CHAPTER

4 Wave Propagation in Free Space

In Chapters 2 and 3, we learned Maxwell's equations in integral form and in differential form. We now have the knowledge of the fundamental laws of electromagnetics that enable us to embark upon the study of their applications. Many of these applications are based on electromagnetic wave phenomena, and hence it is necessary to gain an understanding of the basic principles of wave propagation, which is our goal in this chapter. In particular, we shall consider wave propagation in free space. We shall then in the next chapter consider the interaction of the wave fields with materials to extend the application of Maxwell's equations to material media and discuss wave propagation in material media.

We shall employ an approach in this chapter that will enable us not only to learn how the coupling between space-variations and time-variations of the electric and magnetic fields, as indicated by Maxwell's equations, results in wave motion, but also to illustrate the basic principle of radiation of waves from an antenna, which will be treated in detail in Chapter 9. In this process, we will also learn several techniques of analysis pertinent to field problems. We shall augment our discussion of radiation and propagation of waves by considering such examples as the principle of an antenna array and polarization. Finally, we shall discuss power flow and energy storage associated with the wave motion and introduce the Poynting vector.

4.1 THE INFINITE PLANE CURRENT SHEET

In Chapter 3, we learned that the space-variations of the electric and magnetic field components are related to the time-variations of the magnetic and electric field components, respectively, through Maxwell's equations. This interdependence gives rise to the phenomenon of electromagnetic wave propagation. In the general case, electromagnetic wave propagation involves electric and magnetic fields having more than one component, each dependent on all three coordinates, in addition to time. However, a simple and very useful type of wave that serves as a building block in the study of electromagnetic waves consists of electric and magnetic fields that are perpendicular to each other and to the direction of propagation and are uniform in planes perpendicular to the direction of propagation. These waves are known as *uniform plane waves*. By orienting the coordinate axes such that the electric field is in the x-direction, the magnetic field is in the y-direction, and the direction of propagation is in the z-direction, as shown in Figure 4.1, we have

$$\mathbf{E} = E_x(z, t)\mathbf{a}_x \tag{4.1}$$

$$\mathbf{H} = H_y(z, t)\mathbf{a}_y \tag{4.2}$$

Uniform plane waves do not exist in practice because they cannot be produced by finite-sized antennas. At large distances from physical antennas and ground, however, the waves can be approximated as uniform plane waves. Furthermore, the principles of guiding of electromagnetic waves along transmission lines and waveguides and the principles of many other wave phenomena can be studied basically in terms of uniform plane waves. Hence, it is very important that we understand the principles of uniform plane wave propagation.

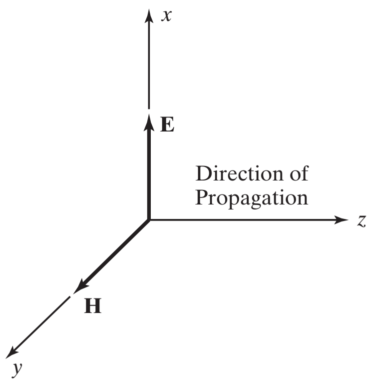

FIGURE 4.1

Directions of electric and magnetic fields and direction of propagation for a simple case of uniform plane wave.

In order to illustrate the phenomenon of interaction of electric and magnetic fields giving rise to uniform plane electromagnetic wave propagation, and the principle of radiation of electromagnetic waves from an antenna, we shall consider a simple, idealized, hypothetical source. This source consists of an infinite sheet lying in the xy-plane, as shown in Figure 4.2. On this infinite plane sheet a uniformly distributed current varying sinusoidally with time flows in the negative x-direction. Since the current is distributed on a surface, we talk of surface current density in order to express the current distribution mathematically. The surface current density, denoted by the symbol $\mathbf{J}_S$, is a vector quantity having the magnitude equal to the current per unit width (A/m) crossing an infinitesimally long line, on the surface, oriented so as to maximize the current. The direction of $\mathbf{J}_S$ is then normal to the line and toward the side of the current flow. In the present case, the surface current density is given by

$$\mathbf{J}_S = -J_{S0} \cos \omega t \; \mathbf{a}_x \qquad \text{for } z = 0 \tag{4.3}$$

where J_{S0} is a constant and ω is the radian frequency of the sinusoidal time-variation of the current density.

Because of the uniformity of the surface current density on the infinite sheet, if we consider any line of width w parallel to the y-axis, as shown in Figure 4.2, the

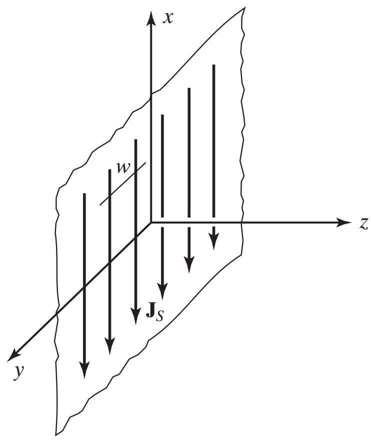

FIGURE 4.2

Infinite plane sheet in the xy-plane carrying surface current of uniform density.

current crossing that line is simply given by w times the current density, that is, $wJ_{S0}\cos\omega t$. If the current density is nonuniform, we have to perform an integration along the width of the line in order to find the current crossing the line. In view of the sinusoidal time-variation of the current density, the current crossing the width w actually alternates between negative x- and positive x-directions, that is, downward and upward. The time history of the current flow for one period of the sinusoidal variation is illustrated in Figure 4.3, with the lengths of the lines indicating the magnitudes of the current.

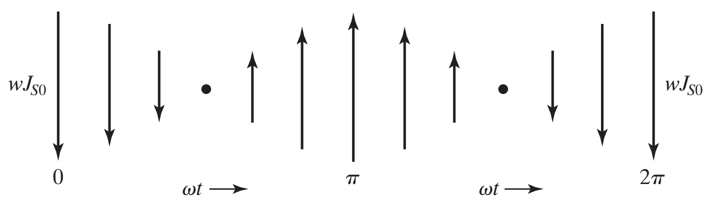

FIGURE 4.3

Time history of current flow across a line of width w parallel to the y-axis for the current sheet of Figure 4.2.

4.2 MAGNETIC FIELD ADJACENT TO THE CURRENT SHEET

In the previous section, we introduced the infinite current sheet lying in the xy-plane and upon which a surface current flows with density given by

$$\mathbf{J}_S = -J_{S0}\cos\omega t\,\mathbf{a}_x \tag{4.4}$$

Our goal is to find the electromagnetic field due to this time-varying current distribution. In order to do this, we have to solve Faraday's and Ampere's circuital laws simultaneously. Since we have here only an x-component of the current density independent of x and y, the equations of interest are

$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t} \tag{4.5}$$

$$\frac{\partial H_y}{\partial z} = -\left(J_x + \frac{\partial D_x}{\partial t}\right) \tag{4.6}$$

The quantity J_x on the right side of (4.6) represents volume current density, whereas we now have a surface current density. Furthermore, in the free space on either side of the current sheet the current density is zero and the differential equations reduce to

$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t} \tag{4.7}$$

$$\frac{\partial H_y}{\partial z} = -\frac{\partial D_x}{\partial t} \tag{4.8}$$

To obtain the solutions for E_x and H_y on either side of the current sheet, we therefore have to solve these two differential equations simultaneously.

To obtain a start on the solution, however, we need to consider the surface current distribution and find the magnetic field immediately adjacent to the current sheet. This is done by making use of Ampere's circuital law in integral form given by

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt}\int_S \mathbf{D} \cdot d\mathbf{S} \tag{4.9}$$

and applying it to a rectangular closed path *abcda*, as shown in Figure 4.4, with the sides *ab* and *cd* lying immediately adjacent to the current sheet, that is, touching the current sheet, and on either side of it. This choice of the rectangular path is not arbitrary but is intentionally chosen to achieve the task of finding the required magnetic field. First, we note from (4.6) that an x-directed current density gives rise to a

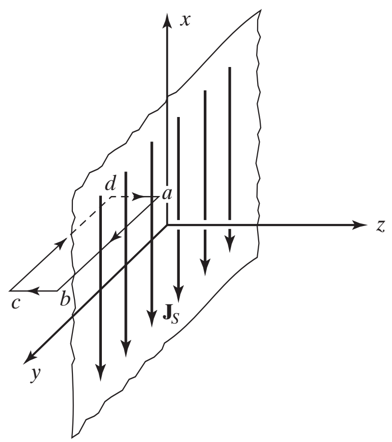

FIGURE 4.4

Rectangular path enclosing a portion of the current on the infinite plane current sheet.

magnetic field in the y-direction. At the source of the current, this magnetic field must also have a differential in the third direction, namely, the z-direction. In fact, from symmetry considerations, we can say that H_y on ab and cd must be equal in magnitude and opposite in direction.

If we now consider the line integral of **H** around the rectangular path $abcda$, we have

$$\int_{abcda} \mathbf{H} \cdot d\mathbf{l} = \int_a^b \mathbf{H} \cdot d\mathbf{l} + \int_b^c \mathbf{H} \cdot d\mathbf{l} + \int_c^d \mathbf{H} \cdot d\mathbf{l} + \int_d^a \mathbf{H} \cdot d\mathbf{l} \tag{4.10}$$

The second and the fourth integrals on the right side of (4.10) are, however, equal to zero, since **H** is normal to the sides bc and da and furthermore bc and da are infinitesimally small. The first and third integrals on the right side of (4.10) are given by

$$\int_a^b \mathbf{H} \cdot d\mathbf{l} = [H_y]_{ab}(ab)$$

$$\int_c^d \mathbf{H} \cdot d\mathbf{l} = -[H_y]_{cd}(cd)$$

Thus,

$$\oint_{abcda} \mathbf{H} \cdot d\mathbf{l} = [H_y]_{ab}(ab) - [H_y]_{cd}(cd) = 2[H_y]_{ab}(ab) \tag{4.11}$$

since $[H_y]_{cd} = -[H_y]_{ab}$.

We have just evaluated the left side of (4.9) for the particular problem under consideration here. To complete the task of finding the magnetic field adjacent to the current sheet, we now evaluate the right side of (4.9), which consists of two terms. The second term is, however, zero, since the area enclosed by the rectangular path is zero in view of the infinitesimally small thickness of the current sheet. The first term is not zero, since there is a current flowing on the sheet. Thus, the first term is simply equal to the current enclosed by the path $abcda$ in the right-hand sense, that is, the current crossing the width ab toward the negative x-direction. This is equal to the surface current density multiplied by the width ab, that is, $J_{S0} \cos \omega t\ (ab)$. Thus, substituting for the quantities on either side of (4.9), we have

$$2[H_y]_{ab}(ab) = J_{S0} \cos \omega t\ (ab)$$

or

$$[H_y]_{ab} = \frac{J_{S0}}{2} \cos \omega t \tag{4.12}$$

It then follows that

$$[H_y]_{cd} = -\frac{J_{S0}}{2} \cos \omega t \tag{4.13}$$

Thus, immediately adjacent to the current sheet the magnetic field intensity has a magnitude $\frac{J_{S0}}{2}\cos\omega t$ and is directed in the positive y-direction on the side $z > 0$ and in the negative y-direction on the side $z < 0$. This is illustrated in Figure 4.5. It is cautioned that this result is true only for points right next to the current sheet, since if we consider points at some distance from the current sheet, the second term on the right side of (4.9) will no longer be zero.

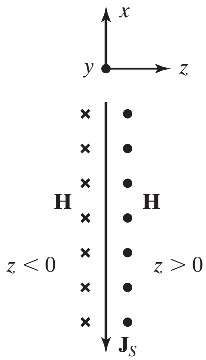

FIGURE 4.5

Magnetic field adjacent to and on either side of the infinite plane current sheet.

The technique we have used here for finding the magnetic field adjacent to the time-varying current sheet by using Ampere's circuital law in integral form is a standard procedure for finding the static electric and magnetic fields due to static charge and current distributions, possessing certain symmetries, by using Gauss' law for the electric field and Ampere's circuital law in integral forms, respectively, as we have already demonstrated in Chapter 2. Since for the static field case the terms involving time derivatives are zero, Ampere's circuital law simplifies to

$$\oint_C \mathbf{H}\cdot d\mathbf{l} = \int_S \mathbf{J}\cdot d\mathbf{S}$$

Hence, if the current distribution were not varying with time, then in order to compute the magnetic field we can choose a rectangular path of any width bc and it would still enclose the same current, namely, the current on the sheet. Thus, the magnetic field would be independent of the distance away from the sheet on either side of it. There are several problems in static fields that can be solved in this manner. We shall not discuss these here; instead, we shall include a few cases in the problems for the interested reader and shall continue with the derivation of the electromagnetic field due to our time-varying current sheet in the following section.

