CHAPTER

3

Maxwell's Equations in Differential Form

In Chapter 2 we introduced Maxwell's equations in integral form. We learned that the quantities involved in the formulation of these equations are the scalar quantities, electromotive force, magnetomotive force, magnetic flux, displacement flux, charge, and current, which are related to the field vectors and source densities through line, surface, and volume integrals. Thus, the integral forms of Maxwell's equations, while containing all the information pertinent to the interdependence of the field and source quantities over a given region in space, do not permit us to study directly the interaction between the field vectors and their relationships with the source densities at individual points. It is our goal in this chapter to derive the differential forms of Maxwell's equations that apply directly to the field vectors and source densities at a given point.

We shall derive Maxwell's equations in differential form by applying Maxwell's equations in integral form to infinitesimal closed paths, surfaces, and volumes, in the limit that they shrink to points. We will find that the differential equations relate the spatial variations of the field vectors at a given point to their temporal variations and to the charge and current densities at that point. In this process we shall also learn two important operations in vector calculus, known as curl and divergence, and two related theorems, known as Stokes' and divergence theorems.

3.1 FARADAY'S LAW

We recall from the previous chapter that Faraday's law is given in integral form by

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S} \tag{3.1}$$

where S is any surface bounded by the closed path C. In the most general case, the electric and magnetic fields have all three components (x, y, and z) and are dependent on all three coordinates (x, y, and z) in addition to time (t). For simplicity, we shall, however, first consider the case in which the electric field has an x-component only, which is dependent only on the z-coordinate, in addition to time. Thus,

$$\mathbf{E} = E_x(z, t)\mathbf{a}_x \tag{3.2}$$

In other words, this simple form of time-varying electric field is everywhere directed in the x-direction and it is uniform in planes parallel to the xy-plane.

Let us now consider a rectangular path C of infinitesimal size lying in a plane parallel to the xz-plane and defined by the points (x, z), $(x, z + \Delta z)$, $(x + \Delta x, z + \Delta z)$, and $(x + \Delta x, z)$, as shown in Figure 3.1. According to Faraday's law, the emf around the closed path C is equal to the negative of the time rate of change of the magnetic flux enclosed by C. The emf is given by the line integral of $\mathbf{E}$ around C. Thus, evaluating the line integrals of $\mathbf{E}$ along the four sides of the rectangular path, we obtain

$$\int_{(x,z)}^{(x,z+\Delta z)} \mathbf{E} \cdot d\mathbf{l} = 0 \qquad \text{since } E_z = 0 \tag{3.3a}$$

$$\int_{(x,z+\Delta z)}^{(x+\Delta x,z+\Delta z)} \mathbf{E} \cdot d\mathbf{l} = [E_x]_{z+\Delta z}\,\Delta x \tag{3.3b}$$

$$\int_{(x+\Delta x,z+\Delta z)}^{(x+\Delta x,z)} \mathbf{E} \cdot d\mathbf{l} = 0 \quad \text{since } E_z = 0 \tag{3.3c}$$

$$\int_{(x+\Delta x,z)}^{(x,z)} \mathbf{E} \cdot d\mathbf{l} = -[E_x]_z\,\Delta x \tag{3.3d}$$

Adding up (3.3a)–(3.3d), we obtain

$$\begin{aligned}\oint_C \mathbf{E} \cdot d\mathbf{l} &= [E_x]_{z+\Delta z}\,\Delta x - [E_x]_z\,\Delta x \\ &= \{[E_x]_{z+\Delta z} - [E_x]_z\}\,\Delta x \end{aligned} \tag{3.4}$$

In (3.3a)–(3.3d) and (3.4), $[E_x]_z$ and $[E_x]_{z+\Delta z}$ denote values of E_x evaluated along the sides of the path for which $z = z$ and $z = z + \Delta z$, respectively.

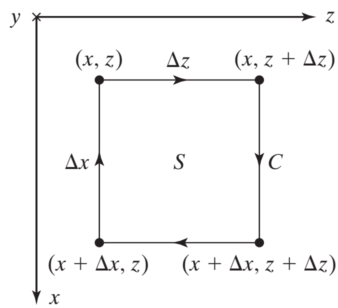

FIGURE 3.1

Infinitesimal rectangular path lying in a plane parallel to the xz-plane.

To find the magnetic flux enclosed by C, let us consider the plane surface S bounded by C. According to the right-hand screw rule, we must use the magnetic flux crossing S toward the positive y-direction, that is, into the page, since the path C is traversed in the clockwise sense. The only component of $\mathbf{B}$ normal to the area S is the y-component. Also, since the area is infinitesimal in size, we can assume B_y to be uniform over the area and equal to its value at (x, z). The required magnetic flux is then given by

$$\int_S \mathbf{B} \cdot d\mathbf{S} = [B_y]_{(x,z)}\, \Delta x\, \Delta z \tag{3.5}$$

Substituting (3.4) and (3.5) into (3.1) to apply Faraday's law to the rectangular path C under consideration, we get

$$\{[E_x]_{z+\Delta z} - [E_x]_z\}\, \Delta x = -\frac{d}{dt}\{[B_y]_{(x,z)}\, \Delta x\, \Delta z\}$$

or

$$\frac{[E_x]_{z+\Delta z} - [E_x]_z}{\Delta z} = -\frac{\partial [B_y]_{(x,z)}}{\partial t} \tag{3.6}$$

If we now let the rectangular path shrink to the point (x, z) by letting Δx and Δz tend to zero, we obtain

$$\lim_{\substack{\Delta x \to 0 \\ \Delta z \to 0}} \frac{[E_x]_{z+\Delta z} - [E_x]_z}{\Delta z} = -\lim_{\substack{\Delta x \to 0 \\ \Delta z \to 0}} \frac{\partial [B_y]_{(x,z)}}{\partial t}$$

or

$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t} \tag{3.7}$$

Equation (3.7) is Faraday's law in differential form for the simple case of $\mathbf{E}$ given by (3.2). It relates the variation of E_x with z (space) at a point to the variation of B_y with t (time) at that point. Since the above derivation can be carried out for any arbitrary point (x, y, z), it is valid for all points. It tells us in particular that a time-varying B_y at a point results in an E_x at that point having a differential in the z-direction. This is to be expected since if this is not the case, $\oint \mathbf{E} \cdot d\mathbf{l}$ around the infinitesimal rectangular path would be zero.

Example 3.1

Given $\mathbf{B} = B_0 \cos \omega t\, \mathbf{a}_y$ and it is known that $\mathbf{E}$ has an x-component only, let us find E_x.

From (3.6), we have

$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t} = -\frac{\partial}{\partial t}(B_0 \cos \omega t) = \omega B_0 \sin \omega t$$

$$E_x = \omega B_0 z \sin \omega t$$

We note that the uniform magnetic field gives rise to an electric field varying linearly with z.

Proceeding further, we can verify this result by evaluating $\oint \mathbf{E} \cdot d\mathbf{l}$ around the rectangular path of Example 2.8. This rectangular path is reproduced in Figure 3.2. The required line integral is given by

$$\begin{aligned}\oint_C \mathbf{E} \cdot d\mathbf{l} &= \int_{z=0}^{b} [E_z]_{x=0}\, dz + \int_{x=0}^{a} [E_x]_{z=b}\, dx \\ &\quad + \int_{z=b}^{0} [E_z]_{x=a}\, dz + \int_{x=a}^{0} [E_x]_{z=0}\, dx \\ &= 0 + [\omega B_0 b \sin \omega t]a + 0 + 0 \\ &= abB_0\omega \sin \omega t\end{aligned}$$

which agrees with the result of Example 2.8.

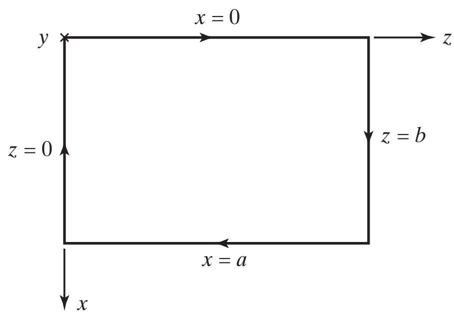

FIGURE 3.2

Rectangular path of Example 2.8.

We shall now proceed to generalize (3.7) for the arbitrary case of the electric field having all three components (x, y, and z), each of them depending on all three coordinates (x, y, and z), in addition to time (t), that is,

$$\mathbf{E} = E_x(x, y, z, t)\mathbf{a}_x + E_y(x, y, z, t)\mathbf{a}_y + E_z(x, y, z, t)\mathbf{a}_z \tag{3.8}$$

To do this, let us consider the three infinitesimal rectangular paths in planes parallel to the three mutually orthogonal planes of the Cartesian coordinate system, as shown in Figure 3.3. Evaluating $\oint \mathbf{E} \cdot d\mathbf{l}$ around the closed paths *abcda*, *adefa*, and *afgba*, we get

$$\begin{aligned}\oint_{abcda} \mathbf{E} \cdot d\mathbf{l} &= [E_y]_{(x,\, z)}\, \Delta y + [E_z]_{(x,\, y+\Delta y)}\, \Delta z \\ &\quad - [E_y]_{(x,\, z+\Delta z)}\, \Delta y - [E_z]_{(x,\, y)}\, \Delta z\end{aligned} \tag{3.9a}$$

$$\begin{aligned}\oint_{adefa} \mathbf{E} \cdot d\mathbf{l} &= [E_z]_{(x,\, y)}\, \Delta z + [E_x]_{(y,\, z+\Delta z)}\, \Delta x \\ &\quad - [E_z]_{(x+\Delta x,\, y)}\, \Delta z - [E_x]_{(y,\, z)}\, \Delta x\end{aligned} \tag{3.9b}$$

$$\oint_{afgba} \mathbf{E} \cdot d\mathbf{l} = [E_x]_{(y,\,z)}\,\Delta x + [E_y]_{(x+\Delta x,\,z)}\,\Delta y$$

$$- [E_x]_{(y+\Delta y,\,z)}\,\Delta x - [E_y]_{(x,\,z)}\,\Delta y \tag{3.9c}$$

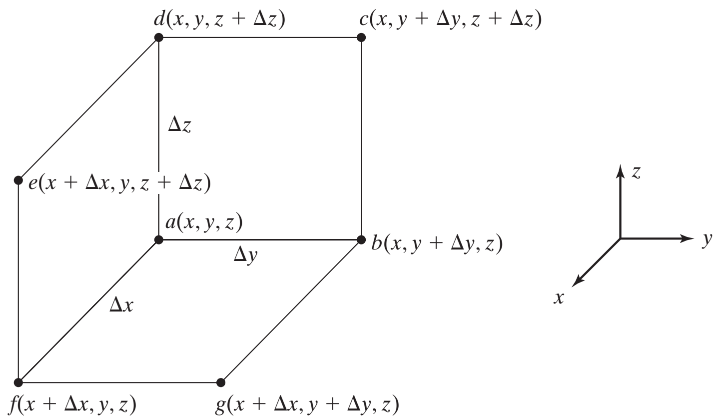

FIGURE 3.3

Infinitesimal rectangular paths in three mutually orthogonal planes.

