CHAPTER

2 Maxwell's Equations in Integral Form

In Chapter 1 we learned the simple rules of vector algebra and familiarized ourselves with the basic concepts of fields, particularly those associated with electric and magnetic fields. We now have the necessary background to introduce the additional tools required for the understanding of the various quantities associated with Maxwell's equations and then discuss Maxwell's equations. In particular, our goal in this chapter is to learn Maxwell's equations in integral form as a prerequisite to the derivation of their differential forms in the next chapter. Maxwell's equations in integral form govern the interdependence of certain field and source quantities associated with regions in space, that is, contours, surfaces, and volumes. The differential forms of Maxwell's equations, however, relate the characteristics of the field vectors at a given point to one another and to the source densities at that point.

Maxwell's equations in integral form are a set of four laws resulting from several experimental findings and a purely mathematical contribution. We shall, however, consider them as postulates and learn to understand their physical significance as well as their mathematical formulation. The source quantities involved in their formulation are charges and currents. The field quantities have to do with the line and surface integrals of the electric and magnetic field vectors. We shall therefore first introduce line and surface integrals and then consider successively the four Maxwell's equations in integral form.

2.1 THE LINE INTEGRAL

Let us consider in a region of electric field $\mathbf{E}$ the movement of a test charge q from the point A to the point B along the path C, as shown in Figure 2.1(a). At each and every point along the path the electric field exerts a force on the test charge and, hence, does a certain amount of work in moving the charge to another point an infinitesimal distance away. To find the total amount of work done from A to B, we divide the path into a number of infinitesimal segments $\Delta\mathbf{l}_1, \Delta\mathbf{l}_2, \Delta\mathbf{l}_3, \ldots, \Delta\mathbf{l}_n$, as shown in Figure 2.1(b), find the infinitesimal amount of work done for each segment and then add up the contributions from all the segments. Since the segments are infinitesimal in length, we can consider each of them to be straight and the electric field at all points within a segment to be the same and equal to its value at the start of the segment.

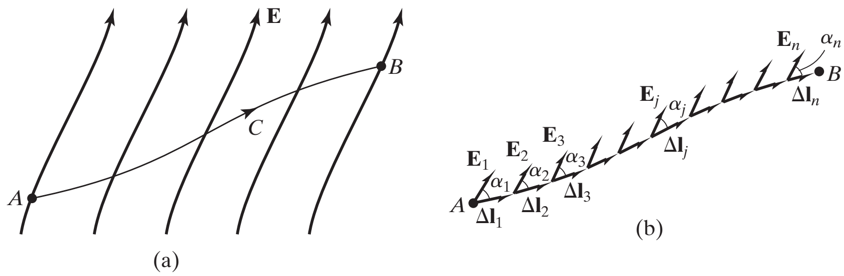

FIGURE 2.1

For evaluating the total amount of work done in moving a test charge along a path C from point A to point B in a region of electric field.

If we now consider one segment, say the jth segment, and take the component of the electric field for that segment along the length of that segment, we obtain the result $E_j \cos \alpha_j$, where α_j is the angle between the direction of the electric field vector $\mathbf{E}_j$ at the start of that segment and the direction of that segment. Since the electric field intensity has the meaning of force per unit charge, the electric force along the direction of the jth segment is then equal to $qE_j \cos \alpha_j$, where q is the value of the test charge. To obtain the work done in carrying the test charge along the length of the jth segment, we then multiply this electric force component by the length Δl_j of that segment. Thus for the jth segment, we obtain the result for the work done by the electric field as

$$\Delta W_j = qE_j \cos \alpha_j \, \Delta l_j \tag{2.1}$$

If we do this for all the infinitesimal segments and add up all the contributions, we get the total work done by the electric field in moving the test charge from A to B as

$$\begin{aligned} W_A^B &= \Delta W_1 + \Delta W_2 + \Delta W_3 + \cdots + \Delta W_n \\ &= qE_1 \cos \alpha_1 \, \Delta l_1 + qE_2 \cos \alpha_2 \, \Delta l_2 + qE_3 \cos \alpha_3 \, \Delta l_3 \\ &\quad + \cdots + qE_n \cos \alpha_n \, \Delta l_n \\ &= q \sum_{j=1}^{n} E_j \cos \alpha_j \, \Delta l_j \end{aligned} \tag{2.2}$$

In vector notation we make use of the dot product operation between two vectors to write this quantity as

$$W_A^B = q \sum_{j=1}^{n} \mathbf{E}_j \cdot \Delta \mathbf{l}_j \tag{2.3}$$

Example 2.1

Let us consider the electric field given by

$$\mathbf{E} = y\mathbf{a}_y$$

and determine the work done by the field in carrying 3 μC of charge from the point $A(0, 0, 0)$ to the point $B(1, 1, 0)$ along the parabolic path $y = x^2$, $z = 0$ shown in Figure 2.2(a).

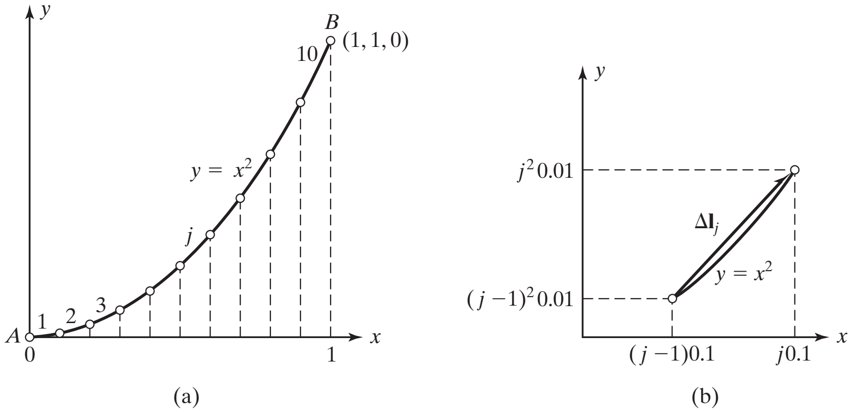

FIGURE 2.2

(a) Division of the path $y = x^2$ from $A\ (0, 0, 0)$ to $B\ (1, 1, 0)$ into ten segments. (b) The length vector corresponding to the jth segment of part (a) approximated as a straight line.

For convenience, we shall divide the path into ten segments having equal widths along the x direction, as shown in Figure 2.2(a). We shall number the segments 1, 2, 3, . . . , 10. The coordinates of the starting and ending points of the jth segment are as shown in Figure 2.2(b). The electric field at the start of the jth segment is given by

$$\mathbf{E}_j = (j - 1)^2\, 0.01\mathbf{a}_y$$

The length vector corresponding to the jth segment, approximated as a straight line connecting its starting and ending points, is

$$\begin{aligned}\Delta\mathbf{l}_j &= 0.1\mathbf{a}_x + [j^2 - (j-1)^2]\, 0.01\mathbf{a}_y \\ &= 0.1\mathbf{a}_x + (2j - 1)\, 0.01\mathbf{a}_y\end{aligned}$$

The required work is then given by

$$\begin{aligned}W_A^B &= 3 \times 10^{-6} \sum_{j=1}^{10} \mathbf{E}_j \cdot \Delta\mathbf{l}_j \\ &= 3 \times 10^{-6} \sum_{j=1}^{10} [(j-1)^2 0.01\mathbf{a}_y] \cdot [0.1\mathbf{a}_x + (2j-1)0.01\mathbf{a}_y] \\ &= 3 \times 10^{-10} \sum_{j=1}^{10} (j-1)^2(2j-1) \\ &= 3 \times 10^{-10}[0 + 3 + 20 + 63 + 144 + 275 + 468 + 735 \\ &\quad + 1088 + 1539] \\ &= 3 \times 10^{-10} \times 4335 = 1.3005\ \mu\mathbf{J}\end{aligned}$$

The result that we have obtained in Example 2.1, for W_A^B, is approximate since we divided the path from A to B into a finite number of segments. By dividing it into larger and larger numbers of segments, we can obtain more and more accurate results. In fact, the problem can be conveniently formulated for a computer solution and by varying the number of segments from a small value to a large value, the convergence of the result can be verified. The value to which the result converges is that for which $n = \infty$. The summation in (2.3) then becomes an integral, which represents exactly the work done by the field and is given by

$$W_A^B = q \int_A^B \mathbf{E} \cdot d\mathbf{l} \tag{2.4}$$

The integral on the right side of (2.4) is known as the *line integral of* **E** *from A to B*.

Example 2.2

We shall illustrate the evaluation of the line integral by computing the exact value of the work done by the electric field in Example 2.1.

To do this, we note that at any arbitrary point $(x, y, 0)$ on the curve $y = x^2$, $z = 0$, the infinitesimal length vector tangential to the curve is given by

$$\begin{aligned} d\mathbf{l} &= dx\,\mathbf{a}_x + dy\,\mathbf{a}_y \\ &= dx\,\mathbf{a}_x + d(x^2)\,\mathbf{a}_y \\ &= dx\,\mathbf{a}_x + 2x\,dx\,\mathbf{a}_y \end{aligned}$$

The value of $\mathbf{E} \cdot d\mathbf{l}$ at the point $(x, y, 0)$ is

$$\begin{aligned} \mathbf{E} \cdot d\mathbf{l} &= y\mathbf{a}_y \cdot (dx\,\mathbf{a}_x + dy\,\mathbf{a}_y) \\ &= x^2\mathbf{a}_y \cdot (dx\,\mathbf{a}_x + 2x\,dx\,\mathbf{a}_y) \\ &= 2x^3\,dx \end{aligned}$$

Thus, the required work is given by

$$\begin{aligned} W_A^B &= q\int_A^B \mathbf{E} \cdot d\mathbf{l} = 3 \times 10^{-6} \int_{(0,0,0)}^{(1,1,0)} 2x^3\,dx \\ &= 3 \times 10^{-6} \left[\frac{2x^4}{4}\right]_{x=0}^{x=1} = 1.5\ \mu\mathbf{J} \end{aligned}$$

Dividing both sides of (2.4) by q, we note that the line integral of **E** from A to B has the physical meaning of work per unit charge done by the field in moving the test charge from A to B. This quantity is known as the *voltage between A and B* and is denoted by the symbol $[V]_A^B$, having the units of volts. Thus,

$$[V]_A^B = \int_A^B \mathbf{E} \cdot d\mathbf{l} \tag{2.5}$$

When the path under consideration is a closed path, as shown in Figure 2.3, the line integral is written with a circle associated with the integral sign in the manner $\oint_C \mathbf{E} \cdot d\mathbf{l}$. The line integral of a vector around a closed path is known as the *circulation* of that vector. In particular, the line integral of **E** around a closed path is the work per unit charge done by the field in moving a test charge around the closed path. It is the voltage around the closed path and is also known as the *electromotive force*. We shall now consider an example of evaluating the line integral of a vector around a closed path.

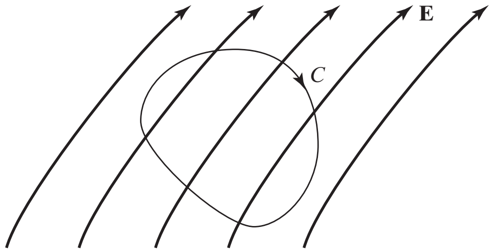

FIGURE 2.3

Closed path C in a region of electric field.