4.3 SUCCESSIVE SOLUTION OF MAXWELL'S EQUATIONS*

In the preceding section, we found the magnetic field adjacent to the infinite plane sheet of current introduced in Section 4.1. Now, to find the solutions for the fields everywhere on either side of the current sheet, let us first consider the region $z > 0$.

*This section may be omitted without loss of continuity.

In this region, the fields simultaneously satisfy the two differential equations (4.7) and (4.8) and with the constraint that the magnetic field at $z = 0$ is given by (4.12). To find the solutions for these differential equations, we have a choice of starting with the solution for H_y given by (4.12) and solving them successively and repeatedly in a step-by-step manner until the solutions satisfy both differential equations or of combining the two differential equations into one and then solving the single equation subject to the constraint at $z = 0$. Although it is somewhat longer and tedious, we shall use the first approach in this section in order to obtain a feeling for the mechanism of interaction between the electric and magnetic fields. We shall consider the second and more conventional approach in the following section.

To simplify the task of the repetitive solution of the two differential equations (4.7) and (4.8), we shall employ the phasor technique. Thus, by letting

$$E_x(z, t) = \text{Re}\,[\bar{E}_x(z)e^{j\omega t}] \tag{4.14}$$

$$H_y(z, t) = \text{Re}\,[\bar{H}_y(z)e^{j\omega t}] \tag{4.15}$$

where Re stands for *real part of* and $\bar{E}_x(z)$ and $\bar{H}_y(z)$ are the phasors corresponding to the time functions $E_x(z, t)$ and $H_y(z, t)$, respectively, and replacing the time functions in (4.7) and (4.8) by the corresponding phasor functions and $\partial/\partial t$ by $j\omega$, we obtain the differential equations for the phasor functions as

$$\frac{\partial \bar{E}_x}{\partial z} = -j\omega \bar{B}_y = -j\omega\mu_0 \bar{H}_y \tag{4.16}$$

$$\frac{\partial \bar{H}_y}{\partial z} = -j\omega \bar{D}_x = -j\omega\epsilon_0 \bar{E}_x \tag{4.17}$$

We also note that since (4.12) can be written as

$$[H_y]_{ab} = \text{Re}\left(\frac{J_{S0}}{2}e^{j\omega t}\right)$$

the solution for the phasor $\bar{H}_y$ at $z = 0$ is given by

$$[\bar{H}_y]_{z=0} = \frac{J_{S0}}{2} \tag{4.18}$$

We start with (4.18) and solve (4.16) and (4.17) successively and repeatedly, and after obtaining the final solutions for $\bar{E}_x$ and $\bar{H}_y$, we put them in (4.14) and (4.15), respectively, to obtain the solutions for the real fields.

Thus, starting with (4.18) and substituting it in (4.16), we get

$$\frac{\partial \bar{E}_x}{\partial z} = -j\omega\mu_0 \frac{J_{S0}}{2}$$

Integrating both sides of this equation with respect to z, we have

$$\bar{E}_x = -j\omega\mu_0 \frac{J_{S0}z}{2} + \bar{C}$$

where $\bar{C}$ is the constant of integration. This constant of integration must, however, be equal to $[\bar{E}_x]_{z=0}$, since the first term on the right side tends to zero as $z \rightarrow 0$. Thus,

$$\bar{E}_x = -j\omega\mu_0 \frac{J_{S0}z}{2} + [\bar{E}_x]_{z=0} \tag{4.19}$$

Now, substituting (4.19) into (4.17), we obtain

$$\begin{aligned}
\frac{\partial \bar{H}_y}{\partial z} &= -j\omega\epsilon_0\left\{-j\omega\mu_0\frac{J_{S0}z}{2} + [\bar{E}_x]_{z=0}\right\} \\
&= -j\omega\epsilon_0[\bar{E}_x]_{z=0} - \omega^2\mu_0\epsilon_0\frac{J_{S0}z}{2} \\
\bar{H}_y &= -j\omega\epsilon_0 z[\bar{E}_x]_{z=0} - \omega^2\mu_0\epsilon_0\frac{J_{S0}z^2}{4} + [\bar{H}_y]_{z=0} \\
&= -j\omega\epsilon_0 z[\bar{E}_x]_{z=0} - \omega^2\mu_0\epsilon_0\frac{J_{S0}z^2}{4} + \frac{J_{S0}}{2} \\
&= -j\omega\epsilon_0 z[\bar{E}_x]_{z=0} + \frac{J_{S0}}{2}\left(1 - \frac{\omega^2\mu_0\epsilon_0 z^2}{2}\right)
\end{aligned} \tag{4.20}$$

We have thus obtained a second-order solution for $\bar{H}_y$, which, however, does not satisfy (4.16) together with the solution for $\bar{E}_x$ given by (4.19). Hence, we must continue the step-by-step solution by substituting (4.20) into (4.16) and finding a higher-order solution for $\bar{E}_x$, and so on. Thus, by substituting (4.20) into (4.16), we get

$$\begin{aligned}
\frac{\partial \bar{E}_x}{\partial z} &= -j\omega\mu_0\left\{-j\omega\epsilon_0 z[\bar{E}_x]_{z=0} + \frac{J_{S0}}{2}\left(1 - \frac{\omega^2\mu_0\epsilon_0 z^2}{2}\right)\right\} \\
&= -\omega^2\mu_0\epsilon_0 z[\bar{E}_x]_{z=0} - j\omega\mu_0\frac{J_{S0}}{2}\left(1 - \frac{\omega^2\mu_0\epsilon_0 z^2}{2}\right) \\
\bar{E}_x &= -\omega^2\mu_0\epsilon_0\frac{z^2}{2}[\bar{E}_x]_{z=0} - j\omega\mu_0\frac{J_{S0}}{2}\left(z - \frac{\omega^2\mu_0\epsilon_0 z^3}{6}\right) + [\bar{E}_x]_{z=0} \\
&= [\bar{E}_x]_{z=0}\left(1 - \frac{\omega^2\mu_0\epsilon_0 z^2}{2}\right) - \frac{j\omega\mu_0 J_{S0}}{2}\left(z - \frac{\omega^2\mu_0\epsilon_0 z^3}{6}\right)
\end{aligned} \tag{4.21}$$

From (4.17), we then have

$$\begin{aligned}
\frac{\partial \bar{H}_y}{\partial z} &= -j\omega\epsilon_0[\bar{E}_x]_{z=0}\left(1 - \frac{\omega^2\mu_0\epsilon_0 z^2}{2}\right) - \frac{\omega^2\mu_0\epsilon_0 J_{S0}}{2}\left(z - \frac{\omega^2\mu_0\epsilon_0 z^3}{6}\right) \\
\bar{H}_y &= -j\omega\epsilon_0[\bar{E}_x]_{z=0}\left(z - \frac{\omega^2\mu_0\epsilon_0 z^3}{6}\right) \\
&\quad - \frac{\omega^2\mu_0\epsilon_0 J_{S0}}{2}\left(\frac{z^2}{2} - \frac{\omega^2\mu_0\epsilon_0 z^4}{24}\right) + [\bar{H}_y]_{z=0} \\
&= -j\omega\epsilon_0[\bar{E}_x]_{z=0}\left(z - \frac{\omega^2\mu_0\epsilon_0 z^3}{6}\right) \\
&\quad + \frac{J_{S0}}{2}\left(1 - \frac{\omega^2\mu_0\epsilon_0 z^2}{2} + \frac{\omega^4\mu_0^2\epsilon_0^2 z^4}{24}\right)
\end{aligned} \tag{4.22}$$

Continuing in this manner, we will get infinite series expressions for $\bar{E}_x$ and $\bar{H}_y$ as follows:

$$\bar{E}_x = [\bar{E}_x]_{z=0}\left[1 - \frac{(\beta z)^2}{2!} + \frac{(\beta z)^4}{4!} - \cdots\right] - j\frac{\eta_0 J_{S0}}{2}\left[\beta z - \frac{(\beta z)^3}{3!} + \frac{(\beta z)^5}{5!} - \cdots\right] \quad (4.23)$$

$$\bar{H}_y = -j\frac{1}{\eta_0}[\bar{E}_x]_{z=0}\left[\beta z - \frac{(\beta z)^3}{3!} + \frac{(\beta z)^5}{5!} - \cdots\right] + \frac{J_{S0}}{2}\left[1 - \frac{(\beta z)^2}{2!} + \frac{(\beta z)^4}{4!} - \cdots\right] \quad (4.24)$$

where we have introduced the notations

$$\beta = \omega\sqrt{\mu_0\epsilon_0} \quad (4.25)$$

$$\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \quad (4.26)$$

It is left to the student to verify that the two expressions (4.23) and (4.24) simultaneously satisfy the two differential equations (4.16) and (4.17). Now, noting that

$$\cos\beta z = 1 - \frac{(\beta z)^2}{2!} + \frac{(\beta z)^4}{4!} - \cdots$$

$$\sin\beta z = \beta z - \frac{(\beta z)^3}{3!} + \frac{(\beta z)^5}{5!} + \cdots$$

and substituting into (4.23) and (4.24), we have

$$\bar{E}_x = [\bar{E}_x]_{z=0}\cos\beta z - j\frac{\eta_0 J_{S0}}{2}\sin\beta z \quad (4.27)$$

$$\bar{H}_y = -j\frac{1}{\eta_0}[\bar{E}_x]_{z=0}\sin\beta z + \frac{J_{S0}}{2}\cos\beta z \quad (4.28)$$

We now obtain the expressions for the real fields by putting (4.27) and (4.28) into (4.14) and (4.15), respectively. Thus,

$$\begin{aligned}
E_x(z,t) &= \text{Re}\left\{[\bar{E}_x]_{z=0}\cos\beta z\, e^{j\omega t} - j\frac{\eta_0 J_{S0}}{2}\sin\beta z\, e^{j\omega t}\right\} \\
&= \cos\beta z\,\text{Re}\,\{[\bar{E}_x]_{z=0}\, e^{j\omega t}\} + \frac{\eta_0 J_{S0}}{2}\sin\beta z\,\text{Re}\,[e^{j(\omega t-\pi/2)}] \\
&= \cos\beta z\,(C\cos\omega t + D\sin\omega t) + \frac{\eta_0 J_{S0}}{2}\sin\beta z\sin\omega t
\end{aligned} \quad (4.29)$$

$$
\begin{aligned}
H_y(z, t) &= \text{Re}\left\{-j\frac{1}{\eta_0}[\bar{E}_x]_{z=0} \sin \beta z \, e^{j\omega t} + \frac{J_{S0}}{2} \cos \beta z \, e^{j\omega t}\right\} \\
&= \frac{1}{\eta_0} \sin \beta z \, \text{Re}\{[\bar{E}_x]_{z=0} \, e^{j(\omega t - \pi/2)}\} + \frac{J_{S0}}{2} \cos \beta z \, \text{Re}\,[e^{j\omega t}] \\
&= \frac{1}{\eta_0} \sin \beta z \, (C \sin \omega t - D \cos \omega t) + \frac{J_{S0}}{2} \cos \beta z \cos \omega t
\end{aligned}
\tag{4.30}
$$

where we have replaced the quantity $\text{Re}\{[\bar{E}_x]_{z=0}e^{j\omega t}\}$ by $(C \cos \omega t + D \sin \omega t)$, in which C and D are arbitrary constants to be determined. Making use of trigonometric identities and proceeding further, we write (4.29) and (4.30) as

$$
\begin{aligned}
E_x(z, t) &= \frac{2C + \eta_0 J_{S0}}{4} \cos(\omega t - \beta z) + \frac{2C - \eta_0 J_{S0}}{4} \cos(\omega t + \beta z) \\
&\quad + \frac{D}{2} \sin(\omega t - \beta z) + \frac{D}{2} \sin(\omega t + \beta z)
\end{aligned}
\tag{4.31}
$$

$$
\begin{aligned}
H_y(z, t) &= \frac{2C + \eta J_{S0}}{4\eta_0} \cos(\omega t - \beta z) - \frac{2C - \eta_0 J_{S0}}{4\eta_0} \cos(\omega t + \beta z) \\
&\quad + \frac{D}{2\eta_0} \sin(\omega t - \beta z) - \frac{D}{2\eta_0} \sin(\omega t + \beta z)
\end{aligned}
\tag{4.32}
$$

Equation (4.32) is the solution for H_y that together with the solution for E_x given by (4.31) satisfies the two differential equations (4.7) and (4.8) and that reduces to (4.12) for $z = 0$. Likewise, we can obtain the solutions for H_y and E_x for the region $z < 0$ by starting with $[H_y]_{z=0-}$ given by (4.13) and proceeding in a similar manner. We shall, however, proceed with the evaluation of the constants C and D in (4.31) and (4.32). In order to do this, we first have to understand the meanings of the functions $\cos(\omega t \mp \beta z)$ and $\sin(\omega t \mp \beta z)$. We shall do this in Section 4.5.