In (3.9a)–(3.9c) the subscripts associated with the field components in the various terms on the right sides of the equations denote the value of the coordinates that remain constant along the sides of the closed paths corresponding to the terms. Now, evaluating $\int \mathbf{B} \cdot d\mathbf{S}$ over the surfaces *abcd*, *adef*, and *afgb*, keeping in mind the right-hand screw rule, we have

$$\int_{abcd} \mathbf{B} \cdot d\mathbf{S} = [B_x]_{(x,\,y,\,z)}\,\Delta y\,\Delta z \tag{3.10a}$$

$$\int_{adef} \mathbf{B} \cdot d\mathbf{S} = [B_y]_{(x,\,y,\,z)}\,\Delta z\,\Delta x \tag{3.10b}$$

$$\int_{afgb} \mathbf{B} \cdot d\mathbf{S} = [B_z]_{(x,\,y,\,z)}\,\Delta x\,\Delta y \tag{3.10c}$$

Applying Faraday's law to each of the three paths by making use of (3.9a)–(3.9c) and (3.10a)–(3.10c) and simplifying, we obtain

$$\frac{[E_z]_{(x,\,y+\Delta y)} - [E_z]_{(x,\,y)}}{\Delta y} - \frac{[E_y]_{(x,\,z+\Delta z)} - [E_y]_{(x,\,z)}}{\Delta z} = -\frac{\partial [B_x]_{(x,\,y,\,z)}}{\partial t} \tag{3.11a}$$

$$\frac{[E_x]_{(y,\,z+\Delta z)} - [E_x]_{(y,\,z)}}{\Delta z} - \frac{[E_z]_{(x+\Delta x,\,y)} - [E_z]_{(x,\,y)}}{\Delta x} = -\frac{\partial [B_y]_{(x,\,y,\,z)}}{\partial t} \tag{3.11b}$$

$$\frac{[E_y]_{(x+\Delta x,\, z)} - [E_y]_{(x,\, z)}}{\Delta x} - \frac{[E_x]_{(y+\Delta y,\, z)} - [E_x]_{(y,\, z)}}{\Delta y} = -\frac{\partial [B_z]_{(x,\, y,\, z)}}{\partial t} \tag{3.11c}$$

If we now let all three paths shrink to the point a by letting Δx, Δy, and Δz tend to zero, (3.11a)–(3.11c) reduce to

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t} \tag{3.12a}$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{\partial B_y}{\partial t} \tag{3.12b}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{\partial B_z}{\partial t} \tag{3.12c}$$

Equations (3.12a)–(3.12c) are the differential equations governing the relationships between the space variations of the electric field components and the time variations of the magnetic field components at a point. An examination of one of the three equations is sufficient to reveal the physical meaning of these relationships. For example, (3.12a) tells us that a time-varying B_x at a point results in an electric field at that point having y- and z-components such that their net right-lateral differential normal to the x-direction is nonzero. The right-lateral differential of E_y normal to the x-direction is its derivative in the $\mathbf{a}_y \times \mathbf{a}_x$, or $-\mathbf{a}_z$-direction, that is, $\partial E_y/\partial(-z)$ or $-\partial E_y/\partial z$. The right-lateral differential of E_z normal to the x-direction is its derivative in the $\mathbf{a}_z \times \mathbf{a}_x$, or $\mathbf{a}_y$-direction, that is, $\partial E_z/\partial y$. Thus, the net right-lateral differential of the y- and z-components of the electric field normal to the x-direction is $(-\partial E_y/\partial z) + (\partial E_z/\partial y)$, or $(\partial E_z/\partial y - \partial E_y/\partial z)$. An example in which the net right-lateral differential is zero, although the individual derivatives are nonzero, is shown in Figure 3.4(a), whereas Figure 3.4(b) shows an example in which the net right-lateral differential is nonzero.

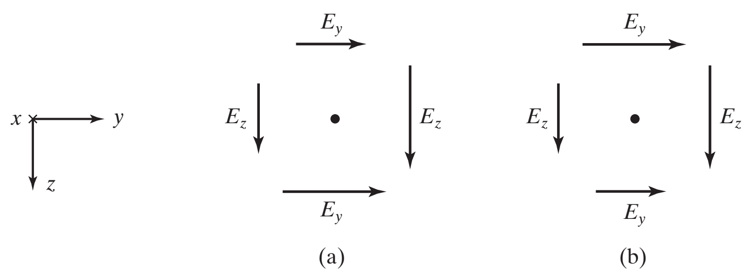

FIGURE 3.4

For illustrating (a) zero, and (b) nonzero net right-lateral differential of E_y and E_z normal to the x-direction.

Equations (3.12a)–(3.12c) can be combined into a single vector equation as given by

$$\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\mathbf{a}_z$$

$$= -\frac{\partial B_x}{\partial t}\mathbf{a}_x - \frac{\partial B_y}{\partial t}\mathbf{a}_y - \frac{\partial B_z}{\partial t}\mathbf{a}_z \qquad (3.13)$$

This can be expressed in determinant form as

$$\begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = -\frac{\partial \mathbf{B}}{\partial t} \qquad (3.14)$$

or as

$$\left(\mathbf{a}_x\frac{\partial}{\partial x} + \mathbf{a}_y\frac{\partial}{\partial y} + \mathbf{a}_z\frac{\partial}{\partial z}\right) \times (E_x\mathbf{a}_x + E_y\mathbf{a}_y + E_z\mathbf{a}_z) = -\frac{\partial \mathbf{B}}{\partial t} \qquad (3.15)$$

The left side of (3.14) or (3.15) is known as the *curl of* $\mathbf{E}$, denoted as $\nabla \times \mathbf{E}$ (del cross $\mathbf{E}$), where ∇ (del) is the vector operator given by

$$\nabla = \mathbf{a}_x\frac{\partial}{\partial x} + \mathbf{a}_y\frac{\partial}{\partial y} + \mathbf{a}_z\frac{\partial}{\partial z} \qquad (3.16)$$

Thus, we have

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad (3.17)$$

Equation (3.17) is Maxwell's equation in differential form corresponding to Faraday's law. We shall discuss curl further in Section 3.3.

Example 3.2

Given $\mathbf{A} = y\mathbf{a}_x - x\mathbf{a}_y$, find $\nabla \times \mathbf{A}$.

From the determinant expansion for the curl of a vector, we have

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix}$$

$$= \mathbf{a}_x\left[-\frac{\partial}{\partial z}(-x)\right] + \mathbf{a}_y\left[\frac{\partial}{\partial z}(y)\right] + \mathbf{a}_z\left[\frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial y}(y)\right]$$

$$= -2\mathbf{a}_z$$

3.2 AMPERE'S CIRCUITAL LAW

In the previous section we derived the differential form of Faraday's law from its integral form. In this section we shall derive the differential form of Ampere's circuital law from its integral form in a completely analogous manner. We recall from Section 2.4 that Ampere's circuital law in integral form is given by

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt}\int_S \mathbf{D} \cdot d\mathbf{S} \tag{3.18}$$

where S is any surface bounded by the closed path C. For simplicity, we shall first consider the case in which the magnetic field has a y-component only, which is dependent only on the z-coordinate, in addition to time. Thus,

$$\mathbf{H} = H_y(z, t)\mathbf{a}_y \tag{3.19}$$

In other words, this simple form of the time-varying magnetic field is everywhere directed in the y-direction and is uniform in planes parallel to the xy-plane.

Let us now consider a rectangular path C of infinitesimal size lying in a plane parallel to the yz-plane and defined by the points (y, z), $(y, z + \Delta z)$, $(y + \Delta y, z + \Delta z)$, and $(y + \Delta y, z)$, as shown in Figure 3.5. According to Ampere's circuital law, the mmf around the closed path C is equal to the total current enclosed by C. The mmf is given by the line integral of $\mathbf{H}$ around C. Thus, evaluating the line integrals of $\mathbf{H}$ along the four sides of the rectangular path, we obtain

$$\begin{aligned}
\oint_C \mathbf{H} \cdot d\mathbf{l} &= \int_{(y,z)}^{(y+\Delta y,\, z)} \mathbf{H} \cdot d\mathbf{l} + \int_{(y+\Delta y,\, z)}^{(y+\Delta y,\, z+\Delta z)} \mathbf{H} \cdot d\mathbf{l} \\
&\quad + \int_{(y+\Delta y,\, z+\Delta z)}^{(y,\, z+\Delta z)} \mathbf{H} \cdot d\mathbf{l} + \int_{(y,\, z+\Delta z)}^{(y,z)} \mathbf{H} \cdot d\mathbf{l} \\
&= [H_y]_z\, \Delta y + 0 - [H_y]_{z+\Delta z}\, \Delta y + 0 \\
&= -\{[H_y]_{z+\Delta z} - [H_y]_z\}\, \Delta z
\end{aligned} \tag{3.20}$$

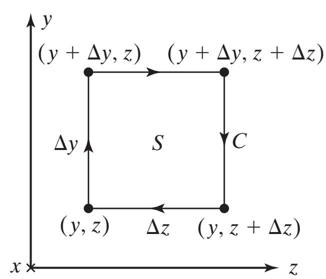

FIGURE 3.5

Infinitesimal rectangular path lying in a plane parallel to the yz-plane.

To find the total current enclosed by C, we consider the plane surface S bounded by C. According to the right-hand screw rule, we must find the current crossing S toward the positive x-direction, that is, into the page, since the path is traversed in the clockwise sense. This current consists of two parts:

$$\int_S \mathbf{J} \cdot d\mathbf{S} = [J_x]_{(y,\,z)} \Delta y \, \Delta z \tag{3.21a}$$

$$\frac{d}{dt} \int_S \mathbf{D} \cdot d\mathbf{S} = \frac{d}{dt}\{[D_x]_{(y,\,z)} \Delta y \, \Delta z\} = \frac{\partial [D_x]_{(y,\,z)}}{\partial t} \Delta y \, \Delta z \tag{3.21b}$$

where we have assumed that since the area is infinitesimal in size, J_x and D_x are uniform over the area and equal to their values at (y, z).

Substituting (3.20), (3.21a), and (3.21b) into (3.18) to apply Ampere's circuital law to the rectangular path C under consideration, we get

$$-\{[H_y]_{z+\Delta z} - [H_y]_z\} \, \Delta y = \left[J_x + \frac{\partial D_x}{\partial t} \right]_{(y,\,z)} \Delta y \, \Delta z$$

or

$$\frac{[H_y]_{z+\Delta z} - [H_y]_z}{\Delta z} = -\left[J_x + \frac{\partial D_x}{\partial t} \right]_{(y,\,z)} \tag{3.22}$$

If we now let the rectangular path shrink to the point (y, z) by letting Δy and Δz tend to zero, we obtain

$$\lim_{\substack{\Delta y \to 0 \\ \Delta z \to 0}} \frac{[H_y]_{z+\Delta z} - [H_y]_z}{\Delta z} = -\lim_{\substack{\Delta y \to 0 \\ \Delta z \to 0}} \left[J_x + \frac{\partial D_x}{\partial t} \right]_{(y,\,z)}$$

or

$$\frac{\partial H_y}{\partial z} = -J_x - \frac{\partial D_x}{\partial t} \tag{3.23}$$

Equation (3.23) is Ampere's circuital law in differential form for the simple case of $\mathbf{H}$ given by (3.19). It relates the variation of H_y with z (space) at a point to the current density J_x and to the variation of D_x with t (time) at that point. Since the above derivation can be carried out for any arbitrary point (x, y, z), it is valid at all points. It tells us in particular that a current density J_x or a time-varying D_x or a nonzero combination of the two quantities at a point results in an H_y at that point having a differential in the z-direction. This is to be expected since if this is not the case, $\oint \mathbf{H} \cdot d\mathbf{l}$ around the infinitesimal rectangular path would be zero.

Example 3.3

Given $\mathbf{E} = E_0 z \sin \omega t \, \mathbf{a}_x$ and it is known that $\mathbf{J}$ is zero and $\mathbf{B}$ has a y-component only, let us find B_y.

