PART II

Essential/Elective Elements

CHAPTER 7

Transmission Lines for Communications

In Chapter 6, we introduced the transmission line and studied propagation and bouncing of waves along a line, a topic applicable to digital electronics. In this chapter, we are concerned with the steady-state analysis of transmission-line systems excited by sinusoidally time-varying sources, a topic that is generally applicable to communication systems. We recall from Chapter 6 that the phenomenon on a transmission line excited by a source connected to the line at a certain instant of time, say, $t = 0$, consists of the transient bouncing of $(+)$ and $(-)$ waves along the line for $t > 0$. In the steady state, the situation is equivalent to the superposition of one $(+)$ wave, which is the sum of all the transient $(+)$ waves, and one $(-)$ wave, which is the sum of all the transient $(-)$ waves. Thus, the general solutions for the line voltage and line current in the sinusoidal steady state are superpositions of voltages and currents, respectively, of sinusoidal $(+)$ and $(-)$ waves. We shall first write these general solutions and then discuss several topics pertinent to sinusoidal steady-state analysis of transmission-line systems.

We introduce the standing-wave concept by first considering the particular case of a short-circuited line and then the general case of a line terminated by an arbitrary load. We discuss several techniques of transmission-line matching. In this connection, we introduce the Smith chart, a useful graphical aid in the solution of transmission-line problems. Finally, we extend our treatment of sinusoidal steady-state analysis to lossy lines and also consider two special cases of pulses on lossy lines.

Although the concepts and techniques discussed in this chapter are based on the analysis of transmission-line systems, many of these are also applicable to the analysis of other, analogous systems. Examples are uniform plane wave propagation involving multiple media, as in Section 4.7, and discontinuities in waveguides, considered in Chapters 8 and 9.

7.1 SHORT-CIRCUITED LINE

General solution in the sinusoidal steady state

From (6.20a) and (6.20b), we write the general solutions for the line voltage and line current in the sinusoidal steady state to be

$$V(z,t) = A\cos\left[\omega\left(t - \frac{z}{v_p}\right) + \theta\right] + B\cos\left[\omega\left(t + \frac{z}{v_p}\right) + \phi\right] \tag{7.1a}$$

$$I(z,t) = \frac{1}{Z_0}\left\{A\cos\left[\omega\left(t - \frac{z}{v_p}\right) + \theta\right] - B\cos\left[\omega\left(t + \frac{z}{v_p}\right) + \phi\right]\right\} \tag{7.1b}$$

The corresponding expressions for the phasor line voltage and phasor line current are

$$\bar{V}(z) = \bar{V}^+ e^{-j\beta z} + \bar{V}^- e^{j\beta z} \tag{7.2a}$$

$$\bar{I}(z) = \frac{1}{Z_0}(\bar{V}^+ e^{-j\beta z} - \bar{V}^- e^{j\beta z}) \tag{7.2b}$$

where $\bar{V}^+ = Ae^{j\theta}$ and $\bar{V}^- = Be^{j\phi}$ and we have substituted β for ω/v_p. For sinusoidal steady-state problems, it is convenient to use a distance variable d that increases as we go from the load toward the generator as opposed to z, which increases from the generator toward the load, as shown in Fig. 7.1. The wave that progresses away from the generator is still denoted as the $(+)$ wave, and the wave that progresses toward the generator is still denoted as the $(-)$ wave. In terms of d, the solutions for $\bar{V}$ and $\bar{I}$ are then given by

$$\bar{V}(d) = \bar{V}^+ e^{j\beta d} + \bar{V}^- e^{-j\beta d} \tag{7.3a}$$

$$\bar{I}(d) = \frac{1}{Z_0}(\bar{V}^+ e^{j\beta d} - \bar{V}^- e^{-j\beta d}) \tag{7.3b}$$

Let us now consider a lossless line short circuited at the far end $d = 0$, as shown in Fig. 7.2. We shall assume that sinusoidally time-varying traveling waves exist on the line due to a source that is not shown in the figure and that conditions have reached steady state. We wish to determine the characteristics

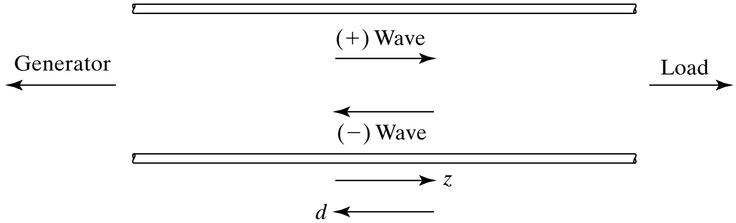

FIGURE 7.1

For illustrating the distance variable d used for sinusoidal steady state analysis of traveling waves.

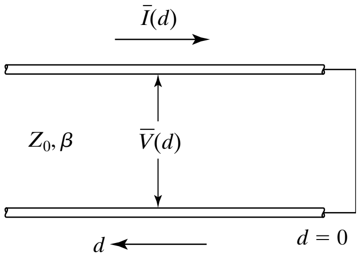

FIGURE 7.2
Transmission line short circuited at the far end.

of the waves satisfying the boundary condition at the short circuit. Since the voltage across a short circuit has to be always equal to zero, this boundary condition is given by

$$\overline{V}(0) = 0 \tag{7.4}$$

Applying it to the general solution for $\overline{V}(d)$ given by (7.3a), we obtain

$$\overline{V}(0) = \overline{V}^+ e^{j\beta(0)} + \overline{V}^- e^{-j\beta(0)} = 0$$

or

$$\overline{V}^- = -\overline{V}^+ \tag{7.5}$$

Thus, we find that the short circuit gives rise to a $(-)$ or reflected wave whose voltage is exactly the negative of the $(+)$ or incident wave voltage, at the short circuit.

Substituting (7.5) into (7.3a) and (7.3b), we get the particular solutions for the complex voltage and current on the short-circuited line to be

$$\overline{V}(d) = \overline{V}^+ e^{j\beta d} - \overline{V}^+ e^{-j\beta d} = 2j\overline{V}^+ \sin \beta d \tag{7.6a}$$

$$\overline{I}(d) = \frac{1}{Z_0}(\overline{V}^+ e^{j\beta d} + \overline{V}^+ e^{-j\beta d}) = 2\frac{\overline{V}^+}{Z_0} \cos \beta d \tag{7.6b}$$

The real voltage and current are then given by

$$\begin{aligned} V(d,t) &= \text{Re}[\overline{V}(d)e^{j\omega t}] \\ &= \text{Re}(2e^{j\pi/2}|\overline{V}^+|e^{j\theta} \sin \beta d \; e^{j\omega t}) \\ &= -2|\overline{V}^+| \sin \beta d \sin(\omega t + \theta) \end{aligned} \tag{7.7a}$$

$$\begin{aligned} I(d,t) &= \text{Re}[\overline{I}(d)e^{j\omega t}] \\ &= \text{Re}\left(2\frac{|\overline{V}^+|}{Z_0} e^{j\theta} \cos \beta d \; e^{j\omega t}\right) \\ &= 2\frac{|\overline{V}^+|}{Z_0} \cos \beta d \cos(\omega t + \theta) \end{aligned} \tag{7.7b}$$

where we have replaced $\overline{V}^+$ by $|\overline{V}^+|e^{j\theta}$ and j by $e^{j\pi/2}$. The instantaneous power flow down the line is given by

$$\begin{aligned} P(d,t) &= V(d,t)I(d,t) \\ &= -\frac{4|\overline{V}^+|^2}{Z_0}\sin\beta d\cos\beta d\sin(\omega t+\theta)\cos(\omega t+\theta) \qquad (7.7c) \\ &= -\frac{|\overline{V}^+|^2}{Z_0}\sin 2\beta d\sin 2(\omega t+\theta) \end{aligned}$$

These results for the voltage, current, and power flow on the short-circuited line are illustrated in Fig. 7.3, which shows the variation of each of these quantities with distance from the short circuit for several values of time. The numbers $1, 2, 3, \ldots, 9$ beside the curves in Fig. 7.3 represent the order of the curves corresponding to values of $(\omega t + \theta)$ equal to $0, \pi/4, \pi/2, \ldots, 2\pi$. From (7.7a), (7.7b), and (7.7c) and from the sketches of Fig. 7.3, we can infer the following:

1. The line voltage is zero for $\sin\beta d = 0$, or $\beta d = 0, \pi, 2\pi, \ldots$, or $d = 0, \lambda/2, \lambda, \ldots$, for all values of time. If we short circuit the line at these values of d, there will be no effect on the voltage and current at any other value of d.
2. The line current is zero for $\cos\beta d = 0$, or $\beta d = \pi/2, 3\pi/2, 5\pi/2, \ldots$, or $d = \lambda/4, 3\lambda/4, 5\lambda/4, \ldots$, for all values of time. If we open circuit the line at these values of d, there will be no effect on the voltage and current at any other value of d.
3. The power flow is zero for $\sin 2\beta d = 0$, or $2\beta d = 0, \pi, 2\pi, \ldots$, or $d = 0, \lambda/4, \lambda/2, \ldots$, for all values of time.

Thus, the phenomenon on the short-circuited line is one in which the voltage, current, and power flow oscillate sinusoidally with time with different amplitudes at different locations on the line, unlike in the case of traveling waves in which a given point on the waveform progresses in distance with time. Since there is no feeling of wave motion down the line, these waves are known as *standing waves*. In particular, they represent *complete standing waves* in view of the zero amplitudes of the voltage, current, and power flow at certain locations on the line, as just discussed and as shown in Fig. 7.3. Complete standing waves are the result of $(+)$ and $(-)$ traveling waves of equal amplitudes. Whatever power is incident on the short circuit by the $(+)$ wave is reflected entirely in the form of the $(-)$ wave since the short circuit cannot absorb any power. Although there is instantaneous power flow at values of d between the voltage and current nodes, there is no time-average power flow for any value of d, as can be seen from

$$\begin{aligned} \langle P\rangle &= \frac{1}{T}\int_{t=0}^{T} P(d,t)\,dt = \frac{\omega}{2\pi}\int_{t=0}^{2\pi/\omega} P(d,t)\,dt \\ &= \frac{\omega}{2\pi}\frac{|\overline{V}^+|^2}{Z_0}\sin 2\beta d\int_{t=0}^{2\pi/\omega}\sin 2(\omega t+\theta)\,dt \\ &= 0 \end{aligned}$$

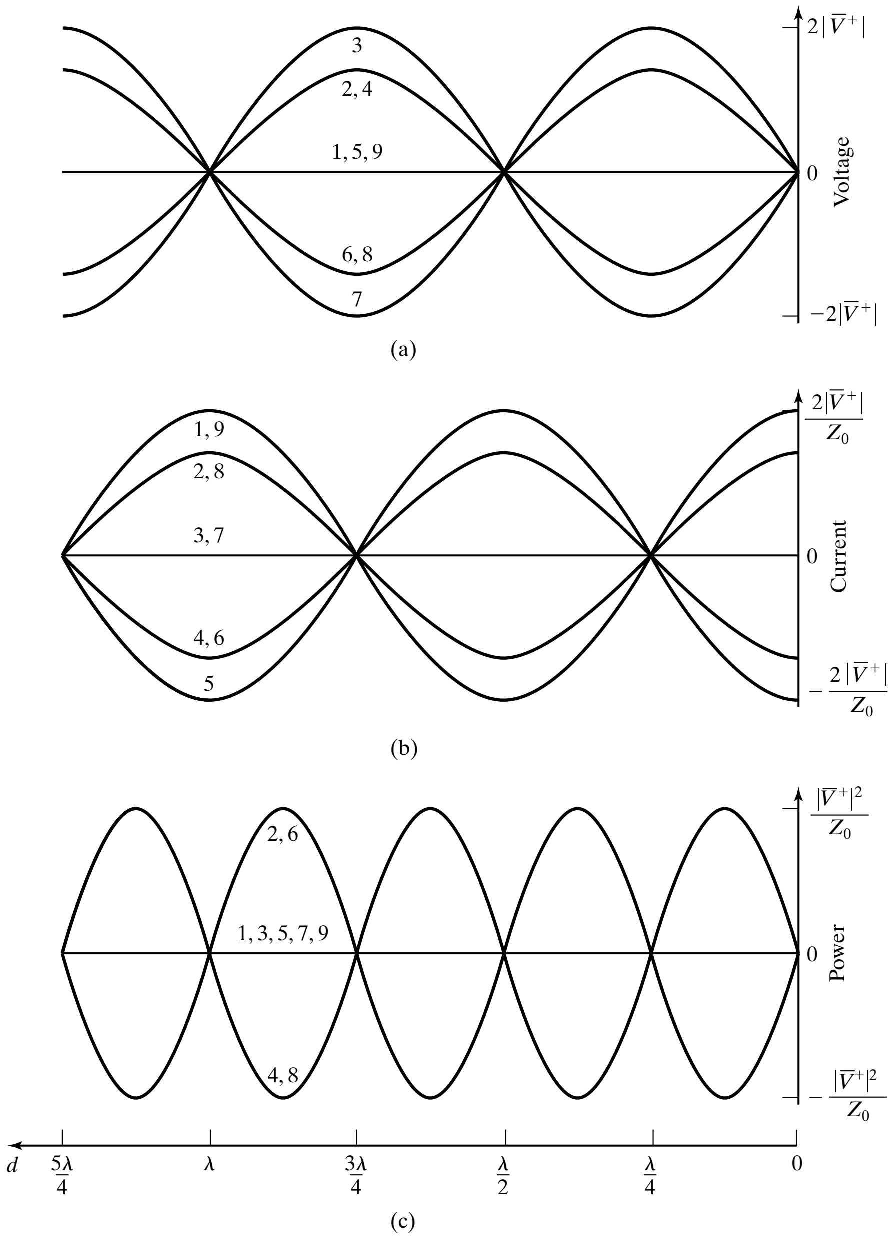

FIGURE 7.3

Time variations of voltage, current, and power flow associated with standing waves on a short-circuited transmission line.

Standing-wave patterns

From (7.6a) and (7.6b) or (7.7a) and (7.7b), or from Figs. 7.3(a) and 7.3(b), we find that the amplitudes of the sinusoidal time variations of the line voltage and line current as functions of distance along the line are

$$|\bar{V}(d)| = 2|\bar{V}^+||\sin \beta d| = 2|\bar{V}^+|\left|\sin \frac{2\pi}{\lambda} d\right| \quad (7.8a)$$

$$|\bar{I}(d)| = \frac{2|\bar{V}^+|}{Z_0}|\cos \beta d| = \frac{2|\bar{V}^+|}{Z_0}\left|\cos \frac{2\pi}{\lambda} d\right| \quad (7.8b)$$

Sketches of these quantities versus d are shown in Fig. 7.4. These are known as the *standing-wave patterns*. They are the patterns of line voltage and line current one would obtain by connecting an ac voltmeter between the conductors of the line and an ac ammeter in series with one of the conductors of the line and observing their readings at various points along the line. Alternatively, one can sample the electric and magnetic fields by means of probes. Standing-wave patterns should not be misinterpreted as the voltage and current remaining constant with time at a given point. On the other hand, the voltage and current at every point on the line vary sinusoidally with time, as shown in the insets of Fig. 7.4, with the amplitudes of these sinusoidal variations equal to the magnitudes indicated by the standing-wave patterns. Since the distance between successive nodes of voltage or current is equal to $\lambda/2$, a measurement of this distance provides the knowledge of the wavelength. Furthermore, if the phase velocity in the line is known, the frequency of the source can be computed, and vice versa, since $v_p = \lambda f$.

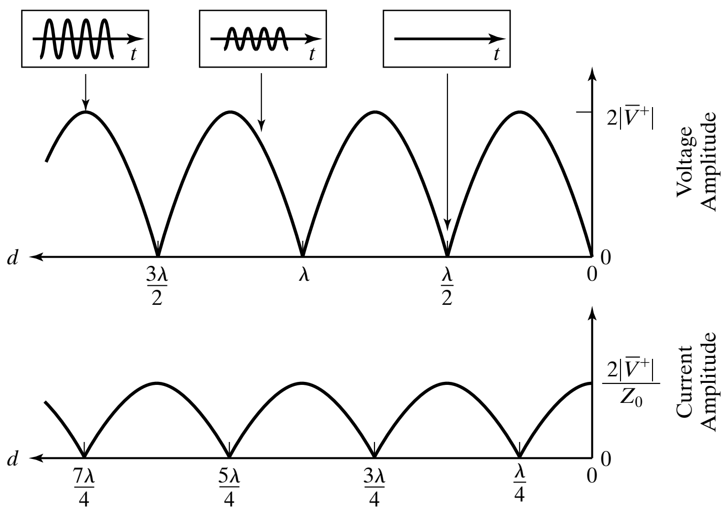

FIGURE 7.4

Standing-wave patterns for voltage and current on a short-circuited line. The insets show time variations of the voltage at points along the line.

Natural oscillations

Since there is no power flow across a voltage node or a current node of the standing-wave patterns, a constant amount of total energy is locked up in every $\lambda/4$ section between two such adjacent nodes with exchange of energy taking place between the electric and magnetic fields. Thus, once the line is excited by applying a source of energy, then each $\lambda/4$ section of the line between the voltage and current nodes acts as a resonator entirely independent of the remainder of the line. In fact, the $\lambda/4$ section can be removed from the line by cutting it, that is, open circuiting it, at the current node and short circuiting it at the voltage node, and still be made to maintain forever the oscillations of voltage and current. Such oscillations are called *natural oscillations*. Similarly, sections of lengths equal to multiples of $\lambda/4$ can be removed by always cutting the line at current nodes and short circuiting it at voltage nodes, without disturbing the oscillations.

For a fixed physical length of the line, its electrical length, that is, its length in terms of wavelength, depends on the frequency. Thus, a line of length equal to one-quarter wavelength at one frequency behaves as a line of length equal to a different multiple of a wavelength at a different frequency. Let us now consider a line of length l, one end of which is open-circuited and the other end short-circuited, and assume that some energy is stored in this line. Suppose we now pose the question: "What are all the possible standing-wave patterns on this line?" To answer this, we note that the voltage across the short circuit must always be zero, and, hence, the current there must have maximum amplitude. Similarly, the current at the open-circuited end must always be zero, and, hence, the voltage there must have maximum amplitude. We also know that the standing-wave patterns are sinusoidal with the distance between successive nodes corresponding to a half sine wave. Thus, the least possible variation is a quarter cycle of a sine waveform. This corresponds to a wavelength, say, λ_1, equal to $4l$, and the corresponding standing-wave patterns are shown in Fig. 7.5(a).

It is not possible to have a standing-wave pattern for which the wavelength is greater than $4l$ since then the pattern on the line of length l will be less than a quarter cycle of a sine wave. On the other hand, it is possible to have a pattern for which the wavelength is less than $4l$ as long as the conditions of zero voltage (maximum current) at the short circuit and zero current (maximum voltage) at the open circuit are satisfied. Obviously, the next largest wavelength λ_2, less than λ_1, for which this condition is satisfied corresponds to the patterns shown in Fig. 7.5(b). For these patterns, $l = 3\lambda_2/4$, or $\lambda_2 = 4l/3$. The next largest wavelength, λ_3, less than λ_2, corresponds to the patterns shown in Fig. 7.5(c). For these patterns, $l = 5\lambda_3/4$, or $\lambda_3 = 4l/5$.

We can continue in this manner and see that any standing-wave pattern for which the length of the line is an odd multiple of one-quarter wavelength, that is,

$$l = \frac{(2n - 1)\lambda_n}{4} \qquad n = 1, 2, 3, \ldots \tag{7.9}$$

is a valid standing-wave pattern. Alternatively, the wavelengths λ_n, corresponding to the valid standing-wave patterns, are given by

$$\lambda_n = \frac{4l}{2n - 1} \qquad n = 1, 2, 3, \ldots \tag{7.10}$$

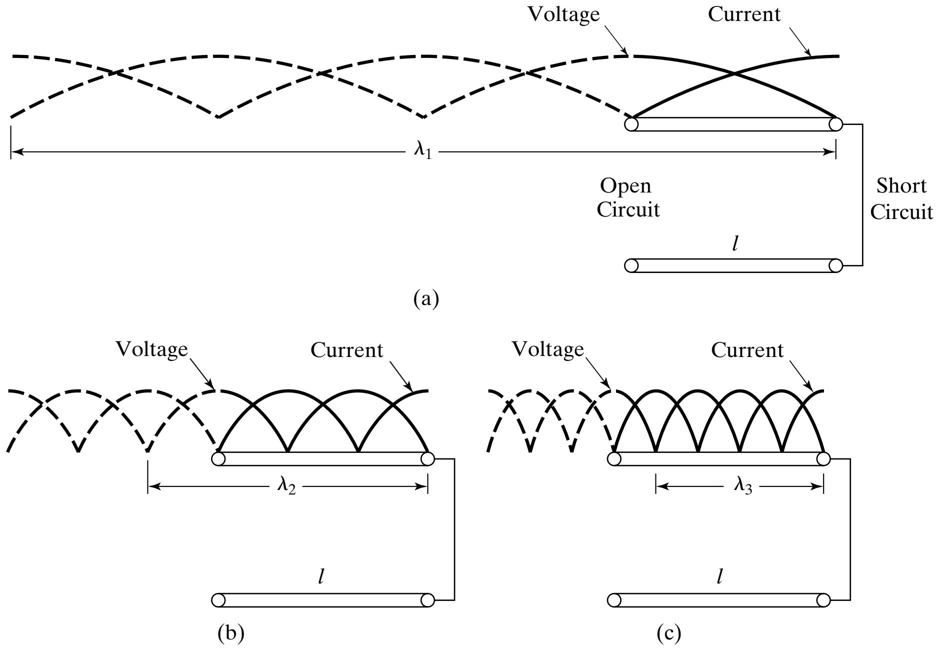

FIGURE 7.5

Standing-wave patterns corresponding to (a) one-quarter cycle, (b) three-quarters cycle, and (c) five-quarters cycle of a sine wave for the voltage and current amplitude distributions for a line of length l open-circuited at one end and short-circuited at the other end.

The corresponding frequencies are

$$f_n = \frac{v_p}{\lambda_n} = \frac{(2n - 1)v_p}{4l} \qquad n = 1, 2, 3, \ldots \tag{7.11}$$

where v_p is the phase velocity. These frequencies are known as the *natural frequencies of oscillation*. The standing-wave patterns are said to correspond to the different natural modes of oscillation. The lowest frequency (corresponding to the longest wavelength) is known as the *fundamental frequency of oscillation*, and the corresponding mode is known as the *fundamental mode*. The quantity n is called the *mode number*. In the most general case of nonsinusoidal voltage and current distributions on the line, the situation corresponds to the superposition of some or all of the infinite number of natural modes.

Considerations similar to those for the line open-circuited at one end and short-circuited at the other end apply to natural oscillations on lines short-circuited at both ends or open-circuited at both ends.

Input impedance

Returning now to the expressions for the phasor line voltage and the phasor line current given by (7.6a) and (7.6b), respectively, we define the ratio of these two quantities as the line impedance $\bar{Z}(d)$ at that point seen looking toward the short circuit. Thus,

$$\bar{Z}(d) = \frac{\bar{V}(d)}{\bar{I}(d)} = \frac{2j\bar{V}^+ \sin \beta d}{2(\bar{V}^+/Z_0) \cos \beta d} = jZ_0 \tan \beta d \tag{7.12}$$

In particular, the input impedance $\bar{Z}_{\text{in}}$ of a short-circuited line of length l is given by

$$\boxed{\bar{Z}_{\text{in}} = jZ_0 \tan \beta l = jZ_0 \tan \frac{2\pi f}{v_p} l} \tag{7.13}$$

We note from (7.13) that the input impedance of the short-circuited line is purely reactive. As the frequency is varied from a low value upward, the input reactance changes from inductive to capacitive and back to inductive, and so on, as illustrated in Fig. 7.6. The input reactance is zero for values of frequency equal to multiples of $v_p/2l$. These are the frequencies for which l is equal to multiples of $\lambda/2$, so that the line voltage is zero at the input and hence the input sees a short circuit. The input reactance is infinity for values of frequency equal to odd multiples of $v_p/4l$. These are the frequencies for which l is equal to odd multiples of $\lambda/4$, so that the line current is zero at the input and hence the input sees an open circuit.