4.4 SOLUTION BY WAVE EQUATION

In Section 4.3, we found the solutions to the two simultaneous differential equations (4.7) and (4.8) by solving them successively and repeatedly in a step-by-step manner. In this section, we shall consider an alternative and more conventional method by combining the two equations into a single equation and then solving it. We recall that the two simultaneous differential equations to be satisfied in the free space on either side of the current sheet are

$$
\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t} = -\mu_0 \frac{\partial H_y}{\partial t} \tag{4.33}
$$

$$
\frac{\partial H_y}{\partial z} = -\frac{\partial D_x}{\partial t} = -\epsilon_0 \frac{\partial E_x}{\partial t} \tag{4.34}
$$

Differentiating (4.33) with respect to z and then substituting for $\partial H_y/\partial z$ from (4.34), we obtain

$$\frac{\partial^2 E_x}{\partial z^2} = -\mu_0 \frac{\partial}{\partial z}\left(\frac{\partial H_y}{\partial t}\right) = -\mu_0 \frac{\partial}{\partial t}\left(\frac{\partial H_y}{\partial z}\right) = -\mu_0 \frac{\partial}{\partial t}\left(-\epsilon_0 \frac{\partial E_x}{\partial t}\right)$$

or

$$\frac{\partial^2 E_x}{\partial z^2} = \mu_0 \epsilon_0 \frac{\partial^2 E_x}{\partial t^2} \tag{4.35}$$

We have thus eliminated H_y from (4.33) and (4.34) and obtained a single second-order partial differential equation involving E_x only.

Equation (4.35) is known as the *wave equation*. A technique of solving this equation is the *separation of variables* technique. Since it is a differential equation involving two variables, z and t, the technique consists of assuming that the required solution is the product of two functions, one of which is a function of z only and the second is a function of t only. Denoting these functions to be Z and T, respectively, we have

$$E_x(z, t) = Z(z)T(t) \tag{4.36}$$

Substituting (4.36) into (4.35) and dividing throughout by $\mu_0\epsilon_0 Z(z)T(t)$, we obtain

$$\frac{1}{\mu_0\epsilon_0 Z}\frac{d^2 Z}{dz^2} = \frac{1}{T}\frac{d^2 T}{dt^2} \tag{4.37}$$

In (4.37), the left side is a function of z only and the right side is a function of t only. In order for this to be satisfied, they both must be equal to a constant. Hence, setting them equal to a constant, say α^2, we have

$$\frac{d^2 Z}{dz^2} = \alpha^2 \mu_0 \epsilon_0 Z \tag{4.38a}$$

$$\frac{d^2 T}{dt^2} = \alpha^2 T \tag{4.38b}$$

We have thus obtained two ordinary differential equations involving separately the two variables z and t; hence, the technique is known as the *separation of variables* technique.

The constant α^2 in (4.38a) and (4.38b) is not arbitrary, since for the case of the sinusoidally time-varying current source the fields must also be sinusoidally time-varying with the same frequency, although not necessarily in phase with the source. Thus, the solution for $T(t)$ must be of the form

$$T(t) = A \cos \omega t + B \sin \omega t \tag{4.39}$$

where A and B are arbitrary constants to be determined. Substitution of (4.39) into (4.38b) gives us $\alpha^2 = -\omega^2$. The solution for (4.38a) is then given by

$$\begin{aligned} Z(z) &= A' \cos \omega\sqrt{\mu_0\epsilon_0}z + B' \sin \omega\sqrt{\mu_0\epsilon_0}z \\ &= A' \cos \beta z + B' \sin \beta z \end{aligned} \tag{4.40}$$

where A' and B' are arbitrary constants to be determined and we have defined

$$\beta = \omega\sqrt{\mu_0\epsilon_0} \tag{4.41}$$

The solution for E_x is then given by

$$\begin{aligned} E_x &= (A' \cos \beta z + B' \sin \beta z)(A \cos \omega t + B \sin \omega t) \\ &= C \cos \beta z \cos \omega t + D \cos \beta z \sin \omega t \\ &\quad + C' \sin \beta z \cos \omega t + D' \sin \beta z \sin \omega t \end{aligned} \tag{4.42}$$

The corresponding solution for H_y can be obtained by substituting (4.42) into one of the two equations (4.33) and (4.34). Thus, using (4.34), we get

$$\begin{aligned} \frac{\partial H_y}{\partial z} &= -\epsilon_0[-\omega C \cos \beta z \sin \omega t + \omega D \cos \beta z \cos \omega t \\ &\quad - \omega C' \sin \beta z \sin \omega t + \omega D' \sin \beta z \cos \omega t] \\ H_y &= \frac{\omega\epsilon_0}{\beta}[C \sin \beta z \sin \omega t - D \sin \beta z \cos \omega t \\ &\quad - C' \cos \beta z \sin \omega t + D' \cos \beta z \cos \omega t] \end{aligned}$$

Defining

$$\eta_0 = \frac{\beta}{\omega\epsilon_0} = \frac{\omega\sqrt{\mu_0\epsilon_0}}{\omega\epsilon_0} = \sqrt{\frac{\mu_0}{\epsilon_0}} \tag{4.43}$$

we have

$$\begin{aligned} H_y &= \frac{1}{\eta_0}[C \sin \beta z \sin \omega t - D \sin \beta z \cos \omega t \\ &\quad - C' \cos \beta z \sin \omega t + D' \cos \beta z \cos \omega t] \end{aligned} \tag{4.44}$$

Equation (4.44) is the general solution for H_y valid on both sides of the current sheet. In order to deduce the arbitrary constants, we first recall that the magnetic field adjacent to the current sheet is given by

$$H_y = \begin{cases} \dfrac{J_{S0}}{2} \cos \omega t & \text{for } z = 0+ \\ -\dfrac{J_{S0}}{2} \cos \omega t & \text{for } z = 0- \end{cases} \tag{4.45}$$

Thus, for $z > 0$,

$$\frac{1}{\eta_0}[-C' \sin \omega t + D' \cos \omega t] = \frac{J_{S0}}{2} \cos \omega t$$

or

$$C' = 0 \quad \text{and} \quad D' = \frac{\eta_0 J_{S0}}{2}$$

giving us

$$H_y = \frac{J_{S0}}{2}\cos\beta z\cos\omega t + \frac{1}{\eta_0}\sin\beta z\,(C\sin\omega t - D\cos\omega t) \tag{4.46}$$

$$E_x = \frac{\eta_0 J_{S0}}{2}\sin\beta z\sin\omega t + \cos\beta z\,(C\cos\omega t + D\sin\omega t) \tag{4.47}$$

Making use of trigonometric identities and proceeding further, we write (4.47) and (4.46) as

$$\begin{aligned} E_x(z, t) = {} & \frac{2C + \eta_0 J_{S0}}{4}\cos(\omega t - \beta z) + \frac{2C - \eta_0 J_{S0}}{4}\cos(\omega t + \beta z) \\ & + \frac{D}{2}\sin(\omega t - \beta z) + \frac{D}{2}\sin(\omega t + \beta z) \end{aligned} \tag{4.48}$$

$$\begin{aligned} H_y(z, t) = {} & \frac{2C + \eta_0 J_{S0}}{4\eta_0}\cos(\omega t - \beta z) - \frac{2C - \eta_0 J_{S0}}{4\eta_0}\cos(\omega t + \beta z) \\ & + \frac{D}{2\eta_0}\sin(\omega t - \beta z) - \frac{D}{2\eta_0}\sin(\omega t + \beta z) \end{aligned} \tag{4.49}$$

Equation (4.49) is the solution for H_y that together with the solution for E_x given by (4.48) satisfies the two differential equations (4.7) and (4.8) and that reduces to (4.12) for $z = 0$. Similarly, we can obtain the solutions for H_y and E_x for the region $z < 0$ by using the value of $[H_y]_{z=0-}$ to evaluate C' and D' in (4.44). We shall, however, proceed with the evaluation of the constants C and D in (4.48) and (4.49). In order to do this, we first have to understand the meanings of the functions $\cos(\omega t \mp \beta z)$ and $\sin(\omega t \mp \beta z)$. We shall do this in the following section.

4.5 UNIFORM PLANE WAVES

In the previous two sections, we derived the solutions for E_x and H_y, due to the infinite plane sheet of sinusoidally time-varying uniform current density, for the region $z > 0$. These solutions consist of the functions $\cos(\omega t \mp \beta z)$ and $\sin(\omega t \mp \beta z)$, which are dependent on both time and distance. Let us first consider the function $\cos(\omega t - \beta z)$. To understand the behavior of this function, we note that for a fixed value of time it varies in a cosinusoidal manner with the distance z. Let us therefore consider three values of time, $t = 0$, $t = \pi/4\omega$, and $t = \pi/2\omega$, and examine the sketches of this function versus z for these three times. By noting that

$$\text{for } t = 0, \quad \cos(\omega t - \beta z) = \cos(-\beta z) = \cos\beta z$$

$$\text{for } t = \frac{\pi}{4\omega}, \quad \cos(\omega t - \beta z) = \cos\left(\frac{\pi}{4} - \beta z\right)$$

$$\text{for } t = \frac{\pi}{2\omega}, \quad \cos(\omega t - \beta z) = \cos\left(\frac{\pi}{2} - \beta z\right) = \sin\beta z$$

we draw the sketches of the three functions as shown in Figure 4.6.

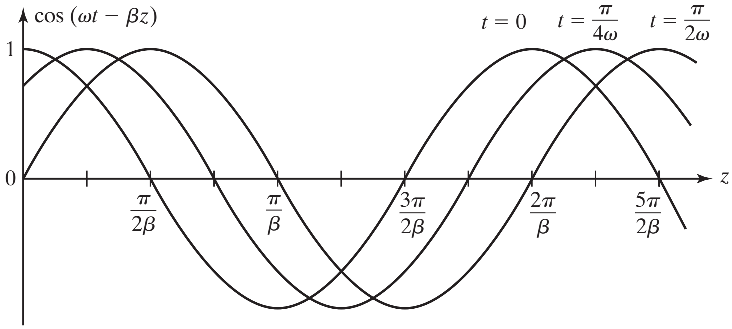

FIGURE 4.6

Sketches of the function $\cos(\omega t - \beta z)$ versus z for three values of t.

It is evident from Figure 4.6 that the sketch of the function for $t = \pi/4\omega$ is a replica of the function for $t = 0$ except that it is shifted by a distance of $\pi/4\beta$ toward the positive z-direction. Similarly, the sketch of the function for $t = \pi/2\omega$ is a replica of the function for $t = 0$ except that it is shifted by a distance of $\pi/2\beta$ toward the positive z-direction. Thus as time progresses, the function shifts bodily to the right, that is, toward increasing values of z. In fact, we can even find the velocity with which the function is traveling by dividing the distance moved by the time elapsed. This gives

$$\begin{aligned}\text{velocity} &= \frac{\pi/\beta - \pi/2\beta}{\pi/2\omega - 0} = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{\mu_0\epsilon_0}} \\ &= \frac{1}{\sqrt{\mu_0\epsilon_0}} = \frac{1}{\sqrt{4\pi \times 10^{-7} \times 10^{-9}/36\pi}} \\ &= 3 \times 10^8\,\text{m/s}\end{aligned}$$

which is the velocity of light in free space, denoted c. Thus, the function $\cos(\omega t - \beta z)$ represents a *traveling wave* moving with a velocity ω/β toward the direction of increasing z. The wave is also known as the *positive going wave*, or *(+) wave.*

Similarly, by considering three values of time, $t = 0$, $t = \pi/4\omega$, and $t = \pi/2\omega$, for the function $\cos(\omega t + \beta z)$, we obtain the sketches shown in Figure 4.7. An examination of these sketches reveals that $\cos(\omega t + \beta z)$ represents a *traveling wave* moving with a velocity ω/β toward the direction of decreasing values of z. The wave is also known as the *negative going wave*, or *(−) wave.* Since the sine functions are cosine functions shifted in phase by $\pi/2$, it follows that $\sin(\omega t - \beta z)$ and $\sin(\omega t + \beta z)$ represent traveling waves moving in the positive and negative z-directions, respectively.