From (3.23), we have

$$\frac{\partial H_y}{\partial z} = -J_x - \frac{\partial D_x}{\partial t} = 0 - \frac{\partial}{\partial t}(\epsilon_0 E_0 z \sin \omega t) = -\omega\epsilon_0 E_0 z \cos \omega t$$

$$H_y = -\omega\epsilon_0 E_0 \frac{z^2}{2} \cos \omega t$$

$$B_y = \mu_0 H_y = -\omega\mu_0\epsilon_0 E_0 \frac{z^2}{2} \cos \omega t$$

We note that the electric field varying linearly with z gives rise to a magnetic field proportional to z^2. In Example 3.1, however, an electric field varying linearly with z was found to result from a uniform magnetic field, according to Faraday's law in differential form. The inconsistency of these two results implies that neither the combination of E_x and B_y in Example 3.1 nor the combination of E_x and B_y in this example simultaneously satisfies the two Maxwell's equations in differential form given by (3.7) and (3.23). The pair of E_x and B_y in Example 3.1 satisfies only (3.7), whereas the pair of E_x and B_y in this example satisfies only (3.23). In the following chapter we shall find a pair of solutions for E_x and B_y that simultaneously satisfies the two Maxwell's equations.

Example 3.4

Let us consider the current distribution given by

$$\mathbf{J} = J_0 \mathbf{a}_x \quad \text{for } -a < z < a$$

as shown in Figure 3.6(a), where J_0 is a constant, and find the magnetic field everywhere.

Since the current density is independent of x and y, the field is also independent of x and y. Also, since the current density is not a function of time, the field is static. Hence, $(\partial D_x/\partial t) = 0$, and we have

$$\frac{\partial H_y}{\partial z} = -J_x$$

Integrating both sides with respect to z, we obtain

$$H_y = -\int_{-\infty}^{z} J_x \, dz + C$$

where C is the constant of integration.

The variation of J_x with z is shown in Figure 3.6(b). Integrating $-J_x$ with respect to z, that is, finding the area under the curve of Figure 3.6(b) as a function of z, and taking its negative, we obtain the result shown by the dashed curve in Figure 3.6(c) for $-\int_{-\infty}^{z} J_x \, dz$. From symmetry considerations, the field must be equal and opposite on either side of the current region $-a < z < a$. Hence, we choose the constant of integration C to be equal to $J_0 a$, thereby obtaining the final result for H_y as shown by the solid curve in Figure 3.6(c). Thus, the magnetic field intensity due to the current distribution is given by

$$\mathbf{H} = \begin{cases} J_0 a\mathbf{a}_y & \text{for } z < -a \\ -J_0 z\mathbf{a}_y & \text{for } -a < z < a \\ -J_0 a\mathbf{a}_y & \text{for } z > a \end{cases}$$

The magnetic flux density, $\mathbf{B}$, is equal to $\mu_0\mathbf{H}$.

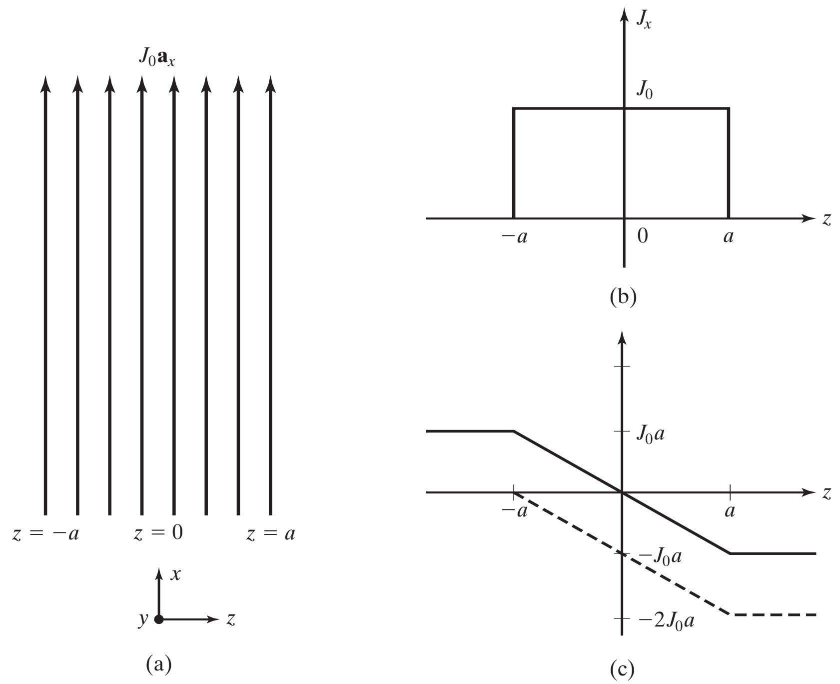

FIGURE 3.6

The determination of magnetic field due to a current distribution.

We now generalize (3.23) for the arbitrary case of a magnetic field having all three components, each of them depending on all three coordinates, in addition to t, that is,

$$\mathbf{H} = H_x(x, y, z, t)\mathbf{a}_x + H_y(x, y, z, t)\mathbf{a}_y + H_z(x, y, z, t)\mathbf{a}_z \tag{3.24}$$

We do this in exactly the same manner as for the case of Faraday's law by considering the three infinitesimal rectangular paths shown in Figure 3.3. Applying Ampere's circuital law to each of the three paths and simplifying, we obtain

$$\frac{[H_z]_{(x,\,y+\Delta y)} - [H_z]_{(x,\,y)}}{\Delta y} - \frac{[H_y]_{(x,\,z+\Delta z)} - [H_y]_{(x,\,z)}}{\Delta z} = \left[J_x + \frac{\partial D_x}{\partial t}\right]_{(x,\,y,\,z)} \tag{3.25a}$$

$$\frac{[H_x]_{(y,\,z+\Delta z)} - [H_x]_{(y,\,z)}}{\Delta z} - \frac{[H_z]_{(x+\Delta x,\,y)} - [H_z]_{(x,\,y)}}{\Delta x} = \left[J_y + \frac{\partial D_y}{\partial t}\right]_{(x,\,y,\,z)} \tag{3.25b}$$

$$\frac{[H_y]_{(x+\Delta x,\,z)} - [H_y]_{(x,\,z)}}{\Delta x} - \frac{[H_x]_{(y+\Delta y,\,z)} - [H_x]_{(y,\,z)}}{\Delta y} = \left[J_z + \frac{\partial D_z}{\partial t}\right]_{(x,\,y,\,z)} \tag{3.25c}$$

If we now let all three paths shrink to the point a by letting Δx, Δy, and Δz tend to zero, (3.25a)–(3.25c) reduce to

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = J_x + \frac{\partial D_x}{\partial t} \tag{3.26a}$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = J_y + \frac{\partial D_y}{\partial t} \tag{3.26b}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_z + \frac{\partial D_z}{\partial t} \tag{3.26c}$$

Equations (3.26a)–(3.26c) are the differential equations governing the relationships between the space variations of the magnetic field components, the components of the current density and the time variations of the electric field components, at a point. They can be interpreted physically in a manner analogous to the interpretation of (3.12a)–(3.12c) in the case of Faraday's law.

Equations (3.26a)–(3.26c) can be combined into a single vector equation in determinant form as given by

$$\begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{3.27}$$

or

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{3.28}$$

Equation (3.28) is Maxwell's equation in differential form corresponding to Ampere's circuital law. The quantity $\partial\mathbf{D}/\partial t$ is known as the *displacement current density.* We shall discuss curl further in the following section.

3.3 CURL AND STOKES' THEOREM

In Sections 3.1 and 3.2 we derived the differential forms of Faraday's and Ampere's circuital laws from their integral forms. These differential forms involve a new vector quantity, namely, the *curl* of a vector. In this section we shall introduce the basic definition of curl and then present a physical interpretation of the curl. In order to do this, let us, for simplicity, consider Ampere's circuital law in differential form without the displacement current density term, that is,

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{3.29}$$

We wish to express $\nabla \times \mathbf{H}$ at a point in the current region in terms of $\mathbf{H}$ at that point. If we consider an infinitesimal surface $\Delta\mathbf{S}$ at the point and take the dot product of both sides of (3.29) with $\Delta\mathbf{S}$, we get

$$(\nabla \times \mathbf{H}) \cdot \Delta\mathbf{S} = \mathbf{J} \cdot \Delta\mathbf{S} \tag{3.30}$$

But $\mathbf{J} \cdot \Delta\mathbf{S}$ is simply the current crossing the surface $\Delta\mathbf{S}$, and according to Ampere's circuital law in integral form without the displacement current term,

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \mathbf{J} \cdot \Delta\mathbf{S} \tag{3.31}$$

where C is the closed path bounding $\Delta\mathbf{S}$. Comparing (3.30) and (3.31), we have

$$(\nabla \times \mathbf{H}) \cdot \Delta\mathbf{S} = \oint_C \mathbf{H} \cdot d\mathbf{l}$$

or

$$(\nabla \times \mathbf{H}) \cdot \Delta S\, \mathbf{a}_n = \oint_C \mathbf{H} \cdot d\mathbf{l} \tag{3.32}$$

where $\mathbf{a}_n$ is the unit vector normal to ΔS and directed toward the side of advance of a right-hand screw as it is turned around C. Dividing both sides of (3.32) by ΔS, we obtain

$$(\nabla \times \mathbf{H}) \cdot \mathbf{a}_n = \frac{\oint_C \mathbf{H} \cdot d\mathbf{l}}{\Delta S} \tag{3.33}$$

The maximum value of $(\nabla \times \mathbf{H}) \cdot \mathbf{a}_n$, and hence that of the right side of (3.33), occurs when $\mathbf{a}_n$ is oriented parallel to $\nabla \times \mathbf{H}$, that is, when the surface ΔS is oriented normal to the current density vector $\mathbf{J}$. This maximum value is simply $|\nabla \times \mathbf{H}|$. Thus,

$$|\nabla \times \mathbf{H}| = \left[\frac{\oint_C \mathbf{H} \cdot d\mathbf{l}}{\Delta S}\right]_{\max} \tag{3.34}$$

Since the direction of $\nabla \times \mathbf{H}$ is the direction of $\mathbf{J}$, or that of the unit vector normal to ΔS, we can then write

$$\nabla \times \mathbf{H} = \left[\frac{\oint_C \mathbf{H} \cdot d\mathbf{l}}{\Delta S}\right]_{\max} \mathbf{a}_n \tag{3.35}$$

Equation (3.35) is only approximate since (3.32) is exact only in the limit that ΔS tends to zero. Thus,

$$\nabla \times \mathbf{H} = \lim_{\Delta S \to 0}\left[\frac{\oint_C \mathbf{H} \cdot d\mathbf{l}}{\Delta S}\right]_{\max} \mathbf{a}_n \tag{3.36}$$

Equation (3.36) is the expression for $\nabla \times \mathbf{H}$ at a point in terms of $\mathbf{H}$ at that point. Although we have derived this for the $\mathbf{H}$ vector, it is a general result and, in fact, is often the starting point for the introduction of curl.

Equation (3.36) tells us that in order to find the curl of a vector at a point in that vector field, we first consider an infinitesimal surface at that point and compute the closed line integral or circulation of the vector around the periphery of this surface by orienting the surface such that the circulation is maximum. We then divide the circulation by the area of the surface to obtain the maximum value of the circulation per unit area. Since we need this maximum value of the circulation per unit area in the limit that the area tends to zero, we do this by gradually shrinking the area and making sure that each time we compute the circulation per unit area an orientation for the area that maximizes this quantity is maintained. The limiting value to which the maximum circulation per unit area approaches is the magnitude of the curl. The limiting direction to which the normal vector to the surface approaches is the direction of the curl. The task of computing the curl is simplified if we consider one component of the field at a time and compute the curl corresponding to that component since then it is sufficient if we always maintain the orientation of the surface normal to that component axis. In fact, this is what we did in Sections 3.1 and 3.2, which led us to the determinant form of curl.