These properties of the input impedance of a short-circuited line (and, similarly, of an open-circuited line) have several applications. We shall here discuss two of these applications.

Location of short circuit

1. *Determination of the location of a short circuit (or open circuit) in a line.* The principle behind this lies in the fact that as the frequency of a generator

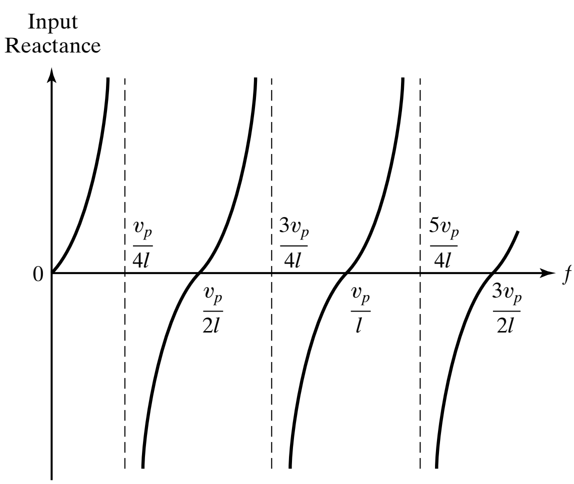

FIGURE 7.6

Variation of the input reactance of a short-circuited transmission line with frequency.

connected to the input of a short-circuited (or open-circuited) line is varied continuously upward, the current drawn from it undergoes alternatively maxima and minima corresponding to zero input reactance and infinite input reactance conditions, respectively. Since the difference between a pair of consecutive frequencies for which the input reactance values are zero and infinity is $v_p/4l$, as can be seen from Fig. 7.6, it follows that the difference between successive frequencies for which the currents drawn from the generator are maxima and minima is $v_p/4l$. As a numerical example, if for an air–dielectric line, it is found that as the frequency is varied from 50 MHz upward, the current reaches a minimum for 50.01 MHz and then a maximum for 50.04 MHz, then the distance l of the short circuit from the generator is given by

$$\frac{v_p}{4l} = (50.04 - 50.01) \times 10^6 = 0.03 \times 10^6 = 3 \times 10^4$$

Since $v_p = 3 \times 10^8$ m/s, it follows that

$$l = \frac{3 \times 10^8}{4 \times 3 \times 10^4} = 2500 \text{ m} = 2.5 \text{ km}$$

Alternatively, if the length l is known, we can compute v_p for the dielectric of the line, from which the permittivity of the dielectric can be found, provided that the value of μ (usually equal to μ_0) is known.

Resonant system

2. *Construction of resonant circuits at microwave frequencies*. The principle behind this lies in the fact that the input reactance of a short-circuited line of a given length can be inductive or capacitive, depending on the frequency, and hence, two short-circuited lines connected together form a resonant system. To obtain the characteristic equation for the resonant frequencies of such a system, let us consider the system shown in Fig. 7.7, which is made up of two short-circuited line sections of characteristic impedances Z_{01} and Z_{02}, lengths l_1 and l_2, and phase velocities v_{p1} and v_{p2}. Denoting the voltages and currents just to the

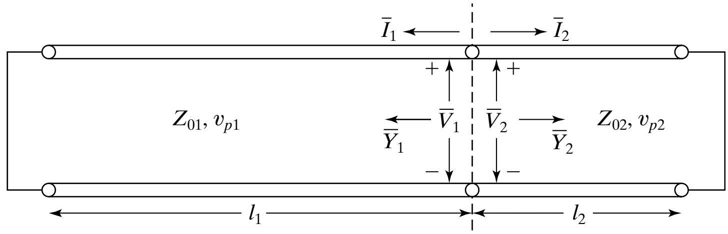

FIGURE 7.7

Resonant system formed by connecting together two short-circuited line sections.

left and just to the right of the junction to be $\overline{V}_1$ and $\overline{I}_1$ and $\overline{V}_2$ and $\overline{I}_2$, respectively, as shown in the figure, we write the boundary conditions at the junction as

$$\overline{V}_1 = \overline{V}_2 \tag{7.14a}$$

$$\overline{I}_1 + \overline{I}_2 = 0 \tag{7.14b}$$

Combining the two, we have

$$\frac{\overline{I}_1}{\overline{V}_1} + \frac{\overline{I}_2}{\overline{V}_2} = 0$$

or

$$\overline{Y}_1 + \overline{Y}_2 = 0 \tag{7.15}$$

where $\overline{Y}_1$ and $\overline{Y}_2$ are the input admittances of the sections to the left and to the right, respectively, of the junction and seen looking toward the short circuits. Equation (7.15) is the condition for resonance of the system. To express it in terms of the line parameters, we note that

$$\overline{Y}_1 = \frac{1}{\overline{Z}_1} = \frac{1}{jZ_{01}\tan\beta_1 l_1} = \frac{1}{jZ_{01}\tan(2\pi f/v_{p1})l_1} \tag{7.16a}$$

$$\overline{Y}_2 = \frac{1}{\overline{Z}_2} = \frac{1}{jZ_{02}\tan\beta_2 l_2} = \frac{1}{jZ_{02}\tan(2\pi f/v_{p2})l_2} \tag{7.16b}$$

Substituting (7.16a) and (7.16b) into (7.15) and simplifying, we obtain the characteristic equation for the resonant frequencies to be

$$Z_{01}\tan\frac{2\pi f}{v_{p1}}l_1 + Z_{02}\tan\frac{2\pi f}{v_{p2}}l_2 = 0 \tag{7.17}$$

We shall illustrate the computation of the resonant frequencies by means of an example.

Example 7.1 Finding the resonant frequencies for a transmission-line resonant system

For the system of Fig. 7.7, let us assume $Z_{01} = 2Z_{02} = 60\ \Omega$, $l_1 = 5$ cm, $l_2 = 2$ cm, and $v_{p1} = v_{p2} = c/2$, and obtain the four lowest resonant frequencies of the system.

Substituting the numerical values of the parameters into the characteristic equation (7.17), we obtain

$$\tan\frac{0.2\pi f}{c} + \frac{1}{2}\tan\frac{0.08\pi f}{c} = 0$$

This equation is of the form

$$\tan kx + m \tan x = 0$$

where $k = 2.5$, $m = 0.5$, and $x = 0.08\pi f/c$. In general, an equation of this type can be solved by plotting $\tan kx$ and $-m \tan x$ to scale versus x and finding the points of intersection. Alternatively, a programmable calculator or a computer can be used. Thus, the first four solutions are x equal to 0.3165π, 0.5563π, 0.8353π, and 1.1647π.

From the values of x obtained from the computer program, we obtain the lowest four resonant frequencies to be 1.1869×10^9, 2.0861×10^9, 3.1324×10^9, and 4.3676×10^9 Hz, or 1.1869, 2.0861, 3.1324, and 4.3676 GHz.

K7.1. Phasor line voltage and line current; General solutions; Short-circuited line; Complete standing waves; Standing-wave patterns; Natural oscillations; Input impedance; Resonant systems.

D7.1. For each of the following characteristics of standing waves on a lossless short-circuited line, find the frequency of the source exciting the line: **(a)** the distance between successive nodes of voltage amplitude is 50 cm and the dielectric is air; **(b)** the distance between successive nodes of current amplitude is 50 cm and the dielectric is nonmagnetic with $\varepsilon = 9\varepsilon_0$; and **(c)** the distance between successive nodes of instantaneous power flow is 50 cm and the dielectric is air.

Ans. **(a)** 300 MHz; **(b)** 100 MHz; **(c)** 150 MHz.

D7.2. A lossless coaxial cable of characteristic impedance 50 Ω and having a nonmagnetic ($\mu = \mu_0$), perfect dielectric of permittivity $\varepsilon = 2.25\varepsilon_0$ is short-circuited at the far end. Find the minimum length of the line for which the input impedance is equal to the impedance of each of the following at $f = 100$ MHz: **(a)** an inductor of value equal to 0.5 μH; **(b)** an inductor of value equal to the inductance per unit length of the line; and **(c)** an inductor of value equal to the inductance of the line.

Ans. **(a)** 44.98 cm; **(b)** 40.19 cm; **(c)** 143.03 cm.

D7.3. A lossless transmission line of length $l = 5$ m, characteristic impedance $Z_0 = 100\ \Omega$, and having a nonmagnetic ($\mu = \mu_0$), perfect dielectric is short-circuited at its far end. A variable frequency voltage source in series with an internal impedance $\bar{Z}_g$ is connected at its input and the line voltage and line current at the input terminals are monitored as the source frequency is varied. It is found that the voltage reaches a maximum amplitude of 10 V at 157.5 MHz and then the current reaches a maximum amplitude of 0.2 A at 165 MHz. Find the following: **(a)** the maximum amplitude of the current in the standing-wave pattern on the line at 157.5 MHz; **(b)** the maximum amplitude of the voltage in the standing-wave pattern on the line at 165 MHz; **(c)** the magnitude of $\bar{Z}_g$; and **(d)** the permittivity of the dielectric of the line.

Ans. **(a)** 0.1 A; **(b)** 20 V; **(c)** 50 Ω; **(d)** $4\varepsilon_0$.

7.2 LINE TERMINATED BY ARBITRARY LOAD

We devoted the preceding section to the short-circuited line. In this section, we consider a line terminated by an arbitrary load impedance $\bar{Z}_R$, as shown in Fig. 7.8.

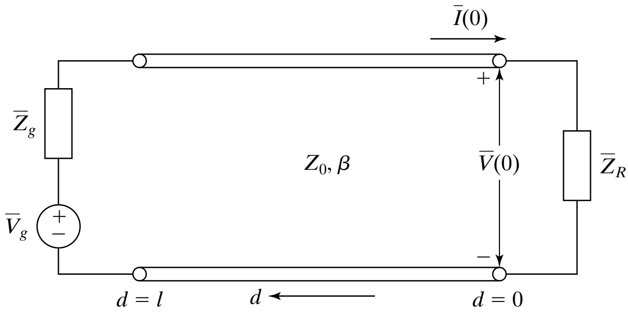

FIGURE 7.8

Line terminated by a complex load impedance.

Then starting with the general solutions for the complex line voltage and line current given by

$$\bar{V}(d) = \bar{V}^+ e^{j\beta d} + \bar{V}^- e^{-j\beta d} \tag{7.18a}$$

$$\bar{I}(d) = \frac{1}{Z_0}(\bar{V}^+ e^{j\beta d} - \bar{V}^- e^{-j\beta d}) \tag{7.18b}$$

and using the boundary condition at $d = 0$, given by

$$\bar{V}(0) = \bar{Z}_R \bar{I}(0) \tag{7.19}$$

we obtain

$$\bar{V}^+ + \bar{V}^- = \frac{\bar{Z}_R}{Z_0}(\bar{V}^+ - \bar{V}^-)$$

or

$$\bar{V}^- = \bar{V}^+ \frac{\bar{Z}_R - Z_0}{\bar{Z}_R + Z_0}$$

Thus, the ratio of $\bar{V}^-$, the reflected wave voltage at the load, to $\bar{V}^+$, the incident wave voltage at the load, that is, the voltage reflection coefficient at the load, denoted by $\bar{\Gamma}_R$, is given by

$$\boxed{\bar{\Gamma}_R = \frac{\bar{V}^-}{\bar{V}^+} = \frac{\bar{Z}_R - Z_0}{\bar{Z}_R + Z_0}} \tag{7.20}$$

The solutions for $\bar{V}(d)$ and $\bar{I}(d)$ can then be written as

$$\bar{V}(d) = \bar{V}^+ e^{j\beta d} + \bar{\Gamma}_R \bar{V}^+ e^{-j\beta d} \tag{7.21a}$$

$$\bar{I}(d) = \frac{1}{Z_0}(\bar{V}^+ e^{j\beta d} - \bar{\Gamma}_R \bar{V}^+ e^{-j\beta d}) \tag{7.21b}$$

Generalized reflection coefficient

We now define the generalized voltage reflection coefficient, $\bar{\Gamma}(d)$, that is, the voltage reflection coefficient at any value of d, as the ratio of the reflected wave voltage to the incident wave voltage at that value of d. From (7.21a), we see that

$$\boxed{\bar{\Gamma}(d) = \frac{\bar{\Gamma}_R \bar{V}^+ e^{-j\beta d}}{\bar{V}^+ e^{j\beta d}} = \bar{\Gamma}_R e^{-j2\beta d}} \tag{7.22}$$

so that

$$|\bar{\Gamma}(d)| = |\bar{\Gamma}_R||e^{-j2\beta d}| = |\bar{\Gamma}_R| \tag{7.23a}$$

and

$$\angle\bar{\Gamma}(d) = \angle\bar{\Gamma}_R + \angle e^{-j2\beta d} = \theta - 2\beta d \tag{7.23b}$$

where θ is the phase angle of $\bar{\Gamma}_R$. Thus, the magnitude of the generalized reflection coefficient remains constant along the line and equal to its value at the load, whereas the phase angle varies linearly with d. In terms of $\bar{\Gamma}(d)$, we can write the solutions for $\bar{V}(d)$ and $\bar{I}(d)$ as

$$\begin{aligned}\bar{V}(d) &= \bar{V}^+ e^{j\beta d}(1 + \bar{\Gamma}_R e^{-j2\beta d}) \\ &= \bar{V}^+ e^{j\beta d}[1 + \bar{\Gamma}(d)]\end{aligned} \tag{7.24a}$$

$$\begin{aligned}\bar{I}(d) &= \frac{\bar{V}^+}{Z_0} e^{j\beta d}(1 - \bar{\Gamma}_R e^{-j2\beta d}) \\ &= \frac{\bar{V}^+}{Z_0} e^{j\beta d}[1 - \bar{\Gamma}(d)]\end{aligned} \tag{7.24b}$$

To study the standing-wave patterns corresponding to (7.24a) and (7.24b), we look at the magnitudes of $\bar{V}(d)$ and $\bar{I}(d)$. These are given by

$$\begin{aligned}|\bar{V}(d)| &= |\bar{V}^+||e^{j\beta d}||1 + \bar{\Gamma}(d)| \\ &= |\bar{V}^+||1 + \bar{\Gamma}_R e^{-j2\beta d}|\end{aligned} \tag{7.25a}$$

$$\begin{aligned}|\bar{I}(d)| &= \frac{|\bar{V}^+|}{Z_0}|e^{j\beta d}||1 - \bar{\Gamma}(d)| \\ &= \frac{|\bar{V}^+|}{Z_0}|1 - \bar{\Gamma}_R e^{-j2\beta d}|\end{aligned} \tag{7.25b}$$

To sketch $|\bar{V}(d)|$ and $|\bar{I}(d)|$, it is sufficient if we consider the quantities $|1 + \bar{\Gamma}_R e^{-j2\beta d}|$ and $|1 - \bar{\Gamma}_R e^{-j2\beta d}|$, since $|\bar{V}^+|$ is simply a constant, determined by the boundary condition at the source end. Each of these quantities consists of two complex numbers, one of which is a constant equal to $(1 + j0)$ and the other of which has a constant magnitude $|\bar{\Gamma}_R|$ but a variable phase angle $(\theta - 2\beta d)$. To evaluate $|1 + \overline{\Gamma}_R e^{-j2\beta d}|$ and $|1 - \overline{\Gamma}_R e^{-j2\beta d}|$, we make use of the constructions in the complex $\overline{\Gamma}$-plane, as shown in Fig. 7.9(a) and (b), respectively. In both diagrams, we draw circles with centers at the origin and having radii equal to $|\overline{\Gamma}_R|$. For $d = 0$, the complex number $\overline{\Gamma}_R e^{-j2\beta d}$ is equal to $\overline{\Gamma}_R$ or $|\overline{\Gamma}_R|e^{j\theta}$, which is represented by point A in Fig. 7.9(a). To add $(1 + j0)$ and $\overline{\Gamma}_R$, we simply draw a line from the point $(-1, 0)$ to the point A. The length of this line gives $|1 + \overline{\Gamma}_R|$, which is proportional to the amplitude of the voltage at $d = 0$. As d increases, point A, representing $\overline{\Gamma}_R e^{-j2\beta d}$, moves around the circle in the clockwise direction. The line joining $(-1, 0)$ to the point A whose length is $|1 + \overline{\Gamma}_R e^{-j2\beta d}|$ executes the motion of a crank. To subtract $\overline{\Gamma}_R$ from $(1 + j0)$, we locate point B in Fig. 7.9(b), which is diametrically opposite to point A in Fig. 7.9(a), and draw a line from $(-1, 0)$ to point B. The length of the line gives $|1 - \overline{\Gamma}_R|$, which is proportional to the amplitude of the current at $d = 0$. As d increases, B moves around the circle in the clockwise direction following the movement of A in Fig. 7.9(a). The line joining $(-1, 0)$ to the point B whose length is $|1 - \overline{\Gamma}_R e^{-j2\beta d}|$ executes the motion of a crank. From these constructions and assuming $-\pi \le \theta < \pi$, we note the following facts:

1. Point A lies along the positive real axis and point B lies along the negative real axis for $(\theta - 2\beta d) = 0, -2\pi, -4\pi, -6\pi, \ldots$, or $d = (\lambda/4\pi)(\theta + 2n\pi)$, where $n = 0, 1, 2, 3, \ldots$. Hence, at these values of d, the voltage amplitude is maximum and equal to $|\overline{V}^+|(1 + |\overline{\Gamma}_R|)$, whereas the current amplitude is minimum and equal to $(|\overline{V}^+|/Z_0)(1 - |\overline{\Gamma}_R|)$. The voltage and current are in phase.
2. Point A lies along the negative real axis and point B lies along the positive real axis for $(\theta - 2\beta d) = -\pi, -3\pi, -5\pi, -7\pi, \ldots$, or $d = (\lambda/4\pi)[\theta + (2n - 1)\pi]$, where $n = 1, 2, 3, 4, \ldots$. Hence, at these values of d, the voltage amplitude is minimum and equal to $|\overline{V}^+|(1 - |\overline{\Gamma}_R|)$, whereas the current

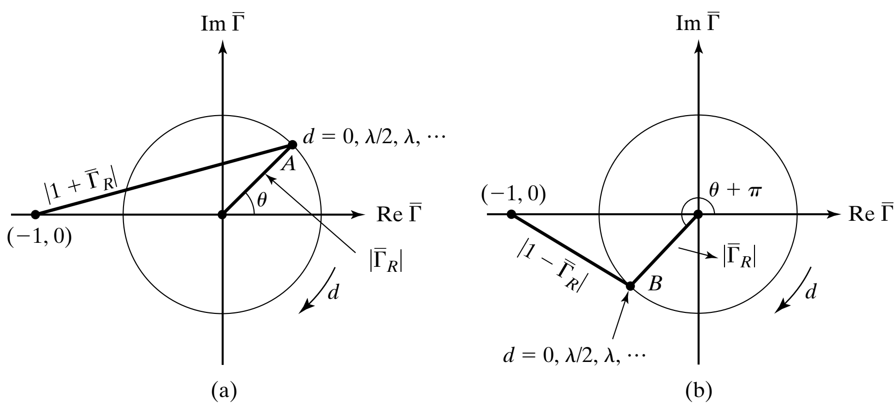

FIGURE 7.9

$\overline{\Gamma}$ plane diagrams for sketching the voltage and current standing-wave patterns for the system of Fig. 7.8.

amplitude is maximum and equal to $(|\bar{V}^+|/Z_0)(1 + |\bar{\Gamma}_R|)$. The voltage and current are in phase.

3. Between maxima and minima, the voltage and current vary in accordance with the lengths of the line joining $(-1, 0)$ to the points A and B, respectively, as they move around the circles. These variations are not sinusoidal with distance. The variations near the minima are sharper than are those near the maxima; hence, the minima can be located more accurately than can the maxima. Also, the voltage and current are not in phase.

Standing-wave parameters

From the preceding discussion, we now sketch the standing-wave patterns for the line voltage and current, as shown in Fig. 7.10. These patterns correspond to partial standing waves, as compared to complete standing waves in the case of the short-circuited line. There are three parameters associated with the standing-wave patterns as follows.

1. *The standing-wave ratio, abbreviated as SWR.* This is the ratio of the maximum voltage amplitude V_{max} to the minimum voltage amplitude V_{min} in the standing-wave pattern. Thus

$$\boxed{\text{SWR} = \frac{V_{max}}{V_{min}} = \frac{|\bar{V}^+|(1 + |\bar{\Gamma}_R|)}{|\bar{V}^+|(1 - |\bar{\Gamma}_R|)} = \frac{1 + |\bar{\Gamma}_R|}{1 - |\bar{\Gamma}_R|}} \tag{7.26}$$

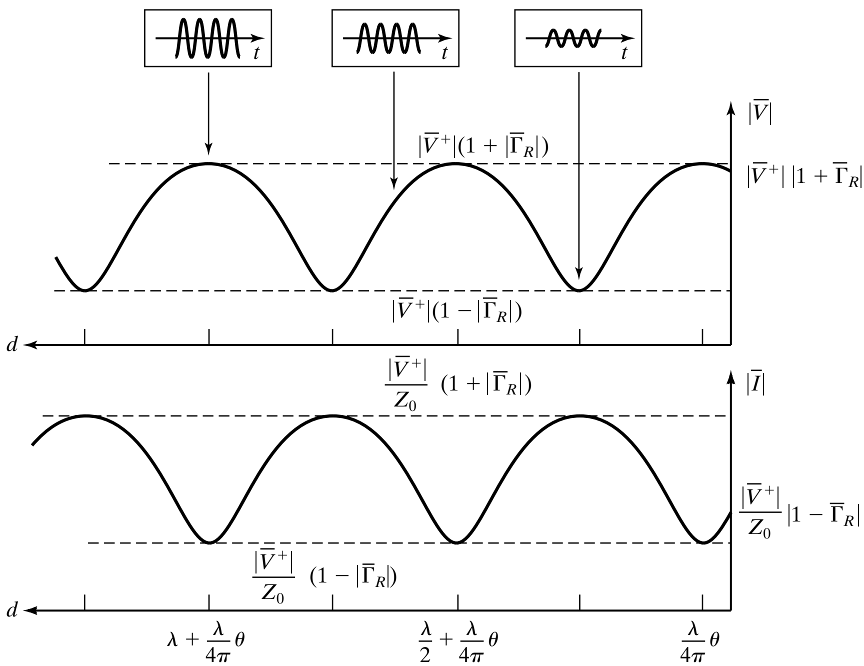

FIGURE 7.10

Voltage and current standing wave patterns for the system of Fig. 7.8. The insets show time variations of voltage at points along the line.

Note also that SWR is equal to the ratio of the maximum current amplitude I_{max} to the minimum current amplitude I_{min} in the standing-wave pattern, since

$$\frac{I_{max}}{I_{min}} = \frac{(|\bar{V}^+|/Z_0)(1 + |\bar{\Gamma}_R|)}{(|\bar{V}^+|/Z_0)(1 - |\bar{\Gamma}_R|)} = \frac{1 + |\bar{\Gamma}_R|}{1 - |\bar{\Gamma}_R|}$$

The SWR is a measure of standing waves on the line. It is an easily measurable parameter. We note the following special cases:

(a) For $\bar{\Gamma}_R = 0$, SWR $= 1$ and the standing-wave pattern is simply a line representing constant amplitude. This is the case for a semi-infinitely long line or for a line terminated by its characteristic impedance.

(b) For $|\bar{\Gamma}_R| = 1$, SWR $= \infty$ and the standing-wave pattern possesses perfect nulls. This is the case for complete standing waves.

2. *The distance of the first voltage minimum from the load, denoted by* d_{min}. The voltage minimum nearest to the load occurs when the phase angle of $\bar{\Gamma}(d) = \bar{\Gamma}_R e^{-j2\beta d}$ is equal to $-\pi$, that is, for $(\theta - 2\beta d)$ equal to $-\pi$. Thus,

$$\theta - 2\beta d_{min} = -\pi \tag{7.27}$$

or

$$\boxed{d_{min} = \frac{\theta + \pi}{2\beta} = \frac{\lambda}{4\pi}(\theta + \pi)} \tag{7.28}$$

where $-\pi \leq \theta < \pi$. If $\theta = 0$, which occurs when $\bar{Z}_R$ is purely real and greater than Z_0, $d_{min} = \lambda/4$ and a voltage maximum exists right at the load. If $\theta = -\pi$, which occurs when $\bar{Z}_R$ is purely real and less than Z_0, $d_{min} = 0$ and a voltage minimum exists right at the load.

