APPENDIX A

Complex Numbers and Phasor Technique

In this appendix, we discuss a mathematical technique known as the phasor technique, pertinent to operations involving sinusoidally time-varying quantities. The technique simplifies the solution of a differential equation in which the steady-state response for a sinusoidally time-varying excitation is to be determined, by reducing the differential equation to an algebraic equation involving phasors. A phasor is a complex number or a complex variable. We first review complex numbers and associated operations.

A complex number has two parts: a real part and an imaginary part. Imaginary numbers are square roots of negative real numbers. To introduce the concept of an imaginary number, we define

$$\boxed{\sqrt{-1} = j} \tag{A.1a}$$

or

$$\boxed{(\pm j)^2 = -1} \tag{A.1b}$$

Thus, $j5$ is the positive square root of -25, $-j10$ is the negative square root of -100, and so on. A complex number is written in the form $a + jb$, where a is the real part and b is the imaginary part. Examples are

$$3 + j4 \qquad -4 + j1 \qquad -2 - j2 \qquad 2 - j3$$

Rectangular form

A complex number is represented graphically in a complex plane by using two orthogonal axes, corresponding to the real and imaginary parts, as shown in Fig. A.1, in which are plotted the numbers just listed. Since the set of orthogonal axes resembles the rectangular coordinate axes, the representation $(a + jb)$ is known as the rectangular form.

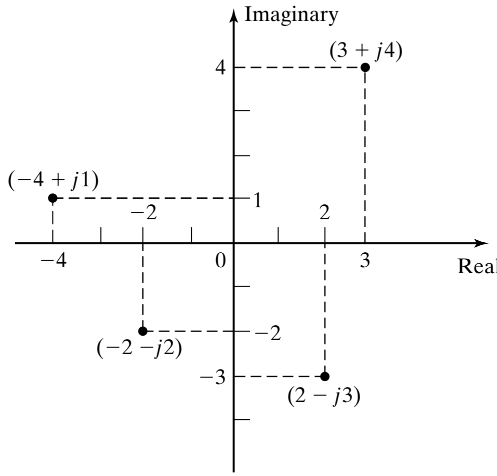

FIGURE A.1

Graphical representation of complex numbers in rectangular form.

Exponential and polar forms

An alternative form of representation of a complex number is the exponential form $Ae^{j\phi}$, where A is the magnitude and ϕ is the phase angle. To convert from one form to another, we first recall that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \tag{A.2}$$

Substituting $x = j\phi$, we have

$$\begin{aligned} e^{j\phi} &= 1 + j\phi + \frac{(j\phi)^2}{2!} + \frac{(j\phi)^3}{3!} + \cdots \\ &= 1 + j\phi - \frac{\phi^2}{2!} - j\frac{\phi^3}{3!} + \cdots \\ &= \left(1 - \frac{\phi^2}{2!} + \cdots\right) + j\left(\phi - \frac{\phi^3}{3!} + \cdots\right) \\ &= \cos\phi + j\sin\phi \end{aligned} \tag{A.3}$$

This is the so-called *Euler's identity*. Thus,

$$\boxed{\begin{aligned} Ae^{j\phi} &= A(\cos\phi + j\sin\phi) \\ &= A\cos\phi + jA\sin\phi \end{aligned}} \tag{A.4}$$

Now, equating the two forms of the complex numbers, we have

$$a + jb = A\cos\phi + jA\sin\phi$$

or

$$a = A\cos\phi \tag{A.5a}$$

$$b = A\sin\phi \tag{A.5b}$$

These expressions enable us to convert from exponential form to rectangular form. To convert from rectangular form to exponential form, we note that

$$a^2 + b^2 = A^2$$

$$\cos\phi = \frac{a}{A} \qquad \sin\phi = \frac{b}{A} \qquad \tan\phi = \frac{b}{a}$$

Thus,

$$A = \sqrt{a^2 + b^2} \tag{A.6a}$$

$$\phi = \tan^{-1}\frac{b}{a} \tag{A.6b}$$

Note that in the determination of ϕ, the signs of $\cos\phi$ and $\sin\phi$ should be considered to see if it is necessary to add π to the angle obtained by taking the inverse tangent of b/a.