We are now ready to discuss the physical interpretation of the curl. We do this with the aid of a simple device known as the *curl meter*. Although the curl meter may take several forms, we shall consider one consisting of a circular disc that floats in water with a paddle wheel attached to the bottom of the disc, as shown in Figure 3.7. A dot at the periphery on top of the disc serves to indicate any rotational motion of the curl meter about its axis, that is, the axis of the paddle wheel. Let us now consider a stream of rectangular cross section carrying water in the z-direction, as shown in Figure 3.7(a). Let us assume the velocity $\mathbf{v}$ of the water to be independent of height but increasing uniformly from a value of zero at the banks to a maximum value v_0 at the center, as shown in Figure 3.7(b), and investigate the behavior of the curl meter when it is placed vertically at different points in the stream. We assume that the size of the curl meter is vanishingly small so that it does not disturb the flow of water as we probe its behavior at different points.

Since exactly in midstream the blades of the paddle wheel lying on either side of the center line are hit by the same velocities, the paddle wheel does not rotate. The curl meter simply slides down the stream without any rotational motion, that is, with the dot on top of the disc maintaining the same position relative to the center of the disc, as shown in Figure 3.7(c). At a point to the left of the midstream the blades of the paddle wheel are hit by a greater velocity on the right side than on the left side so that the paddle wheel rotates in the counterclockwise sense. The curl meter rotates in the counterclockwise direction about its axis as it slides down the stream, as indicated by the changing position of the dot on top of the disc relative to the center of the disc, as shown in Figure 3.7(d). At a point to the right of midstream, the blades of the paddle wheel are hit by a greater velocity on the left side than on the right side so that the paddle wheel rotates in the clockwise sense. The curl meter rotates in the clockwise direction about its axis as it slides down the stream, as indicated by the changing position of the dot on top of the disc relative to the center of the disc, as shown in Figure 3.7(e).

To relate the foregoing discussion of the behavior of the curl meter with the curl of the velocity vector field of the water flow, we note that at a point in midstream, the

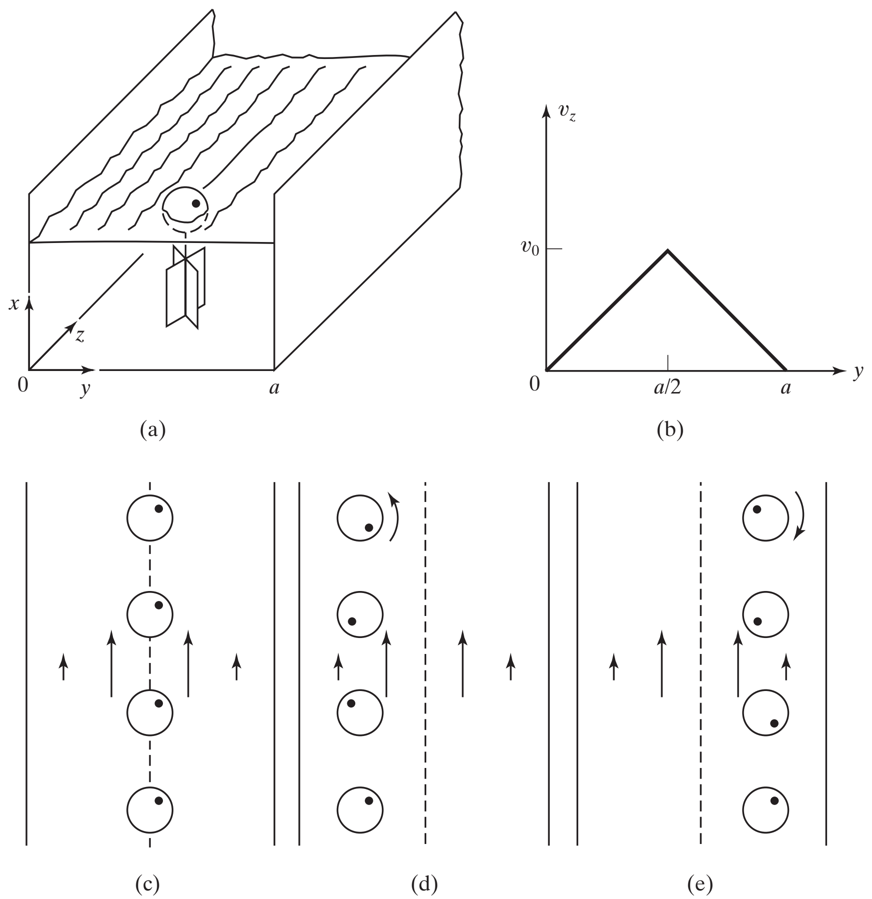

FIGURE 3.7

For explaining the physical interpretation of curl using the curl meter.

circulation of the velocity vector per unit area in the plane normal to the axis of the paddle wheel, that is, parallel to the surface of the stream, is zero and hence the component of the curl along that axis, that is, in the x-direction, is zero. At points on either side of midstream, however, the circulation per unit area is not zero in view of the velocity differential along the y-direction. Hence, the x-component of the curl is nonzero at these points. Furthermore, the x-component of the curl at points on the right side of midstream is opposite in sign to that on the left side of midstream, since the velocity differentials are opposite in sign. These properties are exactly similar to those of the rotational motion of the curl meter.

If we now pick up the curl meter and insert it in the water with its axis parallel to the surface of the stream, the curl meter does not rotate, because its blades are hit with the same force on either side of its axis. This behavior of the curl meter is akin to the property that the horizontal component of the curl of the velocity vector is zero, since the velocity differential along the x-direction is zero.

The foregoing illustration of the physical interpretation of the curl of a vector field can be used to visualize the behavior of electric and magnetic fields. Thus, for example, from

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

we know that at a point in an electromagnetic field at which $\partial\mathbf{B}/\partial t$ is nonzero, there exists an electric field with nonzero circulation per unit area in the plane normal to the vector $\partial\mathbf{B}/\partial t$. Similarly, from

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

we know that at a point in an electromagnetic field at which $\mathbf{J} + \partial\mathbf{D}/\partial t$ is nonzero, there exists a magnetic field with nonzero circulation per unit area in the plane normal to the vector $\mathbf{J} + \partial\mathbf{D}/\partial t$.

We shall now derive a useful theorem in vector calculus, the *Stokes' theorem*. This relates the closed line integral of a vector field to the surface integral of the curl of that vector field. To derive this theorem, let us consider an arbitrary surface S in a magnetic field region and divide this surface into a number of infinitesimal surfaces $\Delta S_1, \Delta S_2, \Delta S_3, \ldots,$ bounded by the contours $C_1, C_2, C_3, \ldots,$ respectively. Then, applying (3.32) to each one of these infinitesimal surfaces and adding up, we get

$$\sum_j (\nabla \times \mathbf{H})_j \cdot \Delta S_j\, \mathbf{a}_{nj} = \oint_{C_1} \mathbf{H} \cdot d\mathbf{l} + \oint_{C_2} \mathbf{H} \cdot d\mathbf{l} + \cdots \tag{3.37}$$

where $\mathbf{a}_{nj}$ are unit vectors normal to the surfaces ΔS_j chosen in accordance with the right-hand screw rule. In the limit that the number of infinitesimal surfaces tends to infinity, the left side of (3.37) approaches to the surface integral of $\nabla \times \mathbf{H}$ over the surface S. The right side of (3.37) is simply the closed line integral of $\mathbf{H}$ around the contour C since the contributions to the line integrals from the portions of the contours interior to C cancel, as shown in Figure 3.8. Thus, we get

$$\int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S} = \oint_C \mathbf{H} \cdot d\mathbf{l} \tag{3.38}$$

Equation (3.38) is Stokes' theorem. Although we have derived it by considering the $\mathbf{H}$ field, it is general and is applicable for any vector field.

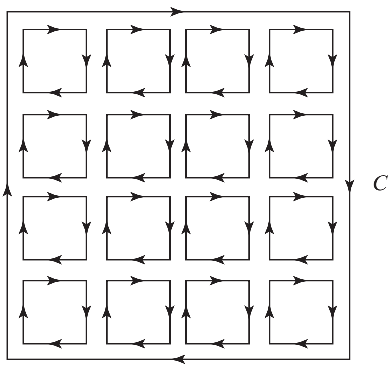

FIGURE 3.8

For deriving Stokes' theorem.

Example 3.5

Let us verify Stokes' theorem by considering

$$\mathbf{A} = y\mathbf{a}_x - x\mathbf{a}_y$$

and the closed path C shown in Figure 3.9.

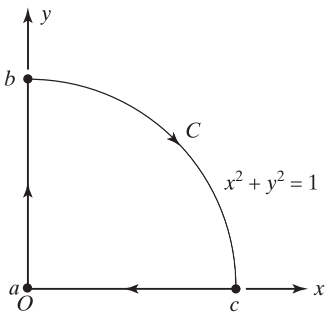

FIGURE 3.9
A closed path for verifying Stokes' theorem.

We first determine $\oint_C \mathbf{A} \cdot d\mathbf{l}$ by evaluating the line integrals along the three segments of the closed path. To do this, we first note that $\mathbf{A} \cdot d\mathbf{l} = y\,dx - x\,dy$. Then, from a to b, $x = 0$, $dx = 0$, $\mathbf{A} \cdot d\mathbf{l} = 0$

$$\int_a^b \mathbf{A} \cdot d\mathbf{l} = 0$$

From b to c, $x^2 + y^2 = 1$, $y = \sqrt{1 - x^2}$

$$2x\,dx + 2y\,dy = 0, \qquad dy = -\frac{x\,dx}{y} = -\frac{x}{\sqrt{1 - x^2}}\,dx$$

$$\mathbf{A} \cdot d\mathbf{l} = \sqrt{1 - x^2}\,dx + \frac{x^2\,dx}{\sqrt{1 - x^2}} = \frac{dx}{\sqrt{1 - x^2}}$$

$$\int_b^c \mathbf{A} \cdot d\mathbf{l} = \int_0^1 \frac{dx}{\sqrt{1 - x^2}} = \left[\sin^{-1} x\right]_0^1 = \frac{\pi}{2}$$

From c to a, $y = 0$, $dy = 0$, $\mathbf{A} \cdot d\mathbf{l} = 0$

$$\int_c^a \mathbf{A} \cdot d\mathbf{l} = 0$$

Thus,

$$\oint_C \mathbf{A} \cdot d\mathbf{l} = \int_a^b \mathbf{A} \cdot d\mathbf{l} + \int_b^c \mathbf{A} \cdot d\mathbf{l} + \int_c^a \mathbf{A} \cdot d\mathbf{l}$$

$$= 0 + \frac{\pi}{2} + 0 = \frac{\pi}{2}$$

Now, to evaluate $\oint_C \mathbf{A} \cdot d\mathbf{l}$ by using Stokes' theorem, we recall from Example 3.2 that

$$\nabla \times \mathbf{A} = \nabla \times (y\mathbf{a}_x - x\mathbf{a}_y) = -2\mathbf{a}_z$$

For the plane surface S enclosed by C,

$$d\mathbf{S} = -dx\,dy\,\mathbf{a}_z$$

Thus,

$$(\nabla \times \mathbf{A}) \cdot d\mathbf{S} = -2\mathbf{a}_z \cdot (-dx\,dy\,\mathbf{a}_z) = 2\,dx\,dy$$

$$\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} 2\,dx\,dy$$

$$= 2(\text{area enclosed by } C) = 2 \times \frac{\pi}{4} = \frac{\pi}{2}$$

thereby verifying Stokes' theorem.

3.4 GAUSS' LAW FOR THE ELECTRIC FIELD

Thus far we have derived Maxwell's equations in differential form corresponding to the two Maxwell's equations in integral form involving the line integrals of **E** and **H**, that is, Faraday's law and Ampere's circuital law, respectively. The remaining two Maxwell's equations in integral form, namely, Gauss' law for the electric field and Gauss' law for the magnetic field, are concerned with the closed surface integrals of **D** and **B**, respectively. We shall in this and the following sections derive the differential forms of these two equations.