Example 2.3

Let us consider the force field

$$\mathbf{F} = x\mathbf{a}_y$$

and evaluate $\oint_C \mathbf{F} \cdot d\mathbf{l}$, where C is the closed path $ABCDA$ shown in Figure 2.4.

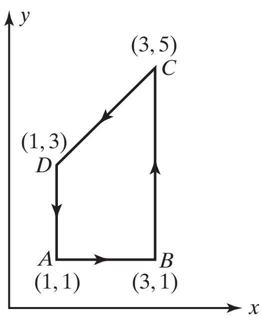

FIGURE 2.4

For evaluating the line integral of a vector field around a closed path.

Noting that

$$\oint_{ABCDA} \mathbf{F} \cdot d\mathbf{l} = \int_A^B \mathbf{F} \cdot d\mathbf{l} + \int_B^C \mathbf{F} \cdot d\mathbf{l} + \int_C^D \mathbf{F} \cdot d\mathbf{l} + \int_D^A \mathbf{F} \cdot d\mathbf{l} \tag{2.6}$$

we simply evaluate each of the line integrals on the right side of (2.6) and add them up to obtain the required quantity. Thus for the side AB,

$$y = 1, \qquad dy = 0, \qquad d\mathbf{l} = dx\,\mathbf{a}_x + (0)\mathbf{a}_y = dx\,\mathbf{a}_x$$

$$\mathbf{F} \cdot d\mathbf{l} = (x\mathbf{a}_y) \cdot (dx\,\mathbf{a}_x) = 0$$

$$\int_A^B \mathbf{F} \cdot d\mathbf{l} = 0$$

For the side BC,

$$x = 3, \qquad dx = 0, \qquad d\mathbf{l} = (0)\mathbf{a}_x + dy\,\mathbf{a}_y = dy\,\mathbf{a}_y$$

$$\mathbf{F} \cdot d\mathbf{l} = (3\mathbf{a}_y) \cdot (dy\,\mathbf{a}_y) = 3\,dy$$

$$\int_B^C \mathbf{F} \cdot d\mathbf{l} = \int_1^5 3\,dy = 12$$

For the side CD,

$$y = 2 + x, \qquad dy = dx, \qquad d\mathbf{l} = dx\,\mathbf{a}_x + dx\,\mathbf{a}_y$$

$$\mathbf{F} \cdot d\mathbf{l} = (x\mathbf{a}_y) \cdot (dx\,\mathbf{a}_x + dx\,\mathbf{a}_y) = x\,dx$$

$$\int_C^D \mathbf{F} \cdot d\mathbf{l} = \int_3^1 x\,dx = -4$$

For the side DA,

$$x = 1, \qquad dx = 0, \qquad d\mathbf{l} = (0)\mathbf{a}_x + dy\,\mathbf{a}_y$$

$$\mathbf{F} \cdot d\mathbf{l} = (\mathbf{a}_y) \cdot (dy\,\mathbf{a}_y) = dy$$

$$\int_D^A \mathbf{F} \cdot d\mathbf{l} = \int_3^1 dy = -2$$

Finally,

$$\oint_{ABCDA} \mathbf{F} \cdot d\mathbf{l} = 0 + 12 - 4 - 2 = 6$$

2.2 THE SURFACE INTEGRAL

Let us consider a region of magnetic field and an infinitesimal surface at a point in that region. Since the surface is infinitesimal, we can assume the magnetic flux density to be uniform on the surface, although it may be nonuniform over a wider region. If the surface is oriented normal to the magnetic field lines, as shown in Figure 2.5(a), then the magnetic flux crossing the surface is simply given by the product of the surface area and the magnetic flux density on the surface, that is, $B\,\Delta S$. If, however, the surface is oriented parallel to the magnetic field lines, as shown in Figure 2.5(b), there is no magnetic flux crossing the surface. If the surface is oriented in such a manner that the normal to the surface makes an angle α with the magnetic field lines, as shown in Figure 2.5(c), then the amount of magnetic flux crossing the surface can be determined by considering that the component of $\mathbf{B}$ normal to the surface is $B \cos \alpha$ and the component tangential to the surface is $B \sin \alpha$. The component of $\mathbf{B}$ normal to the surface results in a flux of $(B \cos \alpha)\,\Delta S$ crossing the surface, whereas the component tangential to the surface does not contribute at all to the flux crossing the surface. Thus, the magnetic flux crossing the surface in this case is $(B \cos \alpha)\,\Delta S$. We can obtain this result

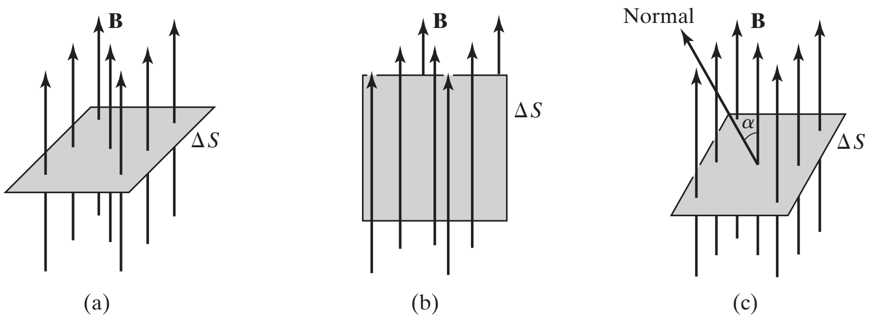

FIGURE 2.5

An infinitesimal surface ΔS in a magnetic field **B** oriented (a) normal to the field, (b) parallel to the field, and (c) with its normal making an angle α to the field.

alternatively by noting that the projection of the surface onto the plane normal to the magnetic field lines is $\Delta S \cos \alpha$.

Let us now consider a large surface S in the magnetic field region, as shown in Figure 2.6. The magnetic flux crossing this surface can be found by dividing the surface into a number of infinitesimal surfaces $\Delta S_1, \Delta S_2, \Delta S_3, \ldots, \Delta S_n$ and applying the result obtained above for each infinitesimal surface and adding up the contributions from all the surfaces. To obtain the contribution from the jth surface, we draw the normal vector to that surface and find the angle α_j between the normal vector and the magnetic flux density vector $\mathbf{B}_j$ associated with that surface. Since the surface is infinitesimal, we can assume $\mathbf{B}_j$ to be the value of $\mathbf{B}$ at the centroid of the surface and we can also erect the normal vector at that point. The contribution to the total magnetic flux from the jth infinitesimal surface is then given by

$$\Delta\psi_j = B_j \cos \alpha_j \, \Delta S_j \tag{2.7}$$

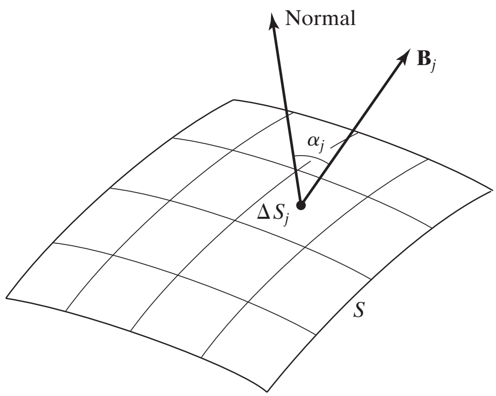

FIGURE 2.6

Division of a large surface S in a magnetic field region into a number of infinitesimal surfaces.

where the symbol ψ represents magnetic flux. The total magnetic flux crossing the surface S is then given by

$$\begin{aligned}[\psi]_S &= \Delta\psi_1 + \Delta\psi_2 + \Delta\psi_3 + \cdots + \Delta\psi_n \\ &= B_1 \cos\alpha_1\,\Delta S_1 + B_2 \cos\alpha_2\,\Delta S_2 + B_3 \cos\alpha_3\,\Delta S_3 \\ &\quad + \cdots + B_n \cos\alpha_n\,\Delta S_n \\ &= \sum_{j=1}^{n} B_j \cos\alpha_j\,\Delta S_j \end{aligned} \tag{2.8}$$

In vector notation we make use of the dot product operation between two vectors to write this quantity as

$$[\psi]_S = \sum_{j=1}^{n} \mathbf{B}_j \cdot \Delta S_j\,\mathbf{a}_{nj} \tag{2.9}$$

where $\mathbf{a}_{nj}$ is the unit vector normal to the surface ΔS_j. In fact, by recalling that the infinitesimal surface can be considered as a vector quantity having magnitude equal to the area of the surface and direction normal to the surface, that is,

$$\Delta\mathbf{S}_j = \Delta S_j\,\mathbf{a}_{nj} \tag{2.10}$$

we can write (2.9) as

$$[\psi]_S = \sum_{j=1}^{n} \mathbf{B}_j \cdot \Delta\mathbf{S}_j \tag{2.11}$$

Example 2.4

Let us consider the magnetic field given by

$$\mathbf{B} = 3xy^2\mathbf{a}_z \text{ Wb/m}^2$$

and determine the magnetic flux crossing the portion of the xy-plane lying between $x = 0$, $x = 1$, $y = 0$, and $y = 1$.

For convenience, we shall divide the surface into 25 equal areas, as shown in Figure 2.7(a). We shall designate the squares as $11, 12, \ldots, 15, 21, 22, \ldots, 55$, where the first digit represents the number of the square in the x-direction and the second digit represents the number of the square in the y-direction. The x- and y-coordinates of the midpoint of the ijth square are $(2i - 1)0.1$ and $(2j - 1)0.1$, respectively, as shown in Figure 2.7(b). The magnetic field at the center of the ijth square is then given by

$$\mathbf{B}_{ij} = 3(2i - 1)(2j - 1)^2\,0.001\mathbf{a}_z$$

Since we have divided the surface into equal areas and since all areas are in the xy-plane,

$$\Delta\mathbf{S}_{ij} = 0.04\,\mathbf{a}_z \qquad \text{for all } i \text{ and } j$$

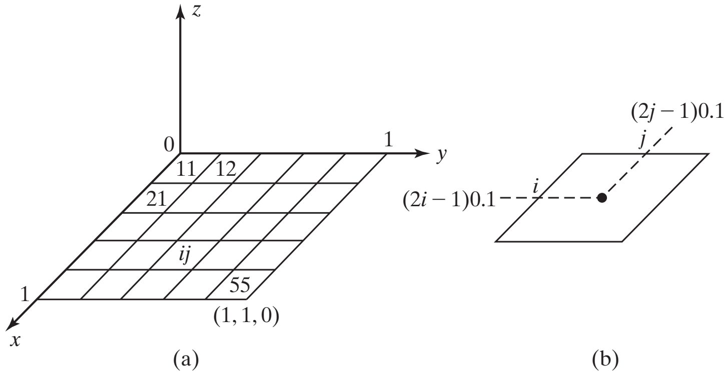

FIGURE 2.7

(a) Division of the portion of the xy-plane lying between $x = 0$, $x = 1$, $y = 0$, and $y = 1$ into 25 squares. (b) The area corresponding to the ijth square.