In terms of graphical representation, A is simply the distance from the origin of the complex plane to the point under consideration, and ϕ is the angle measured counterclockwise from the positive real axis ($\phi = 0$) to the line drawn from the origin to the complex number, as shown in Fig. A.2. Since this representation is akin to the polar coordinate representation of points in two-dimensional space, the complex number is also written as $A\underline{/\phi}$, the polar form.

Turning now to Euler's identity, we see that for $\phi = \pm\pi/2$, $Ae^{\pm j\pi/2} = A\cos\pi/2 \pm jA\sin\pi/2 = \pm jA$. Thus, purely imaginary numbers correspond to $\phi = \pm\pi/2$. This justifies why the vertical axis, which is orthogonal to the real (horizontal) axis, is the imaginary axis.

Conversion from rectangular to exponential or polar form

The complex numbers in rectangular form plotted in Fig. A.1 may now be converted to exponential form (or polar form):

$$\begin{aligned}
3 + j4 &= \sqrt{3^2 + 4^2}\, e^{j\tan^{-1}(4/3)} = 5e^{j0.295\pi} = 5\underline{/53.13^\circ} \\
-4 + j1 &= \sqrt{4^2 + 1^2}\, e^{j[\tan^{-1}(-1/4)+\pi]} = 4.12e^{j0.922\pi} = 4.12\underline{/165.96^\circ} \\
-2 - j2 &= \sqrt{2^2 + 2^2}\, e^{j[\tan^{-1}(1)+\pi]} = 2.83e^{j1.25\pi} = 2.83\underline{/225^\circ} \\
2 - j3 &= \sqrt{2^2 + 3^2}\, e^{j\tan^{-1}(-3/2)} = 3.61e^{-j0.313\pi} = 3.61\underline{/-56.31^\circ}.
\end{aligned}$$

These are shown plotted in Fig. A.3. It can be noted that in converting from rectangular to exponential (or polar) form, the angle ϕ can be correctly determined if the number is first plotted in the complex plane to see in which quadrant it lies. Also note that angles traversed in the clockwise sense from the

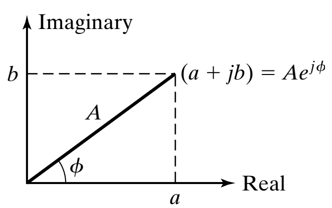

FIGURE A.2

Graphical representation of a complex number in exponential form or polar form.

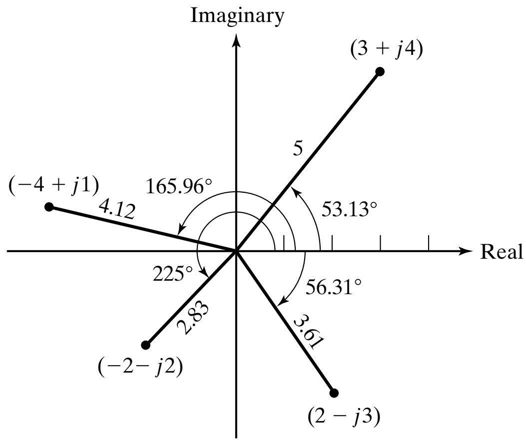

FIGURE A.3

Polar form representation of the complex numbers of Fig. A.1.

positive real axis are negative angles. Furthermore, adding or subtracting an integer multiple of 2π to the angle does not change the complex number.

Arithmetic of complex numbers

Complex numbers are added (or subtracted) by simply adding (or subtracting) their real and imaginary parts separately as follows:

$$(3 + j4) + (2 - j3) = 5 + j1$$
$$(2 - j3) - (-4 + j1) = 6 - j4$$

Graphically, this procedure is identical to the parallelogram law of addition (or subtraction) of two vectors.

Two complex numbers are multiplied by multiplying each part of one complex number by each part of the second complex number and adding the four products according to the rule of addition as follows:

$$\begin{aligned}(3 + j4)(2 - j3) &= 6 - j9 + j8 - j^2(12)\\ &= 6 - j9 + j8 + 12\\ &= 18 - j1\end{aligned}$$

Two complex numbers whose real parts are equal but whose imaginary parts are the negative of each other are known as complex conjugates. Thus, $(a - jb)$ is the complex conjugate of $(a + jb)$, and vice versa. The product of two complex conjugates is a real number:

$$(a + jb)(a - jb) = a^2 - jab + jba + b^2 = a^2 + b^2 \tag{A.7}$$

This property is used in division of one complex number by another by multiplying both the numerator and the denominator by the complex conjugate of the denominator and then performing the division by real number. For example,

$$\frac{3 + j4}{2 - j3} = \frac{(3 + j4)(2 + j3)}{(2 - j3)(2 + j3)} = \frac{-6 + j17}{13} = -0.46 + j1.31$$

