APPENDIX B

Curl, Divergence, Gradient, and Laplacian in Cylindrical and Spherical Coordinate Systems

In Chapter 3, we introduced the curl, divergence, gradient, and Laplacian and derived the expressions for them in the Cartesian coordinate system. In this appendix, we derive the corresponding expressions in the cylindrical and spherical coordinate systems. Considering first the cylindrical coordinate system, we recall from Section 1.3 that the infinitesimal box defined by the three orthogonal surfaces intersecting at point $P(r, \phi, z)$ and the three orthogonal surfaces intersecting at point $Q(r + dr, \phi + d\phi, z + dz)$ is as shown in Fig. B.1.

From the basic definition of the curl of a vector introduced in Section 3.3 and given by

$$\nabla \times \mathbf{A} = \lim_{\Delta S \to 0} \left[\frac{\oint_C \mathbf{A} \cdot d\mathbf{l}}{\Delta S} \right]_{\max} \mathbf{a}_n \tag{B.1}$$

we find the components of $\nabla \times \mathbf{A}$ as follows, with the aid of Fig. B.1:

$$\begin{aligned}
(\nabla \times \mathbf{A})_r &= \lim_{\substack{d\phi \to 0 \\ dz \to 0}} \frac{\oint_{abcda} \mathbf{A} \cdot d\mathbf{l}}{\text{area } abcd} \\
&= \lim_{\substack{d\phi \to 0 \\ dz \to 0}} \frac{\begin{Bmatrix} [A_\phi]_{(r,z)}\, r\, d\phi + [A_z]_{(r,\phi+d\phi)}\, dz \\ -\, [A_\phi]_{(r,z+dz)}\, r\, d\phi - [A_z]_{(r,\phi)}\, dz \end{Bmatrix}}{r\, d\phi\, dz} \\
&= \lim_{d\phi \to 0} \frac{[A_z]_{(r,\phi+d\phi)} - [A_z]_{(r,\phi)}}{r\, d\phi} + \lim_{dz \to 0} \frac{[A_\phi]_{(r,z)} - [A_\phi]_{(r,z+dz)}}{dz} \\
&= \frac{1}{r} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}
\end{aligned} \tag{B.2a}$$

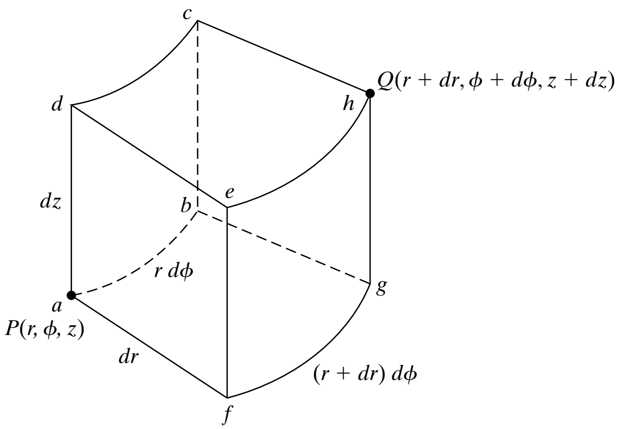

FIGURE B.1

Infinitesimal box formed by incrementing the coordinates in the cylindrical coordinate system.

$$
\begin{aligned}
(\nabla \times \mathbf{A})_\phi &= \lim_{\substack{dz\to 0\\ dr\to 0}} \frac{\oint_{adefa} \mathbf{A}\cdot d\mathbf{l}}{\text{area } adef} \\
&= \lim_{\substack{dz\to 0\\ dr\to 0}} \frac{\left\{\begin{matrix} [A_z]_{(r,\phi)}\, dz + [A_r]_{(\phi,\, z+dz)}\, dr \\ -[A_z]_{(r+dr,\phi)}\, dz - [A_r]_{(\phi,z)}\, dr \end{matrix}\right\}}{dr\, dz} \qquad \text{(B.2b)} \\
&= \lim_{dz\to 0} \frac{[A_r]_{(\phi,z+dz)} - [A_r]_{(\phi,z)}}{dz} + \lim_{dr\to 0} \frac{[A_z]_{(r,\phi)} - [A_z]_{(r+dr,\phi)}}{dr} \\
&= \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}
\end{aligned}
$$