The required magnetic flux is then given by

$$\begin{aligned}
[\psi]_S &= \sum_{i=1}^{5}\sum_{j=1}^{5}\mathbf{B}_{ij} \cdot \Delta\mathbf{S}_{ij} \\
&= \sum_{i=1}^{5}\sum_{j=1}^{5} 3(2i - 1)(2j - 1)^2 0.001\mathbf{a}_z \cdot 0.04\mathbf{a}_z \\
&= 0.00012\sum_{i=1}^{5}\sum_{j=1}^{5}(2i - 1)(2j - 1)^2 \\
&= 0.00012(1 + 3 + 5 + 7 + 9)(1 + 9 + 25 + 49 + 81) \\
&= 0.495 \text{ Wb}
\end{aligned}$$

The result that we have obtained for $[\psi]_S$ in Example 2.4 is approximate since we have divided the surface S into a finite number of areas. By dividing it into larger and larger numbers of squares, we can obtain more and more accurate results. In fact, the problem can be conveniently formulated for a computer solution, and by varying the number of squares from a small value to a large value, the convergence of the result can be verified. The value to which the result converges is that for which the number of squares in each direction is infinity. The summation in (2.11) then becomes an integral that represents exactly the magnetic flux crossing the surface and is given by

$$[\psi]_S = \int_S \mathbf{B} \cdot d\mathbf{S} \tag{2.12}$$

where the symbol S associated with the integral sign denotes that the integration is performed over the surface S. The integral on the right side of (2.12) is known as the *surface integral of* **B** *over* S. The surface integral is a double integral since dS is equal to the product of two differential lengths. In fact, the work in Example 2.4 indicates that as i and j tend to infinity, the double summation becomes a double integral involving the variables of integration x and y.

Example 2.5

We shall illustrate the evaluation of the surface integral by computing the exact value of the magnetic flux in Example 2.4.

To do this, we note that at any arbitrary point (x, y) on the surface, the infinitesimal surface vector is given by

$$d\mathbf{S} = dx\, dy\, \mathbf{a}_z$$

The value of $\mathbf{B} \cdot d\mathbf{S}$ at the point (x, y) is

$$\begin{aligned} \mathbf{B} \cdot d\mathbf{S} &= 3xy^2\, \mathbf{a}_z \cdot dx\, dy\, \mathbf{a}_z \\ &= 3xy^2\, dx\, dy \end{aligned}$$

Thus, the required magnetic flux is given by

$$\begin{aligned} [\psi]_S &= \int_S \mathbf{B} \cdot d\mathbf{S} \\ &= \int_{x=0}^{1} \int_{y=0}^{1} 3xy^2\, dx\, dy = 0.5 \text{ Wb} \end{aligned}$$

When the surface under consideration is a closed surface, the surface integral is written with a circle associated with the integral sign in the manner $\oint_S \mathbf{B} \cdot d\mathbf{S}$. The surface integral of $\mathbf{B}$ over the closed surface S is simply the magnetic flux emanating from the volume bounded by the surface. We shall now consider an example of evaluating the closed surface integral.

Example 2.6

Let us consider the vector field

$$\mathbf{A} = (x + 2)\mathbf{a}_x + (1 - 3y)\mathbf{a}_y + 2z\mathbf{a}_z$$

and evaluate $\oint_S \mathbf{A} \cdot d\mathbf{S}$ where S is the surface of the cubical box bounded by the planes

$$\begin{aligned} x &= 0, \quad x = 1 \\ y &= 0, \quad y = 1 \\ z &= 0, \quad z = 1 \end{aligned}$$

as shown in Figure 2.8.

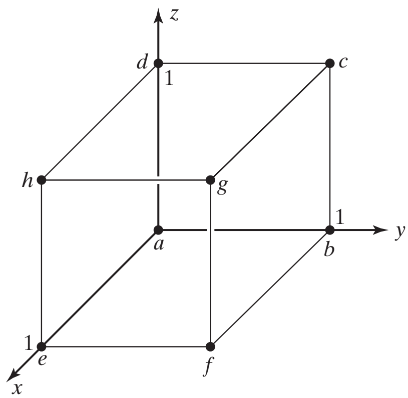

FIGURE 2.8

For evaluating the surface integral of a vector field over a closed surface.

Noting that

$$\oint_S \mathbf{A} \cdot d\mathbf{S} = \int_{abcd} \mathbf{A} \cdot d\mathbf{S} + \int_{efgh} \mathbf{A} \cdot d\mathbf{S} + \int_{aehd} \mathbf{A} \cdot d\mathbf{S} + \int_{bfgc} \mathbf{A} \cdot d\mathbf{S}$$
$$+ \int_{aefb} \mathbf{A} \cdot d\mathbf{S} + \int_{dhgc} \mathbf{A} \cdot d\mathbf{S} \tag{2.13}$$

we simply evaluate each of the surface integrals on the right side of (2.13) and add them up to obtain the required quantity. In doing so, we recognize that since the quantity we want is the flux of **A** out of the box, we should direct the normal vectors toward the outside of the box. Thus for the surface *abcd*,

$$x = 0, \qquad \mathbf{A} = 2\mathbf{a}_x + (1 - 3y)\mathbf{a}_y + 2z\mathbf{a}_z, \qquad d\mathbf{S} = -dy\,dz\,\mathbf{a}_x$$
$$\mathbf{A} \cdot d\mathbf{S} = -2\,dy\,dz$$
$$\int_{abcd} \mathbf{A} \cdot d\mathbf{S} = \int_{z=0}^{1} \int_{y=0}^{1} (-2)\,dy\,dz = -2$$

For the surface *efgh*,

$$x = 1, \qquad \mathbf{A} = 3\mathbf{a}_x + (1 - 3y)\mathbf{a}_y + 2z\mathbf{a}_z, \qquad d\mathbf{S} = dy\,dz\,\mathbf{a}_x$$
$$\mathbf{A} \cdot d\mathbf{S} = 3\,dy\,dz$$
$$\int_{efgh} \mathbf{A} \cdot d\mathbf{S} = \int_{z=0}^{1} \int_{y=0}^{1} 3\,dy\,dz = 3$$

For the surface *aehd*,

$$y = 0, \qquad \mathbf{A} = (x + 2)\mathbf{a}_x + 1\mathbf{a}_y + 2z\mathbf{a}_z, \qquad d\mathbf{S} = -dz\,dx\,\mathbf{a}_y$$
$$\mathbf{A} \cdot d\mathbf{S} = -dz\,dx$$
$$\int_{aehd} \mathbf{A} \cdot d\mathbf{S} = \int_{x=0}^{1} \int_{z=0}^{1} (-1)\,dz\,dx = -1$$

For the surface *bfgc*,

$$y = 1, \qquad \mathbf{A} = (x + 2)\mathbf{a}_x - 2\mathbf{a}_y + 2z\mathbf{a}_z, \qquad d\mathbf{S} = dz\,dx\,\mathbf{a}_y$$

$$\mathbf{A} \cdot d\mathbf{S} = -2\,dz\,dx$$

$$\int_{bfgc} \mathbf{A} \cdot d\mathbf{S} = \int_{x=0}^{1}\int_{z=0}^{1} (-2)\,dz\,dx = -2$$

For the surface *aefb*,

$$z = 0, \qquad \mathbf{A} = (x + 2)\mathbf{a}_x + (1 - 3y)\mathbf{a}_y + 0\mathbf{a}_z, \qquad d\mathbf{S} = -dx\,dy\,\mathbf{a}_z$$

$$\mathbf{A} \cdot d\mathbf{S} = 0$$

$$\int_{aefb} \mathbf{A} \cdot d\mathbf{S} = 0$$

For the surface *dhgc*,

$$z = 1, \qquad \mathbf{A} = (x + 2)\mathbf{a}_x + (1 - 3y)\mathbf{a}_y + 2\mathbf{a}_z, \qquad d\mathbf{S} = dx\,dy\,\mathbf{a}_z$$

$$\mathbf{A} \cdot d\mathbf{S} = 2\,dx\,dy$$

$$\int_{dhgc} \mathbf{A} \cdot d\mathbf{S} = \int_{y=0}^{1}\int_{x=0}^{1} 2\,dx\,dy = 2$$

Finally,

$$\oint_S \mathbf{A} \cdot d\mathbf{S} = -2 + 3 - 1 - 2 + 0 + 2 = 0$$

2.3 FARADAY'S LAW

In the previous sections we introduced the line and surface integrals. We are now ready to consider Maxwell's equations in integral form. The first equation, which we shall discuss in this section, is a consequence of an experimental finding by Michael Faraday in 1831 that time-varying magnetic fields give rise to electric fields and hence it is known as *Faraday's law*. Faraday discovered that when the magnetic flux enclosed by a loop of wire changes with time, a current is produced in the loop, indicating that a voltage or an *electromotive force*, abbreviated as emf, is induced around the loop. The variation of the magnetic flux can result from the time variation of the magnetic flux enclosed by a fixed loop or from a moving loop in a static magnetic field or from a combination of the two, that is, a moving loop in a time-varying magnetic field.

Thus far we have merely stated Faraday's finding without regard to the polarity of the induced emf around the loop or that of the magnetic flux enclosed by the loop. To clarify the point, let us consider a planar circular loop in the plane of the paper as shown in Figure 2.9. Then, we can talk of emf induced in the clockwise sense or in the counterclockwise sense. The emf induced in the clockwise sense is the line integral of $\mathbf{E}$ ($\oint \mathbf{E} \cdot d\mathbf{l}$) evaluated by traversing the loop in the clockwise direction, as shown in Figures 2.9(a) and 2.9(b). The emf induced in the counterclockwise sense is the line integral of $\mathbf{E}$ ($\oint \mathbf{E} \cdot d\mathbf{l}$) evaluated by traversing the loop in the counterclockwise direction, as shown in Figures 2.9(c) and 2.9(d). One is, of course, the negative of the other. Similarly, we can talk of enclosed magnetic flux directed into the paper or out of the paper. The enclosed magnetic flux into the paper is the surface integral of $\mathbf{B}$ ($\int \mathbf{B} \cdot d\mathbf{S}$) evaluated over the plane surface bounded by the loop and with the normal to the surface directed into the paper, as shown in Figures 2.9(a) and 2.9(c). The enclosed magnetic flux out of the paper is the surface integral of $\mathbf{B}$ ($\int \mathbf{B} \cdot d\mathbf{S}$) evaluated over the plane surface bounded by the loop and with the normal to the surface directed out of the paper, as shown in Figures 2.9(b) and 2.9(d). One is, of course, the negative of the other.

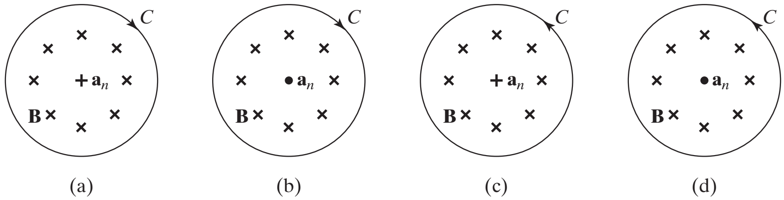

FIGURE 2.9

Four possible pairs of directions of traversal around a planar circular loop and normal to the surface bounded by the loop.