3. *The wavelength* λ. Since the distance between successive voltage minima is equal to $\lambda/2$, the wavelength is twice the distance between successive voltage minima.

For a numerical example involving a complex $\bar{Z}_R$, let us consider $\bar{Z}_R = (15 - j20)\ \Omega$ and $Z_0 = 50\ \Omega$. Then

$$\begin{aligned}
\bar{\Gamma}_R &= \frac{\bar{Z}_R - Z_0}{\bar{Z}_R + Z_0} = \frac{(15 - j20) - 50}{(15 - j20) + 50} \\
&= \frac{-7 - j4}{13 - j4} = \frac{8.06\underline{/-150.26^\circ}}{13.60\underline{/-17.10^\circ}} \\
&= 0.593\underline{/-133.16^\circ} \\
&= 0.593e^{-j0.74\pi}
\end{aligned}$$

$$\text{SWR} = \frac{1 + |\overline{\Gamma}_R|}{1 - |\overline{\Gamma}_R|} = \frac{1 + 0.593}{1 - 0.593} = 3.914$$

$$d_{\min} = \frac{\lambda}{4\pi}(\theta + \pi) = \frac{\lambda}{4\pi}(-0.74\pi + \pi) = 0.065\lambda$$

Determination of unknown load impedance

Conversely to the computation of standing-wave parameters for a given load impedance, an unknown load impedance can be determined from standing-wave measurements on a line of known characteristic impedance. An application in practice is the determination of the input impedance of an antenna by making standing-wave measurements on the line feeding the antenna. To outline the basis, we note that by rearranging (7.26) and (7.28), we obtain

$$\boxed{|\overline{\Gamma}_R| = \frac{\text{SWR} - 1}{\text{SWR} + 1}} \tag{7.29}$$

and

$$\boxed{\theta = \frac{4\pi d_{\min}}{\lambda} - \pi} \tag{7.30}$$

Thus, the measurement of SWR, $d_{\min}$, and λ provides both the magnitude and phase angle of $\overline{\Gamma}_R$. Then, since from (7.20)

$$\boxed{\overline{Z}_R = Z_0 \frac{1 + \overline{\Gamma}_R}{1 - \overline{\Gamma}_R}} \tag{7.31}$$

we can compute the value of $\overline{Z}_R$.

Slotted-line measurements

A traditional method of performing standing-wave measurements in the laboratory is by using a *slotted line*. The slotted line is essentially a rigid coaxial line with air dielectric and having a length of about 1 meter (or at least a half-wavelength long). The center conductor is supported by dielectric inserts. A narrow longitudinal slot is cut in the outer conductor, as shown in Fig. 7.11(a). The

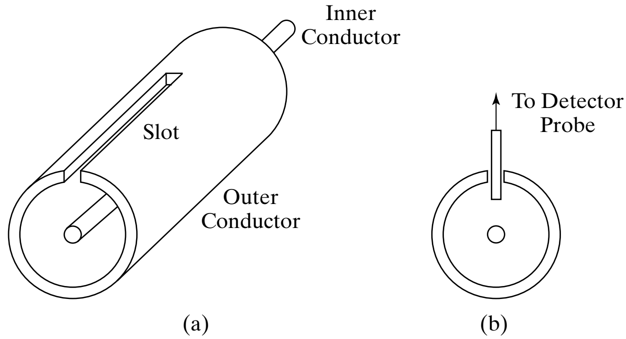

FIGURE 7.11

(a) Slotted line. (b) Cross-sectional view of the slotted-line illustrating the probe arrangement.

width of the slot is so small that it has negligible influence on the current flow on the outer conductor and, hence, on the field configurations between the conductors. A probe of small length, shown in Fig. 7.11(b), intercepts a portion of the electric field between the inner and outer conductors, and a small voltage proportional to the line voltage at the probe's location is developed between the probe and the outer conductor. The signal frequency voltage thus developed is detected by some sort of detector, and the resulting output is used as an indicator of the amplitude of the line voltage at the probe's location. The amount of energy picked up by the probe is small enough not to disturb appreciably the fields within the line. The probe and the associated detector components are mounted on a carriage arranged to slide mechanically along the longitudinal slot. As the probe is moved along the slot, the detector indication provides a measure of the variation of the voltage as a function of position on the line. Since the SWR is the ratio of V_{max} to V_{min}, the quantity of interest is the ratio of the two readings rather than the absolute values of the readings themselves. Therefore, absolute calibration of the detector is not required, provided that the detector response is linear in the range of voltages to be measured.

Since it is not always possible to measure the distances of the standing-wave pattern minima from the location of the load, the following procedure is employed. First, the line is terminated by a short circuit in the place of the load. One of the nulls in the resulting standing-wave pattern is taken as the reference point, as shown in Fig. 7.12(a). This establishes that the location of the load is an integral multiple of half-wavelengths from the reference point. Next, the short circuit is removed and the load is connected. The voltage minimum then shifts away from the reference point, as shown in Fig. 7.12(b). By measuring this shift, either away from the load or toward the load, the value of d_{min} can be established. If the shift d_a away from the load is measured, then we can see from Fig. 7.12(b)

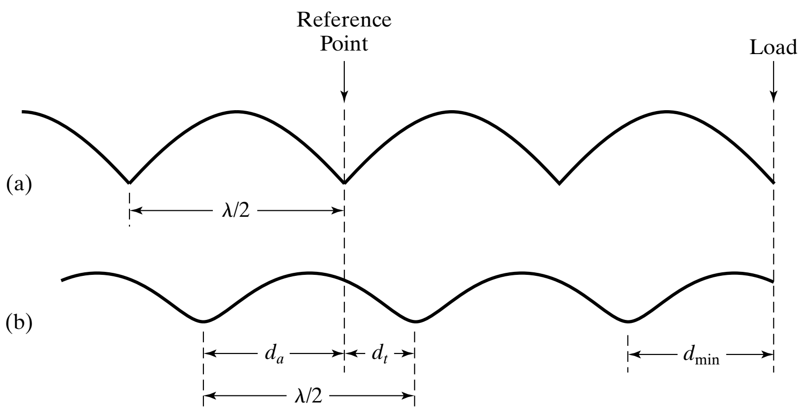

FIGURE 7.12

For illustrating the procedure employed for the determination of d_{min}, the distance of the first voltage minimum of the standing-wave pattern from the load, by making measurements away from the load.

that $d_{\min}$ is simply equal to d_a. On the other hand, if the shift d_t toward the load is measured, then $d_{\min}$ is equal to $\lambda/2 - d_t$, where $\lambda/2$ is given by the distance between consecutive nulls either in the case of short circuit or with the unknown load as the termination.

We shall illustrate the computation of $\bar{Z}_R$ from standing-wave measurements by means of an example.

Example 7.2 Finding the load impedance for a transmission line from standing-wave measurements

Let us assume that measurements performed on a slotted line of characteristic impedance $Z_0 = 50\ \Omega$ provided the following data. First, with the short circuit as the termination, voltage minima were found to be 20 cm apart. Next, with one of the minima marked as the reference point and the short circuit replaced by the unknown load, the SWR was found to be 3.0 and a voltage minimum was found to be at 5.80 cm from the reference point on the side toward the load. We wish to compute the value of the unknown load impedance.

From the value of the SWR, we obtain by using (7.29)

$$|\bar{\Gamma}_R| = \frac{3 - 1}{3 + 1} = 0.5$$

Since the distance between successive voltage minima is 20 cm, $\lambda/2$ is equal to 20 cm, or λ is equal to 40 cm. Since the voltage minimum shifted toward the load from the reference point, $d_{\min}$ is equal to $\lambda/2$ minus the shift, or $20 - 5.8 = 14.2$ cm. Then, from (7.30), we get

$$\theta = \frac{4\pi}{40} \times 14.2 - \pi = 0.42\pi$$

Thus

$$\bar{\Gamma}_R = 0.5e^{j0.42\pi}$$

Finally, using (7.31), we compute the value of the load impedance to be

$$\begin{aligned}
\bar{Z}_R &= 50\,\frac{1 + 0.5e^{j0.42\pi}}{1 - 0.5e^{j0.42\pi}} \\
&= 50\,\frac{1.1243 + j0.4843}{0.8757 - j0.4843} \\
&= 50\,\frac{1.2242\underline{/23.303^\circ}}{1.0007\underline{/-28.945^\circ}} \\
&= 61.17\underline{/52.248^\circ} \\
&= (37.45 + j48.365)\ \Omega
\end{aligned}$$

Line impedance

Returning now to the solutions for the complex line voltage and current given by (7.24a) and (7.24b), respectively, we find that the line impedance $\bar{Z}(d)$, that is, the impedance at any value of d seen looking toward the load, is given by

$$\boxed{\begin{aligned}\bar{Z}(d) &= \frac{\bar{V}(d)}{\bar{I}(d)} = \frac{\bar{V}^{+}e^{j\beta d}[1 + \bar{\Gamma}(d)]}{(\bar{V}^{+}/Z_0)e^{j\beta d}[1 - \bar{\Gamma}(d)]} \\ &= Z_0\frac{1 + \bar{\Gamma}(d)}{1 - \bar{\Gamma}(d)}\end{aligned}} \tag{7.32}$$

The following properties of the line impedance are of interest:

1. At the location of a voltage maximum of the standing-wave pattern, $1 + \bar{\Gamma}(d)$ and $1 - \bar{\Gamma}(d)$ are purely real and equal to their maximum and minimum magnitudes $1 + |\bar{\Gamma}_R|$ and $1 - |\bar{\Gamma}_R|$, respectively. Hence, $\bar{Z}(d)$ is purely real and maximum, say, $R_{\max}$, equal to $Z_0[(1 + |\bar{\Gamma}_R|)/(1 - |\bar{\Gamma}_R|)]$, or Z_0 (SWR).
2. At the location of a voltage minimum of the standing-wave pattern, $1 + \bar{\Gamma}(d)$ and $1 - \bar{\Gamma}(d)$ are purely real and equal to their minimum and maximum magnitudes $1 - |\bar{\Gamma}_R|$ and $1 + |\bar{\Gamma}_R|$, respectively. Hence, $\bar{Z}(d)$ is purely real and minimum, say, $R_{\min}$, equal to $Z_0[(1 - |\bar{\Gamma}_R|)/(1 + |\bar{\Gamma}_R|)]$, or $Z_0/(\text{SWR})$.
3. Between voltage maxima and minima, $1 + \bar{\Gamma}(d)$ and $1 - \bar{\Gamma}(d)$ are both complex and out of phase. Hence, $\bar{Z}(d)$ is complex, with magnitude lying between Z_0 (SWR) and $Z_0/(\text{SWR})$.
4. Since $\bar{\Gamma}(d \pm n\lambda/2) = \bar{\Gamma}(d)e^{\mp j2\beta n\lambda/2} = \bar{\Gamma}(d)e^{\mp j2n\pi} = \bar{\Gamma}(d)$, $n = 1, 2, 3, \ldots,$ $\bar{\Gamma}(d)$ repeats at intervals of $\lambda/2$, and hence, $\bar{Z}(d)$ repeats at intervals of $\lambda/2$.
5. The product of the line impedances at two values of d separated by $\lambda/4$ is given by

$$\begin{aligned}[\bar{Z}(d)]\left[\bar{Z}\left(d \pm \frac{\lambda}{4}\right)\right] &= \left[Z_0\frac{1 + \bar{\Gamma}(d)}{1 - \bar{\Gamma}(d)}\right]\left[Z_0\frac{1 + \bar{\Gamma}(d \pm \lambda/4)}{1 - \bar{\Gamma}(d \pm \lambda/4)}\right] \\ &= Z_0^2\left[\frac{1 + \bar{\Gamma}(d)}{1 - \bar{\Gamma}(d)}\right]\left[\frac{1 + \bar{\Gamma}(d)e^{\mp j2\beta\lambda/4}}{1 - \bar{\Gamma}(d)e^{\mp j2\beta\lambda/4}}\right] \\ &= Z_0^2\left[\frac{1 + \bar{\Gamma}(d)}{1 - \bar{\Gamma}(d)}\right]\left[\frac{1 + \bar{\Gamma}(d)e^{\mp j\pi}}{1 - \bar{\Gamma}(d)e^{\mp j\pi}}\right] \\ &= Z_0^2\left[\frac{1 + \bar{\Gamma}(d)}{1 - \bar{\Gamma}(d)}\right]\left[\frac{1 - \bar{\Gamma}(d)}{1 + \bar{\Gamma}(d)}\right]\end{aligned}$$

or

$$\boxed{[\bar{Z}(d)]\left[\bar{Z}\left(d \pm \frac{\lambda}{4}\right)\right] = Z_0^2} \tag{7.33}$$

This is a useful property, as we shall learn in the following section.

Input impedance

For a line of length l, as in Fig. 7.8, the input impedance is given from (7.32) by

$$\boxed{\bar{Z}_{\text{in}} = \bar{Z}(l) = Z_0 \frac{1 + \bar{\Gamma}(l)}{1 - \bar{\Gamma}(l)}} \tag{7.34}$$

The input impedance is a useful parameter, because, for a given generator voltage and internal impedance, the power flow down the line can be computed by considering the line voltage and current at any value of d, since the line is lossless; in particular, it is convenient to do this at the input end of the line from input impedance considerations. We shall illustrate this by means of an example.

Example 7.3 Finding the power delivered to the load from considerations of line input impedance

Let us consider the system shown in Fig. 7.13, and find the time-average power delivered to the load from input impedance considerations.

We proceed with the solution in the following step-by-step manner:

(a) Compute the reflection coefficient at the load.

$$\bar{\Gamma}_R = \frac{\bar{Z}_R - Z_0}{\bar{Z}_R + Z_0} = \frac{(30 + j40) - 50}{(30 + j40) + 50} = 0.5\underline{/90^\circ}$$

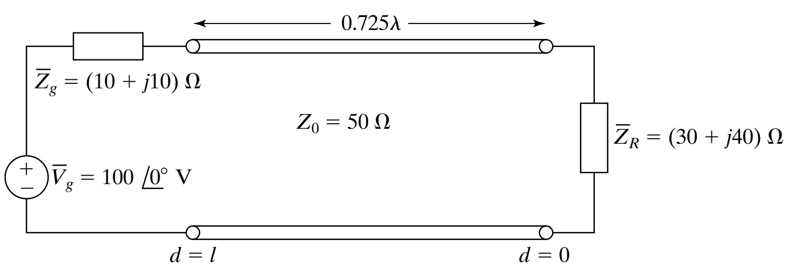

FIGURE 7.13

Transmission-line system for illustrating the computation of power flow from input impedance considerations.

(b) Compute the reflection coefficient $\overline{\Gamma}(l)$ at the input end $d = l$.

$$\begin{aligned}\overline{\Gamma}(l) &= \overline{\Gamma}_R e^{-j2\beta l} \\ &= 0.5\underline{/90^\circ} \times e^{-j(4\pi/\lambda)(0.725\lambda)} \\ &= 0.5\underline{/90^\circ} \times 1\underline{/-162^\circ} \\ &= 0.5\underline{/-72^\circ}\end{aligned}$$

(c) Compute the input impedance.

$$\begin{aligned}\overline{Z}_{\text{in}} = \overline{Z}(l) &= Z_0 \frac{1 + \overline{\Gamma}(l)}{1 - \overline{\Gamma}(l)} \\ &= 50\,\frac{1 + 0.5\underline{/-72^\circ}}{1 - 0.5\underline{/-72^\circ}} = 50\,\frac{1 + (0.1545 - j0.4755)}{1 - (0.1545 - j0.4755)} \\ &= 50\,\frac{1.2486\underline{/-22.385^\circ}}{0.970\underline{/29.353^\circ}} = 64.361\underline{/-51.738^\circ} \\ &= (39.86 - j50.54)\ \Omega\end{aligned}$$

(d) We now have the equivalent circuit at the input, as shown in Fig. 7.14, from which we compute the current $\overline{I}_g = \overline{I}(l)$, drawn from the generator. Thus, we obtain

$$\begin{aligned}\overline{I}(l) = \overline{I}_g &= \frac{\overline{V}_g}{\overline{Z}_g + \overline{Z}_{\text{in}}} = \frac{100\underline{/0^\circ}}{(10 + j10) + (39.86 - j50.54)} \\ &= \frac{100\underline{/0^\circ}}{49.86 - j40.54} = \frac{100\underline{/0^\circ}}{64.261\underline{/-39.114^\circ}} \\ &= 1.5562\underline{/39.114^\circ}\ \text{A}\end{aligned}$$

(e) The voltage across the input impedance is then given by

$$\begin{aligned}\overline{V}(l) &= \overline{Z}_{\text{in}}\overline{I}(l) \\ &= 64.361\underline{/-51.738^\circ} \times 1.5562\underline{/39.114^\circ} \\ &= 100.159\underline{/-12.624^\circ}\ \text{V}\end{aligned}$$

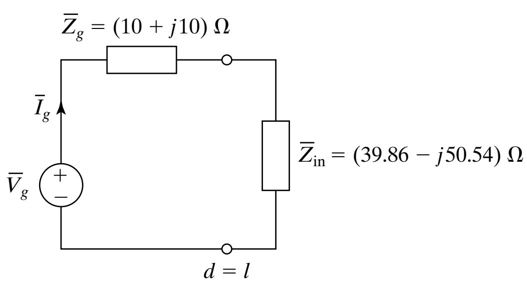

FIGURE 7.14

Equivalent circuit at the input end $d = l$ for the system of Fig. 7.13.

(f) Finally, the time-average power delivered to the input, and hence to the load, is given by

$$\begin{aligned}\langle P\rangle &= \tfrac{1}{2}\mathrm{Re}[\overline{V}(l)\overline{I}^*(l)]\\ &= \tfrac{1}{2}\mathrm{Re}[100.159\underline{/-12.624^\circ} \times 1.5562\underline{/-39.114^\circ}]\\ &= \tfrac{1}{2} \times 100.159 \times 1.5562 \times \cos 51.738^\circ\\ &= 48.26\ \mathrm{W}\end{aligned}$$

Normalized impedance and admittance

Returning to (7.32), we now define the normalized line impedance $\overline{z}(d)$ as the ratio of the line impedance to the line characteristic impedance. Thus,

$$\boxed{\overline{z}(d) = \frac{\overline{Z}(d)}{Z_0} = \frac{1+\overline{\Gamma}(d)}{1-\overline{\Gamma}(d)}} \tag{7.35}$$

Conversely,

$$\boxed{\overline{\Gamma}(d) = \frac{\overline{z}(d)-1}{\overline{z}(d)+1}} \tag{7.36}$$

Finally, the line admittance is given by

$$\overline{Y}(d) = \frac{1}{\overline{Z}(d)} = \frac{1}{Z_0}\frac{1-\overline{\Gamma}(d)}{1+\overline{\Gamma}(d)}$$

or

$$\overline{Y}(d) = Y_0\frac{1-\overline{\Gamma}(d)}{1+\overline{\Gamma}(d)} \tag{7.37}$$

where $Y_0 = 1/Z_0$ is the characteristic admittance of the line. The normalized line admittance is

$$\boxed{\overline{y}(d) = \frac{\overline{Y}(d)}{Y_0} = \frac{1-\overline{\Gamma}(d)}{1+\overline{\Gamma}(d)}} \tag{7.38}$$

and, conversely,

$$\boxed{\overline{\Gamma}(d) = \frac{1-\overline{y}(d)}{1+\overline{y}(d)}} \tag{7.39}$$

We shall use these relationships in the following sections.

K7.2. Arbitrary load; Generalized reflection coefficient; Partial standing waves; Standing-wave ratio; Standing-wave parameters; Standing-wave measurements; Line impedance; Power flow; Normalized line impedance; Normalized line admittance.

D7.4. A line of characteristic impedance 60 Ω is terminated by a load consisting of the series combination of $R = 30\ \Omega$, $L = 1\ \mu$H, and $C = 100\ p$F. Find the values of SWR and $d_{\min}$ for each of the following radian frequencies of the source: **(a)** $\omega = 10^8$; **(b)** $\omega = 2 \times 10^8$; and **(c)** $\omega = 0.8 \times 10^8$.

Ans. **(a)** 2, 0; **(b)** 14.94, 0.309λ; **(c)** 3.324, 0.115λ.

D7.5. Standing-wave measurements are performed on a line of characteristic impedance 60 Ω terminated by a load $\bar{Z}_R$. For each of the following sets of standing-wave data, find $\bar{Z}_R$: **(a)** SWR = 1.5, a voltage minimum right at the load; **(b)** SWR = 3.0, two successive voltage minima at 3 cm and 9 cm from the load; and **(c)** SWR = 2.0, two successive voltage minima at 3 cm and 7 cm from the load.

Ans. **(a)** $(40 + j0)\ \Omega$; **(b)** $(180 + j0)\ \Omega$; **(c)** $(48 + j36)\ \Omega$.

D7.6. An air-dielectric line of characteristic impedance $Z_0 = 75\ \Omega$ is terminated by a load impedance $(45 + j60)\ \Omega$. Find the input impedance of the line for each of the following pairs of values of the frequency f and the length l of the line: **(a)** $f = 15$ MHz, $l = 5$ m; **(b)** $f = 50$ MHz, $l = 3$ m; and **(c)** $f = 37.5$ MHz, $l = 5$ m.

Ans. **(a)** $(45 - j60)\ \Omega$; **(b)** $(45 + j60)\ \Omega$; **(c)** $(225 + j0)\ \Omega$.

7.3 TRANSMISSION-LINE MATCHING

In the preceding section, we discussed standing waves on a line terminated by an arbitrary load. In the presence of standing waves, that is, when the load impedance is not equal to the characteristic impedance, it follows from (7.34) that the input impedance of the line will vary with frequency, because the electrical length of the line and, hence, $\bar{\Gamma}(l) = \bar{\Gamma}_R e^{-j2\beta l}$, changes. This sensitivity to frequency increases with the electrical length of the line. To show this, let the length of the line be $l = n\lambda$. If the frequency is changed by an amount Δf, then the change in n is given by

$$\Delta n = \Delta\left(\frac{l}{\lambda}\right) = \Delta\left(\frac{lf}{v_p}\right) = \frac{l}{v_p}\Delta f = \frac{n\lambda}{v_p}\Delta f = n\frac{\Delta f}{f} \tag{7.40}$$

Thus Δn, the change in the number of wavelengths corresponding to the line length, is proportional to n. The variation of the input impedance with frequency puts a limitation on the performance of a transmission-line system from the point of view of communication. For this and other reasons pertaining to power flow, it is desirable to eliminate standing waves on the line by connecting a *matching* device near the load such that the line views an effective impedance equal to its own characteristic impedance on the generator side of the matching device as shown in Fig. 7.15. The matching device should not at the same time absorb any power. It should be noted that *matching*, as referred to here, is not related to maximum power transfer since the condition for maximum power transfer is that the

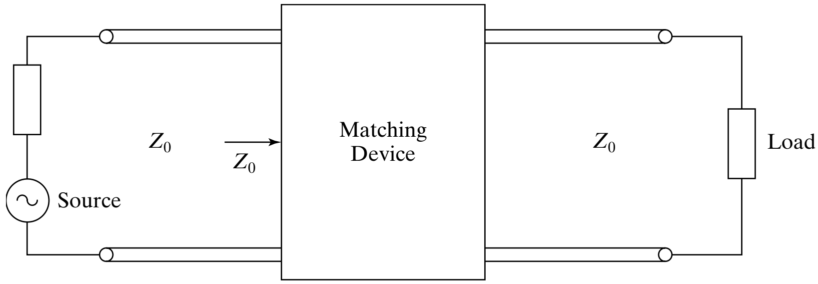

FIGURE 7.15

For illustrating the principle behind transmission-line *matching*.

line input impedance must be the complex conjugate of the generator internal impedance. In the following, we discuss three techniques of matching.