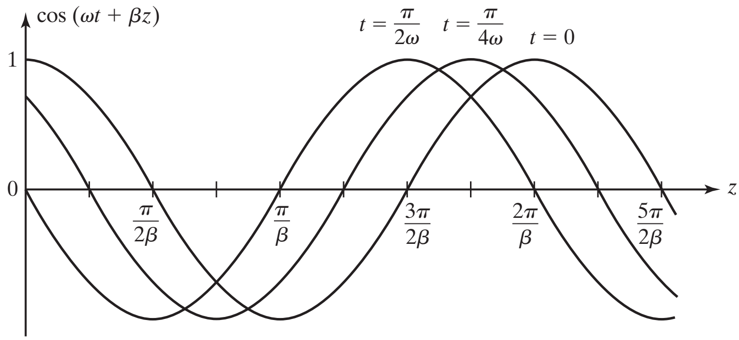

FIGURE 4.7

Sketches of the function cos $(\omega t + \beta z)$ versus z for three values of t.

Returning to the solutions for E_x and H_y given by (4.31) and (4.32) or (4.48) and (4.49), we now know that these solutions consist of superpositions of traveling waves propagating away from and toward the current sheet. In the region $z > 0$, however, we have to rule out traveling waves propagating toward the current sheet, because such a situation requires a source of waves to the right of the sheet or an object that reflects the wave back toward the sheet. Thus, we have

$$D = 0$$

$$2C - \eta_0 J_{S0} = 0 \quad \text{or} \quad C = \frac{\eta_0 J_{S0}}{2}$$

which give us finally

$$\left.\begin{aligned} E_x &= \frac{\eta_0 J_{S0}}{2}\cos(\omega t - \beta z) \\ H_y &= \frac{J_{S0}}{2}\cos(\omega t - \beta z) \end{aligned}\right\} \quad \text{for } z > 0 \tag{4.50}$$

Having found the solutions for the fields in the region $z > 0$, we can now consider the solutions for the fields in the region $z < 0$. From our discussion of the functions $\cos(\omega t \mp \beta z)$, we know that these solutions must be of the form $\cos(\omega t + \beta z)$, since this function represents a traveling wave progressing in the negative z-direction, that is, away from the sheet in the region $z < 0$. Recalling that the magnetic field adjacent to the current sheet and to the left of it is given by

$$[H_y]_{z=0-} = -\frac{J_{S0}}{2}\cos\omega t$$

we get

$$H_y = -\frac{J_{S0}}{2}\cos(\omega t + \beta z) \qquad \text{for } z < 0 \tag{4.51a}$$

The corresponding E_x can be obtained by simply substituting the result just obtained for H_y into one of the two differential equations (4.7) and (4.8). Thus using (4.7), we obtain

$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t} = -\frac{\mu_0 J_{S0}}{2}\omega \sin(\omega t + \beta z)$$

$$E_x = \frac{\mu_0 J_{S0}}{2}\frac{\omega}{\beta}\cos(\omega t + \beta z)$$

$$= \frac{\eta_0 J_{S0}}{2}\cos(\omega t + \beta z) \qquad \text{for } z < 0 \tag{4.51b}$$

Combining (4.50) and (4.51), we find that the solution for the electromagnetic field due to the infinite plane current sheet in the xy-plane characterized by

$$\mathbf{J}_S = -J_{S0} \cos \omega t \; \mathbf{a}_x$$

is given by

$$\mathbf{E} = \frac{\eta_0 J_{S0}}{2}\cos(\omega t \mp \beta z)\,\mathbf{a}_x \qquad \text{for } z \gtrless 0 \tag{4.52a}$$

$$\mathbf{H} = \pm\frac{J_{S0}}{2}\cos(\omega t \mp \beta z)\,\mathbf{a}_y \qquad \text{for } z \gtrless 0 \tag{4.52b}$$

These results are illustrated in Figure 4.8, which shows sketches of the current density on the sheet and the distance-variation of the electric and magnetic fields on either side of the current sheet for a few values of t. It can be seen from these sketches that the phenomenon is one of electromagnetic waves *radiating* away from the current sheet to either side of it, in step with the time-variation of the current density on the sheet.

The solutions that we have just obtained for the fields due to the time-varying infinite plane current sheet are said to correspond to *uniform plane electromagnetic waves* propagating away from the current sheet to either side of it. The terminology arises from the fact that the fields are *uniform* (i.e., they do not vary with position) over the *planes* $z =$ constant. Thus, the phase of the fields, that is, the quantity $(\omega t \pm \beta z)$, as well as the amplitudes of the fields, is uniform over the planes $z =$ constant. The magnitude of the rate of change of phase with distance z for any fixed time is β. The quantity β is therefore known as the *phase constant*. Since the velocity of propagation of the wave, that is, ω/β, is the velocity with which a given constant phase progresses along the z-direction, that is, along the direction of propagation, it is known as the *phase velocity* and is denoted by the symbol v_p. Thus,

$$v_p = \frac{\omega}{\beta} \tag{4.53}$$

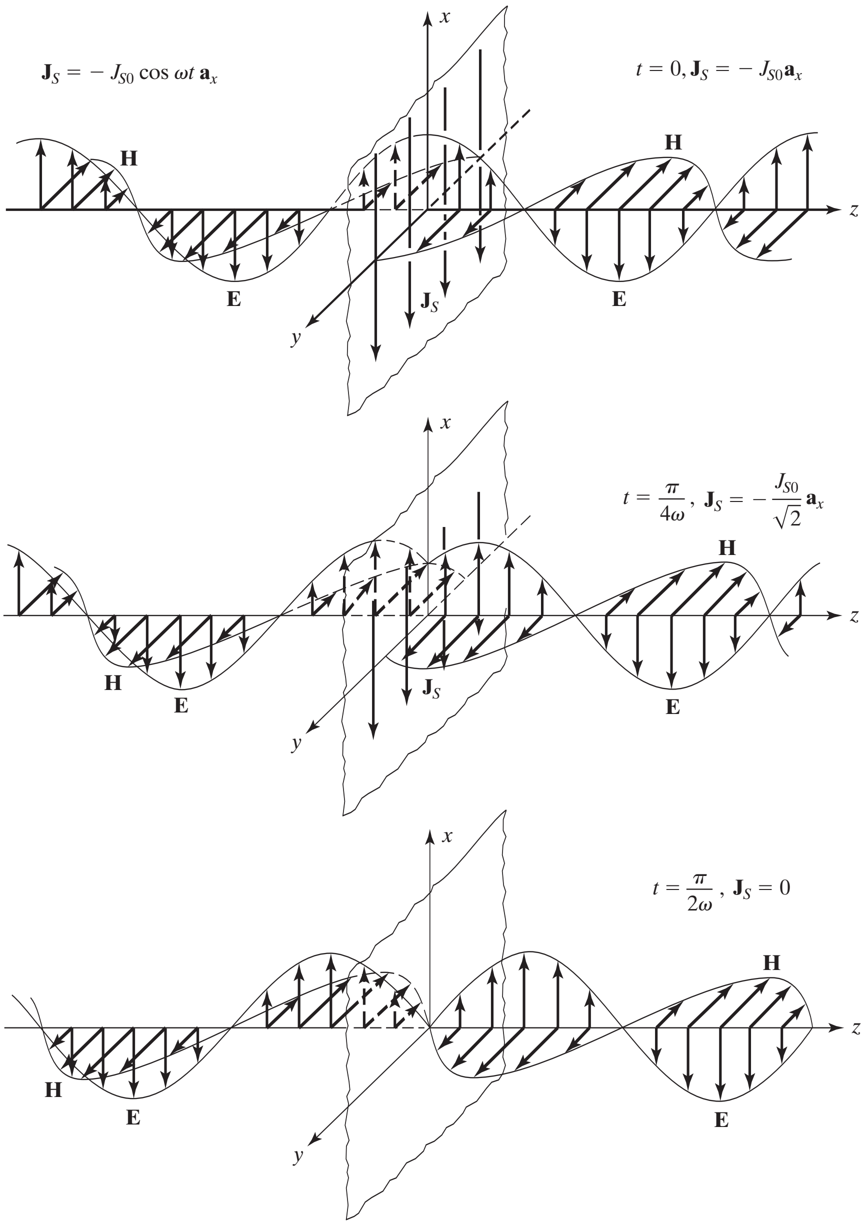

FIGURE 4.8

Time history of uniform plane electromagnetic wave radiating away from an infinite plane current sheet in free space.

The distance in which the phase changes by 2π radians for a fixed time is $2\pi/\beta$. This quantity is known as the *wavelength* and is denoted by the symbol λ. Thus,

$$\lambda = \frac{2\pi}{\beta} \tag{4.54}$$

Substituting (4.53) into (4.54), we obtain

$$\lambda = \frac{2\pi}{\omega/v_p} = \frac{v_p}{f}$$

or

$$\lambda f = v_p \tag{4.55}$$

Equation (4.55) is a simple relationship between the wavelength λ, which is a parameter governing the variation of the field with distance for a fixed time, and the frequency f, which is a parameter governing the variation of the field with time for a fixed value of z. Since for free space $v_p = 3 \times 10^8$ m/s, we have

$$\begin{aligned} \lambda \text{ in meters} \times f \text{ in Hz} &= 3 \times 10^8 \\ \lambda \text{ in meters} \times f \text{ in MHz} &= 300 \end{aligned} \tag{4.56}$$

Other properties of uniform plane waves evident from (4.52) are that the electric and magnetic fields have components lying in the planes of constant phase and perpendicular to each other and to the direction of propagation. In fact, the cross product of **E** and **H** results in a vector that is directed along the direction of propagation, as can be seen by noting that

$$\begin{aligned} \mathbf{E} \times \mathbf{H} &= E_x \mathbf{a}_x \times H_y \mathbf{a}_y \\ &= \pm \frac{\eta_0 J_{S0}^2}{4} \cos^2(\omega t \mp \beta z)\, \mathbf{a}_z \qquad \text{for } z \gtrless 0 \end{aligned} \tag{4.57}$$

Finally, we note that the ratio of E_x to H_y is given by

$$\frac{E_x}{H_y} = \begin{cases} \eta_0 \text{ for } z > 0, \text{ that is, for the } (+) \text{ wave} \\ -\eta_0 \text{ for } z < 0, \text{ that is, for the } (-) \text{ wave} \end{cases} \tag{4.58}$$

The quantity η_0, which is equal to $\sqrt{\mu_0/\epsilon_0}$, is known as the *intrinsic impedance* of free space. Its value is given by

$$\begin{aligned} \eta_0 &= \sqrt{\frac{(4\pi \times 10^{-7}) \text{ H/m}}{(10^{-9}/36\pi) \text{ F/m}}} = \sqrt{(144\pi^2 \times 10^2) \text{ H/F}} \\ &= 120\pi\ \Omega = 377\ \Omega \end{aligned} \tag{4.59}$$

Example 4.1

The electric field of a uniform plane wave is given by $\mathbf{E} = 10 \cos (3\pi \times 10^8 t - \pi z)\, \mathbf{a}_x$ V/m. Let us identify the various parameters associated with the uniform plane wave.

We recognize that

$$\omega = 3\pi \times 10^8 \text{ rad/s}$$

$$f = \frac{\omega}{2\pi} = 1.5 \times 10^8 \text{ Hz} = 150 \text{ MHz}$$

$$\beta = \pi \text{ rad/m}$$

$$\lambda = \frac{2\pi}{\beta} = 2 \text{ m}$$

$$v_p = \frac{\omega}{\beta} = \frac{3\pi \times 10^8}{\pi} = 3 \times 10^8 \text{ m/s}$$

Also, $\lambda f = v_p = 2 \times 1.5 \times 10^8 = 3 \times 10^8$ m/s. From (4.58), and since the given field represents a (+) wave,

$$\mathbf{H} = \frac{E_x}{\eta_0}\mathbf{a}_y = \frac{10}{377} \cos (3\pi \times 10^8 t - \pi z)\, \mathbf{a}_y \text{ A/m}$$

Example 4.2

An antenna array consists of two or more antenna elements spaced appropriately and excited with currents having the appropriate amplitudes and phases in order to obtain a desired radiation characteristic. To illustrate the principle of an antenna array, let us consider two infinite plane parallel current sheets, spaced $\lambda/4$ apart and carrying currents of equal amplitudes but out of phase by $\pi/2$, as given by the densities

$$\mathbf{J}_{S1} = -J_{S0} \cos \omega t\, \mathbf{a}_x \qquad z = 0$$

$$\mathbf{J}_{S2} = -J_{S0} \sin \omega t\, \mathbf{a}_x \qquad z = \frac{\lambda}{4}$$

and find the electric field due to the array of the two current sheets.