If we do not pay any attention to the polarities, we can write four equations relating the emf around the loop to the magnetic flux enclosed by the loop. These are

$$[\text{emf}]_{\text{clockwise}} = \frac{d}{dt}[\text{magnetic flux}]_{\text{into the paper}} \tag{2.14a}$$

$$[\text{emf}]_{\text{clockwise}} = \frac{d}{dt}[\text{magnetic flux}]_{\text{out of the paper}} \tag{2.14b}$$

$$[\text{emf}]_{\text{counterclockwise}} = \frac{d}{dt}[\text{magnetic flux}]_{\text{into the paper}} \tag{2.14c}$$

$$[\text{emf}]_{\text{counterclockwise}} = \frac{d}{dt}[\text{magnetic flux}]_{\text{out of the paper}} \tag{2.14d}$$

The fourth equation is, however, consistent with the first and the third equation is consistent with the second. Thus, we are left with a choice between the first and the second. Only one of them can be correct, since they provide contradictory results for the emf. Faraday's experiments showed that the second equation is the one that should be used. Alternatively, if we wish to work with clockwise-induced emf and magnetic flux into the paper (or with counterclockwise-induced emf and magnetic flux out of the paper), we must include a minus sign in front of the time derivative. This is, in fact, what is done conventionally. The convention is to use that normal to the surface which is directed toward the advancing direction of a right-hand screw when it is turned in the sense in which the loop is traversed, as shown in Figures 2.10(a) and 2.10(b). This is known as the *right-hand screw rule* and is applied consistently for all electromagnetic field laws. Hence, it is well worthwhile digesting it at this early stage.

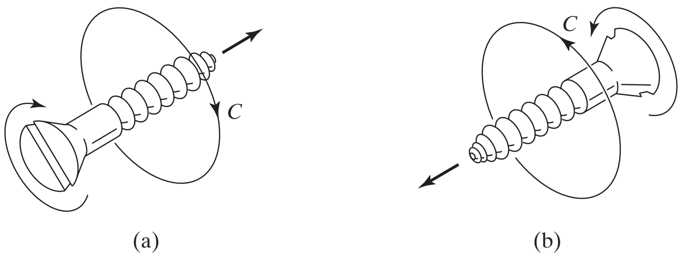

FIGURE 2.10

Right-hand screw rule convention employed in the formulation of electromagnetic field laws.

We can now express Faraday's law mathematically as

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S} \tag{2.15}$$

where S is a surface bounded by C. For the law to be unique, the surface S need not be a plane surface and can be any curved surface bounded by C, as depicted in Figure 2.11. This tells us that the magnetic flux through all possible surfaces bounded by C must be the same. We shall make use of this later. In fact, if C is not a planar loop, we cannot have a plane surface bounded by C. A further point of interest is that C need not represent a loop of wire but can be an imaginary closed path. It means that the time-varying magnetic flux induces an electric field in the region and this results in an emf around the closed path. If a wire is placed in the position occupied by the closed path, the emf will produce a current in the loop simply because the charges in the wire are constrained to move along the wire. Let us now consider some examples.

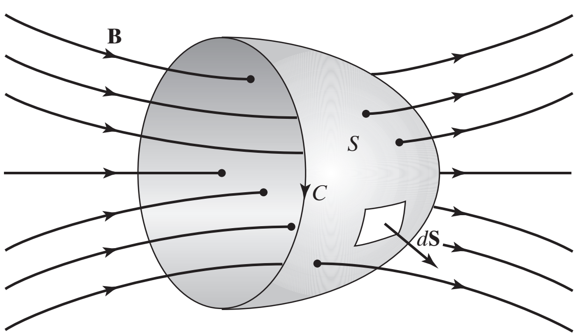

FIGURE 2.11

For illustrating Faraday's law.

Example 2.7

A rectangular loop of wire with three sides fixed and the fourth side movable is situated in a plane perpendicular to a uniform magnetic field $\mathbf{B} = B_0\mathbf{a}_z$, as illustrated in Figure 2.12. The movable side consists of a conducting bar moving with a velocity v_0 in the y-direction. It is desired to find the emf induced in the loop.

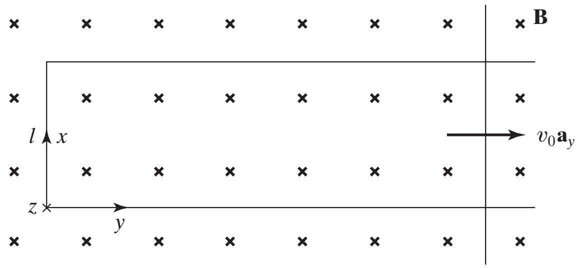

FIGURE 2.12

A rectangular loop of wire with a movable side situated in a uniform magnetic field.

Letting the position of the movable side at any time t be $y_0 + v_0t$, we obtain the magnetic flux enclosed by the loop and directed into the paper as

$$\begin{aligned}\psi &= (\text{area of the loop})B_0 \\ &= l(y_0 + v_0t)B_0\end{aligned}$$

The emf induced in the loop in the clockwise sense is then given by

$$\begin{aligned}\oint \mathbf{E} \cdot d\mathbf{l} &= -\frac{d}{dt}\psi \\ &= -\frac{d}{dt}[l(y_0 + v_0t)B_0] \\ &= -B_0lv_0\end{aligned}$$

Thus, if the bar is moving to the right, the induced emf produces a current in the counterclockwise sense. Note that this polarity of the current is such that it gives rise to a magnetic field directed out of the paper inside the loop. The flux of this magnetic field is in opposition to the flux of the original magnetic field and hence tends to decrease it. This observation is in accordance with *Lenz's law*, which states that the induced emf is such that it acts to oppose the *change* in the magnetic flux producing it. The minus sign on the right side of Faraday's law ensures that Lenz's law is always satisfied.

It is also of interest to note that the induced emf can also be interpreted as due to the electric field induced in the moving bar by virtue of its motion perpendicular to the magnetic field. Thus, a charge Q in the bar experiences a force $\mathbf{F} = Q\mathbf{v} \times \mathbf{B}$ or $Qv_0\mathbf{a}_y \times B_0\mathbf{a}_z = Qv_0B_0\mathbf{a}_x$. To an observer moving with the bar, this force appears as an electric force due to an electric field $\mathbf{F}/Q = v_0B_0\mathbf{a}_x$. Viewed from inside the loop, this electric field is in the counterclockwise direction and hence the induced emf is v_0B_0l in that sense, as deduced above from Faraday's law. This concept of induced emf is known as the *motional emf concept*, which is employed widely in the study of electromechanics.

Example 2.8

A time-varying magnetic field is given by

$$\mathbf{B} = B_0 \cos \omega t \, \mathbf{a}_y$$

where B_0 is a constant. It is desired to find the induced emf around a rectangular loop in the xz-plane, as shown in Figure 2.13.

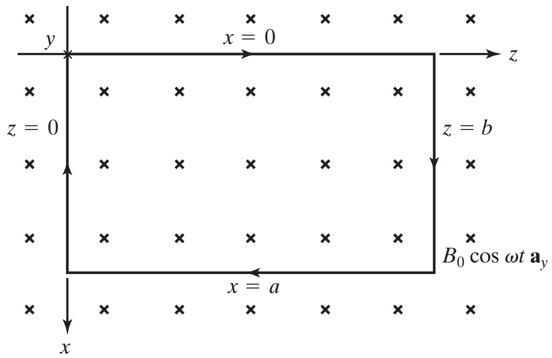

FIGURE 2.13
A rectangular loop in the xz-plane situated in a time-varying magnetic field.

The magnetic flux enclosed by the loop and directed into the paper is given by

$$\psi = \int_S \mathbf{B} \cdot d\mathbf{S} = \int_{z=0}^{b} \int_{x=0}^{a} B_0 \cos \omega t \, \mathbf{a}_y \cdot dx \, dz \, \mathbf{a}_y$$

$$= B_0 \cos \omega t \int_{z=0}^{b} \int_{x=0}^{a} dx \, dz = abB_0 \cos \omega t$$

The induced emf in the clockwise sense is then given by

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{S}$$

$$= -\frac{d}{dt}[abB_0 \cos \omega t] = abB_0 \omega \sin \omega t$$

The time variations of the magnetic flux enclosed by the loop and the induced emf around the loop are shown in Figure 2.14. It can be seen that when the magnetic flux enclosed by the loop is decreasing with time, the induced emf is positive, thereby producing a clockwise current if the loop were a wire. This polarity of the current gives rise to a magnetic field directed into the paper inside the loop and hence acts to increase the magnetic flux enclosed by the loop. When the magnetic flux enclosed by the loop is increasing with time, the induced emf is negative, thereby producing a counterclockwise current around the loop. This polarity of the current gives rise to a magnetic field directed out of the paper inside the loop and hence acts to decrease the magnetic flux enclosed by the loop. These observations are once again consistent with Lenz's law.

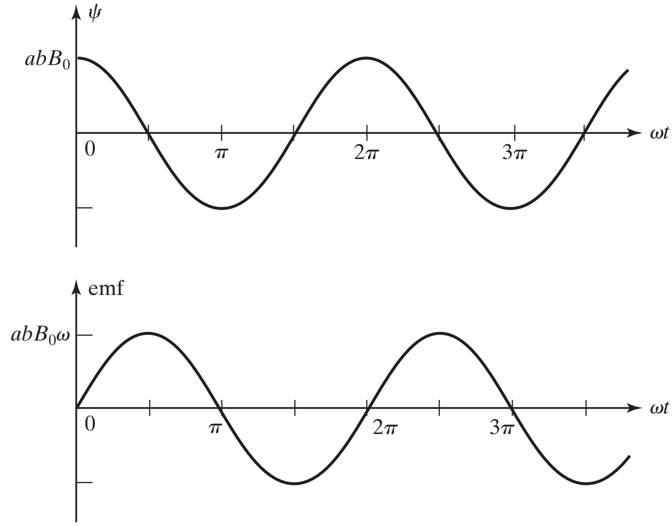

FIGURE 2.14

Time variations of magnetic flux ψ enclosed by the loop of Figure 2.13, and the resulting induced emf around the loop.

2.4 AMPERE'S CIRCUITAL LAW

In the previous section we introduced Faraday's law, one of Maxwell's equations, in integral form. In this section we introduce another of Maxwell's equations in integral form. This equation, known as *Ampere's circuital law*, is a combination of an experimental finding of Oersted that electric currents generate magnetic fields and a mathematical contribution of Maxwell that time-varying electric fields give rise to magnetic fields. It is this contribution of Maxwell that led to the prediction of electromagnetic wave propagation even before the phenomenon was discovered experimentally. In mathematical form, Ampere's circuital law is analogous to Faraday's law and is given by

$$\oint_C \frac{\mathbf{B}}{\mu_0} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt}\int_S \epsilon_0 \mathbf{E} \cdot d\mathbf{S} \tag{2.16}$$

where S is any surface bounded by C, as shown in Figure 2.15. Here again, in order to evaluate the surface integrals on the right side of (2.16), we choose that normal to the surface which is directed toward the advancing direction of a right-hand screw when it is turned in the sense of C, just as in the case of Faraday's law. Also, both integrals on the right side of (2.16) must be evaluated on the same surface, whatever be the surface chosen.