A. Quarter-Wave Transformer Matching

Quarter-wave transformer matching

The quarter-wave transformer, or QWT, matching technique makes use of a section of length $\lambda/4$ of a line of characteristic impedance Z_q different from that of the main line, as shown in Fig. 7.16. The principle is based on the property of line impedance given by (7.33). With reference to the notation of Fig. 7.16, we first note that to achieve a match, $\bar{Z}_1$ must be equal to Z_0. Then, since, from (7.33), $\bar{Z}_1\bar{Z}_2 = Z_q^2$, $\bar{Z}_2 = Z_q^2/\bar{Z}_1 = Z_q^2/Z_0$ must be purely real. We recall from the discussion of line impedance in Section 7.2 that the line impedance is purely real at locations of voltage maxima and minima of the standing-wave pattern. Therefore, within the first half-wavelength from the load, there are two solutions for d_q and hence for Z_q.

If we choose a voltage minimum for the first solution, then from (7.28)

$$\boxed{d_q^{(1)} = \frac{\lambda}{4\pi}(\theta + \pi)} \tag{7.41}$$

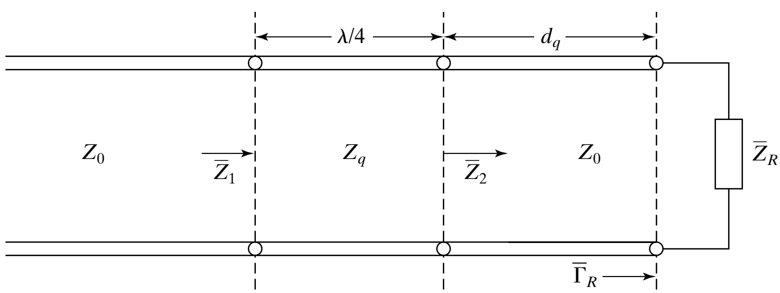

FIGURE 7.16

For illustrating the quarter-wave transformer matching technique.

where θ is the phase angle of $\overline{\Gamma}_R$, and the superscript (1) refers to solution 1. The value of the line impedance is $Z_0(1 - |\overline{\Gamma}_R|)/(1 + |\overline{\Gamma}_R|)$. Hence, the value of Z_q is given by

$$Z_0 \cdot Z_0 \frac{1 - |\overline{\Gamma}_R|}{1 + |\overline{\Gamma}_R|} = Z_q^2$$

or

$$\boxed{Z_q^{(1)} = Z_0 \sqrt{\frac{1 - |\overline{\Gamma}_R|}{1 + |\overline{\Gamma}_R|}}} \tag{7.42}$$

For the second solution, the value of d_q corresponds to the location of a voltage maximum that occurs at $\pm\lambda/4$ from the location of the voltage minimum. Thus,

$$\boxed{d_q^{(2)} = d_q^{(1)} \pm \frac{\lambda}{4}} \tag{7.43}$$

whichever is positive and less than $\lambda/2$. The corresponding line impedance is $Z_0(1 + |\overline{\Gamma}_R|)/(1 - |\overline{\Gamma}_R|)$, so that

$$\boxed{Z_q^{(2)} = Z_0 \sqrt{\frac{1 + |\overline{\Gamma}_R|}{1 - |\overline{\Gamma}_R|}}} \tag{7.44}$$

B. Single-Stub Matching

Single-stub matching

Another technique of transmission-line matching known as *stub matching* consists of connecting small sections of short-circuited lines (stubs) of appropriate lengths in parallel with the line, at appropriate distances from the load. In the single-stub matching technique, one stub is used and a match is achieved by varying the location of the stub and the length of the stub. We shall assume the characteristic impedance of the stub to be the same as that of the line and use the notation shown in Fig. 7.17, in which $\overline{z}_R$ is the normalized load impedance, $\overline{y}_1$ and $\overline{y}_1'$ are the normalized line admittances just to the left and just to the right, respectively, of the stub, and b is the normalized input susceptance of the stub. The solution to the single-stub matching problem then consists of finding the values of d_s and l_s for a given value of $\overline{z}_R$ and hence of $\overline{\Gamma}_R$.

First, we observe that to achieve a match, $\overline{y}_1$ must be equal to $(1 + j0)$. Then proceeding to the right of the stub, we can write the following steps:

$$\overline{y}_1' = 1 - jb \tag{7.45a}$$

$$\overline{\Gamma}_1' = \frac{1 - \overline{y}_1'}{1 + \overline{y}_1'} = \frac{jb}{2 - jb} \tag{7.45b}$$

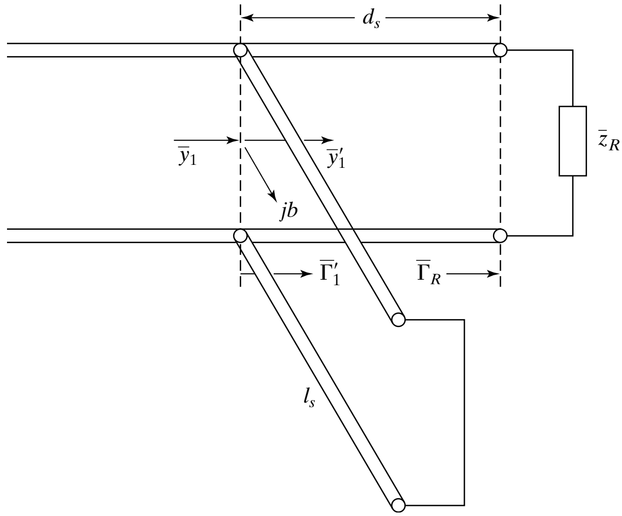

FIGURE 7.17

For illustrating the single-stub matching technique.

$$\begin{aligned}\bar{\Gamma}_R &= \bar{\Gamma}'_1 e^{j2\beta d_s} = \frac{jb}{2 - jb} e^{j2\beta d_s} \\ &= \frac{|b|}{\sqrt{4 + b^2}} e^{j(\pm\pi/2 + \tan^{-1} b/2 + 2\beta d_s + 2n\pi)} \qquad \text{for} \quad b \gtrless 0\end{aligned} \tag{7.45c}$$

where n is an integer (positive or negative). Thus,

$$|\bar{\Gamma}_R| = \frac{|b|}{\sqrt{4 + b^2}} \tag{7.46a}$$

$$\theta = \pm\frac{\pi}{2} + \tan^{-1}\frac{b}{2} + 2\beta d_s + 2n\pi \qquad \text{for} \quad b \gtrless 0 \tag{7.46b}$$

so that

$$\boxed{\begin{aligned} b &= \pm\frac{2|\bar{\Gamma}_R|}{\sqrt{1 - |\bar{\Gamma}_R|^2}} && (7.47) \\ d_s &= \frac{\lambda}{4\pi}\left(\theta \mp \frac{\pi}{2} - \tan^{-1}\frac{b}{2} - 2n\pi\right) \qquad \text{for} \quad b \gtrless 0 && (7.48)\end{aligned}}$$

Thus, two solutions are possible for b as given by (7.47) and the corresponding solutions for d_s are given by (7.48), where the integer value for n is chosen such that $0 \le d_s < \lambda/2$. Finally, to find the solutions for the stub length, we note from (7.13) that the normalized input impedance of a short-circuited line of length l_s is $j \tan \beta l_s$, so that

$$\frac{1}{jb} = j \tan \beta l_s$$

$$\tan \beta l_s = -\frac{1}{b}$$

$$\boxed{l_s = \begin{cases} \dfrac{\lambda}{2\pi}\left[\tan^{-1}\left(-\dfrac{1}{b}\right)\right] + \dfrac{\lambda}{2} & \text{for} \quad b > 0 \\ \dfrac{\lambda}{2\pi}\left[\tan^{-1}\left(-\dfrac{1}{b}\right)\right] & \text{for} \quad b < 0 \end{cases}} \tag{7.49}$$

C. Double-Stub Matching

Double-stub matching

In the single-stub matching technique, it is necessary to vary the distance between the stub and the load, as well as the length of the stub, in order to achieve a match for different loads or for different frequencies. This can be inconvenient for some arrangements of lines. When two stubs are used, it is possible to fix their locations and achieve a match for a wide range of loads by adjusting the lengths of the stubs. To discuss the principle behind this *double-stub matching* technique, we make use of the notation shown in Fig. 7.18, in which all admittances and susceptances are normalized quantities with respect to the characteristic admittance of the line. The solution to the double-stub

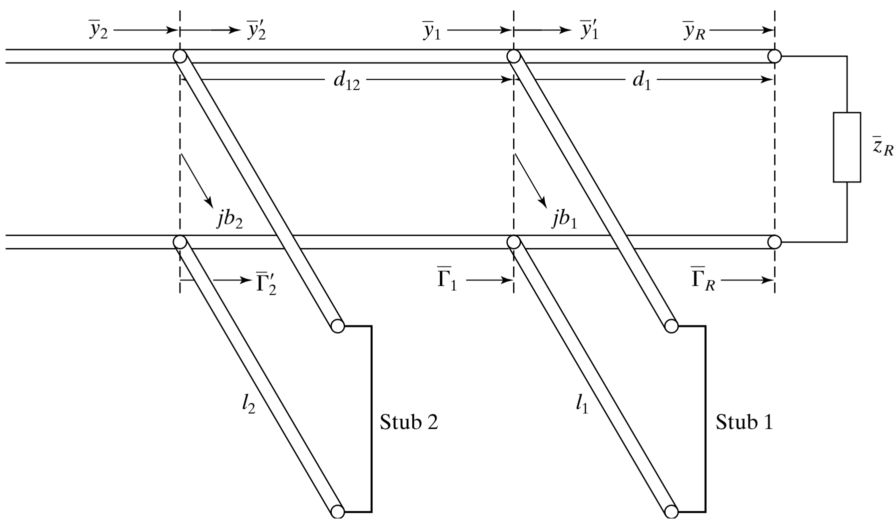

FIGURE 7.18

For illustrating the double-stub matching technique.

matching problem then consists of finding the values of l_1 and l_2 for a given set of values of $\bar{z}_R$ (and hence, of $\bar{\Gamma}_R$), d_1, and d_{12}.

First, we observe that to achieve a match, $\bar{y}_2$ must be equal to $(1 + j0)$. Then proceeding to the right in a step-by-step manner, we obtain an expression for $\bar{y}'_1$ in terms of b_1, b_2, and d_{12} as follows:

$$\bar{y}'_2 = \bar{y}_2 - jb_2 = 1 - jb_2 \tag{7.50a}$$

$$\bar{\Gamma}'_2 = \frac{1 - \bar{y}'_2}{1 + \bar{y}'_2} = \frac{jb_2}{2 - jb_2} \tag{7.50b}$$

$$\bar{\Gamma}_1 = \bar{\Gamma}'_2 e^{j2\beta d_{12}} = \frac{jb_2}{2 - jb_2} e^{j2\beta d_{12}} \tag{7.50c}$$

$$\begin{aligned} \bar{y}_1 &= \frac{1 - \bar{\Gamma}_1}{1 + \bar{\Gamma}_1} \\ &= \frac{4 - j(4b_2 \cos 2\beta d_{12} - 2b_2^2 \sin 2\beta d_{12})}{4 - 4b_2 \sin 2\beta d_{12} + 4b_2^2 \sin^2 \beta d_{12}} \end{aligned} \tag{7.50d}$$

$$\begin{aligned} \bar{y}'_1 &= \bar{y}_1 - jb_1 \\ &= \frac{1}{1 - b_2 \sin 2\beta d_{12} + b_2^2 \sin^2 \beta d_{12}} \\ &\quad + j\left(\frac{b_2^2 \sin 2\beta d_{12} - 2b_2 \cos 2\beta d_{12}}{2 - 2b_2 \sin 2\beta d_{12} + 2b_2^2 \sin^2 \beta d_{12}} - b_1\right) \end{aligned} \tag{7.50e}$$

For given values of $\bar{z}_R$ and d_1, $\bar{y}'_1$ can be computed in the usual manner, and the real and imaginary parts can be equated to the real and imaginary parts, respectively, on the right side of (7.50e). Noting that b_1 does not appear in the real part expression, we can first compute b_2 by solving the equation for the real parts. Thus, letting the real part of $\bar{y}'_1$ as computed from $\bar{z}_R$ and d_1 to be g', we have

$$\frac{1}{1 - b_2 \sin 2\beta d_{12} + b_2^2 \sin^2 \beta d_{12}} = g' \tag{7.51}$$

Rearranging and solving for b_2, we obtain

$$b_2 = \frac{\sin 2\beta d_{12} \pm \sqrt{\sin^2 2\beta d_{12} - 4(1 - 1/g') \sin^2 \beta d_{12}}}{2 \sin^2 \beta d_{12}}$$

or

$$\boxed{b_2 = \frac{\cos \beta d_{12} \pm \sqrt{1/g' - \sin^2 \beta d_{12}}}{\sin \beta d_{12}}} \tag{7.52}$$

We now see that a solution does not exist for b_2 if $g' > 1/\sin^2 \beta d_{12}$, and hence, it is not possible to achieve a match for loads that result in the real part of $\overline{y}_1'$ being greater than $1/\sin^2 \beta d_{12}$. A simple way to get around this problem is to increase d_1 by $\lambda/4$ (see Problem 7.24). Assuming that the condition $g' < 1/\sin^2 \beta d_{12}$ is achieved, we then compute two possible values for b_2 as given by (7.52). From the equation for the imaginary parts of $\overline{y}_1'$, the corresponding values of b_1 are then given by

$$\boxed{b_1 = \frac{b_2^2 \sin 2\beta d_{12} - 2b_2 \cos 2\beta d_{12}}{2 - 2b_2 \sin 2\beta d_{12} + 2b_2^2 \sin^2 \beta d_{12}} - b'} \tag{7.53}$$

where b' is the imaginary part of $\overline{y}_1'$ as computed from $\overline{z}_R$ and d_1. Finally, the lengths of the two stubs are computed from b_1 and b_2 as in the case of the single-stub matching technique.

To consider a numerical example for the solution of all three types of matching techniques, let $R_L = 30\ \Omega$, $X_L = -40\ \Omega$, and for the double-stub matching case, $d_1 = 0$ and $d_{12} = 0.375\lambda$. Then the solutions obtained by using the appropriate equations for the three techniques are listed in Table 7.1.

Note that $d_{12} = 0.375\lambda$ is an odd multiple of $\lambda/8$. Values of odd multiples of $\lambda/8$ are commonly used for d_{12}. Also, if the specified value of Z_R is such that $g' > 1/\sin^2 \beta d_{12}$, then the value of d_1 is increased by $\lambda/4$ and the double-stub matching is continued.

Bandwidth

For any transmission-line matched system, the match is disturbed as the frequency is varied from that at which the various electrical lengths and distances are equal to the computed values for achieving the match. For example, in the QWT matched system, the electrical length of the QWT departs from one-quarter wavelength as the frequency is varied from that at which the match is achieved, and the system is no longer matched even if the load does not vary with frequency. A plot of the SWR in the main line to the left of the QWT versus frequency is typically of the shape shown in Fig. 7.19, where f_0 is the design frequency at which the system is matched, and hence the frequency at which the SWR is unity. One can then specify a tolerable value of SWR, say, S, so that there exists an acceptable bandwidth of operation, $f_2 - f_1$. Similar considerations apply to the single-stub and double-stub matched systems.

TABLE 7.1 Solutions for Transmission-Line Matching Example

Technique	Solution Number	Solution
QWT	1	$d_q = 0.125\lambda$, $Z_q = 28.86751\ \Omega$
QWT	2	$d_q = 0.375\lambda$, $Z_q = 86.60254\ \Omega$
Single stub	1	$d_s = 0.20833\lambda$, $l_s = 0.38641\lambda$
Single stub	2	$d_s = 0.04167\lambda$, $l_s = 0.11359\lambda$
Double stub	1	$l_1 = 0.13483\lambda$, $l_2 = 0.32726\lambda$
Double stub	2	$l_1 = 0.05614\lambda$, $l_2 = 0.05996\lambda$

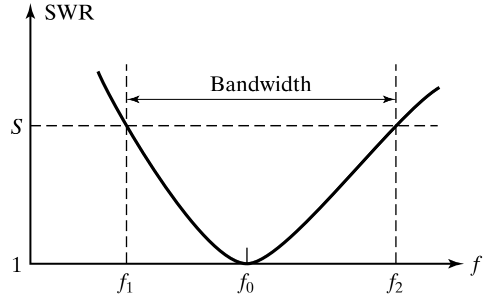

FIGURE 7.19

The SWR versus frequency curve illustrating the bandwidth between the two frequencies f_1 and f_2, on either side of the design frequency f_0, at which the SWR is a specified value S (>1).

SWR versus frequency computation

To discuss a procedure by means of which the SWR versus frequency curve can be computed for all three types of matching techniques discussed, let us consider a transmission-line system having n discontinuities, as shown for $n = 2$ in Fig. 7.20. At each discontinuity, there can exist a stub and a change in characteristic impedance. We shall consider a specification of zero for the length of the stub to mean no stub is present instead of a stub of zero length. This does not result in a conflict, since, for any matched system using short-circuited stubs, values of zero cannot be obtained for stub lengths, because then the value of SWR would be infinity. With this understanding, Fig. 7.20 can be used to represent all three types of matching systems by specifying values for the various parameters, as shown in Table 7.2.

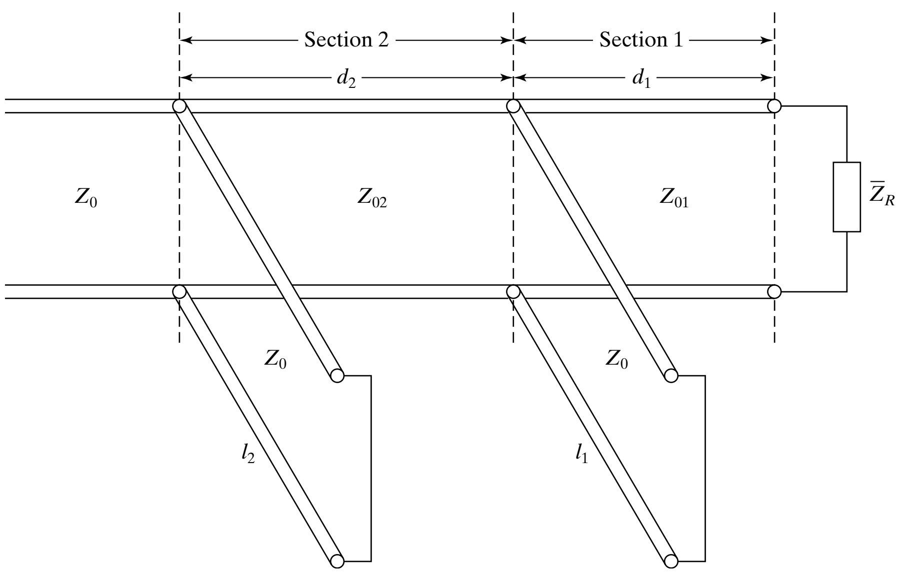

FIGURE 7.20

Transmission-line system for computing the SWR versus frequency curve for QWT, single-stub, and double-stub matched systems.

TABLE 7.2 Values of Parameters for Using the System of Fig. 7.20 for Three Different Cases

System	n	Z_{01}	d_1	l_1	Z_{02}	d_2	l_2
QWT	2	Z_0	d_q	0[a]	Z_q	1/4	0[a]
Single stub	1	Z_0	d_s	l_s	—	—	—
Double stub	2	Z_0	d_1	l_1	Z_0	d_{12}	l_2

[a] Value of zero means no stub present.

Then to compute the SWR in the main line at a given frequency, we first note that, since $\lambda \propto 1/f$, the electrical length of a line section or of a stub is proportional to f. Thus, at a frequency f, the electrical length is equal to f/f_0 times its value at f_0. For a given f/f_0, the procedure consists of starting at the load and computing in succession the line admittance to the left of the stub at each discontinuity, from a knowledge of the line admittance at the output of the line section to the right of that stub, until the line admittance to the left of the last discontinuity is found and used to compute the required SWR. In carrying out this procedure, we observe the following:

1. To compute the normalized admittance, say, $\bar{y}_i$, at the input (left) end of a line section of length l from the normalized admittance, say, $\bar{y}_o$, at the output (right) end of that section, we use the formula

$$\bar{y}_i = \frac{1 - \bar{\Gamma}_i}{1 + \bar{\Gamma}_i} = \frac{1 - \bar{\Gamma}_o e^{-j2\beta l}}{1 + \bar{\Gamma}_o e^{-j2\beta l}}$$
$$= \frac{1 - [(1 - \bar{y}_o)/(1 + \bar{y}_o)]e^{-j2\beta l}}{1 + [(1 - \bar{y}_o)/(1 + \bar{y}_o)]e^{-j2\beta l}}$$

or

$$\boxed{\bar{y}_i = \frac{j \sin \beta l + \bar{y}_o \cos \beta l}{\cos \beta l + j \bar{y}_o \sin \beta l}} \tag{7.54}$$

where $\bar{\Gamma}_i$ and $\bar{\Gamma}_o$ are the reflection coefficients at the input and output ends, respectively.

2. To compute the line admittance to the left of a stub, we add the input admittance of the stub to the line admittance to the right of the stub.

The computation of SWR versus f/f_0 can be done by using a computer program. For values of the input parameters pertinent to the first of the two solutions for the double-stub matching case in Table 7.1, the computed values of SWR are listed in Table 7.3. The frequency variation of Z_R is taken into account by assuming Z_R to be the series combination of a single resistor and a single reactive element.

TABLE 7.3 Computed Values of SWR Versus Frequency

f/f_0	SWR
0.90	1.9249
0.92	1.7124
0.94	1.5117
0.96	1.325
0.98	1.1543
1.00	1.0006
1.02	1.1583
1.04	1.3459
1.06	1.5663
1.08	1.8236
1.10	2.1216

K7.3. Matching; Quarter-wave transformer; Single stub; Double stub; Bandwidth.

D7.7. For a line of characteristic impedance 75 Ω, find the location nearest to the load and the characteristic impedance of a quarter-wave transformer required to achieve a match for each of the following values of $\overline{\Gamma}_R$: **(a)** 1/9; **(b)** $-j0.5$; and **(c)** $j1/3$.

Ans. **(a)** 83.85 Ω, 0; **(b)** 43.30 Ω, 0.125λ; **(c)** 106.07 Ω, 0.125λ.

D7.8. For each of the following values of $\overline{Z}_R$ terminating a line of characteristic impedance 60 Ω, find the lowest value of d_s and the corresponding smallest value of the length l_s of a single short-circuited stub of characteristic impedance 60 Ω required to achieve a match between the line and the load: **(a)** $\overline{Z}_R = 30\ \Omega$ and **(b)** $\overline{Z}_R = (12 - j24)\ \Omega$.

Ans. **(a)** 0.098λ, 0.348λ; **(b)** 0, 0.074λ.

D7.9. For each of the following sets of values of d_1, d_{12}, and $\bar{z}_R$, associated with the double-stub matching technique, determine whether or not it is possible to achieve a match between the line and the load: **(a)** $d_1 = 0, d_{12} = 3\lambda/8, \bar{z}_R = 0.3 + j0.4$; **(b)** $d_1 = \lambda/8, d_{12} = 3\lambda/8, \bar{z}_R = 0.5$; and **(c)** $d_1 = \lambda/4, d_{12} = 5\lambda/8, \bar{z}_R = 2.5 - j5.0$.

Ans. **(a)** Yes; **(b)** yes; **(c)** no.

7.4 THE SMITH CHART: 1. BASIC PROCEDURES

In the preceding section, we considered transmission-line matching techniques and computer solutions of matching problems. In this section, we discuss some basic procedures using the Smith chart. Introduced in 1939 by P. H. Smith,[1] the Smith chart continues to be a popular graphical aid in the solution of transmission-line problems, including simulation on personal computers.[2]

[1]P. H. Smith, "Transmission-Line Calculator," *Electronics*, January 1939, pp. 29–31.

[2]See, for example, M. Felton, "Moving the Smith Chart to a Low-Cost Computer," *Microwave Journal*, October 1983, pp. 131–133, and N. N. Rao, "PC-Assisted Instruction of Introductory Electromagnetics," *IEEE Transactions on Education*, February 1990, pp. 51–59.