We recall from Section 2.5 that Gauss' law for the electric field is given by

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho\,dv \tag{3.39}$$

where V is the volume enclosed by the closed surface S. To derive the differential form of this equation, let us consider a rectangular box of infinitesimal sides Δx, Δy, and Δz and defined by the six surfaces $x = x$, $x = x + \Delta x$, $y = y$, $y = y + \Delta y$, $z = z$, and $z = z + \Delta z$, as shown in Figure 3.10, in a region of electric field

$$\mathbf{D} = D_x(x, y, z, t)\mathbf{a}_x + D_y(x, y, z, t)\mathbf{a}_y + D_z(x, y, z, t)\mathbf{a}_z \tag{3.40}$$

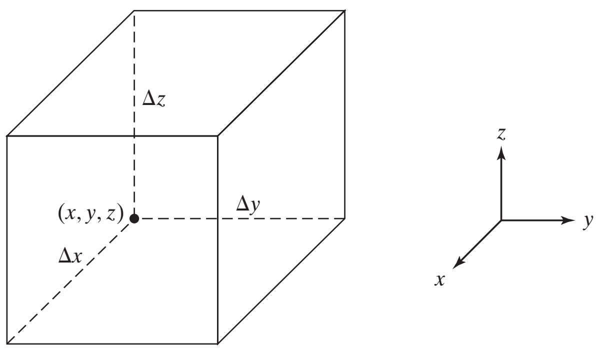

FIGURE 3.10

An infinitesimal rectangular box.

and charge of density $\rho(x, y, z, t)$. According to Gauss' law for the electric field, the displacement flux emanating from the box is equal to the charge enclosed by the box. The displacement flux is given by the surface integral of **D** over the surface of the box, which is comprised of six plane surfaces. Thus, evaluating the displacement flux emanating out of the box over each of the six plane surfaces of the box, we have

$$\int \mathbf{D} \cdot d\mathbf{S} = -[D_x]_x \, \Delta y \, \Delta z \qquad \text{for the surface } x = x \tag{3.41a}$$

$$\int \mathbf{D} \cdot d\mathbf{S} = [D_x]_{x+\Delta x} \, \Delta y \, \Delta z \qquad \text{for the surface } x = x + \Delta x \tag{3.41b}$$

$$\int \mathbf{D} \cdot d\mathbf{S} = -[D_y]_y \, \Delta z \, \Delta x \qquad \text{for the surface } y = y \tag{3.41c}$$

$$\int \mathbf{D} \cdot d\mathbf{S} = [D_y]_{y+\Delta y} \, \Delta z \, \Delta x \qquad \text{for the surface } y = y + \Delta y \tag{3.41d}$$

$$\int \mathbf{D} \cdot d\mathbf{S} = -[D_z]_z \, \Delta x \, \Delta y \qquad \text{for the surface } z = z \tag{3.41e}$$

$$\int \mathbf{D} \cdot d\mathbf{S} = [D_z]_{z+\Delta z} \, \Delta x \, \Delta y \qquad \text{for the surface } z = z + \Delta z \tag{3.41f}$$

Adding up (3.41a)–(3.41f), we obtain the total displacement flux emanating from the box to be

$$\begin{aligned} \oint_S \mathbf{D} \cdot d\mathbf{S} = {} & \{[D_x]_{x+\Delta x} - [D_x]_x\} \, \Delta y \, \Delta z \\ & + \{[D_y]_{y+\Delta y} - [D_y]_y\} \, \Delta z \, \Delta x \\ & + \{[D_z]_{z+\Delta z} - [D_z]_z\} \, \Delta x \, \Delta y \end{aligned} \tag{3.42}$$

Now the charge enclosed by the rectangular box is given by

$$\int_V \rho \, dv = \rho(x, y, z, t) \cdot \Delta x \, \Delta y \, \Delta z = \rho \, \Delta x \, \Delta y \, \Delta z \tag{3.43}$$

where we have assumed ρ to be uniform throughout the volume of the box and equal to its value at (x, y, z), since the box is infinitesimal in volume.

Substituting (3.42) and (3.43) into (3.39) to apply Gauss' law for the electric field to the surface of the box under consideration, we get

$$\begin{aligned} \{[D_x]_{x+\Delta x} - [D_x]_x\} \, \Delta y \, \Delta z + \{[D_y]_{y+\Delta y} - [D_y]_y\} \, \Delta z \, \Delta x \\ + \{[D_z]_{z+\Delta z} - [D_z]_z\} \, \Delta x \, \Delta y = \rho \, \Delta x \, \Delta y \, \Delta z \end{aligned}$$

or

$$\frac{[D_x]_{x+\Delta x} - [D_x]_x}{\Delta x} + \frac{[D_y]_{y+\Delta y} - [D_y]_y}{\Delta y} + \frac{[D_z]_{z+\Delta z} - [D_z]_z}{\Delta z} = \rho \tag{3.44}$$

If we now let the box shrink to the point (x, y, z) by letting Δx, Δy, and Δz tend to zero, we obtain

$$\lim_{\Delta x \to 0} \frac{[D_x]_{x+\Delta x} - [D_x]_x}{\Delta x} + \lim_{\Delta y \to 0} \frac{[D_y]_{y+\Delta y} - [D_y]_y}{\Delta y} + \lim_{\Delta z \to 0} \frac{[D_z]_{z+\Delta z} - [D_z]_z}{\Delta z} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0 \\ \Delta z \to 0}} \rho$$

or

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \rho \tag{3.45}$$

Equation (3.45) tells us that the net longitudinal differential of the components of **D**, that is, the algebraic sum of the derivatives of the components of **D** along their respective directions is equal to the charge density at that point. Conversely, a charge density at a point results in an electric field, having components of **D** such that their net longitudinal differential is nonzero. An example in which the net longitudinal differential is zero although some of the individual derivatives are nonzero is shown in Figure 3.11(a). Figure 3.11(b) shows an example in which the net longitudinal differential is nonzero. Equation (3.45) can be written in vector notation as

$$\left(\mathbf{a}_x \frac{\partial}{\partial x} + \mathbf{a}_y \frac{\partial}{\partial y} + \mathbf{a}_z \frac{\partial}{\partial z}\right) \cdot (D_x \mathbf{a}_x + D_y \mathbf{a}_y + D_z \mathbf{a}_z) = \rho \tag{3.46}$$

The left side of (3.46) is known as the *divergence of* **D**, denoted as $\nabla \cdot \mathbf{D}$ (del dot **D**). Thus, we have

$$\nabla \cdot \mathbf{D} = \rho \tag{3.47}$$

Equation (3.47) is Maxwell's equation in differential form corresponding to Gauss' law for the electric field. We shall discuss divergence further in Section 3.6.

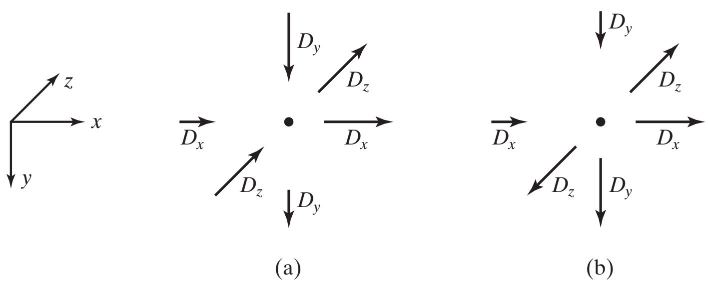

FIGURE 3.11

For illustrating (a) zero, and (b) nonzero net longitudinal differential of the components of **D**.

Example 3.6

Given $\mathbf{A} = 3x\mathbf{a}_x + (y - 3)\mathbf{a}_y + (2 - z)\mathbf{a}_z$, find $\nabla \cdot \mathbf{A}$.

From the expansion for the divergence of a vector, we have

$$\begin{aligned}\nabla \cdot \mathbf{A} &= \left(\mathbf{a}_x \frac{\partial}{\partial x} + \mathbf{a}_y \frac{\partial}{\partial y} + \mathbf{a}_z \frac{\partial}{\partial z}\right) \cdot [3x\mathbf{a}_x + (y - 3)\mathbf{a}_y + (2 - z)\mathbf{a}_z] \\ &= \frac{\partial}{\partial x}(3x) + \frac{\partial}{\partial y}(y - 3) + \frac{\partial}{\partial z}(2 - z) \\ &= 3 + 1 - 1 = 3\end{aligned}$$

Example 3.7

Let us consider the charge distribution given by

$$\rho = \begin{cases} -\rho_0 & \text{for } -a < x < 0 \\ \rho_0 & \text{for } 0 < x < a \end{cases}$$

as shown in Figure 3.12(a), where ρ_0 is a constant, and find the electric field everywhere.

Since the charge density is independent of y and z, the field is also independent of y and z, thereby giving us $\partial D_y/\partial y = \partial D_z/\partial z = 0$ and reducing Gauss' law for the electric field to

$$\frac{\partial D_x}{\partial x} = \rho$$

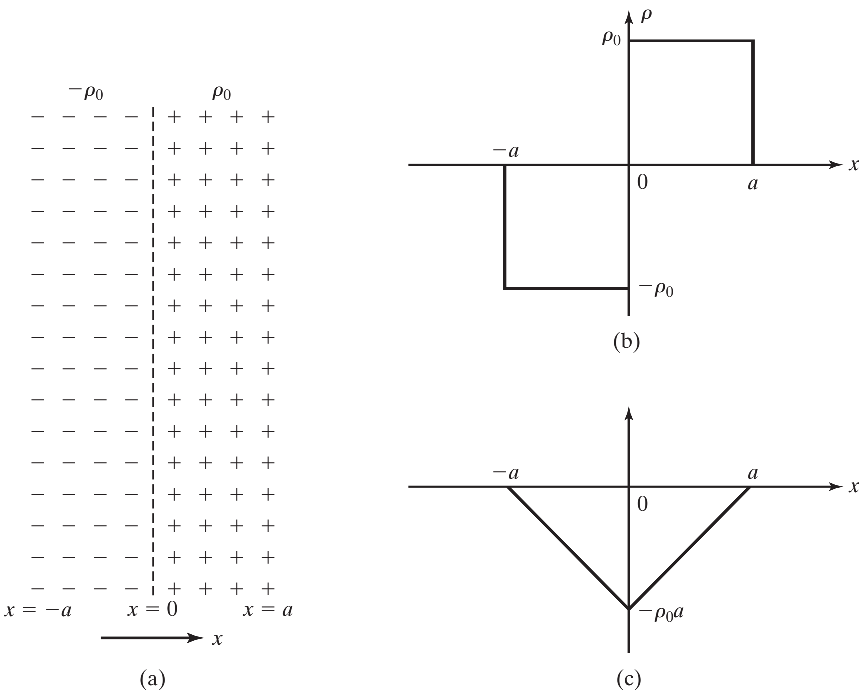

FIGURE 3.12

The determination of electric field due to a charge distribution.

Integrating both sides with respect to x, we obtain

$$D_x = \int_{-\infty}^{x} \rho \, dx + C$$

where C is the constant of integration.

The variation of ρ with x is shown in Figure 3.12(b). Integrating ρ with respect to x, that is, finding the area under the curve of Figure 3.12(b) as a function of x, we obtain the result shown in Figure 3.12(c) for $\int_{-\infty}^{x} \rho \, dx$. The constant of integration C is zero since the symmetry of equal and opposite fields on the two sides of the charge distribution, considered as a superposition of a series of thin slabs of charge, is already satisfied by the plot of Figure 3.12(c). Thus, the displacement flux density due to the charge distribution is given by

$$\mathbf{D} = \begin{cases} 0 & \text{for } x < -a \\ -\rho_0(x + a)\mathbf{a}_x & \text{for } -a < x < 0 \\ \rho_0(x - a)\mathbf{a}_x & \text{for } 0 < x < a \\ 0 & \text{for } x > a \end{cases}$$

The electric field intensity, $\mathbf{E}$, is equal to $\mathbf{D}/\epsilon_0$.