$$
\begin{aligned}
(\nabla \times \mathbf{A})_z &= \lim_{\substack{dr\to 0\\ d\phi\to 0}} \frac{\oint_{afgba} \mathbf{A}\cdot d\mathbf{l}}{\text{area } afgb} \\
&= \lim_{\substack{dr\to 0\\ d\phi\to 0}} \frac{\left\{\begin{matrix} [A_r]_{(\phi,z)}\, dr + [A_\phi]_{(r+dr,z)}(r + dr)\, d\phi \\ - [A_r]_{(\phi+d\phi,z)}\, dr - [A_\phi]_{(r,z)} r\, d\phi \end{matrix}\right\}}{r\, dr\, d\phi} \qquad \text{(B.2c)} \\
&= \lim_{dr\to 0} \frac{[rA_\phi]_{(r+dr,z)} - [rA_\phi]_{(r,z)}}{r\, dr} + \lim_{d\phi\to 0} \frac{[A_r]_{(\phi,z)} - [A_r]_{(\phi+d\phi,z)}}{r\, d\phi} \\
&= \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi) - \frac{1}{r}\frac{\partial A_r}{\partial \phi}
\end{aligned}
$$

Combining (B.2a), (B.2b), and (B.2c), we obtain the expression for the curl of a vector in cylindrical coordinates as

$$\begin{aligned}\nabla \times \mathbf{A} &= \left[\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right]\mathbf{a}_r + \left[\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right]\mathbf{a}_\phi \\ &\quad + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial \phi}\right]\mathbf{a}_z \\ &= \begin{vmatrix} \dfrac{\mathbf{a}_r}{r} & \mathbf{a}_\phi & \dfrac{\mathbf{a}_z}{r} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix}\end{aligned} \tag{B.3}$$

To find the expression for the divergence, we use the basic definition of the divergence of a vector, introduced in Section 3.3 and given by

$$\nabla \cdot \mathbf{A} = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta v} \tag{B.4}$$

Evaluating the right side of (B.4) for the box of Fig. B.1, we obtain

$$\begin{aligned}\nabla \cdot \mathbf{A} &= \lim_{\substack{dr \to 0 \\ d\phi \to 0 \\ dz \to 0}} \frac{\left\{\begin{aligned}&[A_r]_{r+dr}(r + dr)\, d\phi\, dz - [A_r]_r\, r\, d\phi\, dz + [A_\phi]_{\phi+d\phi}\, dr\, dz \\ &- [A_\phi]_\phi\, dr\, dz + [A_z]_{z+dz}\, r\, dr\, d\phi - [A_z]_z\, r\, dr\, d\phi\end{aligned}\right\}}{r\, dr\, d\phi\, dz} \\ &= \lim_{dr \to 0} \frac{[rA_r]_{r+dr} - [rA_r]_r}{r\, dr} + \lim_{d\phi \to 0} \frac{[A_\phi]_{\phi+d\phi} - [A_\phi]_\phi}{r\, d\phi} \\ &\quad + \lim_{dz \to 0} \frac{[A_z]_{z+dz} - [A_z]_z}{dz} \\ &= \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}\end{aligned} \tag{B.5}$$

To obtain the expression for the gradient of a scalar, we recall from Section 1.3 that in cylindrical coordinates,

$$d\mathbf{l} = dr\, \mathbf{a}_r + r\, d\phi\, \mathbf{a}_\phi + dz\, \mathbf{a}_z \tag{B.6}$$