We apply the result given by (4.52) to each current sheet separately and then use superposition to find the required total electric field due to the array of the two current sheets. Thus, for the current sheet in the $z = 0$ plane, we have

$$\mathbf{E}_1 = \begin{cases} \dfrac{\eta_0 J_{S0}}{2} \cos (\omega t - \beta z)\, \mathbf{a}_x & \text{for } z > 0 \\ \dfrac{\eta_0 J_{S0}}{2} \cos (\omega t + \beta z)\, \mathbf{a}_x & \text{for } z < 0 \end{cases}$$

For the current sheet in the $z = \lambda/4$ plane, we have

$$\mathbf{E}_2 = \begin{cases} \dfrac{\eta_0 J_{S0}}{2} \sin\left[\omega t - \beta\left(z - \dfrac{\lambda}{4}\right)\right]\mathbf{a}_x & \text{for } z > \dfrac{\lambda}{4} \\ \dfrac{\eta_0 J_{S0}}{2} \sin\left[\omega t + \beta\left(z - \dfrac{\lambda}{4}\right)\right]\mathbf{a}_x & \text{for } z < \dfrac{\lambda}{4} \end{cases}$$

$$= \begin{cases} \dfrac{\eta_0 J_{S0}}{2} \sin\left(\omega t - \beta z + \dfrac{\pi}{2}\right)\mathbf{a}_x & \text{for } z > \dfrac{\lambda}{4} \\ \dfrac{\eta_0 J_{S0}}{2} \sin\left(\omega t + \beta z - \dfrac{\pi}{2}\right)\mathbf{a}_x & \text{for } z < \dfrac{\lambda}{4} \end{cases}$$

$$= \begin{cases} \dfrac{\eta_0 J_{S0}}{2} \cos\,(\omega t - \beta z)\,\mathbf{a}_x & \text{for } z > \dfrac{\lambda}{4} \\ -\dfrac{\eta_0 J_{S0}}{2} \cos\,(\omega t + \beta z)\,\mathbf{a}_x & \text{for } z < \dfrac{\lambda}{4} \end{cases}$$

Now, using superposition, we find the total electric field due to the two current sheets to be

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$$

$$= \begin{cases} \eta_0 J_{S0} \cos\,(\omega t - \beta z)\,\mathbf{a}_x & \text{for } z > \dfrac{\lambda}{4} \\ \eta_0 J_{S0} \sin \omega t \sin \beta z \,\mathbf{a}_x & \text{for } 0 < z < \dfrac{\lambda}{4} \\ 0 & \text{for } z < 0 \end{cases}$$

Thus, the total field is zero in the region $z < 0$, and hence there is no radiation toward that side of the array. In the region $z > \lambda/4$ the total field is twice that of the field due to a single sheet. The phenomenon is illustrated in Figure 4.9, which shows sketches of the individual fields E_{x1} and E_{x2} and the total field $E_x = E_{x1} + E_{x2}$ for a few values of t. The result that we have obtained here for the total field due to the array of two current sheets, spaced $\lambda/4$ apart and fed with currents of equal amplitudes but out of phase by $\pi/2$, is said to correspond to an *endfire* radiation pattern.

In Section 1.4, we introduced polarization of sinusoidally time-varying fields, which is of relevance here in wave propagation. To extend the discussion, in the case of circular and elliptical polarizations, since the circle or the ellipse can be traversed in one of two opposite senses relative to the direction of the wave propagation, we talk of right-handed or clockwise polarization and left-handed or counterclockwise polarization. The convention is that if in a given constant phase plane, the tip of the field vector of a circularly polarized wave rotates with time in the clockwise sense as seen looking along the direction of propagation of the wave, the wave is said to be right circularly polarized. If the tip of the field vector rotates in the counterclockwise sense, the wave is said to be left circularly polarized. Similar considerations hold for elliptically polarized waves, which arise due to the superposition of two linearly polarized waves in the general case.

For example, for a uniform plane wave propagating in the $+z$-direction and having the electric field,

$$\mathbf{E} = 10 \sin\,(3\pi \times 10^8 t - \pi z)\,\mathbf{a}_x + 10 \cos\,(3\pi \times 10^8 t - \pi z)\,\mathbf{a}_y \text{ V/m} \tag{4.60}$$

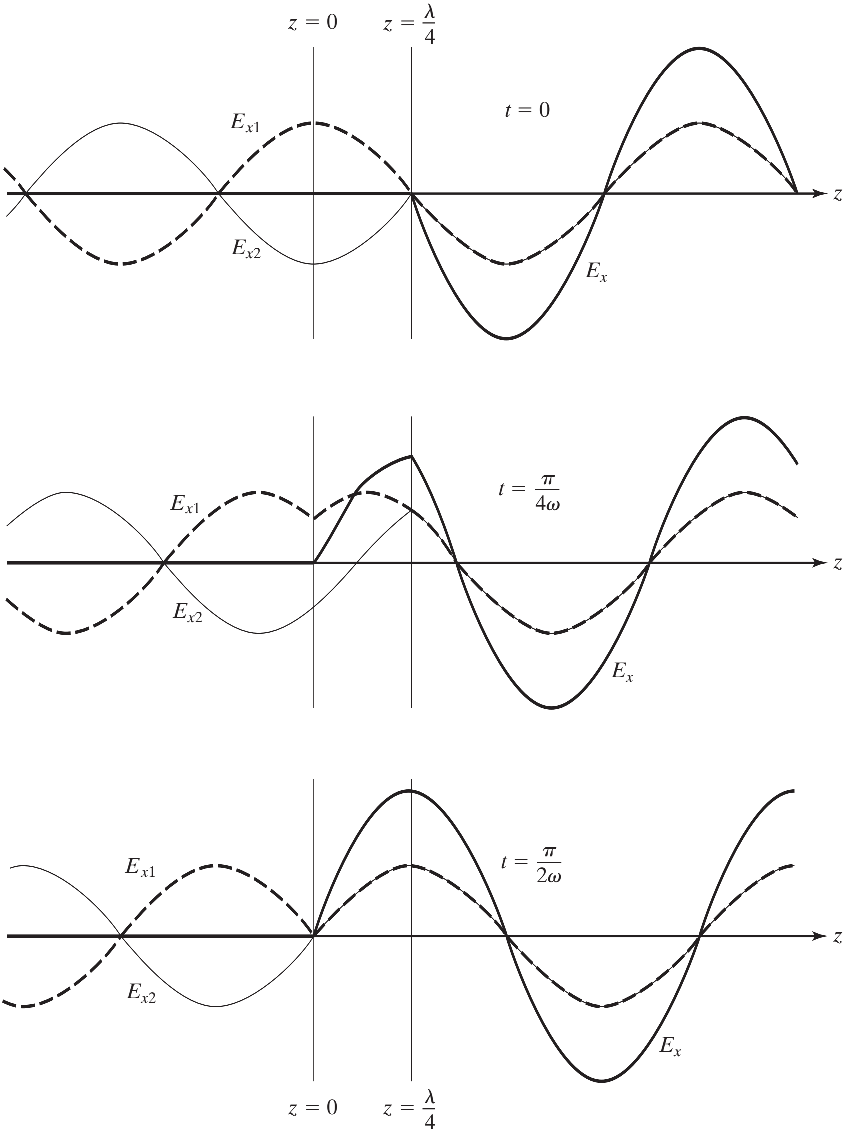

FIGURE 4.9

Time history of individual fields and the total field due to an array of two infinite plane parallel current sheets.

the two components of **E** are equal in amplitude, perpendicular, and out of phase by 90°. Therefore, the wave is circularly polarized. To determine if the polarization is right-handed or left-handed, we look at the electric field vectors in the $z = 0$ plane for two values of time, $t = 0$ and $t = \frac{1}{6} \times 10^{-8}$ s ($3\pi \times 10^8 t = \pi/2$). These are shown in Figure 4.10. As time progresses, the tip of the vector rotates in the counterclockwise sense, as seen looking in the $+z$-direction. Hence, the wave is left circularly polarized.

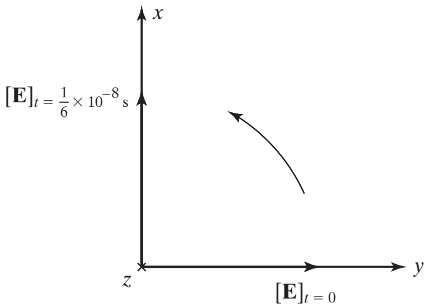

FIGURE 4.10

For the determination of the sense of circular polarization for the field of equation (4.60).

Thus far, we have considered a source of single frequency. We found that wave propagation in free space is characterized by a phase velocity v_p equal to c ($= 3 \times 10^8$ m/s) and intrinsic impedance η_0 ($= 377\ \Omega$), independent of frequency. Let us now consider a nonsinusoidal excitation for the current sheet. Then, since the propagation characteristics are the same for each frequency component of the nonsinusoidal excitation, the resulting fields at any given value of z will have the same shape as that of the source with time, that is, they propagate without change in shape with time. Thus, for an infinite plane current sheet of surface current density given by

$$\mathbf{J}_S(t) = -J_S(t)\mathbf{a}_x \qquad \text{for } z = 0 \tag{4.61}$$

the solution for the electromagnetic field is given by

$$\mathbf{E}(z, t) = \frac{\eta_0}{2} J_S\left(t \mp \frac{z}{v_p}\right)\mathbf{a}_x \qquad \text{for } z \gtrless 0 \tag{4.62a}$$

$$\mathbf{H}(z, t) = \pm\frac{1}{2} J_S\left(t \mp \frac{z}{v_p}\right)\mathbf{a}_y \qquad \text{for } z \gtrless 0 \tag{4.62b}$$

The time variation of the electric field component E_x in a given $z =$ constant plane is the same as the current density variation delayed by the time $|z|/v_p$ and multiplied by $\eta_0/2$. The time variation of the magnetic field component in a given $z =$ constant plane is the same as the current density variation delayed by $|z|/v_p$ and multiplied by $\pm\frac{1}{2}$, depending on $z \gtrless 0$. Using these properties, one can construct plots of the field components versus time for fixed values of z and versus z for fixed values of t.

Example 4.3

Let us consider the function $J_S(t)$ in (4.61) to be that given in Figure 4.11. We wish to find and sketch (a) E_x versus t for $z = 300$ m, (b) H_y versus t for $z = -450$ m, (c) E_x versus z for $t = 1\ \mu$s, and (d) H_y versus z for $t = 2.5\ \mu$s.

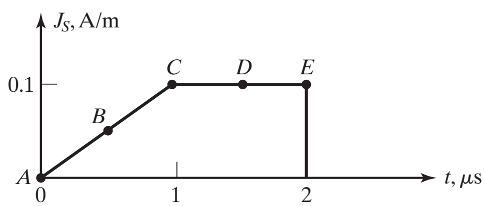

FIGURE 4.11

Plot of J_S versus t for Example 4.3.

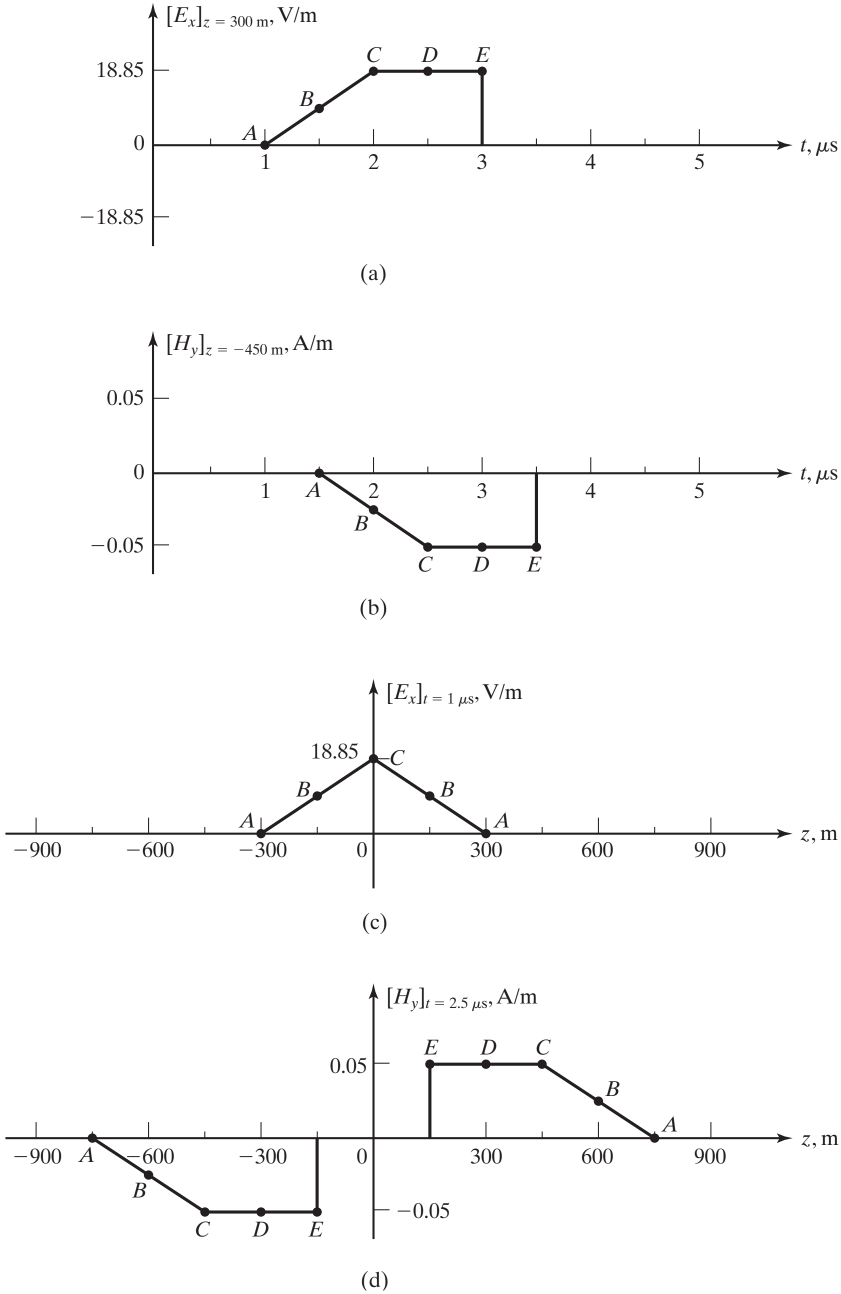

FIGURE 4.12

Plots of field components versus t for fixed values of z and versus z for fixed values of t for Example 4.3.