3.5 GAUSS' LAW FOR THE MAGNETIC FIELD

In the previous section we derived the differential form of Gauss' law for the electric field from its integral form. In this section we shall derive the differential form of Gauss' law for the magnetic field from its integral form. We recall from Section 2.6 that Gauss' law for the magnetic field in integral form is given by

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0 \tag{3.48}$$

where S is any closed surface. This equation states that the magnetic flux emanating from a closed surface is zero. Thus, considering an infinitesimal rectangular box as shown in Figure 3.10 in a region of magnetic field

$$\mathbf{B} = B_x(x, y, z, t)\mathbf{a}_x + B_y(x, y, z, t)\mathbf{a}_y + B_z(x, y, z, t)\mathbf{a}_z \tag{3.49}$$

and evaluating the magnetic flux emanating out of the box in a manner similar to that of the evaluation of the displacement flux in the previous section, and substituting in (3.48), we obtain

$$\begin{aligned} \{[B_x]_{x+\Delta x} - [B_x]_x\}\Delta y \, \Delta z + \{[B_y]_{y+\Delta y} - [B_y]_y\} \, \Delta z \, \Delta x \\ + \{[B_z]_{z+\Delta z} - [B_z]_z\} \, \Delta x \, \Delta y = 0 \end{aligned} \tag{3.50}$$

Dividing (3.50) on both sides by $\Delta x \, \Delta y \, \Delta z$ and letting Δx, Δy, and Δz tend to zero, thereby shrinking the box to the point (x, y, z), we obtain

$$\lim_{\Delta x \to 0} \frac{[B_x]_{x+\Delta x} - [B_x]_x}{\Delta x} + \lim_{\Delta y \to 0} \frac{[B_y]_{y+\Delta y} - [B_y]_y}{\Delta y} + \lim_{\Delta z \to 0} \frac{[B_z]_{z+\Delta z} - [B_z]_z}{\Delta z} = 0$$

or

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 \tag{3.51}$$

Equation (3.51) tells us that the net longitudinal differential of the components of **B** is zero. In vector form it is given by

$$\nabla \cdot \mathbf{B} = 0 \tag{3.52}$$

Equation (3.52) is Maxwell's equation in differential form corresponding to Gauss' law for the magnetic field. We shall discuss divergence further in the following section.

Example 3.8

Determine if the vector $\mathbf{A} = y\mathbf{a}_x - x\mathbf{a}_y$ can represent a magnetic field **B**.

From (3.52), we note that a given vector can be realized as a magnetic field **B** if its divergence is zero. For $\mathbf{A} = y\mathbf{a}_x - x\mathbf{a}_y$,

$$\nabla \cdot \mathbf{A} = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(-x) + \frac{\partial}{\partial z}(0) = 0$$

Hence, the given vector can represent a magnetic field **B**.

3.6 DIVERGENCE AND THE DIVERGENCE THEOREM

In Sections 3.4 and 3.5 we derived the differential forms of Gauss' laws for the electric and magnetic fields from their integral forms. These differential forms involve a new quantity, namely, the *divergence* of a vector. The divergence of a vector is a scalar as compared to the vector nature of the curl of a vector. In this section we shall introduce the basic definition of divergence and then present a physical interpretation for the divergence. In order to do this, let us consider Gauss' law for the electric field in differential form, that is,

$$\nabla \cdot \mathbf{D} = \rho \tag{3.53}$$

We wish to express $\nabla \cdot \mathbf{D}$ at a point in the charge region in terms of **D** at that point. If we consider an infinitesimal volume Δv at the point and multiply both sides of (3.53) by Δv, we get

$$(\nabla \cdot \mathbf{D})\, \Delta v = \rho\, \Delta v \tag{3.54}$$

But $\rho\, \Delta v$ is simply the charge contained in the volume Δv, and according to Gauss' law for the electric field in integral form,

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \rho\, \Delta v \tag{3.55}$$

where S is the closed surface bounding Δv. Comparing (3.54) and (3.55), we have

$$(\nabla \cdot \mathbf{D})\,\Delta v = \oint_S \mathbf{D} \cdot d\mathbf{S} \tag{3.56}$$

Dividing both sides of (3.56) by Δv, we obtain

$$\nabla \cdot \mathbf{D} = \frac{\oint_S \mathbf{D} \cdot d\mathbf{S}}{\Delta v} \tag{3.57}$$

Equation (3.57) is only approximate since (3.56) is exact only in the limit that Δv tends to zero. Thus,

$$\nabla \cdot \mathbf{D} = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{D} \cdot d\mathbf{S}}{\Delta v} \tag{3.58}$$

Equation (3.58) is the expression for $\nabla \cdot \mathbf{D}$ at a point in terms of $\mathbf{D}$ at that point. Although we have derived this for the $\mathbf{D}$ vector, it is a general result and, in fact, is often the starting point for the introduction of divergence.

Equation (3.58) tells us that in order to find the divergence of a vector at a point in that vector field, we first consider an infinitesimal volume at that point and compute the surface integral of the vector over the surface bounding that volume, that is, the outward flux of the vector field emanating from that volume. We then divide the flux by the volume to obtain the flux per unit volume. Since we need this flux per unit volume in the limit that the volume tends to zero, we do this by gradually shrinking the volume. The limiting value to which the flux per unit volume approaches is the value of the divergence of the vector field at the point to which the volume is shrunk.

We are now ready to discuss the physical interpretation of the divergence. To simplify this task, we shall consider the differential form of the law of conservation of charge given in integral form by (2.39), or

$$\oint_S \mathbf{J} \cdot d\mathbf{S} = -\frac{d}{dt}\int_V \rho\, dv \tag{3.59}$$

where S is the surface bounding the volume V. Applying (3.59) to an infinitesimal volume Δv, we have

$$\oint_S \mathbf{J} \cdot d\mathbf{S} = -\frac{d}{dt}(\rho\,\Delta v) = -\frac{\partial \rho}{\partial t}\Delta v$$

or

$$\frac{\oint_S \mathbf{J} \cdot d\mathbf{S}}{\Delta v} = -\frac{\partial \rho}{\partial t} \tag{3.60}$$

Now taking the limit on both sides of (3.60) as Δv tends to zero, we obtain

$$\lim_{\Delta v \to 0} \frac{\oint \mathbf{J} \cdot d\mathbf{S}}{\Delta v} = \lim_{\Delta v \to 0} -\frac{\partial \rho}{\partial t} \tag{3.61}$$

or

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \tag{3.62}$$

or

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0 \tag{3.63}$$

Equation (3.63), which is the differential form of the law of conservation of charge, is familiarly known as the *continuity equation.* It tells us that the divergence of the current density vector at a point is equal to the time rate of decrease of the charge density at that point.

Let us now investigate three different cases: (a) positive value, (b) negative value, and (c) zero value of the time rate of decrease of the charge density at a point, that is, the divergence of the current density vector at that point. We shall do this with the aid of a simple device that we shall call the *divergence meter.* The divergence meter can be imagined to be a tiny, elastic balloon enclosing the point and that expands when hit by charges streaming outward from the point and contracts when acted upon by charges streaming inward toward the point. For case (a), that is, when the time rate of decrease of the charge density at the point is positive, there is a net amount of charge streaming out of the point in a given time, resulting in a net current flow outward from the point that will make the imaginary balloon expand. For case (b), that is, when the time rate of decrease of the charge density at the point is negative or the time rate of increase of the charge density is positive, there is a net amount of charge streaming toward the point in a given time, resulting in a net current flow toward the point and the imaginary balloon will contract. For case (c), that is, when the time rate of decrease of the charge density at the point is zero, the balloon will remain unaffected since the charge is streaming out of the point at exactly the same rate as it is streaming into the point. These three cases are illustrated in Figures 3.13(a), (b), and (c), respectively.

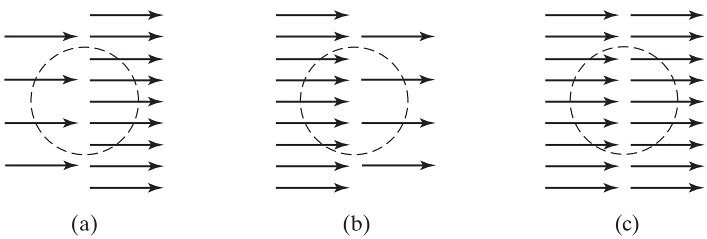

FIGURE 3.13

For explaining the physical interpretation of divergence using the divergence meter.

Generalizing the foregoing discussion to the physical interpretation of the divergence of any vector field at a point, we can imagine the vector field to be a velocity field of streaming charges acting upon the divergence meter and obtain in most cases a qualitative picture of the divergence of the vector field. If the divergence meter expands, the divergence is positive and a source of the flux of the vector field exists at that point. If the divergence meter contracts, the divergence is negative and a sink of the flux of the vector field exists at that point. If the divergence meter remains unaffected, the divergence is zero, and neither a source nor a sink of the flux of the vector field exists at that point. Alternatively, there can exist at the point pairs of sources and sinks of equal strengths.

We shall now derive a useful theorem in vector calculus, *the divergence theorem.* This relates the closed surface integral of the vector field to the volume integral of the divergence of that vector field. To derive this theorem, let us consider an arbitrary volume V in an electric field region and divide this volume into a number of infinitesimal volumes $\Delta v_1, \Delta v_2, \Delta v_3, \ldots$, bounded by the surfaces $S_1, S_2, S_3, \ldots$, respectively. Then, applying (3.56) to each one of these infinitesimal volumes and adding up, we get

$$\sum_j (\nabla \cdot \mathbf{D})_j \, \Delta v_j = \oint_{S_1} \mathbf{D} \cdot d\mathbf{S} + \oint_{S_2} \mathbf{D} \cdot d\mathbf{S} + \cdots \tag{3.64}$$

In the limit that the number of the infinitesimal volumes tends to infinity, the left side of (3.64) approaches to the volume integral of $\nabla \cdot \mathbf{D}$ over the volume V. The right side of (3.64) is simply the closed surface integral of $\mathbf{D}$ over S, since the contribution to the surface integrals from the portions of the surfaces interior to S cancel, as shown in Figure 3.14. Thus, we get

$$\int_V (\nabla \cdot \mathbf{D}) \, dv = \oint_S \mathbf{D} \cdot d\mathbf{S} \tag{3.65}$$

Equation (3.65) is the divergence theorem. Although we have derived it by considering the $\mathbf{D}$ field, it is general and is applicable for any vector field.

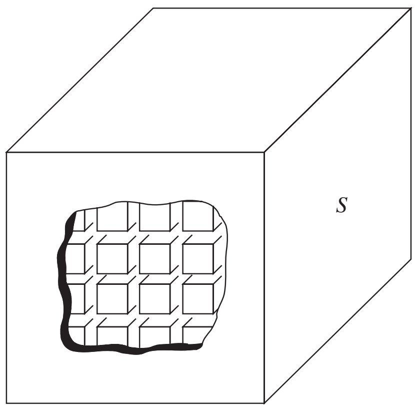

FIGURE 3.14

For deriving the divergence theorem.

Example 3.9

Let us verify the divergence theorem by considering

$$\mathbf{A} = 3x\mathbf{a}_x + (y - 3)\mathbf{a}_y + (2 - z)\mathbf{a}_z$$

and the closed surface of the box bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 2$, $z = 0$, and $z = 3$.