The exponential form is particularly useful for multiplication, division, and other operations, such as raising to the power, since the rules associated with exponential functions are applicable. Thus,

$$(A_1e^{j\phi_1})(A_2e^{j\phi_2}) = A_1A_2e^{j(\phi_1+\phi_2)} \tag{A.8a}$$

$$\frac{A_1e^{j\phi_1}}{A_2e^{j\phi_2}} = \frac{A_1}{A_2}e^{j(\phi_1-\phi_2)} \tag{A.8b}$$

$$(Ae^{j\phi})^n = A^n e^{jn\phi} \tag{A.8c}$$

Let us consider some numerical examples:

(a) $(5e^{j0.295\pi})(3.61e^{-j0.313\pi}) = 18.05e^{-j0.018\pi}$

(b) $\dfrac{5e^{j0.295\pi}}{3.61e^{-j0.313\pi}} = 1.39e^{j0.608\pi}$

(c) $(2.83e^{j1.25\pi})^4 = 64.14e^{j5\pi} = 64.14e^{j\pi}$

(d)
$$\begin{aligned}\sqrt{4.12e^{j0.922\pi}} &= [4.12e^{j(0.922\pi+2k\pi)}]^{1/2}, \qquad k = 0, 1, 2, \ldots \\ &= \sqrt{4.12}\, e^{j(0.461\pi+k\pi)}, \qquad k = 0, 1 \\ &= 2.03e^{j0.461\pi}, \text{ or } 2.03e^{j1.461\pi}\end{aligned}$$

Note that in evaluating the square roots, although k can assume an infinite number of integer values, only the first two need to be considered since the numbers repeat themselves for higher values of integers. Similar considerations apply for cube roots, and so on.

Phasor defined

Having reviewed complex numbers, we are now ready to discuss the phasor technique. The basis behind the phasor technique lies in the fact that since

$$\boxed{Ae^{jx} = A\cos x + jA\sin x} \tag{A.9}$$

we can write

$$\boxed{A\cos x = \text{Re}[Ae^{jx}]} \tag{A.10}$$

where Re stands for "real part of." In particular, if $x = \omega t + \theta$, then we have

$$\begin{aligned}A\cos(\omega t + \theta) &= \text{Re}[Ae^{j(\omega t+\theta)}] \\ &= \text{Re}[Ae^{j\theta}e^{j\omega t}] \\ &= \text{Re}[\overline{A}e^{j\omega t}]\end{aligned} \tag{A.11}$$

where $\overline{A} = Ae^{j\theta}$ is known as the phasor (the overbar denotes that $\overline{A}$ is complex) corresponding to $A\cos(\omega t + \theta)$. Thus, the phasor corresponding to a cosinusoidally time-varying function is a complex number having magnitude same as the amplitude of the cosine function and phase angle equal to the phase of the cosine function for $t = 0$. To find the phasor corresponding to a sine function, we first convert it into a cosine function and proceed as in (A.11). Thus,

$$\begin{aligned} B \sin(\omega t + \phi) &= B \cos(\omega t + \phi - \pi/2) \\ &= \text{Re}[Be^{j(\omega t + \phi - \pi/2)}] \\ &= \text{Re}[Be^{j(\phi - \pi/2)}e^{j\omega t}] \end{aligned} \tag{A.12}$$

Hence, the phasor corresponding to $B \sin(\omega t + \phi)$ is $Be^{j(\phi - \pi/2)}$, or $Be^{j\phi}e^{-j\pi/2}$, or $-jBe^{j\phi}$.