(a) Since $v_p = c = 3 \times 10^8$ m/s , the time delay corresponding to 300 m is 1 μs. Thus, the plot of E_x versus t for $z = 300$ m is the same as that of $J_S(t)$ multiplied by $\eta_0/2$, or 188.5, and delayed by 1 μs, as shown in Figure 4.12(a).

(b) The time delay corresponding to 450 m is 1.5 μs. Thus, the plot of H_y versus t for $z = -450$ m is the same as that of $J_S(t)$ multiplied by $-1/2$ and delayed by 1.5 μs, as shown in Figure 4.12(b).

(c) To sketch E_x versus z for a fixed value of t, say, t_1, we use the argument that a given value of E_x existing at the source at an earlier value of time, say, t_2, travels away from the source by the distance equal to $(t_1 - t_2)$ times v_p. Thus, at $t = 1$ μs, the values of E_x corresponding to points A and B in Figure 4.11 move to the locations $z = \pm 300$ m and $z = \pm 150$ m, respectively, and the value of E_x corresponding to point C exists right at the source. Hence, the plot of E_x versus z for $t = 1$ μs is as shown in Figure 4.12(c). Note that points beyond C in Figure 4.11 correspond to $t > 1$ μs, and therefore they do not appear in the plot of Figure 4.12(c).

(d) Using arguments as in part (c), we see that at $t = 2.5$ μs, the values of H_y corresponding to points A, B, C, D, and E in Figure 4.11 move to the locations $z = \pm 750$ m, ± 600 m, ± 450 m, ± 300 m, and ± 150 m, respectively, as shown in Figure 4.12(d). Note that the plot is an odd function of z, since the factor by which J_{S0} is multiplied to obtain H_y is $\pm\frac{1}{2}$, depending on $z \lessgtr 0$.

4.6 POYNTING VECTOR AND ENERGY STORAGE

In the preceding section, we found the solution for the electromagnetic field due to an infinite plane current sheet situated in the $z = 0$ plane. For a surface current flowing in the negative x-direction, we found the electric field on the sheet to be directed in the positive x-direction. Since the current is flowing against the force due to the electric field, a certain amount of work must be done by the source of the current in order to maintain the current flow on the sheet. Let us consider a rectangular area of length Δx and width Δy on the current sheet, as shown in Figure 4.13. Since the current density is $J_{S0} \cos \omega t$, the charge crossing the width Δy in time dt is $dq = J_{S0}\, \Delta y \cos \omega t\, dt$. The force exerted on this charge by the electric field is given by

$$\mathbf{F} = dq\, \mathbf{E} = J_{S0}\, \Delta y \cos \omega t\, dt\, E_x \mathbf{a}_x \tag{4.63}$$

The amount of work required to be done against the electric field in displacing this charge by the distance Δx is

$$dw = F_x\, \Delta x = J_{S0}\, E_x \cos \omega t\, dt\, \Delta x\, \Delta y \tag{4.64}$$

Thus, the power supplied by the source of the current in maintaining the surface current over the area $\Delta x\, \Delta y$ is

$$\frac{dw}{dt} = J_{S0}\, E_x \cos \omega t\, \Delta x\, \Delta y \tag{4.65}$$

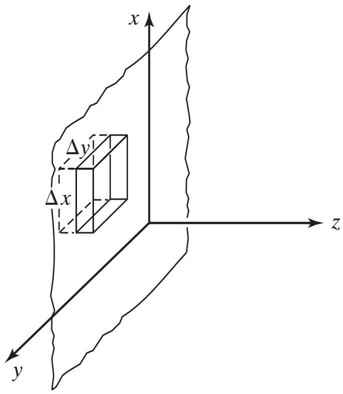

FIGURE 4.13
For the determination of power flow density associated with the electromagnetic field.

Recalling that E_x on the sheet is $\eta_0 \frac{J_{S0}}{2} \cos \omega t$, we obtain

$$\frac{dw}{dt} = \eta_0 \frac{J_{S0}^2}{2} \cos^2 \omega t \; \Delta x \; \Delta y \tag{4.66}$$

We would expect the power given by (4.66) to be carried by the electromagnetic wave, half of it to either side of the current sheet. To investigate this, we note that the quantity $\mathbf{E} \times \mathbf{H}$ has the units of

$$\begin{aligned}\frac{\text{newtons}}{\text{coulomb}} \times \frac{\text{amperes}}{\text{meter}} &= \frac{\text{newtons}}{\text{coulomb}} \times \frac{\text{coulomb}}{\text{second-meter}} \times \frac{\text{meter}}{\text{meter}} \\ &= \frac{\text{newton-meters}}{\text{second}} \times \frac{1}{(\text{meter})^2} = \frac{\text{watts}}{(\text{meter})^2}\end{aligned}$$

which represents power density. Let us then consider the rectangular box enclosing the area $\Delta x \; \Delta y$ on the current sheet and with its sides almost touching the current sheet on either side of it, as shown in Figure 4.13. Recalling that $\mathbf{E} \times \mathbf{H}$ is given by (4.57) and evaluating the surface integral of $\mathbf{E} \times \mathbf{H}$ over the surface of the rectangular box, we obtain the power flow out of the box as

$$\begin{aligned}\oint \mathbf{E} \times \mathbf{H} \cdot d\mathbf{S} &= \eta_0 \frac{J_{S0}^2}{4} \cos^2 \omega t \; \mathbf{a}_z \cdot \Delta x \; \Delta y \; \mathbf{a}_z \\ &\quad + \left(-\eta_0 \frac{J_{S0}^2}{4} \cos^2 \omega t \; \mathbf{a}_z\right) \cdot (-\Delta x \; \Delta y \; \mathbf{a}_z) \\ &= \eta_0 \frac{J_{S0}^2}{2} \cos^2 \omega t \; \Delta x \; \Delta y\end{aligned} \tag{4.67}$$

This result is exactly equal to the power supplied by the current source as given by (4.66).

We now interpret the quantity $\mathbf{E} \times \mathbf{H}$ as the power flow density vector associated with the electromagnetic field. It is known as the *Poynting vector*, after J. H. Poynting, and is denoted by the symbol $\mathbf{P}$. Although we have here introduced the Poynting vector by considering the specific case of the electromagnetic field due to the infinite plane current sheet, the interpretation that $\oint_S \mathbf{E} \times \mathbf{H} \cdot d\mathbf{S}$ is equal to the power flow out of the closed surface S is applicable in the general case.

Example 4.4

Far from a physical antenna, that is, at a distance of several wavelengths from the antenna, the radiated electromagnetic waves are approximately uniform plane waves with their constant phase surfaces lying normal to the radial directions away from the antenna, as shown for two directions in Figure 4.14. We wish to show from the Poynting vector and physical considerations that the electric and magnetic fields due to the antenna vary inversely proportional to the radial distance away from the antenna.

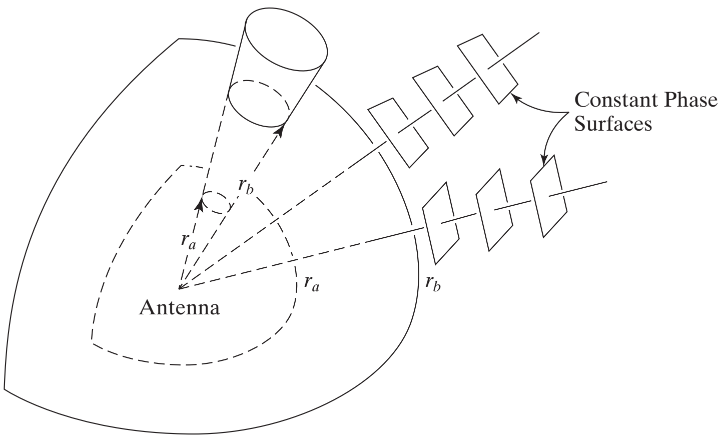

FIGURE 4.14

Radiation of electromagnetic waves far from a physical antenna.

From considerations of electric and magnetic fields of a uniform plane wave, the Poynting vector is directed everywhere in the radial direction indicating power flow radially away from the antenna and is proportional to the square of the magnitude of the electric field intensity. Let us now consider two spherical surfaces of radii r_a and r_b and centered at the antenna and insert a cone through these two surfaces such that the vertex is at the antenna, as shown in Figure 4.14. Then the power crossing the portion of the spherical surface of radius r_b inside the cone must be the same as the power crossing the portion of the spherical surface of radius r_a inside the cone. Since these surface areas are proportional to the square of the radius and since the surface integral of the Poynting vector gives the power, the Poynting vector must be inversely proportional to the square of the radius. This in turn means that the electric field intensity and hence the magnetic field intensity must be inversely proportional to the radius.

Thus, from these simple considerations we have established that far from a radiating antenna the electromagnetic field is inversely proportional to the radial distance away from the antenna. This reduction of the field intensity inversely proportional to the distance is known as the *free space reduction*. For example, let us consider communication from earth to the moon. The distance from the earth to the moon is approximately 38×10^4 km, or 38×10^7 m. Hence, the free space reduction factor for the field intensity is $10^{-7}/38$ or, in terms of decibels, the reduction is $20 \log_{10} 38 \times 10^7$, or 171.6 db.

Returning to the electromagnetic field due to the infinite plane current sheet, let us consider the region $z > 0$. The magnitude of the Poynting vector in this region is given by

$$P_z = E_x H_y = \eta_0 \frac{J_{S0}^2}{4} \cos^2 (\omega t - \beta z) \tag{4.68}$$

The variation of P_z with z for $t = 0$ is shown in Figure 4.15. If we now consider a rectangluar box lying between $z = z$ and $z = z + \Delta z$ planes and having dimensions Δx and Δy in the x- and y-directions, respectively, we would in general obtain a nonzero result for the power flowing out of the box, since $\partial P_z/\partial z$ is not everywhere zero. Thus, there is some energy stored in the volume of the box. We then ask ourselves the question, "Where does this energy reside?" A convenient way of interpretation is to attribute the energy storage to the electric and magnetic fields.

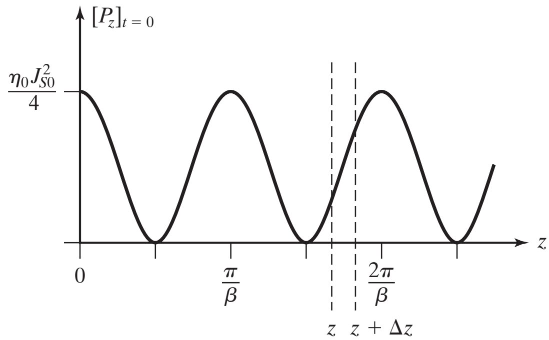

FIGURE 4.15

For the discussion of energy storage in electric and magnetic fields.