We first determine $\oint_S \mathbf{A} \cdot d\mathbf{S}$ by evaluating the surface integrals over the six surfaces of the rectangular box. Thus for the surface $x = 0$,

$$\mathbf{A} = (y - 3)\mathbf{a}_y + (2 - z)\mathbf{a}_z, \qquad d\mathbf{S} = -dy\,dz\,\mathbf{a}_x$$

$$\mathbf{A} \cdot d\mathbf{S} = 0$$

$$\int \mathbf{A} \cdot d\mathbf{S} = 0$$

For the surface $x = 1$,

$$\mathbf{A} = 3\mathbf{a}_x + (y - 3)\mathbf{a}_y + (2 - z)\mathbf{a}_z, \qquad d\mathbf{S} = dy\,dz\,\mathbf{a}_x$$

$$\mathbf{A} \cdot d\mathbf{S} = 3\,dy\,dz$$

$$\int \mathbf{A} \cdot d\mathbf{S} = \int_{z=0}^{3} \int_{y=0}^{2} 3\,dy\,dz = 18$$

For the surface $y = 0$,

$$\mathbf{A} = 3x\mathbf{a}_x - 3\mathbf{a}_y + (2 - z)\mathbf{a}_z, \qquad d\mathbf{S} = -dz\,dx\,\mathbf{a}_y$$

$$\mathbf{A} \cdot d\mathbf{S} = 3\,dz\,dx$$

$$\int \mathbf{A} \cdot d\mathbf{S} = \int_{x=0}^{1} \int_{z=0}^{3} 3\,dz\,dx = 9$$

For the surface $y = 2$,

$$\mathbf{A} = 3x\mathbf{a}_x - \mathbf{a}_y + (2 - z)\mathbf{a}_z, \qquad d\mathbf{S} = dz\,dx\,\mathbf{a}_y$$

$$\mathbf{A} \cdot d\mathbf{S} = -dz\,dx$$

$$\int \mathbf{A} \cdot d\mathbf{S} = \int_{x=0}^{1} \int_{z=0}^{3} -dz\,dx = -3$$

For the surface $z = 0$,

$$\mathbf{A} = 3x\mathbf{a}_x + (y - 3)\mathbf{a}_y + 2\mathbf{a}_z, \qquad d\mathbf{S} = -dx\,dy\,\mathbf{a}_z$$

$$\mathbf{A} \cdot d\mathbf{S} = -2\,dx\,dy$$

$$\int \mathbf{A} \cdot d\mathbf{S} = \int_{y=0}^{2} \int_{x=0}^{1} -2\,dx\,dy = -4$$

For the surface $z = 3$,

$$\mathbf{A} = 3x\mathbf{a}_x + (y - 3)\mathbf{a}_y - \mathbf{a}_z, \qquad d\mathbf{S} = dx\,dy\,\mathbf{a}_z$$

$$\mathbf{A} \cdot d\mathbf{S} = -dx\,dy$$

$$\int \mathbf{A} \cdot d\mathbf{S} = \int_{y=0}^{2} \int_{x=0}^{1} -dx\,dy = -2$$

Thus,

$$\oint_S \mathbf{A} \cdot d\mathbf{S} = 0 + 18 + 9 - 3 - 4 - 2 = 18$$

Now, to evaluate $\oint_S \mathbf{A} \cdot d\mathbf{S}$ by using the divergence theorem, we recall from Example 3.6 that

$$\nabla \cdot \mathbf{A} = \nabla \cdot [3x\mathbf{a}_x + (y - 3)\mathbf{a}_y + (2 - z)\mathbf{a}_z] = 3$$

For the volume enclosed by the rectangular box,

$$\int (\nabla \cdot \mathbf{A})\, dv = \int_{z=0}^{3}\int_{y=0}^{2}\int_{x=0}^{1} 3\,dx\,dy\,dz = 18$$

thereby verifying the divergence theorem.

SUMMARY

We have in this chapter derived the differential forms of Maxwell's equations from their integral forms, which we introduced in the previous chapter. For the general case of electric and magnetic fields having all three components (x, y, z), each of them dependent on all coordinates (x, y, z), and time (t), Maxwell's equations in differential form are given as follows in words and in mathematical form.

Faraday's law. The curl of the electric field intensity is equal to the negative of the time derivative of the magnetic flux density, that is,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{3.66}$$

Ampere's circuital law. The curl of the magnetic field intensity is equal to the sum of the current density due to flow of charges and the displacement current density, which is the time derivative of the displacement flux density, that is,

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{3.67}$$

Gauss' law for the electric field. The divergence of the displacement flux density is equal to the charge density, that is,

$$\nabla \cdot \mathbf{D} = \rho \tag{3.68}$$

Gauss' law for the magnetic field. The divergence of the magnetic flux density is equal to zero, that is,

$$\nabla \cdot \mathbf{B} = 0 \tag{3.69}$$

Auxiliary to (3.66)–(3.69), the continuity equation is given by

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0 \tag{3.70}$$

This equation, which is the differential form of the law of conservation of charge, states that the sum of the divergence of the current density due to flow of charges and the time derivative of the charge density is equal to zero. Also, we recall that

$$\mathbf{D} = \epsilon_0 \mathbf{E} \tag{3.71}$$

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} \tag{3.72}$$

which relate **D** and **H** to **E** and **B**, respectively, for free space.

We have learned that the basic definitions of curl and divergence, which have enabled us to discuss their physical interpretations with the aid of the curl and divergence meters, are

$$\nabla \times \mathbf{A} = \lim_{\Delta S \to 0} \left[\frac{\oint_C \mathbf{A} \cdot d\mathbf{l}}{\Delta S} \right]_{\max} \mathbf{a}_n$$

$$\nabla \cdot \mathbf{A} = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta v}$$

Thus, the curl of a vector field at a point is a vector whose magnitude is the circulation of that vector field per unit area with the area oriented so as to maximize this quantity and in the limit that the area shrinks to the point. The direction of the vector is normal to the area in the aforementioned limit and in the right-hand sense. The divergence of a vector field at a point is a scalar quantity equal to the net outward flux of that vector field per unit volume in the limit that the volume shrinks to the point. In Cartesian coordinates the expansions for curl and divergence are

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \mathbf{a}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \mathbf{a}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \mathbf{a}_z$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Thus, Maxwell's equations in differential form relate the spatial variations of the field vectors at a point to their temporal variations and to the charge and current densities at that point.

We have also learned two theorems associated with curl and divergence. These are the Stokes' theorem and the divergence theorem given, respectively, by

$$\oint_C \mathbf{A} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S}$$

and

$$\oint_S \mathbf{A} \cdot d\mathbf{S} = \int_V (\nabla \cdot \mathbf{A})\, dv$$

Stokes' theorem enables us to replace the line integral of a vector around a closed path by the surface integral of the curl of that vector over any surface bounded by that closed path, and vice versa. The divergence theorem enables us to replace the surface integral of a vector over a closed surface by the volume integral of the divergence of that vector over the volume bounded by the closed surface, and vice versa.

In Chapter 2 we learned that all Maxwell's equations in integral form are not independent. Since Maxwell's equations in differential form are derived from their integral forms, it follows that the same is true for these equations. In fact, by noting that (see Problem 3.32),

$$\nabla \cdot \nabla \times \mathbf{A} \equiv 0 \tag{3.73}$$

and applying it to (3.66), we obtain

$$\nabla \cdot \left(-\frac{\partial \mathbf{B}}{\partial t}\right) = \nabla \cdot \nabla \times \mathbf{E} = 0$$

$$\frac{\partial}{\partial t}(\nabla \cdot \mathbf{B}) = 0$$

$$\nabla \cdot \mathbf{B} = \text{constant with time} \tag{3.74}$$

Similarly, applying (3.73) to (3.67), we obtain

$$\nabla \cdot \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) = \nabla \cdot \nabla \times \mathbf{H} = 0$$

$$\nabla \cdot \mathbf{J} + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{D}) = 0$$

Using (3.70), we then have

$$-\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{D}) = 0$$

$$\frac{\partial}{\partial t}(\nabla \cdot \mathbf{D} - \rho) = 0$$

$$\nabla \cdot \mathbf{D} - \rho = \text{constant with time} \tag{3.75}$$

Since for any given point in space, the constants on the right sides of (3.74) and (3.75) can be made equal to zero at some instant of time, it follows that they are zero forever, giving us (3.69) and (3.68), respectively. Thus (3.69) follows from (3.66), whereas (3.68) follows from (3.67) with the aid of (3.70).

Finally, for the simple, special case in which

$$\mathbf{E} = E_x(z, t)\mathbf{a}_x$$
$$\mathbf{H} = H_y(z, t)\mathbf{a}_y$$

the two Maxwell's curl equations reduce to

$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t} \tag{3.76}$$

$$\frac{\partial H_y}{\partial z} = -J_x - \frac{\partial D_x}{\partial t} \tag{3.77}$$

In fact, we derived these equations first and then the general equations (3.66) and (3.67). We will be using (3.76) and (3.77) in the following chapters to study the phenomenon of electromagnetic wave propagation resulting from the interdependence between the space-variations and time-variations of the electric and magnetic fields.

In fact, Maxwell's equations in differential form lend themselves well for a qualitative discussion of the interdependence of time-varying electric and magnetic fields giving rise to the phenomenon of electromagnetic wave propagation. Recognizing that the operations of curl and divergence involve partial derivatives with respect to space coordinates, we observe that time-varying electric and magnetic fields coexist in space, with the spatial variation of the electric field governed by the temporal variation of the magnetic field in accordance with (3.66), and the spatial variation of the magnetic field governed by the temporal variation of the electric field in addition to the current density in accordance with (3.67). Thus, if in (3.67) we begin with a time-varying current source represented by **J**, or a time-varying electric field represented by $\partial\mathbf{D}/\partial t$, or a combination of the two, then one can visualize that a magnetic field is generated in accordance with (3.67), which in turn generates an electric field in accordance with (3.66), which in turn contributes to the generation of the magnetic field in accordance with (3.67), and so on, as depicted in Figure 3.15. Note that **J** and ρ are coupled, since they must satisfy (3.70). Also, the magnetic field automatically satisfies (3.69), since (3.69) is not independent of (3.66).

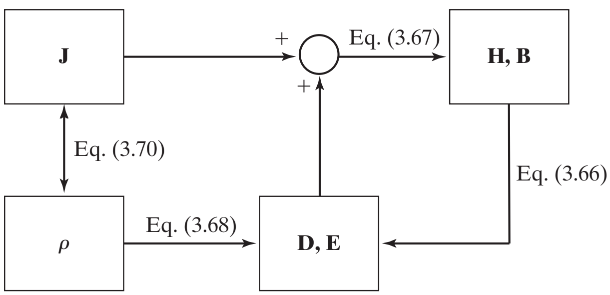

FIGURE 3.15

Generation of interdependent electric and magnetic fields, beginning with sources **J** and ρ.

The process depicted is exactly the phenomenon of electromagnetic waves propagating with a velocity (and other characteristics) determined by the parameters of the medium. In free space, the waves propagate unattenuated with the velocity $1/\sqrt{\mu_0 \varepsilon_0}$, familiarly represented by the symbol c, as we shall learn in Chapter 4. If either the term $\partial\mathbf{B}/\partial t$ in (3.66) or the term $\partial\mathbf{D}/\partial t$ in (1.28) is not present, then wave propagation would not occur. As already stated, it was through the addition of the term $\partial\mathbf{D}/\partial t$ in (3.67) that Maxwell predicted electromagnetic wave propagation before it was confirmed experimentally.

REVIEW QUESTIONS

3.1. State Faraday's law in differential form for the simple case of $\mathbf{E} = E_x(z, t)\mathbf{a}_x$. How is it derived from Faraday's law in integral form?

3.2. Discuss the physical interpretation of Faraday's law in differential form for the simple case of $\mathbf{E} = E_x(z, t)\mathbf{a}_x$.

3.3. State Faraday's law in differential form for the general case of an arbitrary electric field. How is it derived from its integral form?

3.4. What is meant by the net right-lateral differential of the x- and y-components of a vector normal to the z-direction?

3.5. Give an example in which the net right-lateral differential of E_y and E_z normal to the x-direction is zero, although the individual derivatives are nonzero.

3.6. If at a point in space B_y varies with time but B_x and B_z do not, what can we say about the components of $\mathbf{E}$ at that point?