Therefore,

$$\begin{aligned}d\Phi &= \frac{\partial \Phi}{\partial r} dr + \frac{\partial \Phi}{\partial \phi} d\phi + \frac{\partial \Phi}{\partial z} dz \\ &= \left(\frac{\partial \Phi}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial \Phi}{\partial \phi}\mathbf{a}_\phi + \frac{\partial \Phi}{\partial z}\mathbf{a}_z\right) \cdot (dr\, \mathbf{a}_r + r\, d\phi\, \mathbf{a}_\phi + dz\, \mathbf{a}_z) \\ &= \nabla \Phi \cdot d\mathbf{l}\end{aligned} \tag{B.7}$$

Thus,

$$\nabla \Phi = \frac{\partial \Phi}{\partial r}\,\mathbf{a}_r + \frac{1}{r}\frac{\partial \Phi}{\partial \phi}\,\mathbf{a}_\phi + \frac{\partial \Phi}{\partial z}\,\mathbf{a}_z \tag{B.8}$$

To derive the expression for the Laplacian of a scalar, we recall from Section 5.1 that

$$\nabla^2 \Phi = \nabla \cdot \nabla \Phi \tag{B.9}$$

Then using (B.5) and (B.8), we obtain

$$\begin{aligned} \nabla^2 \Phi &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial \phi}\left(\frac{1}{r}\frac{\partial \Phi}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \Phi}{\partial z}\right) \\ &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2} \end{aligned} \tag{B.10}$$

Turning now to the spherical coordinate system, we recall from Section 1.3 that the infinitesimal box defined by the three orthogonal surfaces intersecting at $P(r, \theta, \phi)$ and the three orthogonal surfaces intersecting at $Q(r + dr, \theta + d\theta, \phi + d\phi)$ is as shown in Fig. B.2. From the basic definition of the curl of a vector given by (B.1), we then find the components of $\nabla \times \mathbf{A}$ as follows, with the aid of Fig. B.2:

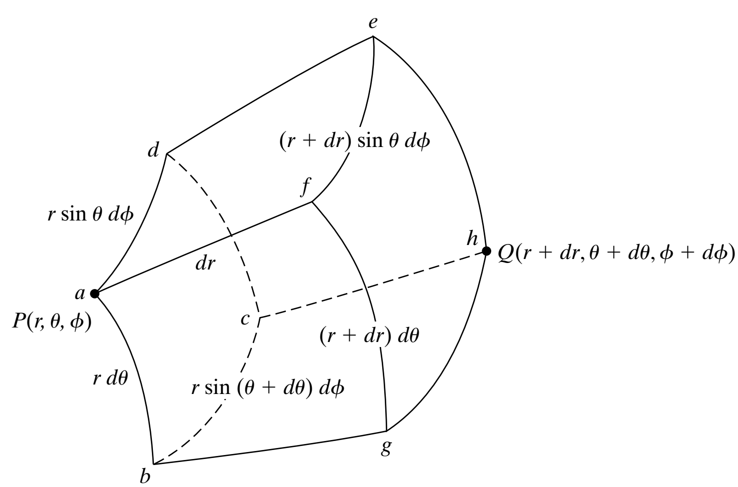

FIGURE B.2

Infinitesimal box formed by incrementing the coordinates in the spherical coordinate system.

$$
\begin{aligned}
(\nabla \times \mathbf{A})_r &= \lim_{\substack{d\theta \to 0 \\ d\phi \to 0}} \frac{\oint_{abcda} \mathbf{A} \cdot d\mathbf{l}}{\text{area } abcd} \\
&= \lim_{\substack{d\theta \to 0 \\ d\phi \to 0}} \frac{\left\{ \begin{array}{l} [A_\theta]_{(r,\phi)} r\, d\theta + [A_\phi]_{(r,\theta+d\theta)} r \sin(\theta + d\theta)\, d\phi \\ -\, [A^\theta]({}_{r,\phi+d\phi}) r\, d\theta - [A_\phi]_{(r,\theta)} r \sin\theta\, d\phi \end{array} \right\}}{r^2 \sin\theta\, d\theta\, d\phi} \\
&= \lim_{d\theta \to 0} \frac{[A_\phi \sin\theta]_{(r,\theta+d\theta)} - [A_\phi \sin\theta]_{(r,\theta)}}{r \sin\theta\, d\theta} \\
&\quad + \lim_{d\phi \to 0} \frac{[A_\theta]_{(r,\phi)} - [A_\theta]_{(r,\phi+d\phi)}}{r \sin\theta\, d\phi} \\
&= \frac{1}{r\sin\theta} \frac{\partial}{\partial\theta}(A_\phi \sin\theta) - \frac{1}{r\sin\theta} \frac{\partial A_\theta}{\partial\phi}
\end{aligned}
\tag{B.11a}
$$