Addition of two sine functions

Let us now consider the addition of two sinusoidally time-varying functions (of the same frequency), for example, $5 \cos \omega t$ and $10 \sin(\omega t - 30°)$, by using the phasor technique. To do this, we proceed as follows:

$$\begin{aligned} 5 \cos \omega t + 10 \sin(\omega t - 30°) &= 5 \cos \omega t + 10 \cos(\omega t - 120°) \\ &= \text{Re}[5e^{j\omega t}] + \text{Re}[10e^{j(\omega t - 2\pi/3)}] \\ &= \text{Re}[5e^{j0}e^{j\omega t}] + \text{Re}[10e^{-j2\pi/3}e^{j\omega t}] \\ &= \text{Re}[5e^{j0}e^{j\omega t} + 10e^{-j2\pi/3}e^{j\omega t}] \\ &= \text{Re}[(5e^{j0} + 10e^{-j2\pi/3})e^{j\omega t}] \\ &= \text{Re}\{[(5 + j0) + (-5 - j8.66)]e^{j\omega t}\} \\ &= \text{Re}[(0 - j8.66)e^{j\omega t}] \\ &= \text{Re}[8.66e^{-j\pi/2}e^{j\omega t}] \\ &= \text{Re}[8.66e^{j(\omega t - \pi/2)}] \\ &= 8.66 \cos(\omega t - 90°) \end{aligned} \tag{A.13}$$

In practice, we need not write all the steps just shown. First, we express all functions in their cosine forms and then recognize the phasor corresponding to each function. For the foregoing example, the complex numbers $5e^{j0}$ and $10e^{-j2\pi/3}$ are the phasors corresponding to $5 \cos \omega t$ and $10 \sin(\omega t - 30°)$, respectively. Then we add the phasors and from the sum phasor write the required cosine function as one having the amplitude the same as the magnitude of the sum phasor and the argument equal to ωt plus the phase angle of the sum phasor. Thus, the steps involved are as shown in the block diagram of Fig. A.4.

Solution of differential equation

We shall now discuss the solution of a differential equation for sinusoidal steady-state response by using the phasor technique. To do this, let us consider the problem of finding the steady-state solution for the current $I(t)$ in the simple RL series circuit driven by the voltage source $V(t) = V_m \cos(\omega t + \phi)$, as shown in Fig. A.5. From Kirchhoff's voltage law, we then have

$$\boxed{RI(t) + L\frac{dI(t)}{dt} = V_m \cos(\omega t + \phi)} \tag{A.14}$$

We know that the steady-state solution for the current must also be a cosine function of time having the same frequency as that of the voltage source,

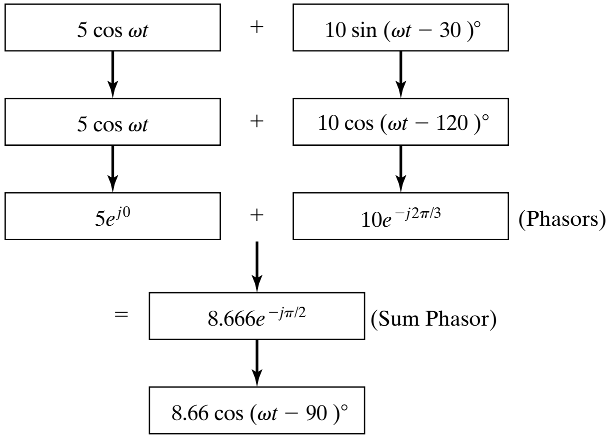

FIGURE A.4

Block diagram of steps involved in the application of the phasor technique to the addition of two sinusoidally time-varying functions.

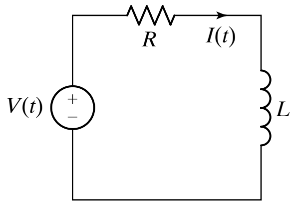

FIGURE A.5

RL series circuit.

but not necessarily in phase with it. Hence, let us assume

$$I(t) = I_m \cos(\omega t + \theta) \tag{A.15}$$

The problem now consists of finding I_m and θ.

Using the phasor concept, we write

$$\begin{aligned} V_m \cos(\omega t + \phi) &= \mathrm{Re}[V_m e^{j(\omega t + \phi)}] \\ &= \mathrm{Re}[V_m e^{j\phi} e^{j\omega t}] \\ &= \mathrm{Re}[\overline{V} e^{j\omega t}] \end{aligned} \tag{A.16a}$$

$$\begin{aligned} I_m \cos(\omega t + \theta) &= \mathrm{Re}[I_m e^{j(\omega t + \theta)}] \\ &= \mathrm{Re}[I_m e^{j\theta} e^{j\omega t}] \\ &= \mathrm{Re}[\overline{I} e^{j\omega t}] \end{aligned} \tag{A.16b}$$

where $\overline{V} = V_m e^{j\phi}$ and $\overline{I} = I_m e^{j\theta}$ are the phasors corresponding to $V(t) = V_m \cos(\omega t + \phi)$ and $I(t) = I_m \cos(\omega t + \theta)$, respectively. Substituting these into the differential equation, we have

$$R\{\mathrm{Re}[\overline{I} e^{j\omega t}]\} + L\frac{d}{dt}\{\mathrm{Re}[\overline{I} e^{j\omega t}]\} = \mathrm{Re}[\overline{V} e^{j\omega t}] \tag{A.17}$$