Construction

The *Smith chart* is a transformation from the complex $\bar{Z}$-plane (or $\bar{Y}$-plane) to the complex $\bar{\Gamma}$-plane. To discuss the basis behind the construction of the Smith chart, we begin with the relationship for the reflection coefficient in terms of the normalized line impedance as given by

$$\bar{\Gamma}(d) = \frac{\bar{z}(d) - 1}{\bar{z}(d) + 1} \tag{7.55}$$

Letting $\bar{z}(d) = r + jx$, we have

$$\bar{\Gamma}(d) = \frac{r + jx - 1}{r + jx + 1} = \frac{(r - 1) + jx}{(r + 1) + jx}$$

and

$$|\bar{\Gamma}(d)| = \left[\frac{(r - 1)^2 + x^2}{(r + 1)^2 + x^2}\right]^{1/2} \leq 1$$

for positive values of r. Thus, for the passive line impedances, the reflection coefficient lies inside or on the circle of unit radius in the $\bar{\Gamma}$-plane. We will hereafter call this circle the *unit circle*. Conversely, each point inside or on the unit circle represents a possible value of reflection coefficient corresponding to a unique value of passive normalized line impedance. Hence, all possible values of passive normalized line impedances can be mapped onto the region bounded by the unit circle.

To determine how the normalized line impedance values are mapped onto the region bounded by the unit circle, we note that

$$\bar{\Gamma} = \frac{r + jx - 1}{r + jx + 1} = \frac{r^2 - 1 + x^2}{(r + 1)^2 + x^2} + j\frac{2x}{(r + 1)^2 + x^2}$$

so that

$$\boxed{\begin{aligned} \mathrm{Re}(\bar{\Gamma}) &= \frac{r^2 - 1 + x^2}{(r + 1)^2 + x^2} \qquad (7.56a) \\ \mathrm{Im}(\bar{\Gamma}) &= \frac{2x}{(r + 1)^2 + x^2} \qquad (7.56b) \end{aligned}}$$

Let us now discuss different cases:

1. $\bar{z}$ is purely real; that is, $x = 0$. Then

$$\mathrm{Re}(\bar{\Gamma}) = \frac{r - 1}{r + 1} \qquad \text{and} \qquad \mathrm{Im}(\bar{\Gamma}) = 0$$

Purely real values of $\bar{z}$ are represented by points on the real axis. For example, $r = 0, \frac{1}{3}, 1, 3$, and ∞ are represented by $\bar{\Gamma} = -1, -\frac{1}{2}, 0, \frac{1}{2}$ and 1, respectively, as shown in Fig. 7.21(a).

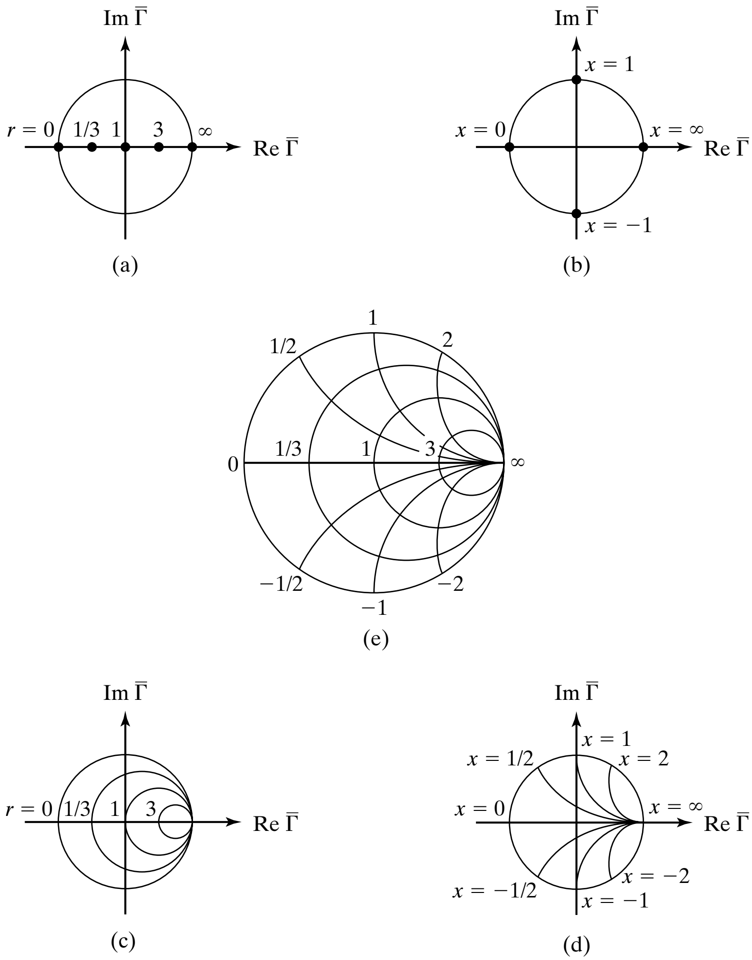

FIGURE 7.21

Development of the Smith chart by transformation from $\bar{z}$ to $\overline{\Gamma}$.

2. $\bar{z}$ is purely imaginary; that is, $r = 0$. Then

$$|\overline{\Gamma}| = \left|\frac{x^2 - 1}{x^2 + 1} + j\frac{2x}{x^2 + 1}\right| = 1$$

and

$$\angle\overline{\Gamma} = \tan^{-1}\frac{2x}{x^2 - 1}$$

Purely imaginary values of $\bar{z}$ are represented by points on the unit circle. For example, $x = 0, 1, \infty, -1,$ and $-\infty$ are represented by $\bar{\Gamma} = 1\underline{/\pi}$, $1\underline{/\pi/2}$, $1\underline{/0^\circ}$, $1\underline{/-\pi/2}$, and $1\underline{/2\pi}$, respectively, as shown in Fig. 7.21(b).

3. $\bar{z}$ is complex, but its real part is constant. Then

$$\left[\mathrm{Re}(\bar{\Gamma}) - \frac{r}{r+1}\right]^2 + [\mathrm{Im}(\bar{\Gamma})]^2$$
$$= \left[\frac{r^2 - 1 + x^2}{(r+1)^2 + x^2} - \frac{r}{r+1}\right]^2 + \left[\frac{2x}{(r+1)^2 + x^2}\right]^2 = \left(\frac{1}{r+1}\right)^2$$

This is the equation of a circle with center at $\mathrm{Re}(\bar{\Gamma}) = r/(r+1)$ and $\mathrm{Im}(\bar{\Gamma}) = 0$ and radius equal to $1/(r+1)$. Thus, loci of constant r are circles in the $\bar{\Gamma}$-plane with centers at $[r/(r+1), 0]$ and radii $1/(r+1)$. For example, for $r = 0, \frac{1}{3}, 1, 3,$ and ∞, the centers of the circles are (0, 0), $(\frac{1}{4}, 0)$, $(\frac{1}{2}, 0)$, $(\frac{3}{4}, 0)$, and (1, 0), respectively, and the radii are $1, \frac{3}{4}, \frac{1}{2}, \frac{1}{4}$, and 0, respectively. These circles are shown in Fig. 7.21(c).

4. $\bar{z}$ is complex, but its imaginary part is constant. Then

$$[\mathrm{Re}(\bar{\Gamma}) - 1]^2 + \left[\mathrm{Im}(\bar{\Gamma}) - \frac{1}{x}\right]^2$$
$$= \left[\frac{r^2 - 1 + x^2}{(r+1)^2 + x^2} - 1\right]^2 + \left[\frac{2x}{(r+1)^2 + x^2} - \frac{1}{x}\right]^2 = \left(\frac{1}{x}\right)^2$$

This is the equation of a circle with center at $\mathrm{Re}(\bar{\Gamma}) = 1$ and $\mathrm{Im}(\bar{\Gamma}) = 1/x$ and radius equal to $1/|x|$. Thus, loci of constant x are circles in the $\bar{\Gamma}$-plane with centers at $(1, 1/x)$ and radii equal to $1/|x|$. For example, for $x = 0, \pm\frac{1}{2}$, ± 1, ± 2, and $\pm\infty$, the centers of the circles are $(1, \infty)$, $(1, \pm 2)$, $(1, \pm 1)$, $(1, \pm\frac{1}{2})$, and (1, 0), respectively, and the radii are $\infty, 2, 1, \frac{1}{2}$, and 0, respectively. Portions of these circles that fall inside the unit circle are shown in Fig. 7.21(d). Portions that fall outside the unit circle represent active impedances.

Combining Figs. 7.21(c) and (d), we obtain the chart of Fig. 7.21(e). In a commercially available form shown in Fig. 7.22, the Smith chart contains circles of constant r and constant x for very small increments of r and x, respectively, so that interpolation between the contours can be carried out accurately. We now consider an example to illustrate some basic procedures using the Smith chart.

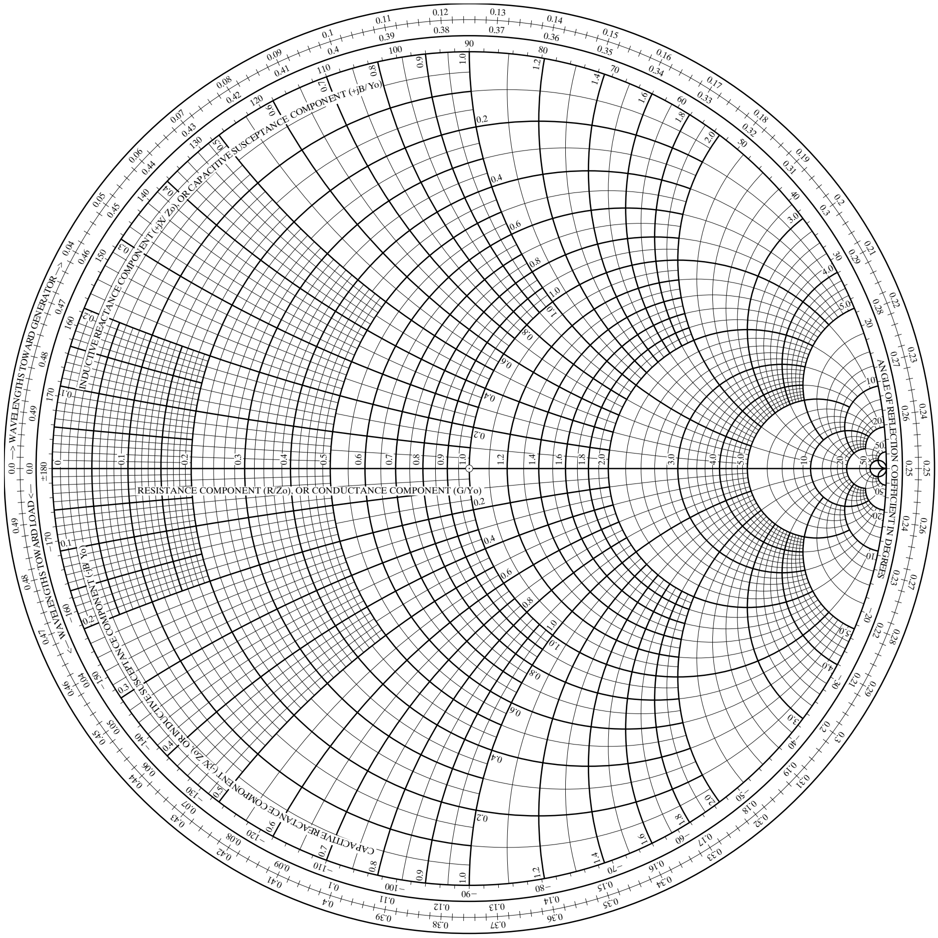

FIGURE 7.22

The Smith chart. (Copyrighted by and reproduced with the permission of Kay Elemetrics Corp., Pine Brook, N.J.)

Example 7.4 For illustrating several basic procedures using the Smith chart

Some basic procedures

A transmission line of characteristic impedance 50 Ω is terminated by a load impedance $\overline{Z}_R = (15 - j20)\ \Omega$. It is desired to find the following quantities by using the Smith chart.

1. Reflection coefficient at the load
2. SWR on the line
3. Distance of the first voltage minimum of the standing-wave pattern from the load
4. Line impedance at $d = 0.05\lambda$
5. Line admittance at $d = 0.05\lambda$
6. Location nearest to the load at which the real part of the line admittance is equal to the line characteristic admittance

We proceed with the solution of the problem in the following step-by-step manner with reference to Fig. 7.23.

(a) Find the normalized load impedance.

$$\bar{z}_R = \frac{\bar{Z}_R}{Z_0} = \frac{15 - j20}{50} = 0.3 - j0.4$$

(b) Locate the normalized load impedance on the Smith chart at the intersection of the 0.3 constant normalized resistance circle and -0.4 constant normalized reactance circle (point A).

(c) Locating point A actually amounts to computing the reflection coefficient at the load since the Smith chart is a transformation in the $\bar{\Gamma}$-plane. The magnitude of the reflection coefficient is the distance from the center (O) of the Smith chart (origin of the $\bar{\Gamma}$-plane) to the point A based on a radius of unity for the outermost circle. For this example, $|\bar{\Gamma}_R| = 0.6$. The phase angle of $\bar{\Gamma}_R$ is the angle measured from the horizontal axis to the right of O (positive real axis in the $\bar{\Gamma}$-plane) to the line OA in the counterclockwise direction. This angle is indicated on the chart along its circumference. For this example, $\underline{/\bar{\Gamma}_R} = 227°$. Thus,

$$\bar{\Gamma}_R = 0.6e^{j1.261\pi}$$

(d) To find the SWR, we recall that at the location of a voltage maximum, the line impedance is purely real and given by

$$R_{\max} = Z_0(\text{SWR}) \tag{7.57}$$

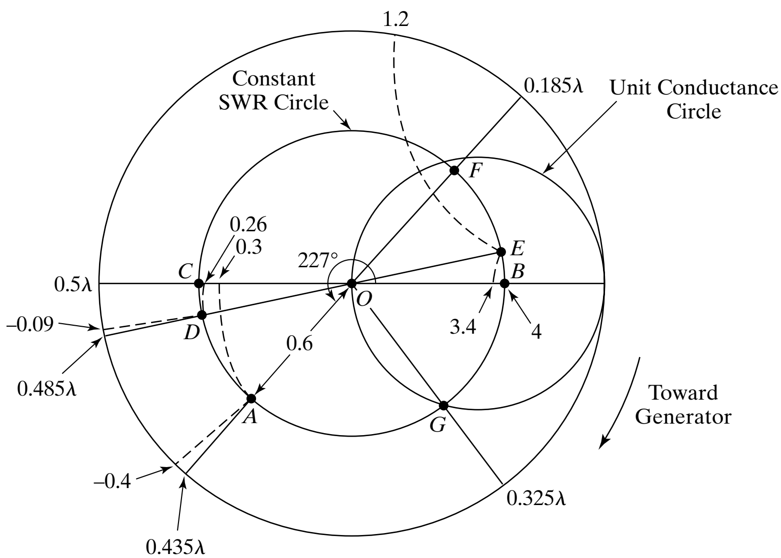

FIGURE 7.23

For illustrating the various procedures to be followed in using the Smith chart.

Thus, the normalized value of R_{max} is equal to the SWR. We therefore move along the line to the location of the voltage maximum, which involves going around the constant $|\bar{\Gamma}|$ circle to the point on the positive real axis. To do this on the Smith chart, we draw a circle passing through A and with center at O. This circle is known as the *constant SWR circle*, since, for points on the circle, $|\bar{\Gamma}|$ and, hence, SWR $= (1 + |\bar{\Gamma}|)/(1 - |\bar{\Gamma}|)$ are constants. Impedance values along this circle are normalized line impedances as seen moving along the line. In particular, since point B (the intersection of the constant SWR circle with the horizontal axis to the right of O) corresponds to voltage maximum, the normalized impedance value at point B, which is purely real and maximum, is equal to the SWR. Thus, for this example, SWR $= 4$.

(e) Just as point B represents the position of a voltage maximum on the line, point C (intersection of the constant SWR circle with the horizontal axis to the left of O, i.e., the negative real axis of the $\bar{\Gamma}$-plane) represents the location of a voltage minimum. Hence, to find the distance of the first voltage minimum from the load, we move along the constant SWR circle starting at point A (load impedance) toward the generator (clockwise direction on the chart) to reach point C. Distance moved along the constant SWR circle in this process can be determined by recognizing that one complete revolution around the chart ($\bar{\Gamma}$-plane diagram) constitutes movement on the line by 0.5λ. However, it is not necessary to compute in this manner since distance scales in terms of λ are provided along the periphery of the chart for movement in both directions. For this example, the distance from the load to the first voltage minimum $= (0.5 - 0.435)\lambda = 0.065\lambda$. Conversely, if the SWR and the location of the voltage minimum are specified, we can find the load impedance by following the foregoing procedures in reverse.

(f) To find the line impedance at $d = 0.05\lambda$, we start at point A and move along the constant SWR circle toward the generator (in the clockwise direction) by a distance of 0.05λ to reach point D. This step is equivalent to finding the reflection coefficient at $d = 0.05\lambda$ knowing the reflection coefficient at $d = 0$ and then computing the normalized line impedance by using (7.35). Thus, from the coordinates corresponding to point D, the normalized line impedance at $d = 0.05\lambda$ is $(0.26 - j0.09)$, and hence the line impedance at $d = 0.05\lambda$ is $50(0.26 - j0.09)$ or $(13 - j4.5)\ \Omega$.

(g) To find the line admittance at $d = 0.05\lambda$, we recall that

$$[\bar{Z}(d)]\left[\bar{Z}\left(d + \frac{\lambda}{4}\right)\right] = Z_0^2$$

so that

$$[\bar{z}(d)]\left[\bar{z}\left(d + \frac{\lambda}{4}\right)\right] = 1$$

or

$$\bar{y}(d) = \bar{z}\left(d + \frac{\lambda}{4}\right) \tag{7.58}$$

Thus, the normalized line admittance at point D is the same as the normalized line impedance at a distance $\lambda/4$ from it. Hence, to find $\bar{y}(0.05\lambda)$, we start at point D and move along the constant SWR circle by a distance $\lambda/4$ to reach point E (we note that this point is diametrically opposite to point D) and read its coordinates. This gives $\bar{y}(0.05\lambda) = (3.4 + j1.2)$. We then have $\bar{Y}(0.05\lambda) = \bar{y}(0.05\lambda) \times Y_0 = (3.4 + j1.2) \times 1/50 = (0.068 + j0.024)$ S.

(h) Relationship (7.58) permits us to use the Smith chart as an admittance chart instead of as an impedance chart. In other words, if we want to find the normalized line admittance $\bar{y}(Q)$ at a point Q on the line, knowing the normalized line admittance $\bar{y}(P)$ at another point P on the line, we can simply locate $\bar{y}(P)$ by entering the chart at coordinates equal to its real and imaginary parts and then moving along the constant SWR circle by the amount of the distance from P to Q in the proper direction to obtain the coordinates equal to the real and imaginary parts of $\bar{y}(Q)$. Thus, it is not necessary first to locate $\bar{z}(P)$ diametrically opposite to $\bar{y}(P)$ on the constant SWR circle, then move along the constant SWR circle to locate $\bar{z}(Q)$, and then find $\bar{y}(Q)$ diametrically opposite to $\bar{z}(Q)$. To find the location nearest to the load at which the real part of the line admittance is equal to the line characteristic admittance, we first locate $\bar{y}(0)$ at point F, diametrically opposite to point A, which corresponds to $\bar{z}(0)$. We then move along the constant SWR circle toward the generator to reach point G on the circle corresponding to constant real part equal to unity. (We call this circle the *unit conductance circle*.) Distance moved from F to G is read off the chart as $(0.325 - 0.185)\lambda = 0.14\lambda$. This is the distance closest to the load at which the real part of the normalized line admittance is equal to unity and, hence, the real part of the line admittance is equal to line characteristic admittance.

K7.4. $\bar{\Gamma}$-plane; Unit circle; Transformation from $\bar{z}$ (or $\bar{y}$) to $\bar{\Gamma}$; Smith chart; Constant SWR circle; Unit conductance circle.

D7.10. Find the values of $\bar{\Gamma}$ in polar form onto which the following normalized impedances are mapped: **(a)** $0.25 + j0$; **(b)** $0 - j0.5$; **(c)** $3 + j3$; **(d)** $-1 + j2$.

Ans. **(a)** $0.6\underline{/180^\circ}$; **(b)** $1\underline{/233.13^\circ}$; **(c)** $0.721\underline{/19.44^\circ}$; **(d)** $1.414\underline{/45^\circ}$.

D7.11. Find the following using the Smith chart: **(a)** the normalized input impedance of a line of length 0.1λ and terminated by a normalized load impedance $(2 + j1)$; **(b)** the normalized input admittance of a short-circuited stub of length 0.17λ; and **(c)** the shortest length of an open-circuited stub having the normalized input admittance $j0.4$.

Ans. **(a)** $1.4 - j1.1$; **(b)** $-j0.55$; **(c)** 0.06λ.

7.5 THE SMITH CHART: 2. APPLICATIONS

In the preceding section, we introduced the Smith chart and discussed some basic procedures. In this section, we first consider by means of examples graphical solutions of transmission-line matching problems using the Smith chart and then discuss further applications.

Example 7.5 Solution of a single-stub matching problem by using the Smith chart

Single-stub matching solution

Let us consider a transmission line of characteristic impedance $Z_0 = 50\ \Omega$ terminated by a load impedance $\bar{Z}_R = (30 - j40)\ \Omega$, and illustrate the solution of the single-stub matching problem by using the Smith chart, assuming Z_0 of the stub to be $50\ \Omega$.

With reference to the notation in Fig. 7.17, we recall that to achieve a match, the stub must be located at a point on the line at which the real part of the normalized line admittance is equal to unity; the imaginary part of the line admittance at that point is then canceled by appropriately choosing the length of the stub. Hence, we proceed with the solution in the following step-by-step manner with reference to Fig. 7.24.

(a) Find the normalized load impedance.

$$\bar{z}_R = \frac{\bar{Z}_R}{Z_0} = \frac{30 - j40}{50} = 0.6 - j0.8$$

Locate the normalized load impedance on the Smith chart at point A.

(b) Draw the constant SWR circle passing through point A. This is the locus of the normalized line impedance as well as the normalized line admittance. Starting at point A, go around the constant SWR circle by half a revolution to reach point B diametrically opposite to point A. Point B corresponds to the normalized load admittance.

(c) Starting at point B, go around the constant SWR circle toward the generator until point C on the unit conductance circle is reached. This point corresponds

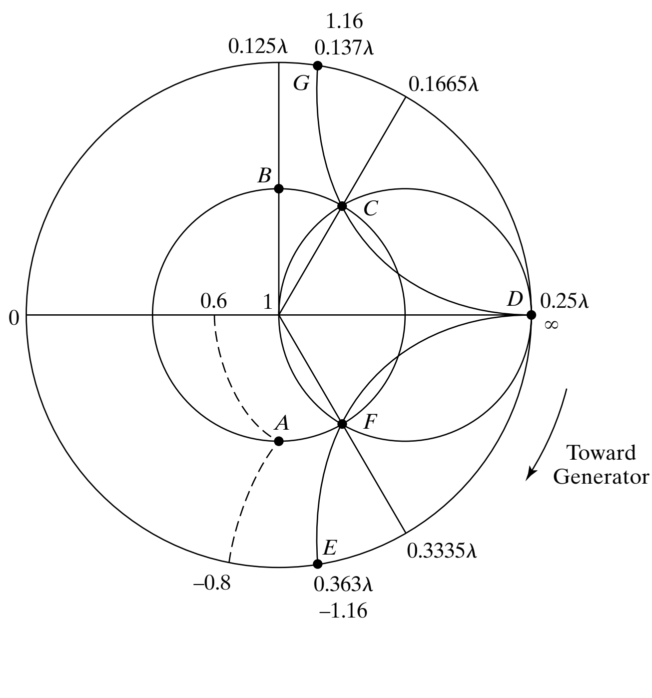

FIGURE 7.24

Solution of single-stub matching problem by using the Smith chart.

to the normalized line admittance having the real part equal to unity; hence, it corresponds to the location of the stub. The distance moved from point B to point C (not from point A to point C) is equal to the distance from the load at which the stub must be located. Thus, the location of the stub from the load = $(0.1665 - 0.125)\lambda = 0.0415\lambda$.

(d) Read off the Smith chart the normalized susceptance value corresponding to point C. This value is 1.16 and it is the imaginary part of the normalized line admittance at the location of the stub. The imaginary part of the line admittance is equal to $1.16 \times Y_0 = (1.16/50)$ S. The input susceptance of the stub must therefore be equal to $-(1.16/50)$ S.