$$
\begin{aligned}
(\nabla \times \mathbf{A})_\theta &= \lim_{\substack{d\phi \to 0 \\ dr \to 0}} \frac{\oint_{adefa} \mathbf{A} \cdot d\mathbf{l}}{\text{area } adef} \\
&= \lim_{\substack{d\phi \to 0 \\ dr \to 0}} \frac{\left\{ \begin{array}{l} [A_\phi]_{(r,\theta)}\, r \sin\theta\, d\phi + [A_r]_{(\theta,\phi+d\phi)}\, dr \\ -[A_\phi]_{(r+dr,\theta)} (r + dr) \sin\theta\, d\phi - [A_r]_{(\theta,\phi)}\, dr \end{array} \right\}}{r \sin\theta\, dr\, d\phi} \\
&= \lim_{d\phi \to 0} \frac{[A_r]_{(\theta,\phi+d\phi)} - [A_r]_{(\theta,\, \phi)}}{r \sin\theta\, d\phi} \\
&\quad + \lim_{dr \to 0} \frac{[rA_\phi]_{(r,\theta)} - [rA_\phi]_{(r+dr,\, \theta)}}{r\, dr} \\
&= \frac{1}{r\sin\theta} \frac{\partial A_r}{\partial\phi} - \frac{1}{r} \frac{\partial}{\partial r}(rA_\phi)
\end{aligned}
\tag{B.11b}
$$

$$
\begin{aligned}
(\nabla \times \mathbf{A})_\phi &= \lim_{\substack{dr \to 0 \\ d\theta \to 0}} \frac{\oint_{afgba} \mathbf{A} \cdot d\mathbf{l}}{\text{area } afgb} \\
&= \lim_{\substack{dr \to 0 \\ d\theta \to 0}} \frac{\left\{ \begin{array}{l} [A_r]_{(\theta,\phi)}\, dr + [A_\theta]_{(r+dr,\phi)} (r + dr)\, d\theta \\ -[A_r]_{(\theta+d\theta,\phi)}\, dr - [A_\theta]_{(r,\phi)}\, r\, d\theta \end{array} \right\}}{r\, dr\, d\theta} \\
&= \lim_{dr \to 0} \frac{[rA_\theta]_{(r+dr,\phi)} - [rA_\theta]_{(r,\phi)}}{r\, dr} \\
&\quad + \lim_{d\theta \to 0} \frac{[A_r]_{(\theta,\phi)}\, dr - [A_r]_{(\theta+d\theta,\phi)}\, dr}{r\, d\theta} \\
&= \frac{1}{r} \frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r} \frac{\partial A_r}{\partial\theta}
\end{aligned}
\tag{B.11c}
$$

Combining (B.11a), (B.11b), and (B.11c), we obtain the expression for the curl of a vector in spherical coordinates as

$$\begin{aligned}
\boldsymbol{\nabla} \times \mathbf{A} &= \frac{1}{r \sin\theta}\left[\frac{\partial}{\partial\theta}(A_\phi \sin\theta) - \frac{\partial A_\theta}{\partial\phi}\right]\mathbf{a}_r \\
&\quad + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)\right]\mathbf{a}_\theta \\
&\quad + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]\mathbf{a}_\phi \\
&= \begin{vmatrix} \dfrac{\mathbf{a}_r}{r^2 \sin\theta} & \dfrac{\mathbf{a}_\theta}{r\sin\theta} & \dfrac{\mathbf{a}_\phi}{r} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ A_r & rA_\theta & r\sin\theta\, A_\phi \end{vmatrix}
\end{aligned} \tag{B.12}$$