To discuss the energy storage in the electric and magnetic fields further, we evaluate the power flow out of the rectangular box. Thus,

$$\begin{aligned}\oint_S \mathbf{P} \cdot d\mathbf{S} &= [P_z]_{z+\Delta z} \, \Delta x \, \Delta y - [P_z]_z \, \Delta x \, \Delta y \\ &= \frac{[P_z]_{z+\Delta z} - [P_z]_z}{\Delta z} \Delta x \, \Delta y \, \Delta z \\ &= \frac{\partial P_z}{\partial z} \Delta v \end{aligned} \tag{4.69}$$

where Δv is the volume of the box. Letting P_z equal E_xH_y and using (4.7) and (4.8), we obtain

$$\begin{aligned}\oint_S \mathbf{P} \cdot d\mathbf{S} &= \frac{\partial}{\partial z}[E_xH_y]\Delta v \\ &= \left(H_y\frac{\partial E_x}{\partial z} + E_x\frac{\partial H_y}{\partial z}\right)\Delta v \\ &= \left(-H_y\frac{\partial B_y}{\partial t} - E_x\frac{\partial D_x}{\partial t}\right)\Delta v \\ &= -\mu_0 H_y\frac{\partial H_y}{\partial t}\Delta v - \epsilon_0 E_x\frac{\partial E_x}{\partial t}\Delta v \\ &= -\frac{\partial}{\partial t}\left(\frac{1}{2}\mu_0 H_y^2\,\Delta v\right) - \frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon_0 E_x^2\,\Delta v\right) \qquad (4.70)\end{aligned}$$

Equation (4.70), which is known as Poynting's theorem, tells us that the power flow out of the box is equal to the sum of the time rates of decrease of the quantities $\frac{1}{2}\epsilon_0 E_x^2\,\Delta v$ and $\frac{1}{2}\mu_0 H_y^2\,\Delta v$. These quantities are obviously the energies stored in the electric and magnetic fields, respectively, in the volume of the box. It then follows that the energy densities associated with the electric and magnetic fields are $\frac{1}{2}\epsilon_0 E_x^2$ and $\frac{1}{2}\mu_0 H_y^2$, respectively. It is left to the student to verify that the quantities $\frac{1}{2}\epsilon_0 E^2$ and $\frac{1}{2}\mu_0 H^2$ do indeed have the units J/m^3. Once again, although we have obtained these results by considering the particular case of the uniform plane wave, they hold in general.

Summarizing our discussion in this section, we have introduced the Poynting vector $\mathbf{P} = \mathbf{E} \times \mathbf{H}$ as the power flow density associated with the electromagnetic field characterized by the electric and magnetic fields, **E** and **H**, respectively. The surface integral of **P** over a closed surface always gives the correct result for the power flow out of that surface. There is energy storage associated with the electric and magnetic fields with the energy densities given by

$$w_e = \frac{1}{2}\epsilon_0 E^2 \qquad (4.71)$$

and

$$w_m = \frac{1}{2}\mu_0 H^2 \qquad (4.72)$$

respectively.

SUMMARY

In this chapter, we studied the principles of uniform plane wave propagation in free space. Uniform plane waves are a building block in the study of electromagnetic wave propagation. They are the simplest type of solutions resulting from the coupling of the electric and magnetic fields in Maxwell's curl equations. We learned that uniform plane waves have their electric and magnetic fields perpendicular to each other and to the direction of propagation. The fields are *uniform* in the *planes* perpendicular to the direction of propagation.

We obtained the uniform plane wave solution to Maxwell's equations by considering an infinite plane current sheet in the xy-plane with uniform surface current density given by

$$\mathbf{J}_S = -J_{S0} \cos \omega t \, \mathbf{a}_x \text{ A/m} \tag{4.73}$$

and deriving the electromagnetic field due to the current sheet to be given by

$$\mathbf{E} = \frac{\eta_0 J_{S0}}{2} \cos(\omega t \mp \beta z) \, \mathbf{a}_x \qquad \text{for } z \gtrless 0 \tag{4.74a}$$

$$\mathbf{H} = \pm \frac{J_{S0}}{2} \cos(\omega t \mp \beta z) \, \mathbf{a}_y \qquad \text{for } z \gtrless 0 \tag{4.74b}$$

In (4.74a) and (4.74b), $\cos(\omega t - \beta z)$ represents wave motion in the positive z-direction, whereas $\cos(\omega t + \beta z)$ represents wave motion in the negative z-direction. Thus, (4.74a) and (4.74b) correspond to waves propagating away from the current sheet to either side of it. Since the fields are independent of x and y, they represent uniform plane waves.

The quantity $\beta\,(=\omega\sqrt{\mu_0\epsilon_0})$ is the phase constant, that is, the magnitude of the rate of change of phase with distance along the direction of propagation, for a fixed time. The phase velocity v_p, that is, the velocity with which a particular constant phase progresses along the direction of propagation, is given by

$$v_p = \frac{\omega}{\beta} \tag{4.75}$$

The wavelength λ, that is, the distance along the direction of propagation in which the phase changes by 2π radians, for a fixed time, is given by

$$\lambda = \frac{2\pi}{\beta} \tag{4.76}$$

The wavelength is related to the frequency f in a simple manner as given by

$$v_p = \lambda f \tag{4.77}$$

which follows from (4.75) and (4.76). The quantity $\eta_0\,(=\sqrt{\mu_0/\epsilon_0})$ is the intrinsic impedance of free space. It is the ratio of the magnitude of **E** to the magnitude of **H** and has a value of $120\pi\ \Omega$.

In the process of deriving the electromagnetic field due to the infinite plane current sheet, we used two approaches and learned several useful techniques. These are discussed in the following:

1. The determination of the magnetic field adjacent to the current sheet by employing Ampere's circuital law in integral form: This is a common procedure used in the computation of static fields due to charge and current distributions possessing certain symmetries. In Chapter 5 we shall derive the *boundary conditions*, that is, the relationships between the fields on either side of an interface between two different media, by applying Maxwell's equations in integral form to closed paths and surfaces straddling the boundary as we have done here in the case of the current sheet.

2. The successive, step-by-step solution of the two Maxwell's curl equations, to obtain the final solution consistent with the two equations, starting with the solution obtained for the field adjacent to the current sheet: This technique provided us a feel for the phenomenon of *radiation* of electromagnetic waves resulting from the time-varying current distribution and the interaction between the electric and magnetic fields. We shall use this kind of approach and the knowledge gained on wave propagation to obtain in Chapter 9 the complete electromagnetic field due to an elemental antenna, which forms the basis for the study of physical antennas.
3. The solution of wave equation by the separation of variables technique: This is the standard technique employed in the solution of partial differential equations involving multiple variables.
4. The application of phasor technique for the solution of the differential equations: The phasor technique is a convenient tool for analyzing sinusoidal steady-state problems as we learned in Chapter 1.

We discussed (a) polarization of sinusoidally time-varying fields, as it pertains to uniform plane wave propagation, and (b) nonsinusoidal excitation giving rise to nonsinusoidal waves propagating in free space without change in shape, in view of phase velocity independent of frequency.

We also learned that there is power flow and energy storage associated with the wave propagation that accounts for the work done in maintaining the current flow on the sheet. The power flow density is given by the Poynting vector

$$\mathbf{P} = \mathbf{E} \times \mathbf{H}$$

and the energy densities associated with the electric and magnetic fields are given, respectively, by

$$w_e = \frac{1}{2}\epsilon_0 E^2$$

$$w_m = \frac{1}{2}\mu_0 H^2$$

The surface integral of the Poynting vector over a given closed surface gives the total power flow out of the volume bounded by that surface.

Finally, we have augmented our study of uniform plane wave propagation in free space by illustrating (a) the principle of an antenna array, and (b) the inverse distance dependence of the fields far from a physical antenna.

REVIEW QUESTIONS

4.1. What is a uniform plane wave?

4.2. Why is the study of uniform plane waves important?

4.3. How is the surface current density vector defined? Distinguish it from the volume current density vector.

4.4. How do you find the current crossing a given line on a sheet of surface current?

4.5. Why is it that Ampere's circuital law in integral form is used to find the magnetic field adjacent to the current sheet of Figure 4.2?

4.6. Why is the path chosen to evaluate the magnetic field in Figure 4.4 rectangular?

4.7. Outline the application of Ampere's circuital law in integral form to find the magnetic field adjacent to the current sheet of Figure 4.2.

4.8. Why is the displacement current enclosed by the rectangular path *abcda* in Figure 4.4 equal to zero?

4.9. How would you use Ampere's circuital law in differential form to find the magnetic field adjacent to the current sheet?

4.10. If the current density on the infinite plane current sheet of Figure 4.2 were directed in the positive y-direction, what would be the directions of the magnetic field adjacent to the current sheet and on either side of it?

4.11. Why are the results given by (4.12) and (4.13) for the magnetic field not valid for points at some distance from the current sheet?

4.12. Under what conditions would a result obtained for the magnetic field adjacent to the infinite plane current sheet of Figure 4.2 be valid at points distant from the current sheet?

4.13. Briefly outline the procedure involved in the successive solution of Maxwell's equations.

4.14. How does the technique of successive solution of Maxwell's equations reveal the interaction between the electric and magnetic fields giving rise to wave propagation?

4.15. State the wave equation for the case of $\mathbf{E} = E_x(z, t)\mathbf{a}_x$. How is it derived?

4.16. Briefly outline the separation of variables technique of solving the wave equation.

4.17. Discuss how the function $\cos(\omega t - \beta z)$ represents a traveling wave propagating in the positive z-direction.

4.18. Discuss how the function $\cos(\omega t + \beta z)$ represents a traveling wave propagating in the negative z-direction.

4.19. Discuss how the solution for the electromagnetic field given by (4.52) corresponds to that of a uniform plane wave.

4.20. Why is the quantity β in $\cos(\omega t - \beta z)$ known as the phase constant?

4.21. What is phase velocity? How is it related to the radian frequency and the phase constant of the wave?

4.22. Define wavelength. How is it related to the phase constant?

4.23. What is the relationship between frequency, wavelength, and phase velocity? What is the wavelength in free space for a frequency of 15 MHz?

4.24. What is the direction of propagation for a uniform plane wave having its electric field in the negative y-direction and its magnetic field in the positive z-direction?

4.25. What is the direction of the magnetic field for a uniform plane wave having its electric field in the positive z-direction and propagating in the positive x-direction?

4.26. What is intrinsic impedance? What is its value for free space?

4.27. Discuss the principle of an antenna array.

4.28. What should be the spacing and the relative phase angle of the current densities for an array of two infinite, plane, parallel current sheets of uniform densities, equal in magnitude, to confine their radiation to the region between the two sheets?

4.29. Discuss polarization of sinusoidally time-varying fields, as it is relevant to propagation of uniform plane waves.

4.30. Discuss the propagation of uniform plane waves arising from an infinite plane current sheet of nonsinusoidally time-varying surface current density.

4.31. Why is a certain amount of work involved in maintaining current flow on the sheet of Figure 4.2? How is this work accounted for?

4.32. What is a Poynting vector? What is its physical significance?

4.33. What is the physical interpretation of the surface integral of the Poynting vector over a closed surface?

4.34. Discuss how the fields far from a physical antenna vary inversely proportional to the distance from the antenna.

4.35. Discuss the interpretation of energy storage in the electric and magnetic fields of a uniform plane wave.

4.36. What are the energy densities associated with the electric and magnetic fields?

PROBLEMS

4.1. An infinite plane sheet lying in the $z = 0$ plane carries a current of uniform density $\mathbf{J}_S = -0.1\,\mathbf{a}_x$ A/m. Find the currents crossing the following straight lines: (a) from $(0, 0, 0)$ to $(0, 2, 0)$; (b) from $(0, 0, 0)$ to $(2, 0, 0)$; (c) from $(0, 0, 0)$ to $(2, 2, 0)$.

4.2. An infinite plane sheet lying in the $z = 0$ plane carries a current of nonuniform density $\mathbf{J}_S = -0.1e^{-|y|}\,\mathbf{a}_x$ A/m. Find the currents crossing the following straight lines: (a) from $(0, 0, 0)$ to $(0, 1, 0)$; (b) from $(0, 0, 0)$ to $(0, \infty, 0)$; (c) from $(0, 0, 0)$ to $(1, 1, 0)$.

4.3. An infinite plane sheet lying in the $z = 0$ plane carries a current of uniform density

$$\mathbf{J}_S = (-0.1 \cos \omega t\,\mathbf{a}_x + 0.1 \sin \omega t\,\mathbf{a}_y) \text{ A/m}$$

Find the currents crossing the following straight lines: (a) from $(0, 0, 0)$ to $(0, 2, 0)$; (b) from $(0, 0, 0)$ to $(2, 0, 0)$; (c) from $(0, 0, 0)$ to $(2, 2, 0)$.