(e) This step consists of finding the length of a short-circuited stub having an input susceptance equal to $-(1.16/50)$ S. We can use the Smith chart for this purpose since this simply consists of finding the distance between two points on a line (the stub in this case) at which the admittances (purely imaginary in this case) are known. Thus, since the short circuit corresponds to a susceptance of infinity, we start at point D and move toward the generator along the constant SWR circle through D (the outermost circle) to reach point E corresponding to $-j1.16$, which is the input admittance of the stub normalized with respect to its own characteristic admittance. The distance moved from D to E is the required length of the stub. Thus, length of the short-circuited stub = $(0.363 - 0.25)\lambda = 0.113\lambda$.

(f) The results obtained for the location and the length of the stub agree with one of the solutions found analytically in Section 7.3. The second solution can be obtained by noting that in step (c), we can go around the constant SWR circle from point B until point F on the unit conductance circle is reached, instead of stopping at point C. The stub location for this solution is $(0.3335 - 0.125)\lambda = 0.2085\lambda$. The required input susceptance of the stub is (1.16/50) S. The length of the stub is the distance from point D to point G in the clockwise direction. This is $(0.137 + 0.25)\lambda = 0.387\lambda$. These values are the same as the second solution obtained in Section 7.3.

Example 7.6 Solution of a double-stub matching problem by using the Smith chart

Double-stub marching solution

For the line of characteristic impedance $Z_0 = 50\ \Omega$ and load impedance $\bar{Z}_R = (30 - j40)\ \Omega$ of Example 7.5, it is desired to solve the double-stub matching problem by using the Smith chart and assuming Z_0 of both stubs to be 50 Ω, the first stub to be located at the load, and distance between stubs equal to 0.375λ.

With reference to the notation of Fig. 7.18, we first note that to achieve a match, $\bar{y}_2' = 1 - jb_2$ must fall on the unit conductance circle. Now since $\bar{y}_2'$ and $\bar{y}_1$ correspond to locations at the end points of the line section between the stubs, for a given $\bar{y}_1$, $\bar{y}_2'$ can be obtained by drawing the constant SWR circle through $\bar{y}_1$ and going toward the generator (clockwise direction) by the distance d_{12} from $\bar{y}_1$. Conversely, to obtain $\bar{y}_1$ for a given $\bar{y}_2'$, we start at $\bar{y}_2'$ and go toward the load (counterclockwise direction) by the distance d_{12} along the constant SWR circle. Hence, for $\bar{y}_2'$ to fall on the unit conductance circle, $\bar{y}_1$ must fall on a circle that is obtained by pivoting the unit conductance circle at the center of the Smith chart (point O) and rotating it toward the load by the distance d_{12}, as shown in Fig. 7.25 for $d_{12} = 3\lambda/8$. We shall call this circle the *auxiliary circle*. Thus, the auxiliary circle is the *locus of* $\bar{y}_1$ *for possible match*.

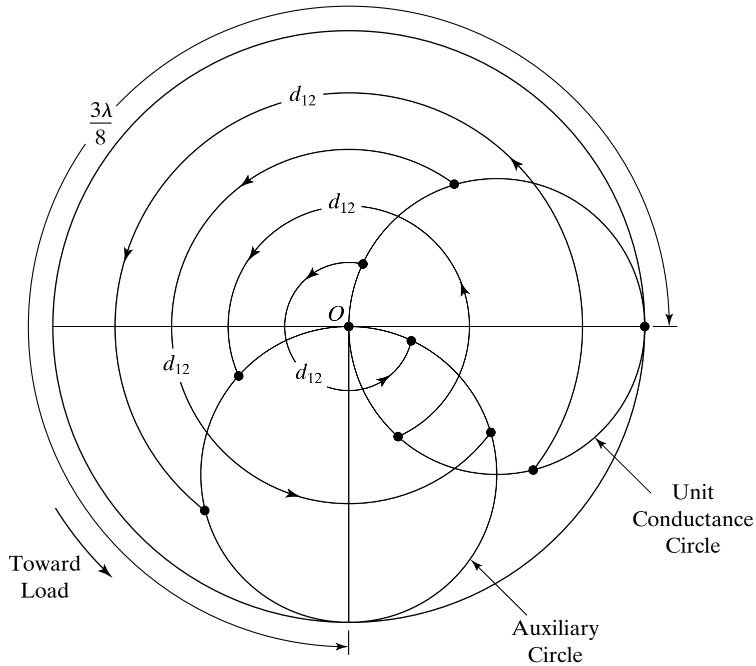

FIGURE 7.25

Rotation of the unit conductance circle by $d_{12}(= 3\lambda/8)$ toward the load about O for illustrating the construction of the auxiliary circle, that is, the locus of $\bar{y}_1$ for possible match for the double-stub matching arrangement of Fig. 7.18.

The matching procedure consists of first locating $\bar{y}_R$ on the Smith chart and then moving along the constant SWR circle through $\bar{y}_R$ toward the generator by the distance d_1 between the load and the first stub, thereby locating $\bar{y}'_1$. The right amount of susceptance is then added to $\bar{y}'_1$ to reach a point on the auxiliary circle. This point corresponds to $\bar{y}_1$ and determines a new constant SWR circle. By going along this new constant SWR circle toward the generator by the distance d_{12}, $\bar{y}'_2$ is located on the unit conductance circle. The amount of susceptance added to $\bar{y}'_1$ is the required normalized input susceptance of the first stub, whereas the negative of the imaginary part of $\bar{y}'_2$ is the required normalized input susceptance of the second stub.

Considering now the numerical values of $\bar{z}_R = (30 - j40)/50 = (0.6 - j0.8)$, $d_1 = 0$, and $d_{12} = 0.375\lambda$, we proceed with the solution in the following step-by-step manner with reference to Fig. 7.26.

(a) Locate $\bar{z}_R = (0.6 - j0.8)$ at point A and draw the constant SWR circle through A.

(b) Locate point B on the constant SWR circle and diametrically opposite to point A. This point corresponds to $\bar{y}_R$. Since d_1 is equal to zero, it also corresponds to $\bar{y}'_1$. If d_1 is not equal to zero, then $\bar{y}'_1$ has to be located by going along the constant SWR circle toward the generator by the distance d_1 from point B.

(c) Draw the auxiliary circle, which is the circle obtained by pivoting the unit conductance circle at the center of the chart and rotating it by the distance $d_{12} = 0.375\lambda$ toward the load.

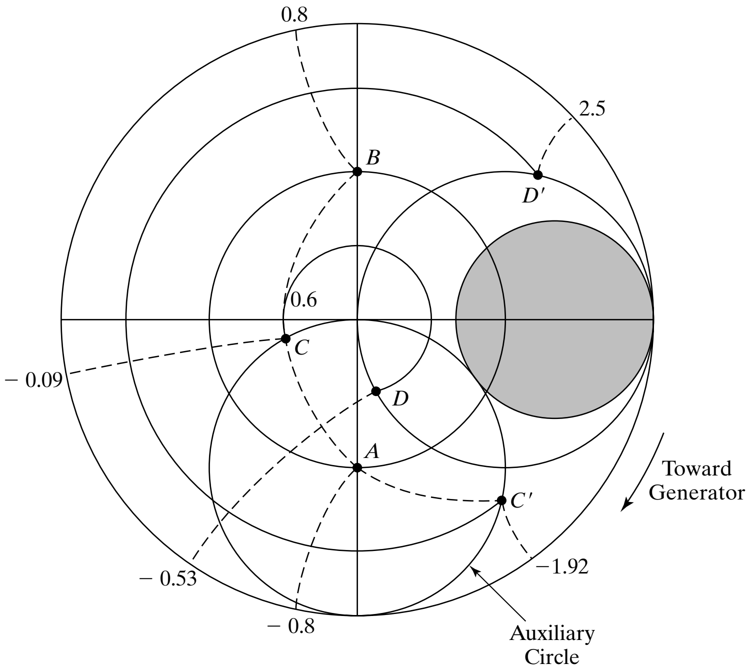

FIGURE 7.26

Solution of the double-stub matching problem by using the Smith chart.

(d) This step consists of adding the right amount of susceptance to $\overline{y}'_1$ to get to a point on the auxiliary circle. Hence, starting at point B, go along the constant conductance circle to reach point C on the auxiliary circle. This point corresponds to $\overline{y}_1$. The required normalized input susceptance of the first stub can now be found by noting that $\overline{y}_1 = \overline{y}'_1 + jb_1$, and, hence,

$$jb_1 = \overline{y}_1 - \overline{y}'_1 = (0.6 - j0.09) - (0.6 + j0.8) = -j0.89$$

(e) Starting at point C, go along the constant SWR circle through C toward the generator by $d_{12} = 0.375\lambda$ to reach point D on the unit conductance circle. This point corresponds to $\overline{y}'_2$. Note that the SWR on the portion of the line between the stubs is different from the SWR to the right of the first stub because of the discontinuity introduced by the stub. The required normalized input susceptance of the second stub can now be found by reading the imaginary part of $\overline{y}'_2$ and taking its negative. Thus,

$$jb_2 = -j[\text{Im}(\overline{y}'_2)] = j0.53$$

(f) This step consists of finding the lengths of the two stubs having the normalized input susceptances found in steps (d) and (e), by using the procedure discussed in Example 7.5. Thus, we obtain

$$l_1\text{, length of first stub} = (0.385 - 0.25)\lambda = 0.135\lambda$$
$$l_2\text{, length of second stub} = (0.077 + 0.25)\lambda = 0.327\lambda$$

which agree with one of the solutions found analytically in Section 7.3.

(g) Finally, the second solution can be obtained by going from point B to point C' on the auxiliary circle and then to point D' on the unit conductance circle, and computing jb_1 and jb_2 as in steps (d) and (e). Thus, we obtain

$$jb_1 = (0.6 - j1.92) - (0.6 + j0.8) = -j2.72$$
$$jb_2 = -j2.5$$

giving us

$$l_1 = (0.306 - 0.25)\lambda = 0.056\lambda$$
$$l_2 = (0.31 - 0.25)\lambda = 0.06\lambda$$

These values are the same as the second solution obtained in Section 7.3.

Before proceeding further, we recall from Section 7.3 that in the double-stub matching technique, it is not possible to achieve a match for loads that result in the real part of $\bar{y}'_1$ being greater than $1/\sin^2 \beta d_{12}$. For $d_{12} = 3\lambda/8$, $1/\sin^2 \beta d_{12} = 2$, and a match cannot be achieved if the real part of $\bar{y}'_1$ is greater than 2. This is easily evident from the Smith chart construction in Fig. 7.26, since if point B falls inside the shaded region (real part > 2), it is not possible to reach a point on the auxiliary circle by moving on the constant conductance circle through B. The shaded region is therefore called the *forbidden region of* $\bar{y}'_1$ *for possible match*. As pointed out in Section 7.3, a solution to the problem is to increase d_1 by $\lambda/4$. This effectively rotates the forbidden region by 180° about the center of the chart, thereby making possible a match.

Transformation across a discontinuity

To illustrate the application of the Smith chart further, we shall now discuss a very useful property of the reflection coefficient and, hence, of the Smith chart. This has to do with the transformation of the reflection coefficient from one side of a discontinuity to the other side of the discontinuity. Let us, for example, consider the system shown in Fig. 7.27, which consists of a junction

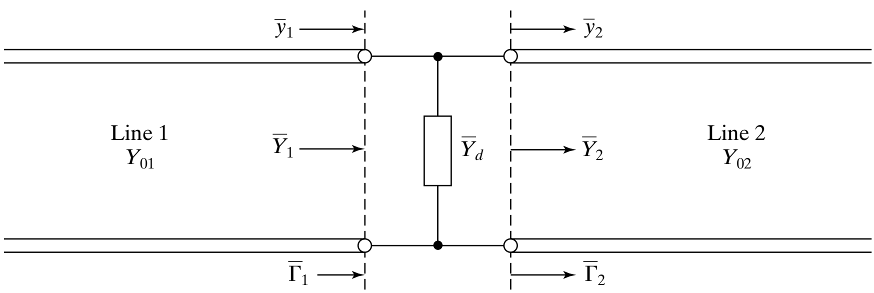

FIGURE 7.27

Transmission-line system for deriving the transformation of $\bar{\Gamma}$ across a discontinuity.

between two lines of characteristic admittances Y_{01} and Y_{02} and in addition, an admittance $\overline{Y}_d$ connected across the junction. If $\overline{Y}_d = 0$, then the system reduces to a simple junction between two lines. If $Y_{01} = Y_{02}$, then the system reduces to an admittance discontinuity in the same line.

Let $\overline{y}_1 = \overline{Y}_1/Y_{01}$ and $\overline{y}_2 = \overline{Y}_2/Y_{02}$ be the normalized admittances to the left and to the right, respectively, of the junction, and let the corresponding reflection coefficients be $\overline{\Gamma}_1$ and $\overline{\Gamma}_2$, respectively, as shown in Fig. 7.27. Then, since $\overline{Y}_1 = \overline{Y}_2 + \overline{Y}_d$, we have

$$\begin{aligned} \overline{y}_1 &= \frac{\overline{Y}_1}{Y_{01}} = \frac{\overline{Y}_2}{Y_{01}} + \frac{\overline{Y}_d}{Y_{01}} \\ &= \frac{Y_{02}}{Y_{01}}\left(\frac{\overline{Y}_2}{Y_{02}} + \frac{\overline{Y}_d}{Y_{02}}\right) \\ &= a(\overline{y}_2 + \overline{y}_d) \end{aligned} \tag{7.59}$$

where $a = Y_{02}/Y_{01}$ is the ratio of the characteristic admittances of the two lines and $\overline{y}_d = \overline{Y}_d/Y_{02}$ is the normalized value of $\overline{Y}_d$ with respect to Y_{02}. Substituting for $\overline{y}_1$ and $\overline{y}_2$ in (7.59) in terms of $\overline{\Gamma}_1$ and $\overline{\Gamma}_2$, respectively, we have

$$\frac{1 - \overline{\Gamma}_1}{1 + \overline{\Gamma}_1} = a\left(\frac{1 - \overline{\Gamma}_2}{1 + \overline{\Gamma}_2} + \overline{y}_d\right) \tag{7.60}$$

Rearranging (7.60), we obtain

$$\boxed{\overline{\Gamma}_1 = \frac{(1 + a - a\overline{y}_d)\overline{\Gamma}_2 + (1 - a - a\overline{y}_d)}{(1 - a + a\overline{y}_d)\overline{\Gamma}_2 + (1 + a + a\overline{y}_d)}} \tag{7.61}$$

Equation (7.61) is of the form of the so-called *bilinear transformation* between two complex planes, a property of which is that circles in one plane are transformed into circles in the second plane. Consequently, loci of $\overline{\Gamma}_2$, which are circles in the $\overline{\Gamma}$-plane, are mapped on to loci of $\overline{\Gamma}_1$, which are also circles in the $\overline{\Gamma}$-plane, and vice versa. Since the Smith chart is a transformation (also bilinear) from $\overline{z}$ or $\overline{y}$ to $\overline{\Gamma}$, this means that loci of $\overline{y}_2$, which are circles, are mapped on to loci of $\overline{y}_1$, which are also circles. Since a circle is defined completely by three points, it is therefore sufficient if we use any three points on the locus of $\overline{y}_2$ and find the corresponding three points for $\overline{y}_1$. By locating the center at the intersection of the perpendicular bisectors of lines joining any two pairs of these three points, we can then draw the circle passing through these points, that is, the locus of $\overline{y}_1$. Although we have demonstrated this property by considering the discontinuity of the form shown in Fig. 7.27, it can be shown that the property holds for the case of any linear, passive, bilateral network serving as the discontinuity. We shall now consider an example.

Example 7.7 Application of the Smith chart to transformation across a discontinuity

Let us consider the system shown in Fig. 7.28, in which a line is terminated by a normalized admittance $\bar{y}_R = (0.6 + j0.8)$ and a normalized susceptance of value $b = 0.8$ connected between the two conductors of the line forms the discontinuity. We wish to find the locus of the normalized admittance $\bar{y}_1$ to the left of the discontinuity as the susceptance slides along the line, and then determine the location, nearest to the load, of the susceptance for which the SWR to the left of it is minimized.

To construct the locus of $\bar{y}_1$, we first locate $\bar{y}_R = (0.6 + j0.8)$ on the Smith chart at point A and draw the constant SWR circle passing through A, as shown in Fig. 7.29. This circle is the locus of $\bar{y}_2$, the normalized admittance just to the right of the discontinuity as the distance between the load and the discontinuity is varied, that is, as the susceptance slides along the line. We then choose any three points on the locus of $\bar{y}_2$ and locate the corresponding three points for $\bar{y}_1 = \bar{y}_2 + j0.8$. Here, we choose the points A, B, and C. Following the constant conductance circles through these points by the amount of normalized susceptance +0.8, we obtain the points D, E, and F, respectively. We then draw the circle passing through these points to obtain the locus of $\bar{y}_1$.

Proceeding further, we note that each point on the locus of $\bar{y}_1$ corresponds to a value of SWR to the left of the susceptance, obtained by following the constant SWR circle through that point to the r value at the $V_{\max}$ point. In particular, it can be seen that minimum SWR is achieved to the left of the susceptance for $\bar{y}_1$ lying at point G, which is the closest point to the center of the chart, and the minimum SWR value is 1.35. The distance from the load at which the susceptance must be connected to achieve this minimum SWR can be found by locating the $\bar{y}_2$ corresponding to the $\bar{y}_1$ at G by following the constant conductance circle through G by the amount -0.8 to reach point H. The distance from point A to point H toward the generator is the required distance. It is equal to $(0.346 - 0.125)\lambda$, or 0.221λ.

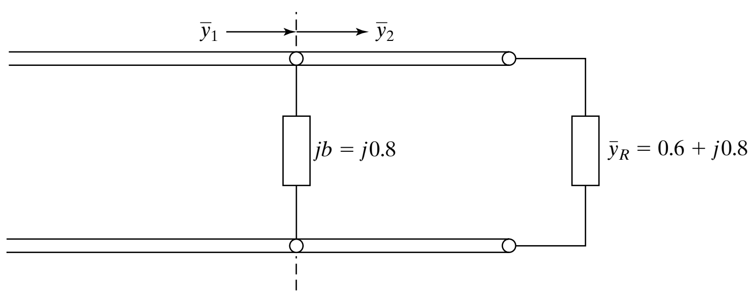

FIGURE 7.28

Transmission-line system in which a susceptance of fixed value sliding along the line forms a discontinuity.

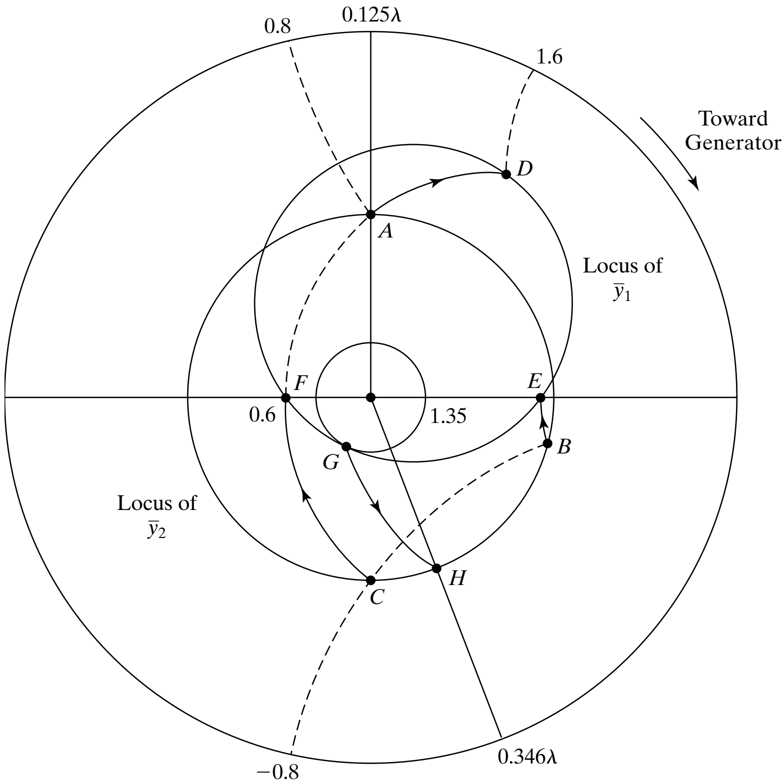

FIGURE 7.29

Construction of the locus of $\overline{y}_1$ for the system of Fig. 7.28 as the susceptance b slides along the line and determination of the minimum SWR that can be achieved to the left of the susceptance and the location of the susceptance to achieve the minimum SWR.

Together with the basic procedures discussed in the previous section, the methods that we have discussed in this section can be extended to solve many other problems using the Smith chart. We include some of these in the problems.

K7.5. Single-stub matching; Double-stub matching; Auxiliary circle; Forbidden region of $\overline{y}_1'$ for possible match; Transformation of $\overline{\Gamma}$ across a discontinuity.

D7.12. A line of characteristic impedance 100 Ω is terminated by a load of impedance $(50 + j65)$ Ω. Find the following using the Smith chart: **(a)** the SWR on the line; **(b)** the minimum SWR that can be achieved on the line by connecting a stub in parallel with the line at the load; and **(c)** the minimum SWR that can be achieved on the line by connecting a stub in series with the line at the load.

Ans. **(a)** 3.0; **(b)** 1.33; **(c)** 2.0.

D7.13. A line of characteristic impedance 50 Ω is terminated by a load of impedance $(100 + j100)$ Ω. Find the following by using the Smith chart: **(a)** the minimum distance at which a reactance of value 50 Ω must be connected in parallel with the line to minimize the SWR to the left of the reactance and the minimum SWR achieved; **(b)** the minimum length of a line section of characteristic impedance 100 Ω between the main line and the load required to minimize the SWR on the main line and the minimum SWR achieved; and **(c)** the characteristic impedance of a $\lambda/4$ section of line inserted between the main line and the load to minimize the SWR on the main line and the minimum SWR achieved.
Ans. **(a)** 0.204λ, 1.63; **(b)** 0.338λ, 1.30; **(c)** 83.7 Ω, 2.42.

7.6 THE LOSSY LINE

Distributed equivalent circuit

Thus far, we have been concerned with lossless lines. We learned in Section 6.1 that the distributed equivalent circuit for a lossless line consists of series inductors and shunt capacitors, representing energy storage in magnetic and electric fields, respectively. A lossy line is characterized by imperfect but good conductors and imperfect dielectric giving rise to power dissipation, thereby modifying the distributed equivalent circuit. The power dissipation in the conductors is taken into account by a resistance in series with the inductor, whereas the power dissipation in the dielectric is taken into account by a conductance in parallel with the capacitor. In addition, the magnetic field inside the conductors is taken into account by an additional inductance in the series branch. Thus, the distributed equivalent circuit for the lossy line is as shown in Fig. 7.30, where $\mathscr{L}$ includes the additional inductance just mentioned. Note that the notation $\mathscr{R}$ for the series resistance and $\mathscr{G}$ for the shunt conductance is not to be confused to mean that $\mathscr{G}$ is the reciprocal of $\mathscr{R}$.

Transmission-line equations and solution

To discuss wave propagation on a lossy line, we first obtain the transmission-line equations by applying Kirchhoff's voltage and current laws to the circuit of Fig. 7.30. Thus, we have

$$V(z + \Delta z, t) - V(z, t) = -\mathscr{R}\,\Delta z\, I(z, t) - \mathscr{L}\,\Delta z \frac{\partial I(z, t)}{\partial t} \tag{7.62a}$$

$$I(z + \Delta z, t) - I(z, t) = -\mathscr{G}\,\Delta z\, V(z, t) - \mathscr{C}\,\Delta z \frac{\partial V(z, t)}{\partial t} \tag{7.62b}$$

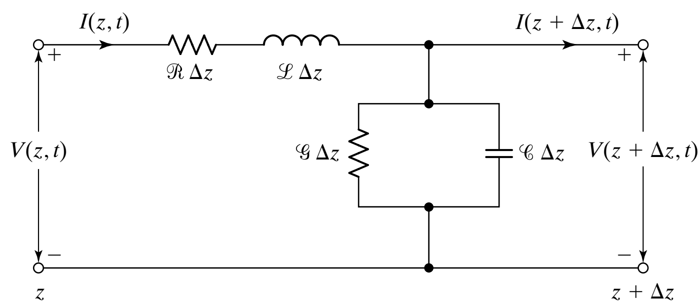

FIGURE 7.30

Distributed equivalent circuit for a lossy transmission line.