To find the expression for the divergence, we use the basic definition of the divergence of a vector given by (B.4), and by evaluating its right side for the box of Fig. B.2, we obtain

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{A} &= \lim_{\substack{dr\to 0\\ d\theta\to 0\\ d\phi\to 0}} \frac{\left\{\begin{array}{l}[A_r]_{r+dr}(r+dr)^2 \sin\theta\, d\theta\, d\phi - [A_r]_r r^2 \sin\theta\, d\theta\, d\phi \\ \;+ [A^\theta]^{\theta+}{}_{d\theta}\, r\sin(\theta + d\theta)\, dr\, d\phi - [A^\theta]^\theta\, r\sin\theta\, dr\, d\phi \\ \;+ [A^\phi]^{\phi+}{}_{d\phi}\, r\, dr\, d\theta - [A_\phi]_\phi r\, dr\, d\theta \end{array}\right\}}{r^2 \sin\theta\, dr\, d\theta\, d\phi} \\
&= \lim_{dr\to 0}\frac{[r^2 A_r]_{r+dr} - [r^2 A_r]_r}{r^2\, dr} + \lim_{d\theta\to 0}\frac{[A_\theta \sin\theta]_{\theta+d\theta} - [A_\theta\sin\theta]_\theta}{r\sin\theta\, d\theta} \\
&\quad + \lim_{d\phi\to 0}\frac{[A_\phi]_{\phi+d\phi} - [A_\phi]_\phi}{r\sin\theta\, d\phi} \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}
\end{aligned} \tag{B.13}$$

To obtain the expression for the gradient of a scalar, we recall from Section 1.3 that in spherical coordinates,

$$d\mathbf{l} = dr\,\mathbf{a}_r + r\,d\theta\,\mathbf{a}_\theta + r\sin\theta\,d\phi\,\mathbf{a}_\phi \tag{B.14}$$

Therefore,

$$\begin{aligned} d\Phi &= \frac{\partial \Phi}{\partial r}\,dr + \frac{\partial \Phi}{\partial \theta}\,d\theta + \frac{\partial \Phi}{\partial \phi}\,d\phi \\ &= \left(\frac{\partial \Phi}{\partial r}\,\mathbf{a}_r + \frac{1}{r}\frac{\partial \Phi}{\partial \theta}\,\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial \Phi}{\partial \phi}\,\mathbf{a}_\phi\right) \\ &\quad \cdot (dr\,\mathbf{a}_r + r\,d\theta\,\mathbf{a}_\theta + r\sin\theta\,d\phi\,\mathbf{a}_\phi) \\ &= \boldsymbol{\nabla}\Phi \cdot d\mathbf{l} \end{aligned} \tag{B.15}$$

Thus,

$$\boldsymbol{\nabla}\Phi = \frac{\partial \Phi}{\partial r}\,\mathbf{a}_r + \frac{1}{r}\frac{\partial \Phi}{\partial \theta}\,\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial \Phi}{\partial \phi}\,\mathbf{a}_\phi \tag{B.16}$$

To derive the expression for the Laplacian of a scalar, we use (B.9), in conjunction with (B.13) and (B.16). Thus, we obtain

$$\begin{aligned} \nabla^2\Phi &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \Phi}{\partial r}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial \Phi}{\partial \theta}\sin\theta\right) \\ &\quad + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\left(\frac{1}{r\sin\theta}\frac{\partial \Phi}{\partial \phi}\right) \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\,\frac{\partial \Phi}{\partial \theta}\right) \\ &\quad + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 \Phi}{\partial \phi^2} \end{aligned} \tag{B.17}$$