4.4. An infinite plane sheet lying in the $z = 0$ plane carries a current of uniform density

$$\mathbf{J}_S = (-0.2 \cos \omega t\,\mathbf{a}_x + 0.2 \sin \omega t\,\mathbf{a}_y) \text{ A/m}$$

Find the magnetic field intensities adjacent to the sheet and on either side of it. What is the polarization of the field?

4.5. An infinite plane sheet lying in the $z = 0$ plane carries a current of nonuniform density $\mathbf{J}_S = -0.2e^{-|y|} \cos \omega t\,\mathbf{a}_x$ A/m. Find the magnetic field intensities adjacent to the current sheet and on either side of it at (a) the point $(0, 1, 0)$ and (b) the point $(2, 2, 0)$.

4.6. Current flows with uniform density $\mathbf{J} = J_0\mathbf{a}_x$ A/m^2 in the region $|z| < a$. Using Ampere's circuital law in integral form and symmetry considerations, find $\mathbf{H}$ everywhere.

4.7. Current flows with nonuniform density $\mathbf{J} = J_0(1 - |z|/a)\mathbf{a}_x$ A/m^2 in the region $|z| < a$, where J_0 is a constant. Using Ampere's circuital law in integral form and symmetry considerations, find $\mathbf{H}$ everywhere.

4.8. For an infinite plane sheet of charge lying in the xy-plane with uniform surface charge density ρ_{S0} C/m^2, find the electric field intensity on both sides of the sheet by using Gauss' law for the electric field in integral form and symmetry considerations.

4.9. Charge is distributed with uniform density $\rho = \rho_0$ C/m^3 in the region $|x| < a$. Using Gauss' law for the electric field in integral form and symmetry considerations, find $\mathbf{E}$ everywhere.

4.10. Charge is distributed with nonuniform density $\rho = \rho_0(1 - |x|/a)$ C/m^3 in the region $|x| < a$, where ρ_0 is a constant. Using Gauss' law for the electric field in integral form and symmetry considerations, find $\mathbf{E}$ everywhere.

4.11. Verify that expressions (4.23) and (4.24) simultaneously satisfy the differential equations (4.16) and (4.17).

4.12. For the infinite plane current sheet in the $z = 0$ plane carrying surface current of density $\mathbf{J}_S = -J_{S0}t\,\mathbf{a}_x$ A/m, where J_{S0} is a constant, find the magnetic field adjacent to the current sheet. Then use the method of successive solution of Maxwell's equations to show that for $z > 0$,

$$E_x = \left(\frac{2C + \eta_0 J_{S0}}{4}\right)(t - z\sqrt{\mu_0\epsilon_0}) + \left(\frac{2C - \eta_0 J_{S0}}{4}\right)(t + z\sqrt{\mu_0\epsilon_0})$$

$$H_y = \left(\frac{2C + \eta_0 J_{S0}}{4\eta_0}\right)(t - z\sqrt{\mu_0\epsilon_0}) - \left(\frac{2C - \eta_0 J_{S0}}{4\eta_0}\right)(t + z\sqrt{\mu_0\epsilon_0})$$

where C is a constant.

4.13. For the infinite plane current sheet in the $z = 0$ plane carrying surface current of density $\mathbf{J}_S = -J_{S0}t^2\mathbf{a}_x$ A/m, where J_{S0} is a constant, find the magnetic field adjacent to the current sheet. Then use the method of successive solution of Maxwell's equations to show that for $z > 0$,

$$E_x = \left(\frac{2C + \eta_0 J_{S0}}{4}\right)(t - z\sqrt{\mu_0\epsilon_0})^2 + \left(\frac{2C - \eta_0 J_{S0}}{4}\right)(t + z\sqrt{\mu_0\epsilon_0})^2$$

$$H_y = \left(\frac{2C + \eta_0 J_{S0}}{4\eta_0}\right)(t - z\sqrt{\mu_0\epsilon_0})^2 - \left(\frac{2C - \eta_0 J_{S0}}{4\eta_0}\right)(t + z\sqrt{\mu_0\epsilon_0})^2$$

where C is a constant.

4.14. Verify that expressions (4.48) and (4.49) simultaneously satisfy the differential equations (4.7) and (4.8), and that (4.49) reduces to (4.12) for $z = 0+$.

4.15. Show that $(t - z\sqrt{\mu_0\epsilon_0})^2$ and $(t + z\sqrt{\mu_0\epsilon_0})^2$ are solutions of the wave equation. With the aid of sketches, discuss the nature of these functions.

4.16. For arbitrary time-variation of the fields, show that the solutions for the differential equations (4.33) and (4.34) are

$$E_x = Af(t - z\sqrt{\mu_0\epsilon_0}) + Bg(t + z\sqrt{\mu_0\epsilon_0})$$

$$H_y = \frac{1}{\eta_0}[Af(t - z\sqrt{\mu_0\epsilon_0}) - Bg(t + z\sqrt{\mu_0\epsilon_0})]$$

where A and B are arbitrary constants. Discuss the nature of the functions $f(t - z\sqrt{\mu_0\epsilon_0})$ and $g(t + z\sqrt{\mu_0\epsilon_0})$.

4.17. In Problems 4.12 and 4.13, evaluate the constant C and obtain the solutions for E_x and H_y in the region $z > 0$. Then write the solutions for E_x and H_y in the region $z < 0$.

4.18. The electric field intensity of a uniform plane wave is given by

$$\mathbf{E} = 37.7 \cos(6\pi \times 10^8 t + 2\pi z)\,\mathbf{a}_y \text{ V/m}.$$

Find (a) the frequency, (b) the wavelength, (c) the phase velocity, (d) the direction of propagation of the wave, and (e) the associated magnetic field intensity vector **H**.

4.19. An infinite plane sheet lying in the $z = 0$ plane carries a surface current of density

$$\mathbf{J}_S = (-0.2 \cos 6\pi \times 10^8 t\,\mathbf{a}_x - 0.1 \cos 12\pi \times 10^8 t\,\mathbf{a}_x) \text{ A/m}$$

Find the expressions for the electric and magnetic fields on either side of the sheet.

4.20. An array is formed by two infinite plane parallel current sheets with the current densities given by

$$\mathbf{J}_{S1} = -J_{S0}\cos\omega t\,\mathbf{a}_x \qquad z = 0$$
$$\mathbf{J}_{S2} = -J_{S0}\cos\omega t\,\mathbf{a}_x \qquad z = \frac{\lambda}{2}$$

where J_{S0} is a constant. Find the electric field intensity in all three regions: (a) $z < 0$; (b) $0 < z < \lambda/2$; (c) $z > \lambda/2$.

4.21. Determine the spacing, relative amplitudes, and phase angles of current densities for an array of two infinite plane parallel current sheets required to obtain a radiation characteristic such that the field radiated to one side of the array is twice that of the field radiated to the other side of the array.

4.22. For two infinite plane parallel current sheets with the current densities given by

$$\mathbf{J}_{S1} = -J_{S0}\cos\omega t\,\mathbf{a}_x \qquad z = 0$$
$$\mathbf{J}_{S2} = -J_{S0}\cos\omega t\,\mathbf{a}_y \qquad z = \frac{\lambda}{2}$$

where J_{S0} is a constant, find the electric field in all three regions: (a) $z < 0$; (b) $0 < z < \lambda/2$; (c) $z > \lambda/2$. Discuss the polarization of the field in all three regions.

4.23. For each of the following fields, determine if the polarization is right- or left-circular.
(a) $E_0\cos(\omega t - \beta y)\,\mathbf{a}_z + E_0\sin(\omega t - \beta y)\,\mathbf{a}_x$
(b) $E_0\cos(\omega t + \beta x)\,\mathbf{a}_y + E_0\sin(\omega t + \beta x)\,\mathbf{a}_z$

4.24. For each of the following fields, determine if the polarization is right- or left-elliptical.
(a) $E_0\cos(\omega t + \beta y)\,\mathbf{a}_x - 2E_0\sin(\omega t + \beta y)\,\mathbf{a}_z$
(b) $E_0\cos(\omega t - \beta x)\,\mathbf{a}_z - E_0\sin(\omega t - \beta x + \pi/4)\,\mathbf{a}_y$

4.25. Express the following uniform plane wave electric field as a superposition of right- and left-circularly polarized fields: $E_0\mathbf{a}_x\cos(\omega t + \beta z)$

4.26. Repeat Problem 4.25 for the following electric field: $E_0\mathbf{a}_x\cos(\omega t - \beta z + \pi/3) - E_0\mathbf{a}_y\cos(\omega t - \beta z + \pi/6)$

4.27. Write the expression for the electric field intensity of a sinusoidally time-varying uniform plane wave propagating in free space and having the following characteristics: (a) $f = 100$ MHz; (b) direction of propagation is the $+z$-direction; and (c) polarization is right circular with the electric field in the $z = 0$ plane at $t = 0$ having an x-component equal to E_0 and a y-component equal to $0.75E_0$.

4.28. An infinite plane sheet lying in the $z = 0$ plane carries a surface current of density $\mathbf{J}_S = -J_S(t)\mathbf{a}_x$, where $J_S(t)$ is the periodic function shown in Figure 4.16. Find and sketch (a) H_y versus t for $z = 0+$, (b) E_x versus t for $z = 150$ m, and (c) E_x versus z for $t = 1\ \mu$s.

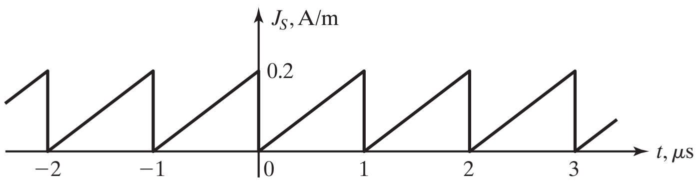

FIGURE 4.16

For Problem 4.28.

4.29. The time-variation of the electric field intensity E_x in the $z = 600$ m plane of a uniform plane wave propagating away from an infinite plane current sheet lying in the $z = 0$ plane is given by the periodic function shown in Figure 4.17. Find and sketch (a) E_x versus t for $z = 200$ m, (b) E_x versus z for $t = 0$, and (c) H_y versus z for $t = \frac{1}{3}\ \mu$s.

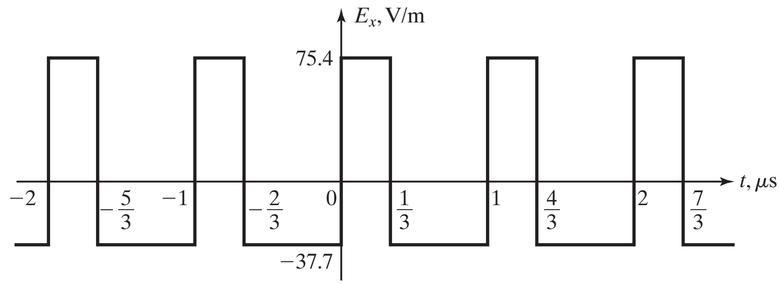

FIGURE 4.17
For Problem 4.29.

4.30. The time-variation of the electric field intensity E_x in the $z = 300$ m plane of a uniform plane wave propagating away from an infinite plane current sheet lying in the $z = 0$ plane is given by the aperiodic function shown in Figure 4.18. Find and sketch (a) E_x versus t for $z = 600$ m, (b) E_x versus z for $t = 1\ \mu$s, and (c) H_y versus z for $t = 2\ \mu$s.

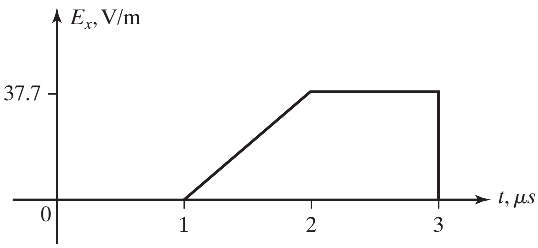

FIGURE 4.18
For Problem 4.30.

4.31. Show that the time-average value of the magnitude of the Poynting vector given by (4.68) is one-half its peak value. For an antenna radiating a time-average power of 150 kW, find the peak value of the electric field intensity at a distance of 100 km from the antenna. Assume the antenna to be radiating equally in all directions.

4.32. The electric field of a uniform plane wave propagating in the positive z-direction is given by

$$\mathbf{E} = E_0 \cos(\omega t - \beta z)\, \mathbf{a}_x + E_0 \sin(\omega t - \beta z)\, \mathbf{a}_y$$

where E_0 is a constant. (a) Find the corresponding magnetic field $\mathbf{H}$. (b) Find the Poynting vector.

4.33. Show that the quantities $\frac{1}{2}\epsilon_0 E^2$ and $\frac{1}{2}\mu_0 H^2$ have the units J/m^3.

4.34. Show that the energy is stored equally in the electric and magnetic fields of a traveling wave.