Dividing both sides of (7.62a) and (7.62b) by Δz and letting $\Delta z \rightarrow 0$, we obtain the transmission-line equations

$$\boxed{\begin{aligned} \frac{\partial V(z,t)}{\partial z} &= -\mathcal{R}I(z,t) - \mathcal{L}\frac{\partial I(z,t)}{\partial t} \\ \frac{\partial I(z,t)}{\partial z} &= -\mathcal{G}V(z,t) - \mathcal{C}\frac{\partial V(z,t)}{\partial t} \end{aligned}} \quad \begin{aligned} &(7.63\text{a}) \\ &(7.63\text{b}) \end{aligned}$$

The corresponding equations in terms of phasor voltage and current are

$$\frac{\partial \bar{V}(z)}{\partial z} = -\mathcal{R}\bar{I}(z) - j\omega\mathcal{L}\bar{I}(z)$$

$$\frac{\partial \bar{I}(z)}{\partial z} = -\mathcal{G}\bar{V}(z) - j\omega\mathcal{C}\bar{V}(z)$$

or

$$\boxed{\begin{aligned} \frac{\partial \bar{V}}{\partial z} &= -(\mathcal{R} + j\omega\mathcal{L})\bar{I} \\ \frac{\partial \bar{I}}{\partial z} &= -(\mathcal{G} + j\omega\mathcal{C})\bar{V} \end{aligned}} \quad \begin{aligned} &(7.64\text{a}) \\ &(7.64\text{b}) \end{aligned}$$

where $\bar{V}$ and $\bar{I}$ are understood to be functions of z.

Combining the two transmission-line equations (7.64a) and (7.64b) by eliminating $\bar{I}$, we obtain the wave equation

$$\begin{aligned} \frac{\partial^2 \bar{V}}{\partial z^2} &= -(\mathcal{R} + j\omega\mathcal{L})\frac{\partial \bar{I}}{\partial z} \\ &= (\mathcal{R} + j\omega\mathcal{L})(\mathcal{G} + j\omega\mathcal{C})\bar{V} \end{aligned}$$

or

$$\frac{\partial^2 \bar{V}}{\partial z^2} = \bar{\gamma}^2 \bar{V} \tag{7.65}$$

where

$$\boxed{\begin{aligned} \bar{\gamma} &= \alpha + j\beta \\ &= \sqrt{(\mathcal{R} + j\omega\mathcal{L})(\mathcal{G} + j\omega\mathcal{C})} \end{aligned}} \tag{7.66}$$

The solution for $\bar{V}(z)$ is given by

$$\bar{V}(z) = \bar{A}e^{-\bar{\gamma}z} + \bar{B}e^{\bar{\gamma}z} \tag{7.67}$$

where $\bar{A} = Ae^{j\theta}$ and $\bar{B} = Be^{j\phi}$ are arbitrary constants. It then follows that

$$\begin{aligned} V(z,t) &= \text{Re}[Ae^{j\theta}e^{-\alpha z}e^{-j\beta z}e^{j\omega t} + Be^{j\phi}e^{\alpha z}e^{j\beta z}e^{j\omega t}] \\ &= Ae^{-\alpha z}\cos(\omega t - \beta z + \theta) + Be^{\alpha z}\cos(\omega t + \beta z + \phi) \end{aligned}$$

Noting that the first and second terms on the right side correspond to waves propagating in the $+z$-and $-z$-directions, respectively, we write (7.67) as

$$\boxed{\bar{V}(z) = \bar{V}^+e^{-\bar{\gamma}z} + \bar{V}^-e^{\bar{\gamma}z}} \tag{7.68}$$

where the superscripts + and − denote (+) and (−) waves, respectively. The quantity β, which is the imaginary part of $\bar{\gamma}$, is, of course, the phase constant, that is, the rate of change of phase with z for a fixed time, for either wave. The quantity α, which is the real part of $\bar{\gamma}$, is the attenuation constant, denoting that the waves get attenuated by the factor e^{α} per unit distance as they propagate in their respective directions. Thus, the quantity $\bar{\gamma}(= \alpha + j\beta)$ is the propagation constant associated with the wave. We recall that the units of α are nepers per meter. Proceeding further, we obtain the corresponding solution for the phasor line current by substituting (7.68) into one of the transmission-line equations. Thus, using (7.64a), we obtain

$$\begin{aligned} \bar{I}(z) &= -\frac{1}{\mathscr{R} + j\omega\mathscr{L}}\frac{\partial\bar{V}}{\partial z} \\ &= -\frac{1}{\mathscr{R} + j\omega\mathscr{L}}[-\bar{\gamma}\bar{V}^+e^{-\bar{\gamma}z} + \bar{\gamma}\bar{V}^-e^{\bar{\gamma}z}] \end{aligned}$$

or

$$\boxed{\bar{I}(z) = \frac{1}{\bar{Z}_0}(\bar{V}^+e^{-\bar{\gamma}z} - \bar{V}^-e^{\bar{\gamma}z})} \tag{7.69}$$

where

$$\bar{Z}_0 = \sqrt{\frac{\mathscr{R} + j\omega\mathscr{L}}{\mathscr{G} + j\omega\mathscr{C}}} \tag{7.70}$$

is the characteristic impedance of the line, which is now complex.

Low-loss line

Equations (7.68) and (7.69) are the general solutions for the phasor line voltage and current, respectively, with the associated propagation constant and characteristic impedance given by (7.66) and (7.70), respectively. Although it is possible to obtain explicit expressions for α and β, as well as for the real and imaginary parts of $\bar{Z}_0$ in terms of ω, $\mathscr{R}$, $\mathscr{L}$, $\mathscr{G}$ and $\mathscr{C}$, such expressions are often not meaningful since $\mathscr{R}$, $\mathscr{L}$, $\mathscr{G}$, and $\mathscr{C}$ are themselves functions of frequency. Hence, in practice, these quantities are obtained from experimental determination of characteristic impedance and propagation constant. However, for the special case of the low-loss line, that is, for $\omega\mathcal{L} \gg \mathcal{R}$ and $\omega\mathcal{C} \gg \mathcal{G}$, we have

$$\begin{aligned}
\bar{\gamma} &= \sqrt{j\omega\mathcal{L}\left(1 + \frac{\mathcal{R}}{j\omega\mathcal{L}}\right)j\omega\mathcal{C}\left(1 + \frac{\mathcal{G}}{j\omega\mathcal{C}}\right)} \\
&\approx j\omega\sqrt{\mathcal{L}\mathcal{C}}\sqrt{1 + \frac{\mathcal{R}}{j\omega\mathcal{L}} + \frac{\mathcal{G}}{j\omega\mathcal{C}}} \\
&\approx j\omega\sqrt{\mathcal{L}\mathcal{C}}\left[1 + \frac{1}{2}\left(\frac{\mathcal{R}}{j\omega\mathcal{L}} + \frac{\mathcal{G}}{j\omega\mathcal{C}}\right)\right] \\
&\approx \frac{1}{2}\left(\mathcal{R}\sqrt{\frac{\mathcal{C}}{\mathcal{L}}} + \mathcal{G}\sqrt{\frac{\mathcal{L}}{\mathcal{C}}}\right) + j\omega\sqrt{\mathcal{L}\mathcal{C}}
\end{aligned}$$

so that

$$\alpha \approx \frac{1}{2}\left(\mathcal{R}\sqrt{\frac{\mathcal{C}}{\mathcal{L}}} + \mathcal{G}\sqrt{\frac{\mathcal{L}}{\mathcal{C}}}\right) \tag{7.71a}$$

$$\beta \approx \omega\sqrt{\mathcal{L}\mathcal{C}} \tag{7.71b}$$

$$v_p = \frac{\omega}{\beta} \approx \frac{1}{\sqrt{\mathcal{L}\mathcal{C}}} \tag{7.71c}$$

Similarly,

$$\begin{aligned}
\bar{Z}_0 &= \sqrt{\frac{j\omega\mathcal{L}(1 + \mathcal{R}/j\omega\mathcal{L})}{j\omega\mathcal{C}(1 + \mathcal{G}/j\omega\mathcal{C})}} \\
&\approx \sqrt{\frac{\mathcal{L}}{\mathcal{C}}}\sqrt{\left(1 + \frac{\mathcal{R}}{j\omega\mathcal{L}}\right)\left(1 - \frac{\mathcal{G}}{j\omega\mathcal{C}}\right)} \\
&\approx \sqrt{\frac{\mathcal{L}}{\mathcal{C}}}\sqrt{\left(1 + \frac{\mathcal{R}}{j\omega\mathcal{L}} - \frac{\mathcal{G}}{j\omega\mathcal{C}}\right)} \\
&\approx \sqrt{\frac{\mathcal{L}}{\mathcal{C}}}\left[1 + \frac{1}{2}\left(\frac{\mathcal{R}}{j\omega\mathcal{L}} - \frac{\mathcal{G}}{j\omega\mathcal{C}}\right)\right] \\
&\approx \sqrt{\frac{\mathcal{L}}{\mathcal{C}}}
\end{aligned} \tag{7.71d}$$

Thus, for the low-loss line, the expressions for β and $\bar{Z}_0$ are essentially the same as those for a lossless line. Note that the low-loss conditions $\omega\mathcal{L} \gg \mathcal{R}$ and $\omega\mathcal{C} \gg \mathcal{G}$ are valid for very high frequencies or for very small values of $\mathcal{R}$ and $\mathcal{G}$ at lower frequencies.

Experimental determination of $\bar{Z}_0$ and $\bar{\gamma}$

As already pointed out, for the general case it is more convenient to determine experimentally the values of $\bar{Z}_0$ and $\bar{\gamma}$ than it is to compute them analytically. The experimental technique is based on the measurements of the input impedance of the line for two values of load impedance. To obtain the expression for the input impedance, we first write the general solutions for the phasor line voltage and current given by (7.68) and (7.69), respectively, in terms of the distance variable d, measured from the load toward the generator, as opposed to z, measured from the generator toward the load. Thus, we have

$$\bar{V}(d) = \bar{V}^{+}e^{\bar{\gamma}d} + \bar{V}^{-}e^{-\bar{\gamma}d} \tag{7.72a}$$

$$\bar{I}(d) = \frac{1}{\bar{Z}_0}(\bar{V}^{+}e^{\bar{\gamma}d} - \bar{V}^{-}e^{-\bar{\gamma}d}) \tag{7.72b}$$

or

$$\boxed{\bar{V}(d) = \bar{V}^{+}e^{\bar{\gamma}d}[1 + \bar{\Gamma}(d)]} \tag{7.73a}$$

$$\boxed{\bar{I}(d) = \frac{\bar{V}^{+}}{\bar{Z}_0}e^{\bar{\gamma}d}[1 - \bar{\Gamma}(d)]} \tag{7.73b}$$

where

$$\boxed{\begin{aligned}\bar{\Gamma}(d) &= \frac{\bar{V}^{-}(d)}{\bar{V}^{+}(d)} = \frac{\bar{V}^{-}e^{-\bar{\gamma}d}}{\bar{V}^{+}e^{\bar{\gamma}d}} \\ &= \bar{\Gamma}_R e^{-2\bar{\gamma}d} = \bar{\Gamma}_R e^{-2\alpha d}e^{-j2\beta d}\end{aligned}} \tag{7.74}$$

is the voltage reflection coefficient at any value of d, and $\bar{\Gamma}_R$ is the voltage reflection coefficient at the load.

The line impedance is given by

$$\boxed{\begin{aligned}\bar{Z}(d) &= \frac{\bar{V}(d)}{\bar{I}(d)} = \bar{Z}_0\frac{1 + \bar{\Gamma}(d)}{1 - \bar{\Gamma}(d)} \\ &= \bar{Z}_0\frac{1 + \bar{\Gamma}_R e^{-2\bar{\gamma}d}}{1 - \bar{\Gamma}_R e^{-2\bar{\gamma}d}}\end{aligned}} \tag{7.75}$$

The input impedance of a line of length l terminated by a load impedance $\bar{Z}_R$, as shown in Fig. 7.31, is then given in terms of $\bar{Z}_R$ by

$$\begin{aligned}\bar{Z}_{\text{in}} = \bar{Z}(l) &= \bar{Z}_0\frac{1 + \bar{\Gamma}_R e^{-2\bar{\gamma}l}}{1 - \bar{\Gamma}_R e^{-2\bar{\gamma}l}} \\ &= \bar{Z}_0\frac{1 + [(\bar{Z}_R - \bar{Z}_0)/(\bar{Z}_R + \bar{Z}_0)]e^{-2\bar{\gamma}l}}{1 - [(\bar{Z}_R - \bar{Z}_0)/(\bar{Z}_R + \bar{Z}_0)]e^{-2\bar{\gamma}l}} \\ &= \bar{Z}_0\frac{(\bar{Z}_R + \bar{Z}_0) + (\bar{Z}_R - \bar{Z}_0)e^{-2\bar{\gamma}l}}{(\bar{Z}_R + \bar{Z}_0) - (\bar{Z}_R - \bar{Z}_0)e^{-2\bar{\gamma}l}} \\ &= \bar{Z}_0\frac{\bar{Z}_R \cosh \bar{\gamma}l + \bar{Z}_0 \sinh \bar{\gamma}l}{\bar{Z}_R \sinh \bar{\gamma}l + \bar{Z}_0 \cosh \bar{\gamma}l}\end{aligned}$$

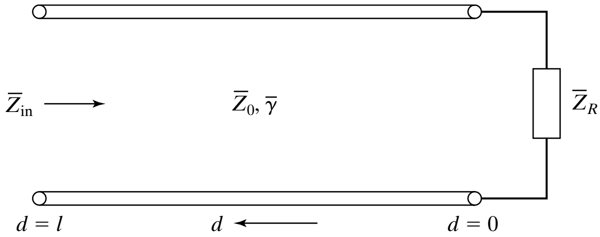

FIGURE 7.31

Lossy line of length 1 terminated by $\bar{Z}_R$.

or

$$\boxed{\bar{Z}_{in} = \bar{Z}_0 \frac{\bar{Z}_R + \bar{Z}_0 \tanh \bar{\gamma} l}{\bar{Z}_R \tanh \bar{\gamma} l + \bar{Z}_0}} \tag{7.76}$$

Let us now consider two values of $\bar{Z}_R$; in particular, $\bar{Z}_R = 0$ and $\bar{Z}_R = \infty$, corresponding to a short circuit and an open circuit, respectively. Then, denoting the corresponding input impedances to be $\bar{Z}_{in}^s$ and $\bar{Z}_{in}^o$, respectively, we have from (7.76),

$$\bar{Z}_{in}^s = \bar{Z}_0 \tanh \bar{\gamma} l \tag{7.77a}$$

$$\bar{Z}_{in}^o = \bar{Z}_0 \coth \bar{\gamma} l \tag{7.77b}$$

from which we obtain

$$\boxed{\bar{Z}_0 = \sqrt{\bar{Z}_{in}^s \bar{Z}_{in}^o}} \tag{7.78}$$

and

$$\boxed{\tanh \bar{\gamma} l = \sqrt{\frac{\bar{Z}_{in}^s}{\bar{Z}_{in}^o}}} \tag{7.79}$$

To illustrate the computation of $\bar{Z}_0$ and $\bar{\gamma}$ by means of a numerical example, let us assume that at a certain frequency, measurements indicated

$$\bar{Z}_{in}^s = (30 - j40)\ \Omega$$

$$\bar{Z}_{in}^o = (30 + j40)\ \Omega$$

Then from (7.78),

$$\bar{Z}_0 = \sqrt{(30 - j40)(30 + j40)} = 50\ \Omega$$

From (7.79),

$$\tanh \bar{\gamma} l = \sqrt{\frac{30 - j40}{30 + j40}} = \sqrt{\frac{50\angle -53.13^\circ}{50\angle 53.13^\circ}}$$
$$= 1\angle -53.13^\circ = 0.6 - j0.8$$
$$\bar{\gamma} l = \tanh^{-1}(0.6 - j0.8)$$

Using the identity

$$\tanh^{-1} x = \frac{1}{2}\ln\frac{1 + x}{1 - x}$$

we then have

$$\bar{\gamma} l = \frac{1}{2}\ln\frac{1.6 - j0.8}{0.4 + j0.8} = \frac{1}{2}\ln\frac{1.789\angle -26.565^\circ}{0.894\angle 63.435^\circ}$$
$$= \frac{1}{2}\ln 2\angle -90^\circ = \frac{1}{2}\ln\left[2e^{j(2n\pi - \pi/2)}\right]$$
$$= \frac{1}{2}[\ln 2 + j(2n\pi - \pi/2)]$$
$$= 0.3466 + j(n\pi - \pi/4) \quad n = 0, 1, 2, \ldots$$

Thus,

$$\alpha l = 0.3466$$
$$\alpha = 0.3466/l$$

whereas

$$\beta l = n\pi - \pi/4 \quad n = 1, 2, \ldots$$

where $n = 0$ is ruled out since it gives negative value for β. Note that β can only be determined to within $n\pi$. However, if the approximate value of β is known, then the correct value of n, and hence of β, can be determined.

In practice, since a perfect open-circuited termination can often be difficult to achieve, it may be desirable to consider the second value of $\bar{Z}_R$ to be arbitrary instead of being equal to ∞. Denoting the corresponding input impedance to be $\bar{Z}_{in}$, we then have from (7.76) and (7.77a)

$$\bar{Z}_{in} = \bar{Z}_0^2 \frac{\bar{Z}_R + \bar{Z}_{in}^s}{\bar{Z}_R \bar{Z}_{in}^s + \bar{Z}_0^2} \tag{7.80}$$

and hence

$$\bar{Z}_0 = \sqrt{\frac{\bar{Z}_R \bar{Z}^s_{\text{in}} \bar{Z}_{\text{in}}}{\bar{Z}_R + \bar{Z}^s_{\text{in}} - \bar{Z}_{\text{in}}}} \tag{7.81}$$

Knowing the value of $\bar{Z}_0$ from (7.81), we can then compute the value of $\bar{\gamma}$ by using (7.77a).

Power flow

We shall conclude this section with a discussion of power flow down the line. From (7.73a) and (7.73b), the time-average power flow down the line is given by

$$\begin{aligned}
\langle P \rangle &= \frac{1}{2}\,\text{Re}[\bar{V}(d)\bar{I}^*(d)] \\
&= \frac{1}{2}\,\text{Re}\left\{\bar{V}^+ e^{\bar{\gamma} d}[1 + \bar{\Gamma}(d)]\frac{(\bar{V}^+)^*}{\bar{Z}_0^*}e^{\bar{\gamma}^* d}[1 - \bar{\Gamma}^*(d)]\right\} \\
&= \frac{1}{2}\,\text{Re}\left\{\frac{|\bar{V}^+|^2}{\bar{Z}_0^*}e^{2\alpha d}[1 - |\bar{\Gamma}(d)|^2 + \bar{\Gamma}(d) - \bar{\Gamma}^*(d)]\right\} \\
&= \frac{1}{2}\,\text{Re}\{|\bar{V}^+|^2 \bar{Y}_0^* e^{2\alpha d}[1 - |\bar{\Gamma}(d)|^2 + j2\,\text{Im}\,\bar{\Gamma}(d)]\}
\end{aligned}$$

or

$$\boxed{\langle P \rangle = \tfrac{1}{2}|\bar{V}^+|^2 e^{2\alpha d}\{G_0[1 - |\bar{\Gamma}(d)|^2] + 2B_0\,\text{Im}\,\bar{\Gamma}(d)\}} \tag{7.82}$$

where

$$\bar{Y}_0 = \frac{1}{\bar{Z}_0} = G_0 + jB_0$$

is the characteristic admittance of the line. For a given source voltage and impedance, we can compute the value of $|\bar{V}^+|$ from line impedance and power flow considerations at the input end of the line and use that value for further computations. We shall illustrate this by means of an example.

Example 7.8 Computation of power flow and power dissipated for a lossy line

Let us consider the low-loss line system shown in Fig. 7.32, and compute the time-average power delivered to the input of the line, the time-average power delivered to the load, and the time-average power dissipated in the line.

We proceed with the solution in a step-by-step manner as follows:

(a) The reflection coefficient at the load end is given by

$$\bar{\Gamma}_R = \frac{\bar{Z}_R - \bar{Z}_0}{\bar{Z}_R + \bar{Z}_0} = \frac{150 - 50}{150 + 50} = 0.5$$

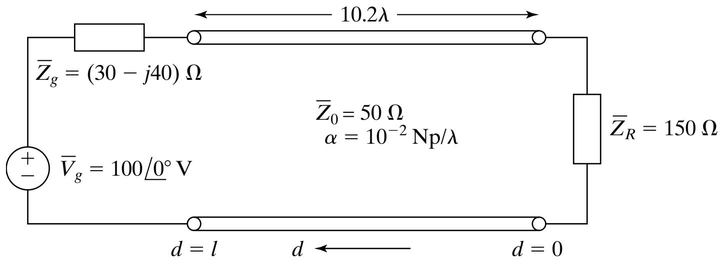

FIGURE 7.32

Lossy transmission-line system for illustrating the computation of power flow at the two ends of the line and the power dissipated in the line.

(b) Noting that α is specified in nepers per wavelength, we obtain the reflection coefficient at the input end $d = l$ as

$$\begin{aligned}\overline{\Gamma}(l) &= \overline{\Gamma}_R e^{-2\overline{\gamma} l} = \overline{\Gamma}_R e^{-2\alpha l} e^{-j2\beta l} \\ &= 0.5e^{-0.204} e^{-j40.8\pi} \\ &= 0.4077\underline{/-144°}\end{aligned}$$

(c) The input impedance of the line is given by

$$\begin{aligned}\overline{Z}_{\text{in}}\overline{Z}(l) &= \overline{Z}_0 \frac{1 + \overline{\Gamma}(l)}{1 - \overline{\Gamma}(l)} \\ &= 50\frac{1 + 0.4077\underline{/-144°}}{1 - 0.4077\underline{/-144°}} = 50\frac{1 + (-0.33 - j0.24)}{1 - (-0.33 - j0.24)} \\ &= 50\frac{0.7117\underline{/-19.708°}}{1.3515\underline{/10.229°}} = 26.33\underline{/-29.937°} \\ &= (22.817 - j13.140)\ \Omega\end{aligned}$$

(d) The current $\overline{I}_g = \overline{I}(l)$ drawn from the voltage generator can be obtained as

$$\begin{aligned}\overline{I}(l) &= \frac{\overline{V}_g}{\overline{Z}_g + \overline{Z}_{\text{in}}} = \frac{100\underline{/0°}}{(30 - j40) + (22.817 - j13.140)} \\ &= \frac{100\underline{/0°}}{52.817 - j53.140} = \frac{100\underline{/0°}}{74.923\underline{/-45.175°}} \\ &= 1.3347\underline{/45.175°}\ \text{A}\end{aligned}$$

(e) The voltage at the input end of the line is given by

$$\begin{aligned}\overline{V}(l) &= \overline{Z}_{\text{in}}\overline{I}(l) \\ &= 26.33\underline{/-29.937°} \times 1.3347\underline{/45.175°} \\ &= 35.143\underline{/15.238°}\ \text{V}\end{aligned}$$

(f) The time-average power flow at the input end of the line is given by

$$\begin{aligned}\langle P(l)\rangle &= \tfrac{1}{2}\text{Re}[\bar{V}(l)\bar{I}^*(l)] \\ &= \tfrac{1}{2}\text{Re}[35.143\underline{/15.238^\circ} \times 1.3347\underline{/-45.175^\circ}] \\ &= \tfrac{1}{2} \times 35.143 \times 1.3347 \times \cos 29.937^\circ \\ &= 20.32 \text{ W}\end{aligned}$$

(g) Noting that $B_0 = 0$, we then obtain the value of $|\bar{V}^+|$ by applying (7.82) to $d = l$. Thus,

$$\begin{aligned}|\bar{V}^+| &= \sqrt{\frac{2\langle P(l)\rangle e^{-2\alpha l}}{G_0[1 - |\bar{\Gamma}(l)|^2]}} \\ &= \sqrt{\frac{2 \times 20.32 \times e^{-0.204}}{0.02(1 - 0.4077^2)}} \\ &= 44.58 \text{ V}\end{aligned}$$

(h) The time-average power delivered to the load is then given by

$$\begin{aligned}\langle P(0)\rangle &= \tfrac{1}{2}|\bar{V}^+|^2 G_0(1 - |\bar{\Gamma}_R|^2) \\ &= \tfrac{1}{2} \times 44.58^2 \times 0.02(1 - 0.25) \\ &= 14.91 \text{ W}\end{aligned}$$

(i) Finally, the time-average power dissipated in the line is

$$\begin{aligned}\langle P_d\rangle &= \langle P(l)\rangle - \langle P(0)\rangle \\ &= 20.32 - 14.91 \\ &= 5.41 \text{ W}\end{aligned}$$

K7.6. Distributed equivalent circuit; Transmission-line equations; Complex propagation constant; Complex characteristic impedance; Input impedance; $\bar{Z}_0$ and $\bar{\gamma}$ from input impedance considerations; Power flow; Power dissipation.