The quantity $\mathbf{J}$ on the right side of (2.16) is the volume current density vector having the magnitude equal to the maximum value of current per unit area (A/m^2) at the point under consideration, as discussed in Section 1.5. Thus, the quantity $\int_S \mathbf{J} \cdot d\mathbf{S}$, being the surface integral of $\mathbf{J}$ over S, has the meaning of current due to flow of charges crossing the surface S bounded by C. It also includes line currents, that is, currents flowing along thin filamentary wires enclosed by C, and surface currents, that is, currents flowing along ribbon-like wires enclosed by C. Thus, $\int_S \mathbf{J} \cdot d\mathbf{S}$, although formulated in

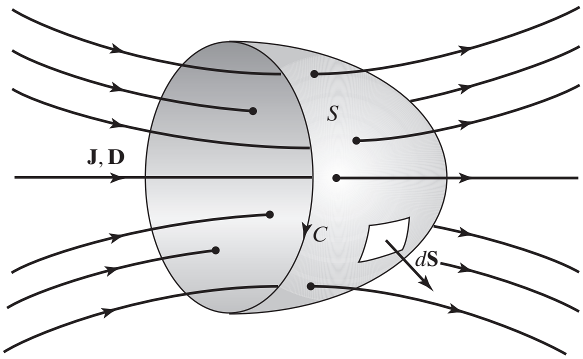

FIGURE 2.15
For illustrating Ampere's circuital law.

terms of the volume current density vector **J**, represents the algebraic sum of all the currents due to flow of charges across the surface *S*.

The quantity $\int_S \epsilon_0 \mathbf{E} \cdot d\mathbf{S}$ on the right side of (2.16) is the flux of the vector field $\epsilon_0\mathbf{E}$ crossing the surface *S*. The vector $\epsilon_0\mathbf{E}$ is known as the *displacement vector* or the *displacement flux density vector* and is denoted by the symbol **D**. By recalling from (1.52) that **E** has the units of (charge) per [(permittivity)(distance)^2], we note that the quantity **D** has the units of charge per unit area, or C/m^2. Hence, the quantity $\int_S \epsilon_0 \mathbf{E} \cdot d\mathbf{S}$, that is, the displacement flux has the units of charge, and the quantity $\frac{d}{dt}\int_S \epsilon_0 \mathbf{E} \cdot d\mathbf{S}$ has the units of $\frac{d}{dt}$ (charge) or current and is known as the *displacement current*. Physically, it is not a current in the sense that it does not represent the flow of charges, but mathematically it is equivalent to a current crossing the surface *S*.

The quantity $\oint_C \frac{\mathbf{B}}{\mu_0} \cdot d\mathbf{l}$ on the left side of (2.16) is the line integral of the vector field $\mathbf{B}/\mu_0$ around the closed path *C*. We learned in Section 2.1 that the quantity $\oint_C \mathbf{E} \cdot d\mathbf{l}$ has the physical meaning of work per unit charge associated with the movement of a test charge around the closed path *C*. The quantity $\oint_C \frac{\mathbf{B}}{\mu_0} \cdot d\mathbf{l}$ does not have a similar physical meaning. This is because magnetic force on a moving charge is directed perpendicular to the direction of motion of the charge as well as to the direction of the magnetic field and hence does not do work in the movement of the charge. The vector $\mathbf{B}/\mu_0$ is known as the *magnetic field intensity vector* and is denoted by the symbol **H**. By recalling from (1.68) that **B** has the units of [(permeability)(current)(length)] per [(distance)^2], we note that the quantity **H** has the units of current per unit distance, or A/m. This gives the units of current or A to $\oint_C \mathbf{H} \cdot d\mathbf{l}$. In analogy with the name *electromotive force* for $\oint_C \mathbf{E} \cdot d\mathbf{l}$, the quantity $\oint_C \mathbf{H} \cdot d\mathbf{l}$ is known as the *magnetomotive force*, abbreviated as mmf.

Replacing $\mathbf{B}/\mu_0$ and $\epsilon_0\mathbf{E}$ in (2.16) by **H** and **D**, respectively, we rewrite Ampere's circuital law as

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt}\int_S \mathbf{D} \cdot d\mathbf{S} \tag{2.17}$$

In words, (2.17) states that "the magnetomotive force around a closed path *C* is equal to the total current, that is, the current due to actual flow of charges plus the displacement current bounded by *C*." When we say "the total current bounded by *C*," we mean "the total current crossing any given surface S bounded by C." This implies that the total current crossing all possible surfaces bounded by C must be the same since for a given C, $\oint_C \mathbf{H} \cdot d\mathbf{l}$ must have a unique value.

Example 2.9

An infinitely long, thin, straight wire situated along the z-axis carries a current I in the z-direction. It is desired to find $\oint_C \mathbf{H} \cdot d\mathbf{l}$ around a circle of radius a lying on the xy-plane and centered at the origin as shown in Figure 2.16.

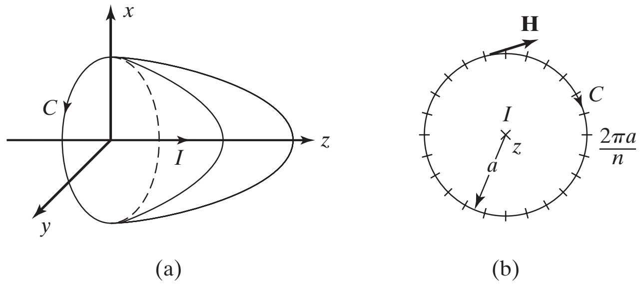

FIGURE 2.16
(a) For illustrating the uniqueness of a wire current enclosed by a closed path for an infinitely long, straight wire. (b) For finding the magnetic field due to the wire.

Let us consider the plane surface enclosed by C. The total current crossing the surface consists entirely of the current I carried by the wire. In fact, since the wire is infinitely long, the total current crossing any of the infinite number of surfaces bounded by C is equal to I. The situation is illustrated in Figure 2.16(a) for a few of the infinite number of surfaces. Thus, noting that the current I is bounded by C in the right-hand sense, and that it is uniquely given, we obtain

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = I \tag{2.18}$$

We can proceed further and evaluate $\mathbf{H}$ at points on the circular path from symmetry considerations. In order for $\oint_C \mathbf{H} \cdot d\mathbf{l}$ to be nonzero, $\mathbf{H}$ must be directed (or have a component) tangential to the circular path and then, from symmetry considerations, it must have the same magnitude at all points on the circle, since the circle is centered at the wire. We, however, know from elementary considerations of the magnetic field due to a current element that $\mathbf{H}$ must be directed entirely tangential to the circular path. Thus, let us divide the circle into a large number of equal segments, say n, as shown in Figure 2.16(b). Since the length of each segment is $2\pi a/n$ and since $\mathbf{H}$ is parallel to the segment, $\mathbf{H} \cdot d\mathbf{l}$ for the segment is $(2\pi a/n)H$ and

$$\begin{aligned}\oint_C \mathbf{H} \cdot d\mathbf{l} &= \frac{2\pi a}{n} H(\text{number of segments}) \\ &= \frac{2\pi a}{n} H \cdot n = 2\pi a H\end{aligned}$$

From (2.18), we then have

$$2\pi a H = I$$

or

$$H = \frac{I}{2\pi a}$$

Thus, the magnetic field intensity due to the infinitely long wire is directed circular to the wire in the right-hand sense and has a magnitude $I/2\pi a$, where a is the distance of the point from the wire. The method we have discussed here is a standard procedure for the determination of the static magnetic field due to current distributions possessing certain symmetries. We shall include some cases in the problems for the interested reader.

If the wire of Example 2.9 is finitely long, say, extending from $-d$ to $+d$ on the z-axis, then, the construction of Figure 2.17 illustrates that for some surfaces the wire pierces through the surface, whereas for some other surfaces it does not. Thus, for this case, there is no unique value of the wire current alone that is enclosed by C. Hence, there must be a displacement current through the surfaces in addition to the wire current so that the total current enclosed by C is uniquely given. In fact, this displacement current is provided by the time-varying electric field due to charges accumulating at one end and depleting at the other end of the current-carrying wire. Thus, considering, for example, the surfaces S_1 and S_3 and setting the total currents through S_1 and S_3 to be equal, we have

$$\int_{S_1} \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt}\int_{S_1} \mathbf{D} \cdot d\mathbf{S} = \int_{S_3} \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt}\int_{S_3} \mathbf{D} \cdot d\mathbf{S} \tag{2.19}$$

Now, since the wire pierces through S_1 in the right-hand sense,

$$\int_{S_1} \mathbf{J} \cdot d\mathbf{S} = I \tag{2.20}$$

The wire does not pierce through S_3. Hence,

$$\int_{S_3} \mathbf{J} \cdot d\mathbf{S} = 0 \tag{2.21}$$

Substituting (2.20) and (2.21) into (2.19), we get

$$I + \frac{d}{dt}\int_{S_1} \mathbf{D} \cdot d\mathbf{S} = 0 + \frac{d}{dt}\int_{S_3} \mathbf{D} \cdot d\mathbf{S} \tag{2.22}$$

or

$$\frac{d}{dt}\int_{S_3} \mathbf{D} \cdot d\mathbf{S} - \frac{d}{dt}\int_{S_1} \mathbf{D} \cdot d\mathbf{S} = I \tag{2.23}$$

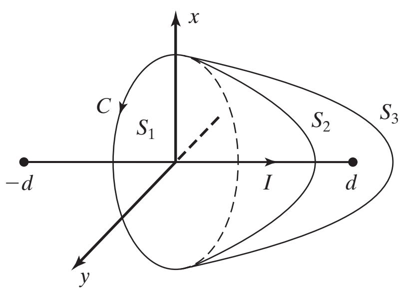

FIGURE 2.17

For illustrating that the wire current enclosed by a closed path is not unique for a finitely long wire.

Reversing the sense of evaluation of the surface integral of **D** over S_1 and changing the minus sign to a plus sign, we obtain

$$\frac{d}{dt}\oint_{S_3+S_1} \mathbf{D} \cdot d\mathbf{S} = I \tag{2.24}$$

Thus, the displacement current emanating from the closed surface $S_1 + S_3$ is equal to I.

Another example in which the wire current enclosed by C is not uniquely defined is shown in Figure 2.18, which is that of a simple circuit consisting of a capacitor driven by an alternating voltage source. Considering two surfaces S_1 and S_2, where S_1 cuts through the wire and S_2 passes between the plates of the capacitor, we have

$$\int_{S_1} \mathbf{J} \cdot d\mathbf{S} = I \tag{2.25}$$

and

$$\int_{S_2} \mathbf{J} \cdot d\mathbf{S} = 0 \tag{2.26}$$

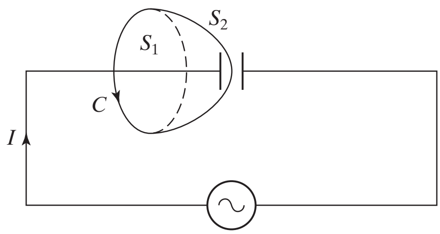

FIGURE 2.18

A capacitor circuit illustrating that the wire current enclosed by a closed path is not unique.

