm. By constructing a field map consisting of curvilinear squares, obtain the approximate values of the line parameters $\mathcal{L}$, $\mathcal{C}$, and $\mathcal{G}$ in terms of μ, ϵ, and σ of the dielectric.

6.32. Consider a transmission line having the cross section shown in Figure 6.21. The inner conductor is a circle of radius a and the outer conductor is a square of sides $2a$. Find the approximate values of $\mathcal{L}$, $\mathcal{C}$, and $\mathcal{G}$, by using the method of curvilinear squares.

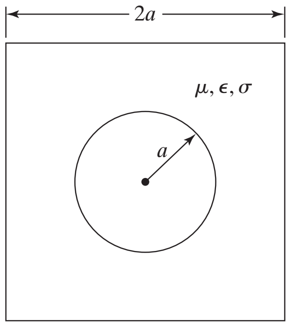

FIGURE 6.21
For Problem 6.32.