Since R and L are constants, and since d/dt and Re can be interchanged, we can simplify this equation in accordance with the following steps:

$$\begin{aligned} \text{Re}[R\bar{I}e^{j\omega t}] + \text{Re}\left[L\frac{d}{dt}(\bar{I}e^{j\omega t})\right] &= \text{Re}[\bar{V}e^{j\omega t}] \\ \text{Re}[R\bar{I}e^{j\omega t}] + \text{Re}[j\omega L\bar{I}e^{j\omega t}] &= \text{Re}[\bar{V}e^{j\omega t}] \\ \text{Re}[(R\bar{I} + j\omega L\bar{I})e^{j\omega t}] &= \text{Re}[\bar{V}e^{j\omega t}] \end{aligned} \tag{A.18}$$

Let us now consider two values of ωt, say, $\omega t = 0$ and $\omega t = \pi/2$. For $\omega t = 0$, we obtain

$$\text{Re}(R\bar{I} + j\omega L\bar{I}) = \text{Re}(\bar{V}) \tag{A.19}$$

For $\omega t = \pi/2$, we obtain

$$\text{Re}[j(R\bar{I} + j\omega L\bar{I})] = \text{Re}[j\bar{V}]$$

or

$$\text{Im}(R\bar{I} + j\omega L\bar{I}) = \text{Im}(\bar{V}) \tag{A.20}$$

where Im stands for "imaginary part of." Now, since the real parts as well as the imaginary parts of $(R\bar{I} + j\omega L\bar{I})$ and $\bar{V}$ are equal, it follows that the two complex numbers are equal. Thus,

$$\boxed{R\bar{I} + j\omega L\bar{I} = \bar{V}} \tag{A.21}$$

By solving this equation, we obtain $\bar{I}$ and hence I_m and θ. Note that by using the phasor technique, we have reduced the problem of solving the differential equation (A.14) to one of solving the phasor (algebraic) equation (A.21). In fact, the phasor equation can be written directly from the differential equation without the necessity of the intermediate steps, by recognizing that the time functions $I(t)$ and $V(t)$ are replaced by their phasors $\bar{I}$ and $\bar{V}$, respectively, and d/dt is replaced by $j\omega$. We have here included the intermediate steps merely to illustrate the basis behind the phasor technique. We shall now consider an example.

Example A.1 Solution of differential equation using phasor technique

For the circuit of Fig. A.5, let us assume that $R = 1\ \Omega$, $L = 10^{-3}$ H, and $V(t) = 10 \cos(1000t + 30°)$ V and obtain the steady-state solution for $I(t)$.

The differential equation for $I(t)$ is given by

$$I + 10^{-3}\frac{dI}{dt} = 10 \cos(1000t + 30°)$$

Replacing the current and voltage by their phasors $\bar{I}$ and $10e^{j\pi/6}$, respectively, and d/dt by $j\omega = j1000$, we obtain the phasor equation

$$\bar{I} + 10^{-3}(j1000\bar{I}) = 10e^{j\pi/6}$$

or

$$\begin{aligned}\bar{I}(1 + j1) &= 10e^{j\pi/6} \\ \bar{I} &= \frac{10e^{j\pi/6}}{1 + j1} = \frac{10e^{j\pi/6}}{\sqrt{2}\, e^{j\pi/4}} \\ &= 7.07e^{-j\pi/12}\end{aligned}$$

Having determined the value of $\bar{I}$, we now find the required solution to be

$$\begin{aligned}I(t) &= \text{Re}[\bar{I}e^{j\omega t}] \\ &= \text{Re}[7.07e^{-j\pi/12}e^{j1000t}] \\ &= 7.07 \cos(1000t - 15^\circ) \text{ A}\end{aligned}$$