If we neglect fringing and assume that the electric field in the capacitor is contained entirely within the region between the plates, then

$$\int_{S_1} \mathbf{D} \cdot d\mathbf{S} = 0 \tag{2.27}$$

For $\oint_C \mathbf{H} \cdot d\mathbf{l}$ to be unique,

$$\int_{S_1} \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt}\int_{S_1} \mathbf{D} \cdot d\mathbf{S} = \int_{S_2} \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt}\int_{S_2} \mathbf{D} \cdot d\mathbf{S} \tag{2.28}$$

Substituting (2.25), (2.26), and (2.27) into (2.28), we obtain

$$\frac{d}{dt}\int_{S_2} \mathbf{D} \cdot d\mathbf{S} = I \tag{2.29}$$

Thus, the displacement current, that is, the time rate of change of the displacement flux between the capacitor plates, is equal to the wire current.

Example 2.10

A time-varying electric field is given by

$$\mathbf{E} = E_0 z \sin \omega t \, \mathbf{a}_x$$

where E_0 is a constant. It is desired to find the induced mmf around a rectangular loop in the yz-plane, as shown in Figure 2.19.

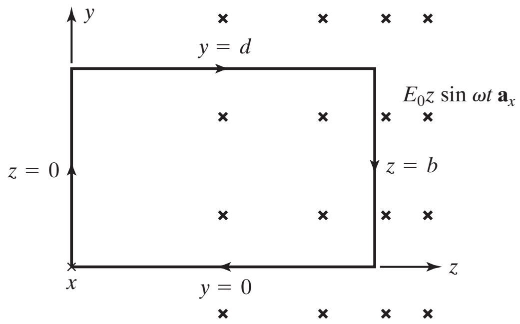

FIGURE 2.19

A rectangular loop in a time-varying electric field.

The total current here is composed entirely of displacement current. The displacement flux enclosed by the loop and directed into the paper is given by

$$\begin{aligned}\int_S \mathbf{D}\cdot d\mathbf{S} &= \int_{z=0}^{b}\int_{y=0}^{d} \epsilon_0 E_0 z \sin\omega t\, \mathbf{a}_x \cdot dy\, dz\, \mathbf{a}_x \\ &= \epsilon_0 E_0 \sin\omega t \int_{z=0}^{b}\int_{y=0}^{d} z\, dy\, dz \\ &= \epsilon_0 \frac{b^2 d}{2} E_0 \sin\omega t\end{aligned}$$

The induced mmf around C is then given by

$$\begin{aligned}\oint_C \mathbf{H}\cdot d\mathbf{l} &= \frac{d}{dt}\int_S \mathbf{D}\cdot d\mathbf{S} \\ &= \frac{d}{dt}\left(\epsilon_0 \frac{b^2 d}{2} E_0 \sin\omega t\right) \\ &= \epsilon_0 \frac{b^2 d}{2} E_0 \omega \cos\omega t\end{aligned}$$

2.5 GAUSS' LAW FOR THE ELECTRIC FIELD

In the previous two sections we learned two of the four Maxwell's equations. These two equations have to do with the line integrals of the electric and magnetic fields around closed paths. The remaining two Maxwell's equations are pertinent to the surface integrals of the electric and magnetic fields over closed surfaces. These are known as *Gauss' laws*.

Gauss' law for the electric field states that "the total displacement flux emanating from a closed surface S is equal to the total charge contained within the volume V bounded by that surface," as illustrated in Figure 2.20. This statement, although familiarly known as Gauss' law, has its origin in experiments conducted by Faraday. In mathematical form, Gauss' law for the electric field is given by

$$\oint_S \mathbf{D}\cdot d\mathbf{S} = \int_V \rho\, dv \tag{2.30}$$

where ρ is the volume charge density associated with points in the volume V.

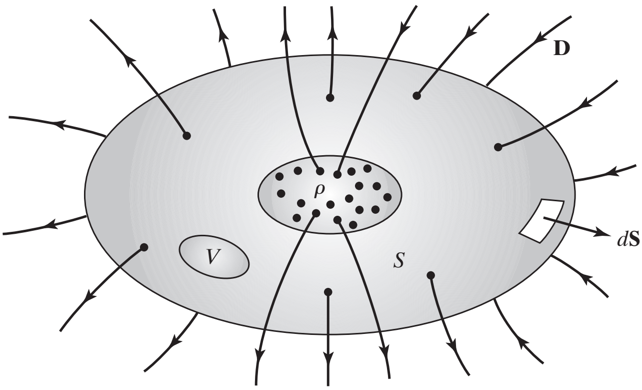

FIGURE 2.20

For illustrating Gauss' law for the electric field.

The volume charge density at a point is defined as the charge per unit volume (C/m^3) at that point in the limit that the volume shrinks to zero. Thus,

$$\rho = \lim_{\Delta v \to 0} \frac{\Delta Q}{\Delta v} \tag{2.31}$$

As an illustration of the computation of the charge contained in a given volume for a specified charge density, let us consider

$$\rho = (x + y + z)\ C/m^3$$

and the cubical volume V bounded by the planes $x = 0, x = 1, y = 0, y = 1, z = 0$, and $z = 1$. Then the charge Q contained within the cubical volume is given by

$$\begin{aligned}
Q &= \int_V \rho\, dv = \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} (x + y + z)\, dx\, dy\, dz \\
&= \int_{x=0}^{1}\int_{y=0}^{1}\left[xz + yz + \frac{z^2}{2}\right]_{z=0}^{1} dx\, dy \\
&= \int_{x=0}^{1}\int_{y=0}^{1}\left(x + y + \frac{1}{2}\right) dx\, dy \\
&= \int_{x=0}^{1}\left[xy + \frac{y^2}{2} + \frac{y}{2}\right]_{y=0}^{1} dx \\
&= \int_{x=0}^{1} (x + 1)\, dx \\
&= \left[\frac{x^2}{2} + x\right]_{x=0}^{1} \\
&= \frac{3}{2}\ \mathrm{C}
\end{aligned}$$

Although the quantity on the right side of (2.30), that is, the charge contained within the volume V bounded by the surface S associated with the quantity on the left side of (2.30), is formulated in terms of the volume charge density, it includes surface charges, line charges, and point charges enclosed by S. Thus it represents the algebraic sum of all the charges contained in the volume V. Let us now consider an example.

Example 2.11

A point charge Q is situated at the origin. It is desired to find $\oint_S \mathbf{D} \cdot d\mathbf{S}$ and $\mathbf{D}$ over the surface of a sphere of radius a centered at the origin.

According to Gauss' law for the electric field, the required displacement flux is given by

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q \tag{2.32}$$

To evaluate $\mathbf{D}$ on the surface of the sphere, we note that in order for $\oint_S \mathbf{D} \cdot d\mathbf{S}$ to be nonzero, $\mathbf{D}$ must be directed normal to the spherical surface. From symmetry considerations, it must have the same magnitude at all points on the spherical surface, since the surface is centered at the origin. Thus, let us divide the spherical surface into a large number of infinitesimal areas, as shown in Figure 2.21. Since $\mathbf{D}$ is normal to each area, $\mathbf{D} \cdot d\mathbf{S}$ for each area is simply equal to $D\,dS$. Hence,

$$\begin{aligned}\oint_S \mathbf{D} \cdot d\mathbf{S} &= D \int_S dS \\ &= D \text{ (surface area of the sphere)} \\ &= 4\pi a^2 D\end{aligned}$$

From (2.32), we then have

$$4\pi a^2 D = Q$$

or

$$D = \frac{Q}{4\pi a^2}$$

Thus, the displacement flux density due to the point charge is directed away from the charge and has a magnitude $Q/4\pi a^2$ where a is the distance of the point from the charge. The method we have discussed here is a standard procedure for the determination of the static electric field due to charge distributions possessing certain symmetries. We shall include some cases in the problems for the interested reader.

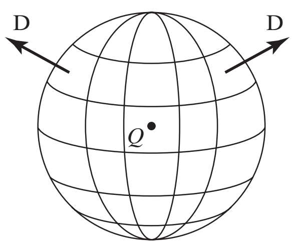

FIGURE 2.21

For evaluating the displacement flux density over the surface of a sphere centered at a point charge.

Gauss' law for the electric field is not independent of Ampere's circuital law if we recognize that, in view of conservation of electric charge, "the total current due to flow of charges emanating from a closed surface S is equal to the time rate of decrease of the charge within the volume V bounded by S," that is,

$$\oint_S \mathbf{J} \cdot d\mathbf{S} = -\frac{d}{dt}\int_V \rho\, dv$$

or

$$\oint_S \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt}\int_V \rho\, dv = 0 \tag{2.33}$$

This statement is known as the *law of conservation of charge*. In fact, it is this consideration that led to the mathematical contribution of Maxwell to Ampere's circuital law. Ampere's circuital law in its original form did not include the displacement current term which resulted in an inconsistency with (2.33) for time-varying fields.

Returning to the discussion of the dependency of Gauss' law on Ampere's circuital law through (2.33), let us consider the geometry of Figure 2.22, consisting of a closed path C and two surfaces S_1 and S_2, both of which are bounded by C. Applying Ampere's circuital law to C and S_1 and to C and S_2, we get

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_{S_1} \mathbf{J} \cdot d\mathbf{S}_1 + \frac{d}{dt}\int_{S_1} \mathbf{D} \cdot d\mathbf{S}_1 \tag{2.34a}$$

and

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = -\int_{S_2} \mathbf{J} \cdot d\mathbf{S}_2 - \frac{d}{dt}\int_{S_2} \mathbf{D} \cdot d\mathbf{S}_2 \tag{2.34b}$$

respectively. Combining (2.34a) and (2.34b), we obtain

$$\oint_{S_1+S_2} \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt}\oint_{S_1+S_2} \mathbf{D} \cdot d\mathbf{S} = 0 \tag{2.35}$$

Now, using (2.33), we have

$$-\frac{d}{dt}\int_V \rho\, dv + \frac{d}{dt}\oint_S \mathbf{D} \cdot d\mathbf{S} = 0$$

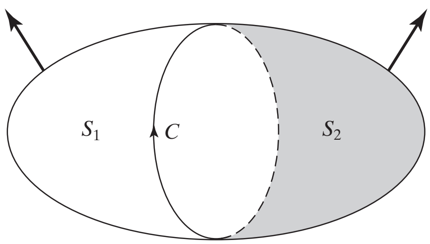

FIGURE 2.22

A closed path C, and two surfaces S_1 and S_2 bounded by C.

or

$$\frac{d}{dt}\left[\oint_S \mathbf{D} \cdot d\mathbf{S} - \int_V \rho\, dv\right] = 0 \tag{2.36}$$

where we have replaced $S_1 + S_2$ by S and where V is the volume enclosed by $S_1 + S_2$. Thus from (2.36), we get

$$\oint_S \mathbf{D} \cdot d\mathbf{S} - \int_V \rho\, dv = \text{constant with time} \tag{2.37}$$

Since there is no experimental evidence that the right side of (2.37) is nonzero, it follows that

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho\, dv$$

thereby giving Gauss' law for the electric field.