3.7. What is the determinant expansion for the curl of a vector?

3.8. What is the significance of the curl of a vector being equal to zero?

3.9. State Ampere's circuital law in differential form for the simple case of $\mathbf{H} = H_y(z, t)\mathbf{a}_y$. How is it derived from Ampere's circuital law in integral form?

3.10. Discuss the physical interpretation of Ampere's circuital law in differential form for the simple case of $\mathbf{H} = H_y(z, t)\mathbf{a}_y$.

3.11. State Ampere's circuital law in differential form for the general case of an arbitrary magnetic field. How is it derived from its integral form?

3.12. What is the significance of a nonzero net right-lateral differential of H_x and H_y normal to the z-direction at a point in space?

3.13. If a pair of $\mathbf{E}$ and $\mathbf{B}$ at a point satisfies Faraday's law in differential form, does it necessarily follow that it also satisfies Ampere's circuital law in differential form, and vice versa?

3.14. State and briefly discuss the basic definition of the curl of a vector.

3.15. What is a curl meter? How does it help visualize the behavior of the curl of a vector field?

3.16. Provide two examples of physical phenomena in which the curl of a vector field is nonzero.

3.17. State Stokes' theorem and discuss its application.

3.18. State Gauss' law for the electric field in differential form. How is it derived from its integral form?

3.19. What is meant by the net longitudinal differential of the components of a vector field?

3.20. Give an example in which the net longitudinal differential of the components of a vector is zero, although the individual derivatives are nonzero.

3.21. What is the expansion for the divergence of a vector?

3.22. State Gauss' law for the magnetic field in differential form. How is it derived from its integral form?

3.23. How can you determine if a given vector can represent a magnetic field?

3.24. State and briefly discuss the basic definition of the divergence of a vector.

3.25. What is a divergence meter? How does it help visualize the behavior of the divergence of a vector field?

3.26. Provide two examples of physical phenomena in which the divergence of a vector field is nonzero.

3.27. State the continuity equation and discuss its physical interpretation.

3.28. Distinguish between the physical interpretations of the divergence and the curl of a vector field by means of examples.

3.29. State the divergence theorem and discuss its application.

3.30. What is the divergence of the curl of a vector?

3.31. Summarize Maxwell's equations in differential form.

3.32. Are all Maxwell's equations in differential form independent? If not, which of them are independent?

3.33. Provide a qualitative explanation of the phenomenon of electromagnetic wave propagation based on Maxwell's equations in differential form.

PROBLEMS

3.1. Given $\mathbf{B} = B_0 z \cos \omega t \, \mathbf{a}_y$ and it is known that $\mathbf{E}$ has only an x-component, find $\mathbf{E}$ by using Faraday's law in differential form. Then verify your result by applying Faraday's law in integral form to the rectangular closed path, in the xz-plane, defined by $x = 0, x = a, z = 0$, and $z = b$.

3.2. Assuming $\mathbf{E} = E_y(z, t)\mathbf{a}_y$ and considering a rectangular closed path in the yz-plane, carry out the derivation of Faraday's law in differential form similar to that in the text.

3.3. Find the curls of the following vector fields:
(a) $zx\mathbf{a}_x + xy\mathbf{a}_y + yz\mathbf{a}_z$; (b) $ye^{-x}\mathbf{a}_x - e^{-x}\mathbf{a}_y$.

3.4. For $\mathbf{A} = xy^2\mathbf{a}_x + x^2\mathbf{a}_y$, (a) find the net right-lateral differential of A_x and A_y normal to the z-direction at the point $(2, 1, 0)$, and (b) find the locus of the points at which the net right-lateral differential of A_x and A_y normal to the z-direction is zero.

3.5. Given $\mathbf{E} = 10 \cos (6\pi \times 10^8 t - 2\pi z) \, \mathbf{a}_x$ V/m, find $\mathbf{B}$ by using Faraday's law in differential form.

3.6. Show that the curl of $\left(\mathbf{a}_x\frac{\partial}{\partial x} + \mathbf{a}_y\frac{\partial}{\partial y} + \mathbf{a}_z\frac{\partial}{\partial z}\right)f$, that is, ∇f, where f is any scalar function of x, y, and z, is zero. Then find the scalar function for which $\nabla f = y\mathbf{a}_x + x\mathbf{a}_y$.

3.7. Given $\mathbf{E} = E_0 z^2 \sin \omega t \, \mathbf{a}_x$ and it is known that $\mathbf{J}$ is zero and $\mathbf{B}$ has only a y-component, find $\mathbf{B}$ by using Ampere's circuital law in differential form. Then find $\mathbf{E}$ from $\mathbf{B}$ by using Faraday's law in differential form. Comment on your result.

3.8. Assuming $\mathbf{H} = H_x(z, t)\,\mathbf{a}_x$ and considering a rectangular closed path in the xz-plane, carry out the derivation of Ampere's circuital law in differential form similar to that in the text.

3.9. Given $\mathbf{B} = \dfrac{10^{-7}}{3}\cos(6\pi \times 10^8 t - 2\pi z)\,\mathbf{a}_y$ Wb/m^2 and it is known that $\mathbf{J} = 0$, find $\mathbf{E}$ by using Ampere's circuital law in differential form. Then find $\mathbf{B}$ from $\mathbf{E}$ by using Faraday's law in differential form. Comment on your result.

3.10. Assuming $\mathbf{J} = 0$, determine which of the following pairs of E_x and H_y simultaneously satisfy the two Maxwell's equations in differential form given by (3.7) and (3.23):

(a) $E_x = 10\cos 2\pi z \cos 6\pi \times 10^8 t \qquad H_y = \dfrac{1}{12\pi}\sin 2\pi z \sin 6\pi \times 10^8 t$

(b) $E_x = (t - z\sqrt{\mu_0\epsilon_0}) \qquad H_y = \sqrt{\dfrac{\epsilon_0}{\mu_0}}\,(t - z\sqrt{\mu_0\epsilon_0})$

(c) $E_x = z^2 \sin \omega t \qquad H_y = -\dfrac{\omega\epsilon_0}{3} z^3 \cos \omega t$

3.11. A current distribution is given by

$$\mathbf{J} = \begin{cases} -J_0\mathbf{a}_x & \text{for } -a < z < 0 \\ J_0\mathbf{a}_x & \text{for } 0 < z < a \end{cases}$$

where J_0 is a constant. Using Ampere's circuital law in differential form and symmetry considerations, find the magnetic field everywhere.

3.12. A current distribution is given by

$$\mathbf{J} = J_0\left(1 - \frac{|z|}{a}\right)\mathbf{a}_x \qquad \text{for } -a < z < a$$

where J_0 is a constant. Using Ampere's circuital law in differential form and symmetry considerations, find the magnetic field everywhere.

3.13. Assume that the velocity of water in the stream of Figure 3.7(a) decreases linearly from a maximum at the top surface to zero at the bottom surface, with the velocity at the top surface given by Figure 3.7(b). Discuss the curl of the velocity vector field with the aid of the curl meter.

3.14. For the vector field $\mathbf{r} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$, discuss the behavior of the curl meter and verify your reasoning by evaluating the curl of $\mathbf{r}$.

3.15. Discuss the curl of the vector field $y\mathbf{a}_x - x\mathbf{a}_y$ with the aid of the curl meter.

3.16. Verify Stokes' theorem for the vector field $\mathbf{A} = y\mathbf{a}_x + z\mathbf{a}_y + x\mathbf{a}_z$ and the closed path comprising the straight lines from $(1, 0, 0)$ to $(0, 1, 0)$, from $(0, 1, 0)$ to $(0, 0, 1)$, and from $(0, 0, 1)$ to $(1, 0, 0)$.

3.17. Verify Stokes' theorem for the vector field $\mathbf{A} = e^{-y}\mathbf{a}_x - xe^{-y}\mathbf{a}_y$ and any closed path of your choice.

3.18. For the vector $\mathbf{A} = yz\mathbf{a}_x + zx\mathbf{a}_y + xy\mathbf{a}_z$, use Stokes' theorem to show that $\oint_C \mathbf{A} \cdot d\mathbf{l}$ is zero for any closed path C. Then evaluate $\int \mathbf{A} \cdot d\mathbf{l}$ from the origin to the point $(1, 1, 2)$ along the curve $x = \sqrt{2}\sin t$, $y = \sqrt{2}\sin t$, $z = (8/\pi)t$.

3.19. Find the divergences of the following vector fields:
(a) $3xy^2\mathbf{a}_x + 3x^2y\mathbf{a}_y + z^3\mathbf{a}_z$; (b) $2xy\mathbf{a}_x - y^2\mathbf{a}_y$.

3.20. For $\mathbf{A} = xy\mathbf{a}_x + yz\mathbf{a}_y + zx\mathbf{a}_z$, (a) find the net longitudinal differential of the components of $\mathbf{A}$ at the point $(1, 1, 1)$, and (b) find the locus of the points at which the net longitudinal differential of the components of $\mathbf{A}$ is zero.

3.21. For each of the following vectors, find the curl and the divergence and discuss your results: (a) $xy\mathbf{a}_x$; (b) $y\mathbf{a}_x$; (c) $x\mathbf{a}_x$; (d) $y\mathbf{a}_x + x\mathbf{a}_y$.

3.22. A charge distribution is given by

$$\rho = \rho_0\left(1 - \frac{|x|}{a}\right) \qquad \text{for } -a < x < a$$

where ρ_0 is a constant. Using Gauss' law for the electric field in differential form and symmetry considerations, find the electric field everywhere.

3.23. A charge distribution is given by

$$\rho = \rho_0\frac{x}{a} \qquad \text{for } -a < x < a$$

where ρ_0 is a constant. Using Gauss' law for the electric field in differential form and symmetry considerations, find the electric field everywhere.

3.24. Given $\mathbf{D} = x^2y\mathbf{a}_x - y^3\mathbf{a}_y$, find the charge density at (a) the point $(2, 1, 0)$ and (b) the point $(3, 2, 0)$.

3.25. Determine which of the following vectors can represent a magnetic flux density vector $\mathbf{B}$: (a) $y\mathbf{a}_x - x\mathbf{a}_y$; (b) $x\mathbf{a}_x + y\mathbf{a}_y$; (c) $z^3 \cos \omega t\ \mathbf{a}_y$.

3.26. Given $\mathbf{J} = e^{-x^2}\mathbf{a}_x$, find the time rate of decrease of the charge density at (a) the point $(0, 0, 0)$ and (b) the point $(1, 0, 0)$.

3.27. For the vector field $\mathbf{r} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$, discuss the behavior of the divergence meter, and verify your reasoning by evaluating the divergence of $\mathbf{r}$.

3.28. Discuss the divergence of the vector field $y\mathbf{a}_x - x\mathbf{a}_y$ with the aid of the divergence meter.

3.29. Verify the divergence theorem for the vector field $\mathbf{A} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$ and the closed surface bounding the volume within the hemisphere of radius unity above the xy-plane and centered at the origin.

3.30. Verify the divergence theorem for the vector field $\mathbf{A} = xy\mathbf{a}_x + yz\mathbf{a}_y + zx\mathbf{a}_z$ and the closed surface of the volume bounded by the planes $x = 0, x = 1, y = 0, y = 1, z = 0$, and $z = 1$.

3.31. For the vector $\mathbf{A} = y^2\mathbf{a}_y - 2yz\mathbf{a}_z$, use the divergence theorem to show that $\oint_S \mathbf{A} \cdot d\mathbf{S}$ is zero for any closed surface S. Then evaluate $\int \mathbf{A} \cdot d\mathbf{S}$ over the surface $x + y + z = 1, x > 0, y > 0, z > 0$.

3.32. Show that $\nabla \cdot \nabla \times \mathbf{A} = 0$ for any $\mathbf{A}$ in two ways: (a) by evaluating $\nabla \cdot \nabla \times \mathbf{A}$ in Cartesian coordinates, and (b) by using Stokes' and divergence theorems.