2.6 GAUSS' LAW FOR THE MAGNETIC FIELD

Gauss' law for the magnetic field states that "the total magnetic flux emanating from a closed surface S is equal to zero." In mathematical form, this is given by

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0 \tag{2.38}$$

In physical terms, (2.38) signifies that magnetic charges do not exist and magnetic flux lines are closed. Whatever magnetic flux enters (or leaves) a certain part of a closed surface must leave (or enter) through the remainder of the closed surface, as illustrated in Figure 2.23.

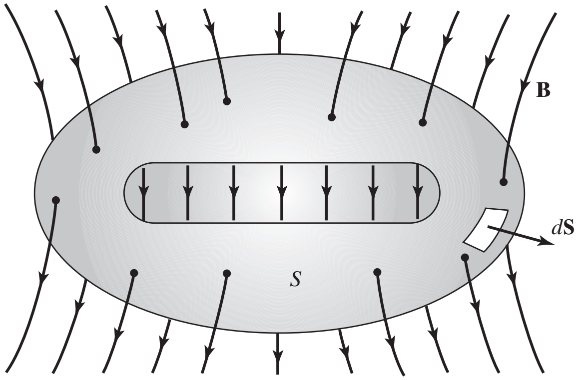

FIGURE 2.23
For illustrating Gauss' law for the magnetic field.

Equation (2.38) is not independent of Faraday's law. This can be shown by considering the geometry of Figure 2.22. Applying Faraday's law to C and S_1, we have

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_{S_1} \mathbf{B} \cdot d\mathbf{S}_1 \tag{2.39}$$

where $d\mathbf{S}_1$ is directed out of the volume bounded by the closed surface $S_1 + S_2$. Applying Faraday's law to C and S_2, we have

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = \frac{d}{dt}\int_{S_2} \mathbf{B} \cdot d\mathbf{S}_2 \tag{2.40}$$

where $d\mathbf{S}_2$ is directed out of the volume bounded by $S_1 + S_2$. Combining (2.39) and (2.40), we obtain

$$-\frac{d}{dt}\int_{S_1} \mathbf{B} \cdot d\mathbf{S}_1 = \frac{d}{dt}\int_{S_2} \mathbf{B} \cdot d\mathbf{S}_2 \tag{2.41}$$

or

$$\frac{d}{dt}\oint_{S_1+S_2} \mathbf{B} \cdot d\mathbf{S} = 0 \tag{2.42}$$

or

$$\oint_{S_1+S_2} \mathbf{B} \cdot d\mathbf{S} = \text{constant with time} \tag{2.43}$$

Since there is no experimental evidence that the right side of (2.43) is nonzero, it follows that

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$$

where we have replaced $S_1 + S_2$ by S.

SUMMARY

We first learned in this chapter how to evaluate line and surface integrals of vector quantities and then we introduced Maxwell's equations in integral form. These equations, which form the basis of electromagnetic field theory, are given as follows in words and in mathematical form and are illustrated in Figures 2.11, 2.15, 2.20, and 2.23, respectively.

Faraday's law. The electromotive force around a closed path C is equal to the negative of the time rate of change of the magnetic flux enclosed by that path, that is,

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S} \tag{2.44}$$

Ampere's circuital law. The magnetomotive force around a closed path C is equal to the sum of the current enclosed by that path due to the actual flow of charges and the displacement current due to the time rate of change of the displacement flux enclosed by that path, that is,

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt}\int_S \mathbf{D} \cdot d\mathbf{S} \tag{2.45}$$

Gauss' law for the electric field. The displacement flux emanating from a closed surface S is equal to the charge enclosed by that surface, that is,

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho \, dv \tag{2.46}$$

Gauss' law for the magnetic field. The magnetic flux emanating from a closed surface S is equal to zero, that is,

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0 \tag{2.47}$$

The vectors **D** and **H**, known as the displacement flux density and the magnetic field intensity vectors, respectively, are related to **E** and **B**, known as the electric field intensity and the magnetic flux density vectors, respectively, in the manner

$$\mathbf{D} = \epsilon_0 \mathbf{E} \tag{2.48}$$

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} \tag{2.49}$$

where ϵ_0 and μ_0 are the permittivity and the permeability of free space, respectively. In evaluating the right sides of (2.44) and (2.45), the normal vectors to the surfaces must be chosen such that they are directed in the right-hand sense, that is, toward the side of advance of a right-hand screw as it is turned around C, as shown in Figures 2.11 and 2.15. We have also learned that (2.47) is not independent of (2.44) and that (2.46) follows from (2.45) with the aid of the law of conservation of charge given by

$$\oint_S \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt} \int_V \rho \, dv = 0 \tag{2.50}$$

In words, (2.50) states that the sum of the current due to the flow of charges across a closed surface S and the time rate of increase of the charge within the volume V bounded by S is equal to zero. In (2.46), (2.47), and (2.50) the surface integrals must be evaluated in order to find the flux outward from the volume bounded by the surface.

Finally, we observe that time-varying electric and magnetic fields are interdependent, since according to Faraday's law (2.44), a time-varying magnetic field produces an electric field, whereas according to Ampere's circuital law (2.45), a time-varying electric field gives rise to a magnetic field. In addition, Ampere's circuital law tells us that an electric current generates a magnetic field. These properties from the basis for the phenomena of radiation and propagation of electromagnetic waves. To provide a simplified, qualitative explanation of radiation from an antenna, we begin with a piece of wire carrying a time-varying current, $I(t)$, as shown in Figure 2.24. Then, the time-varying current generates a time-varying magnetic field $\mathbf{H}(t)$, which surrounds the wire. Time-varying electric and magnetic fields, $\mathbf{E}(t)$ and $\mathbf{H}(t)$, are then produced in succession, as shown by two views in Figure 2.24, thereby giving rise to electromagnetic waves. Thus, just as water waves are produced when a rock is thrown in a pool of water, electromagnetic waves are radiated when a piece of wire in space is excited by a time-varying current.

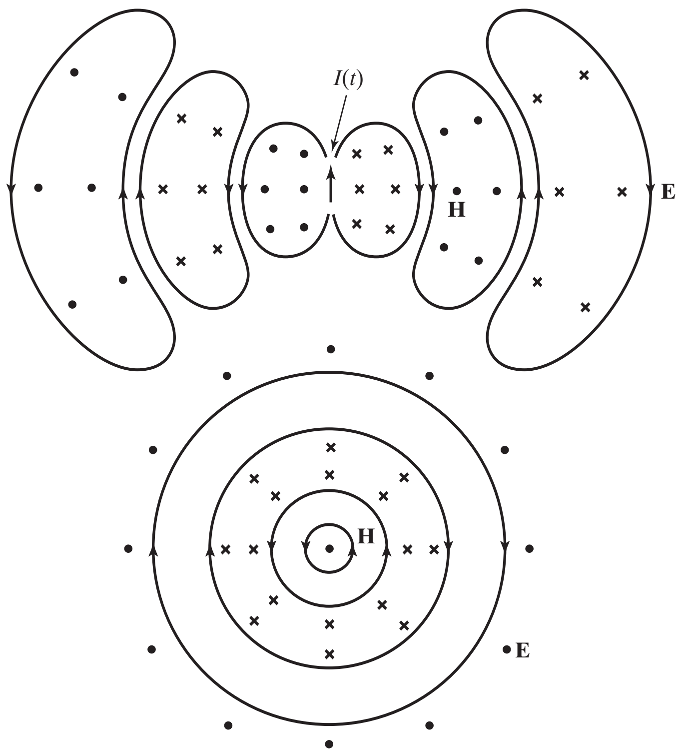

FIGURE 2.24

Two views of a simplified depiction of electromagnetic wave radiation from a piece of wire carrying a time-varying current.

REVIEW QUESTIONS

2.1. How do you find the work done in moving a test charge by an infinitesimal distance in an electric field?

2.2. What is the amount of work involved in moving a test charge normal to the electric field?

2.3. What is the physical interpretation of the line integral of **E** between two points A and B?

2.4. How do you find the approximate value of the line integral of a vector along a given path?

2.5. How do you find the exact value of the line integral?

2.6. What is the physical significance of the line integral of the earth's gravitational field intensity?

2.7. What is the value of the line integral of the earth's gravitational field intensity around a closed path?

2.8. How do you find the magnetic flux crossing an infinitesimal surface?

2.9. What is the magnetic flux crossing an infinitesimal surface oriented parallel to the magnetic flux density vector?

2.10. For what orientation of the infinitesimal surface relative to the magnetic flux density vector is the magnetic flux crossing the surface a maximum?

2.11. How do you find the approximate value of the surface integral over a given surface?

2.12. How do you find the exact value of the surface integral?

2.13. Provide physical interpretations for the closed surface integrals of any two vectors of your choice.

2.14. State Faraday's law.

2.15. Why is it necessary to have the minus sign associated with the time rate of increase of magnetic flux on the right side of Faraday's law?

2.16. What is electromotive force?

2.17. What are the different ways in which an emf is induced around a loop?

2.18. To find the induced emf around a planar loop, is it necessary to consider the magnetic flux crossing the plane surface bounded by the loop?

2.19. Discuss briefly the motional emf concept.

2.20. What is Lenz's law?

2.21. How would you orient a loop antenna in order to obtain maximum signal from an incident electromagnetic wave which has its magnetic field linearly polarized in the north–south direction?

2.22. State three applications of Faraday's law.

2.23. State Ampere's circuital law.

2.24. What are the units of the magnetic field intensity vector?

2.25. What are the units of the displacement flux density vector?

2.26. What is displacement current? Give an example involving displacement current.

2.27. Why is it necessary to have the displacement current term on the right side of Ampere's circuital law?

2.28. When can you say that the current in a wire enclosed by a closed path is uniquely defined? Give two examples.

2.29. Give an example in which the current in a wire enclosed by a closed path is not uniquely defined.

2.30. Is it meaningful to consider two different surfaces bounded by a closed path to compute the two different currents on the right side of Ampere's circuital law to find $\oint \mathbf{H} \cdot d\mathbf{l}$ around the closed path?

2.31. Discuss briefly the application of Ampere's circuital law to determine the magnetic field due to current distributions.

2.32. State Gauss' law for the electric field.

2.33. How is volume charge density defined?

2.34. State the law of conservation of charge.

2.35. How is Gauss' law for the electric field derived from Ampere's circuital law?

2.36. Discuss briefly the application of Gauss' law for the electric field to determine the electric field due to charge distributions.

2.37. State Gauss' law for the magnetic field. How is it derived from Faraday's law?

2.38. What is the physical interpretation of Gauss' law for the magnetic field?

2.39. Summarize Maxwell's equations in integral form. Discuss the interdependence of time-varying electric and magnetic fields, with the aid of an example.

2.40. Which two of the Maxwell's equations are independent?

PROBLEMS

2.1. For the force field $\mathbf{F} = x^2\mathbf{a}_y$, find the approximate value of the line integral of **F** from the origin to the point (1, 3, 0) along a straight line path by dividing the path into ten equal segments.

2.2. For the force field $\mathbf{F} = x^2\mathbf{a}_y$, obtain a series expression for the line integral of **F** from the origin to the point (1, 3, 0) along a straight line path by dividing the path into n equal segments. Express the sum of the series in closed form and compute its value for values of n equal to 5, 10, 100, and ∞.

2.3. For the force field $\mathbf{F} = x^2\mathbf{a}_y$, find the exact value of the line integral of **F** from the origin to the point (1, 3, 0) along a straight line path.

2.4. Given $\mathbf{E} = y\mathbf{a}_x + x\mathbf{a}_y$, find $\int_{(0,0,0)}^{(1,1,0)} \mathbf{E} \cdot d\mathbf{l}$ along the following paths: (a) straight line path $y = x, z = 0$, (b) straight line path from (0, 0, 0) to (1, 0, 0), and then straight line path from (1, 0, 0) to (1, 1, 0), and (c) any path of your choice.

2.5. Show that for any closed path C, $\oint_C d\mathbf{l} = 0$ and hence show that for a uniform field **F**, $\oint_C \mathbf{F} \cdot d\mathbf{l} = 0$.

2.6. Given $\mathbf{F} = y\mathbf{a}_x - x\mathbf{a}_y$, find $\oint_C \mathbf{F} \cdot d\mathbf{l}$ where C is the closed path in the xy-plane consisting of the following: the straight line path from (0, 0, 0) to (−1, 1, 0), the straight line path from (−1, 1, 0) to (0, $\sqrt{2}$, 0), the straight line path from (0, $\sqrt{2}$, 0) to (0, 1, 0), the circular path from (0, 1, 0) to (1, 0, 0) having its center at (0, 0, 0), and the straight line path from (1, 0, 0) to (0, 0, 0).

2.7. Given $\mathbf{F} = xy\mathbf{a}_x + yz\mathbf{a}_y + zx\mathbf{a}_z$, find $\oint_C \mathbf{F} \cdot d\mathbf{l}$ where C is the closed path comprising the straight lines from (0, 0, 0) to (1, 1, 1), from (1, 1, 1) to (1, 1, 0), and from (1, 1, 0) to (0, 0, 0).

2.8. For the magnetic flux density vector $\mathbf{B} = x^2e^{-y}\mathbf{a}_z$ Wb/m^2, find the approximate value of the magnetic flux crossing the portion of the xy-plane lying between $x = 0$, $x = 1$, $y = 0$, and $y = 1$, by dividing the area into 100 equal parts.

2.9. For the magnetic flux density vector $\mathbf{B} = x^2e^{-y}\mathbf{a}_z$ Wb/m^2, obtain a series expression for the magnetic flux crossing the portion of the xy-plane lying between $x = 0$, $x = 1$, $y = 0$, and $y = 1$ by dividing the area into n^2 equal parts. Express the sum of the series in closed form and compute its value for values of n equal to 5, 10, 100, and ∞.

2.10. For the magnetic flux density vector $\mathbf{B} = x^2e^{-y}\mathbf{a}_z$ Wb/m^2, find the exact value of the magnetic flux crossing the portion of the xy-plane lying between $x = 0$, $x = 1$, $y = 0$, and $y = 1$ by evaluating the surface integral of **B**.

2.11. Given $\mathbf{A} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$, find $\int_S \mathbf{A} \cdot d\mathbf{S}$ where S is the hemispherical surface of radius 2 m lying above the xy-plane and having its center at the origin.

2.12. Show that for any closed surface S, $\oint_S d\mathbf{S} = 0$ and hence show that for a uniform field **A**, $\oint_S \mathbf{A} \cdot d\mathbf{S} = 0$.

2.13. Given $\mathbf{J} = 3x\mathbf{a}_x + (y - 3)\mathbf{a}_y + (2 + z)\mathbf{a}_z$ A/m^2, find $\oint_S \mathbf{J} \cdot d\mathbf{S}$, that is, the current flowing out of the surface S of the rectangular box bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 2$, $z = 0$, and $z = 3$.

2.14. Given $\mathbf{E} = (y\mathbf{a}_x - x\mathbf{a}_y) \cos \omega t$ V/m, find the time rate of decrease of the magnetic flux crossing toward the positive z-side and enclosed by the path in the xy-plane from (0, 0, 0) to (1, 0, 0) along $y = 0$, from (1, 0, 0) to (1, 1, 0) along $x = 1$, and from (1, 1, 0) to (0, 0, 0) along $y = x^3$.

2.15. A magnetic field is given in the xz-plane by $\mathbf{B} = \dfrac{B_0}{x}\mathbf{a}_y$ Wb/m^2, where B_0 is a constant. A rigid rectangular loop is situated in the xz-plane and with its corners at the points (x_0, z_0), $(x_0, z_0 + b)$, $(x_0 + a, z_0 + b)$, and $(x_0 + a, z_0)$. If the loop is moving in that plane with a velocity $\mathbf{v} = v_0\mathbf{a}_x$ m/s, where v_0 is a constant, find by using Faraday's law the induced emf around the loop in the sense defined by connecting the above specified points in succession. Discuss your result by using the motional emf concept.

2.16. Assuming the rectangular loop of Problem 2.15 to be stationary, find the induced emf around the loop if $\mathbf{B} = \dfrac{B_0}{x}\cos\omega t\,\mathbf{a}_y$ Wb/m^2.

2.17. Assuming the rectangular loop of Problem 2.15 to be moving with the velocity $\mathbf{v} = v_0\mathbf{a}_x$ m/s, find the induced emf around the loop if $\mathbf{B} = \dfrac{B_0}{x}\cos\omega t\,\mathbf{a}_y$ Wb/m^2.

2.18. For $\mathbf{B} = B_0\cos\omega t\,\mathbf{a}_z$ Wb/m^2, find the induced emf around the closed path comprising the straight lines successively connecting the points (0, 0, 0), (1, 0, 0.01), (1, 1, 0.02), (0, 1, 0.03), (0, 0, 0.04), and (0, 0, 0).

2.19. Repeat Problem 2.18 for the closed path comprising the straight lines successively connecting the points (0, 0, 0), (1, 0, 0.01), (1, 1, 0.02), (0, 1, 0.03), (0, 0, 0.04), (1, 0, 0.05), (1, 1, 0.06), (0, 1, 0.07), (0, 0, 0.08), and (0, 0, 0), with a slight kink in the last straight line at the point (0, 0, 0.04) to avoid touching the point.

2.20. A rigid rectangular loop of area A is situated normal to the xy-plane and symmetrically about the z-axis. It revolves around the z-axis at ω_1 rad/s in the sense defined by the curling of the fingers of the right hand when the z-axis is grabbed with the thumb pointed in the positive z-direction. Find the induced emf around the loop if $\mathbf{B} = B_0\cos\omega_2 t\,\mathbf{a}_x$, where B_0 is a constant, and show that the induced emf has two frequency components $(\omega_1 + \omega_2)$ and $|\omega_1 - \omega_2|$.

2.21. For the revolving loop of Problem 2.20, find the induced emf around the loop if $\mathbf{B} = B_0(\cos\omega_1 t\,\mathbf{a}_x + \sin\omega_1 t\,\mathbf{a}_y)$.

2.22. For the revolving loop of Problem 2.20, find the induced emf around the loop if $\mathbf{B} = B_0(\cos\omega_1 t\,\mathbf{a}_x - \sin\omega_1 t\,\mathbf{a}_y)$.

2.23. A current I_1 flows from infinity to a point charge at the origin through a thin wire along the negative y-axis and a current I_2 flows from the point charge to infinity through another thin wire along the positive y-axis. From considerations of uniqueness of $\oint_C \mathbf{H}\cdot d\mathbf{l}$, find the displacement current emanating from (a) a spherical surface of radius 1 m and having its center at the point (2, 2, 2) and (b) a spherical surface of radius 1 m and having its center at the origin.

2.24. A current density due to flow of charges is given by $\mathbf{J} = y\cos\omega t\,\mathbf{a}_y$ A/m^2. From consideration of uniqueness of $\oint_C \mathbf{H}\cdot d\mathbf{l}$, find the displacement current emanating from the cubical box bounded by the planes $x = 0, x = 1, y = 0, y = 1, z = 0$, and $z = 1$.

2.25. An infinitely long, cylindrical wire of radius a, having the z-axis as its axis, carries current in the positive z-direction with uniform density J_0 A/m^2. Find $\mathbf{H}$ both inside and outside the wire.

2.26. An infinitely long, hollow, cylindrical wire of inner radius a and outer radius b, having the z-axis as its axis, carries current in the positive z-direction with uniform density J_0 A/m^2. Find $\mathbf{H}$ everywhere.

2.27. An infinitely long, straight wire situated along the z-axis carries current I in the positive z-direction. What are the values of $\int_{(1,0,0)}^{(0,1,0)} \mathbf{H}\cdot d\mathbf{l}$ along (a) the circular path of radius 1 m and centered at the origin and (b) along a straight line path?

2.28. Given $\mathbf{D} = y\mathbf{a}_y$, find the charge contained in the volume of the wedge-shaped box defined by the planes $x = 0, x + z = 1, y = 0, y = 1$, and $z = 0$.

2.29. Given $\rho = xe^{-x^2}$ C/m^3, find the displacement flux emanating from the surface of the cubical box defined by the planes $x = 0, x = 1, y = 0, y = 1, z = 0$, and $z = 1$.

2.30. Charge is distributed uniformly along the z-axis with density ρ_{L0} C/m. Using Gauss' law for the electric field, find the electric field intensity due to the line charge.

2.31. Charge is distributed uniformly with density ρ_0 C/m^3 within a spherical volume of radius a m and having its center at the origin. Using Gauss' law for the electric field, find the electric field intensity both inside and outside the charge distribution.

2.32. A point charge Q C is situated at the origin. What are the values of the displacement flux crossing (a) the spherical surface $x^2 + y^2 + z^2 = 1, x > 0, y > 0$, and $z > 0$ and (b) the plane surface $x + y + z = 1, x > 0, y > 0$, and $z > 0$?

2.33. Given $\mathbf{J} = x\mathbf{a}_x$ A/m^2, find the time rate of increase of the charge contained in the cubical volume bounded by the planes $x = 0, x = 1, y = 0, y = 1, z = 0$, and $z = 1$.

2.34. Given $\mathbf{J} = x\mathbf{a}_x$ A/m^2, find the time rate of increase of the charge contained in the volume of the wedge-shaped box that is defined by the planes $x = 0, x + z = 1, y = 0, y = 1$, and $z = 0$.

2.35. Using the property that $\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$, find the absolute value of $\int \mathbf{B} \cdot d\mathbf{S}$ over that portion of the surface $y = \sin x$ bounded by $x = 0$, $x = \pi$, $z = 0$, and $z = 1$, for $\mathbf{B} = y\mathbf{a}_x - x\mathbf{a}_y$.

2.36. Repeat Problem 2.35 for the plane rectangular surface having the vertices at $(0, 0, 0)$, $(0, 0, 1)$, $(1, 1, 1)$, and $(0, 1, 1)$.