D7.14. For a lossy line of length $l = 16.3\lambda$ and characterized by $\bar{Z}_0 = 60\ \Omega$ and $\alpha = 0.02$ Np/λ, find the input impedance for each of the following values of $\bar{Z}_R$: **(a)** $\bar{Z}_R = 0$; **(b)** $\bar{Z}_R = \infty$; and **(c)** $\bar{Z}_R = (36 + j0)\ \Omega$.

Ans. **(a)** $(102.04 - j85.77)\ \Omega$; **(b)** $(20.67 + j17.38)\ \Omega$; **(c)** $(73.17 - j11.39)\ \Omega$.

D7.15. A lossy line of length $l = 10\lambda$ and characterized by $\bar{Z}_0 = 100\ \Omega$ and $\alpha = 10^{-2}$ Np/λ is terminated by a load impedance $\bar{Z}_R$. If a time-average power of 10 W is to be delivered to the load, determine how much time-average power should be delivered to the input terminals of the line for each of the following values of $\bar{Z}_R$: **(a)** $\bar{Z}_R = 100\ \Omega$; **(b)** $\bar{Z}_R = 20\ \Omega$; and **(c)** $\bar{Z}_R = 300\ \Omega$.

Ans. **(a)** 12.214 W; **(b)** 15.436 W; **(c)** 13.556 W.

7.7 PULSES ON LOSSY LINES

In the previous section, we considered the sinusoidal steady state analysis of lossy lines. We learned that as the waves propagate down the line, they get attenuated in addition to undergoing phase shift. Since α and v_p are in general functions of frequency, the different frequency components of an arbitrarily time-varying signal undergo different amounts of attenuation and different amounts of phase shift. Hence, as the signal propagates down the line, it gets distorted. For the general case, the analysis can be performed by employing Fourier techniques. There are, however, two special cases of importance that permit solution without the use of Fourier techniques: distortionless transmission and diffusion. We shall consider these two cases in this section.

A. Distortionless Transmission

Distortionless line

As the name implies, for this case the propagation along the lossy line is distortionless—although it is characterized by attenuation, as shown in Fig. 7.33. As the signal propagates down the line, its shape versus time remains the same, but it diminishes in magnitude. The situation arises for the condition

$$\frac{\mathscr{R}}{\mathscr{L}} = \frac{\mathscr{G}}{\mathscr{C}} \tag{7.83}$$

Substituting (7.83) in (7.66), we observe that

$$\begin{aligned}\overline{\gamma} &= \sqrt{(\mathscr{R} + j\omega\mathscr{L})(\mathscr{G} + j\omega\mathscr{C})}\\ &= \sqrt{\mathscr{R}\left(1 + j\frac{\omega\mathscr{L}}{\mathscr{R}}\right)g\left(1 + j\frac{\omega\mathscr{C}}{\mathscr{G}}\right)}\end{aligned}$$

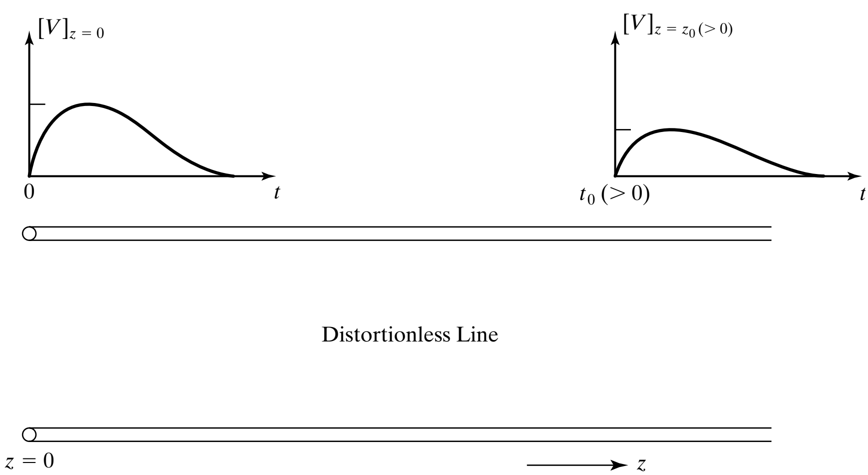

FIGURE 7.33

For illustrating pulse propagation along a lossy, but distortionless, line.

$$
\begin{aligned}
&= \sqrt{\mathcal{R}\mathcal{G}}\left(1 + j\omega\frac{\mathcal{L}}{\mathcal{R}}\right) \\
&= \sqrt{\mathcal{R}\mathcal{G}} + j\omega\mathcal{L}\sqrt{\frac{\mathcal{G}}{\mathcal{R}}} \\
&= \sqrt{\mathcal{R}\mathcal{G}} + j\omega\mathcal{L}\sqrt{\frac{\mathcal{C}}{\mathcal{L}}} \\
&= \sqrt{\mathcal{R}\mathcal{G}} + j\omega\sqrt{\mathcal{L}\mathcal{C}}
\end{aligned}
$$

so that

$$\alpha = \sqrt{\mathcal{R}\mathcal{G}} \tag{7.84a}$$

$$\beta = \omega\sqrt{\mathcal{L}\mathcal{C}} \tag{7.84b}$$

$$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mathcal{L}\mathcal{C}}} \tag{7.84c}$$

Thus, α and v_p are both independent of frequency, provided, of course, that $\mathcal{R}$, $\mathcal{L}$, $\mathcal{G}$, and $\mathcal{C}$ are independent of frequency. Hence, the signal propagates distortionless. Furthermore

$$
\begin{aligned}
\bar{Z}_0 &= \sqrt{\frac{\mathcal{R} + j\omega\mathcal{L}}{\mathcal{G} + j\omega\mathcal{C}}} = \sqrt{\frac{\mathcal{R}(1 + j\omega\mathcal{L}/\mathcal{R})}{\mathcal{G}(1 + j\omega\mathcal{C}/\mathcal{G})}} \\
&= \sqrt{\frac{\mathcal{R}}{\mathcal{G}}} = \sqrt{\frac{\mathcal{L}}{\mathcal{C}}}
\end{aligned}
\tag{7.84d}
$$

so that $\bar{Z}_0$ is also independent of frequency. Note that the expressions (7.84a)–(7.84d) are exact; that is, they do not involve any approximations as in the case of the corresponding expressions [Eqs. (7.71a)–(7.71d)] for the low-loss line, which also exhibits the same approximate frequency behavior. Hence, the present expressions are valid in any frequency range for which the condition (7.83) holds and in which $\mathcal{R}$, $\mathcal{L}$, $\mathcal{G}$, and $\mathcal{C}$ are constants.

Example 7.9 Pulse propagation along a lossy, but distortionless, transmission-line system

Let us consider the distortionless line system shown in Fig. 7.34, in which the switch S is closed at $t = 0$, thereby applying the voltage source pulse of 0.1 μs duration, in series with the 50 Ω internal resistance, to the line. We wish to find and sketch **(a)** the voltage V_R across the load resistor as a function of time, **(b)** the line voltage as a function of z at $t = 0.5$ μs, and **(c)** the line voltage as a function of z at $t = 1.5$ μs.

We proceed with the solution in the following manner.

(a) Initially, the voltage source views an impedance of 50 Ω at $z = 0$; hence, the line voltage at $z = 0$ is equal to $\frac{1}{2}V_g$. Thus, a voltage pulse of amplitude 50 V and duration

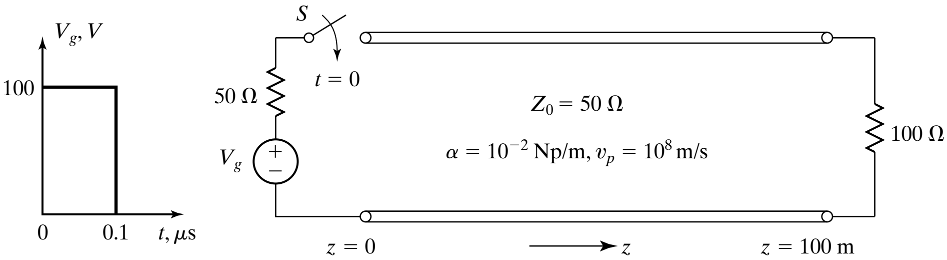

FIGURE 7.34

A distortionless transmission line system.

0.1 μs travels toward the load undistorted in shape but attenuated in accordance with $e^{-\alpha z} = e^{-0.01z}$. Since the one-way travel time on the line is $100/10^8 = 10^{-6}$ s $= 1\ \mu$s, the leading edge of the pulse reaches the load end at $t = 1\ \mu$s and sets up a reflection with reflection coefficient equal to $(100 - 50)/(100 + 50)$, or 1/3. Hence, the voltage across the load is $\left(1 + \frac{1}{3}\right)$ or 4/3 times the incident voltage. Thus, noting that the attenuation in a distance of 100 m is $e^{-0.01\times100} = e^{-1}$, we obtain the voltage V_R to be a pulse of amplitude $50 \times e^{-1} \times \frac{4}{3} = 24.53$ V and duration from 1 to 1.1 μs, as shown in Fig. 7.35(a).

(b) To find the line voltage versus z for $t = 0.5\ \mu$s, we observe that the leading edge of the incident voltage pulse will have reached $z = 50$ m at that time, whereas the trailing edge will have traveled to only $z = 40$ m. Since the attenuation undergone in 50 m is $e^{-0.01\times50} = e^{-0.5}$, whereas the attenuation undergone in 40 m is only $e^{-0.01\times40} = e^{-0.4}$, the line voltage distribution is a pulse stretching from 40 m to 50 m and having a value $50e^{-0.4} = 33.52$ V at 40 m but only $50e^{-0.5} = 30.33$ V at 50 m, as shown in Fig. 7.35(b). The slightly downward-curved shape of the pulse between the two edges can be understood by noting, for example, that at 45 m, the voltage is $50e^{-0.01\times45} = 50e^{-0.45} = 31.88$ V.

(c) At $t = 1.5\ \mu$s, the line voltage consists entirely of the reflected wave voltage, which is 1/3 of the incident wave voltage. The leading edge will have reached $z = 50$ m with a value of $\frac{1}{3} \times 50e^{-1} \times e^{-0.5} = 3.72$ V, whereas the trailing edge occupies the location $z = 60$ m with a value of $\frac{1}{3} \times 50e^{-1} \times e^{-0.4} = 4.11$ V, as shown in Fig. 7.35(c). The slightly downward-curved shape of the pulse between the two edges can once again be understood by noting that the line voltage at $z = 55$ m is $\frac{1}{3} \times 50e^{-1} \times e^{-0.45} = 3.91$ V.

Finally, the nonrectangular shapes of the voltage distributions with z should not be misunderstood as distortion, because at every value of z, the individual wave voltage variation with time is a rectangular pulse of 0.1-μs duration, with amplitude determined by the attenuation undergone and the reflection coefficient(s).

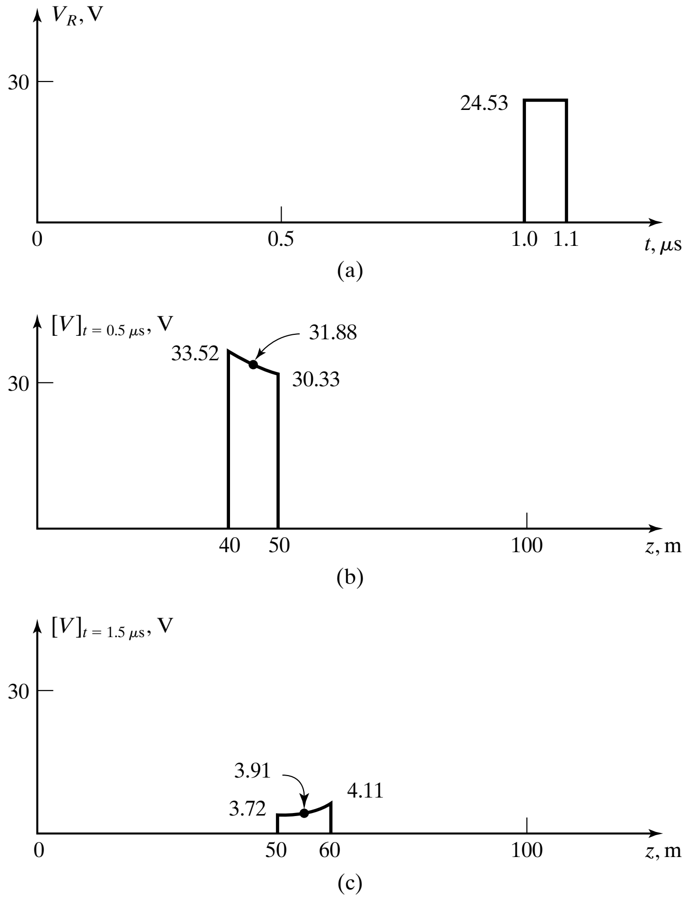

FIGURE 7.35

(a) Time-variation of the voltage V_R across the load, and **(b)** and **(c)** distance variations of line voltage for $t = 0.5\ \mu$s and $t = 1.5\ \mu$s, respectively, for the distortionless line system of Fig. 7.34.

B. Diffusion

Noninductive, leakage-free cable

This case pertains to the historically important noninductive, leakage-free cable first investigated by Lord Kelvin in 1855, but is also relevant to modern lines with large skin-effect losses and to many other physical phenomena, such as heat flow.

The noninductive, leakage-free cable is characterized by $\mathscr{L} = \mathscr{G} = 0$ so that the distributed equivalent circuit consists simply of series resistors and shunt capacitors, as shown by one section in Fig. 7.36. Setting $\mathscr{L} = \mathscr{G} = 0$ in the

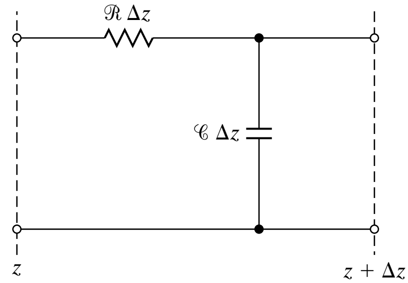

FIGURE 7.36

One section of distributed equivalent circuit for a noninductive, leakage-free cable.

transmission-line equations in time-domain form given by (7.63a) and (7.63b), we then obtain

$$\frac{\partial V(z,t)}{\partial z} = -\mathcal{R}I(z,t) \tag{7.85a}$$

$$\frac{\partial I(z,t)}{\partial z} = -\mathcal{C}\frac{\partial V(z,t)}{\partial t} \tag{7.85b}$$

Differentiating (7.85a) with respect to z and using (7.85b), we obtain the differential equation for the line voltage to be

$$\frac{\partial^2 V}{\partial z^2} = \mathcal{R}\mathcal{C}\frac{\partial V}{\partial t} \tag{7.86}$$

This equation is of the type

$$\frac{\partial^2 \tau}{\partial z^2} = \frac{1}{D}\frac{\partial \tau}{\partial t} \tag{7.87}$$

which is known as the *diffusion equation*. Its solution is of the form

$$\tau = A\int_0^{(\sqrt{1/4Dt})z} e^{-Y^2}\,dY + B \tag{7.88}$$

where A and B are arbitrary constants to be evaluated from boundary conditions, and Y is a dummy variable. Thus the general solution for (7.86) is

$$V(z,t) = A\int_0^{(\sqrt{\mathcal{R}\mathcal{C}/4t})z} e^{-Y^2}\,dY + B \tag{7.89}$$

Let us now consider an initially quiescent cable extending from $z = 0$ to $z = \infty$ and excited at $z = 0$ by a constant voltage source of value V_0 connected

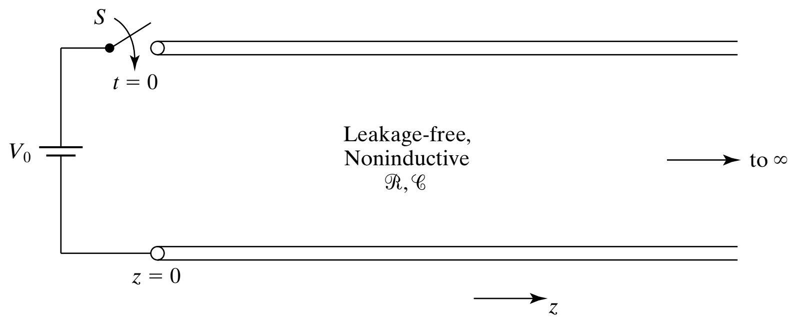

FIGURE 7.37

Semi-infinitely long, noninductive, leakage-free cable, excited by a constant voltage source.

at $t = 0$, as shown in Fig. 7.37, and find the solutions for the line voltage and current for $t > 0$. The boundary conditions for the line voltage are

$$V(0, 0+) = V_0 \tag{7.90a}$$

$$V(\infty, t) = 0 \qquad \text{for } t > 0 \tag{7.90b}$$

Substituting these boundary conditions into (7.89), we have, from (7.90a),

$$A \int_0^0 e^{-Y^2}\, dY + B = V_0$$

or

$$B = V_0$$

and then from (7.90b)

$$A \int_0^\infty e^{-Y^2}\, dY + V_0 = 0$$

or

$$A \frac{\sqrt{\pi}}{2} + V_0 = 0$$

$$A = -\frac{2V_0}{\sqrt{\pi}}$$

Thus the particular solution for $V(z, t)$ for the system of Fig. 7.37 is

$$V(z,t) = -\frac{2V_0}{\sqrt{\pi}}\int_0^{(\sqrt{\mathcal{RC}/4t})z} e^{-Y^2}\,dY + V_0$$
$$= V_0\left(1 - \frac{2}{\sqrt{\pi}}\int_0^{(\sqrt{\mathcal{RC}/4t})z} e^{-Y^2}\,dY\right) \qquad (7.91)$$

The second term inside the parentheses is the well-known *error function* (erf) having the argument $(\sqrt{\mathcal{RC}/4t})z$. Hence, (7.91) can be written as

$$V(z,t) = V_0[1 - \operatorname{erf}(\sqrt{\mathcal{RC}/4t}z)]$$
$$= V_0 \operatorname{erfc}(\sqrt{\mathcal{RC}/4t}z) \qquad (7.92)$$

where erfc is the complementary error function. Substituting (7.91) into (7.85a), we obtain the corresponding solution for $I(z, t)$ to be

$$I(z,t) = -\frac{1}{\mathcal{R}}\frac{\partial V(z,t)}{\partial t}$$
$$= V_0\sqrt{\frac{\mathcal{C}}{\pi\mathcal{R}t}}\,e^{-(\mathcal{RC}/4t)z^2} \qquad (7.93)$$

Note that the boundary conditions of no current anywhere on the line for $t = 0$, that is, before the voltage source is connected to the line, and no current at $z = \infty$ for all t, are satisfied by (7.93).

To discuss the solution for the line voltage given by (7.92), we sketch

$$\frac{V(z,t)}{V_0} = \operatorname{erfc}\left(\sqrt{\frac{\mathcal{RC}}{4t}}z\right) \qquad (7.94)$$

as shown in Fig. 7.38. Since the argument involves both z and t in the manner $(\sqrt{\mathcal{RC}/4t})z$, this sketch represents the shape of the line voltage variation with z for any fixed value of t. In fact, the scale for the abscissa can be converted to one

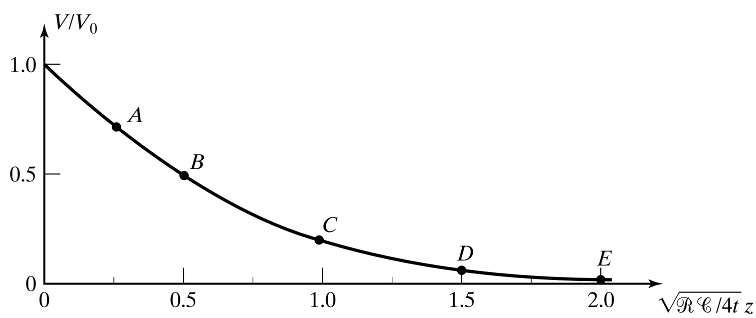

FIGURE 7.38

Sketch of the complementary error function depicting the solution for $V(z,t)/V_0$ for the system of Fig. 7.37.

for z by multiplying the numbers by $\sqrt{4t/\mathcal{RC}}$. Thus, we note that, immediately after closure of the switch, there is voltage everywhere on the line. This corresponds to the phenomenon of *diffusion*—as distinguished from *propagation*, which is characterized by a well-defined velocity. As t increases, a given point on the sketch corresponds to larger and larger values of z, indicating that as time progresses, the line voltage at all values of z increases. For example, since the values of z are doubled as t is quadrupled, the voltage at a given distance from the source and at a particular time after closure of the switch is the same as the voltage at half that distance and at one-fourth of that time. This is depicted in Fig. 7.39(a) for two values of time $\mathcal{RC}z_0^2$ and $4\mathcal{RC}z_0^2$, where z_0 is any value of $z > 0$. The points *A, B, C, D*, and *E* correspond to the points *A, B, C, D*, and *E*, respectively, in Fig. 7.38. The sketch of Fig. 7.38 can also be used to obtain the time-variations of the line voltage for fixed values of z, by noting that for a fixed z, the numbers on the abscissa can be converted to values of time. This is depicted in Fig. 7.39(b) for two values of z, which are $2\sqrt{t_0/\mathcal{RC}}$ and $4\sqrt{t_0/\mathcal{RC}}$, where t_0 is any value of $t > 0$. Again, the points *A, B, C*, and *E* correspond to the points *A, B, C*, and *E*, respectively, in Fig. 7.38.

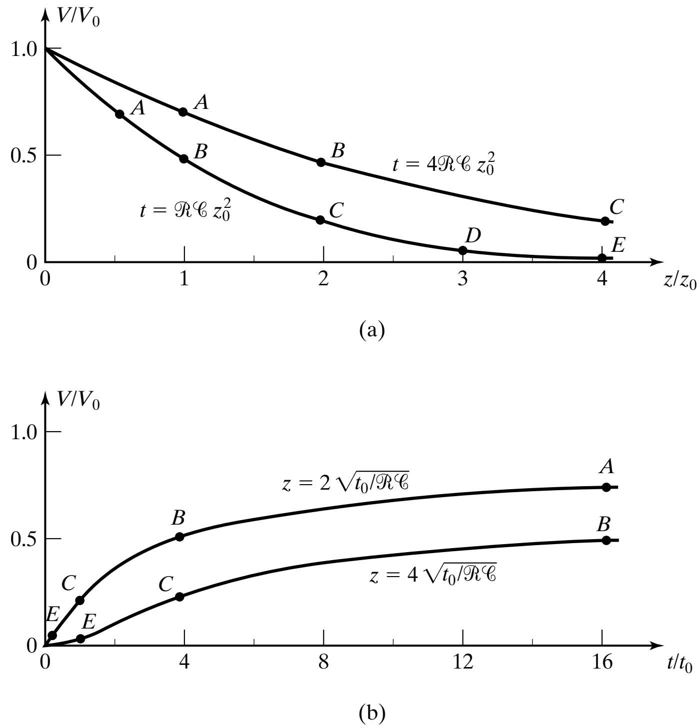

FIGURE 7.39
(a) Line voltage variations with distance for two values of time, and **(b)** line voltage variations with time for two values of distance, for the system of Fig. 7.37. The points *A, B, C, D*, and *E* correspond to the points *A, B, C, D*, and *E*, respectively, in Fig. 7.38.

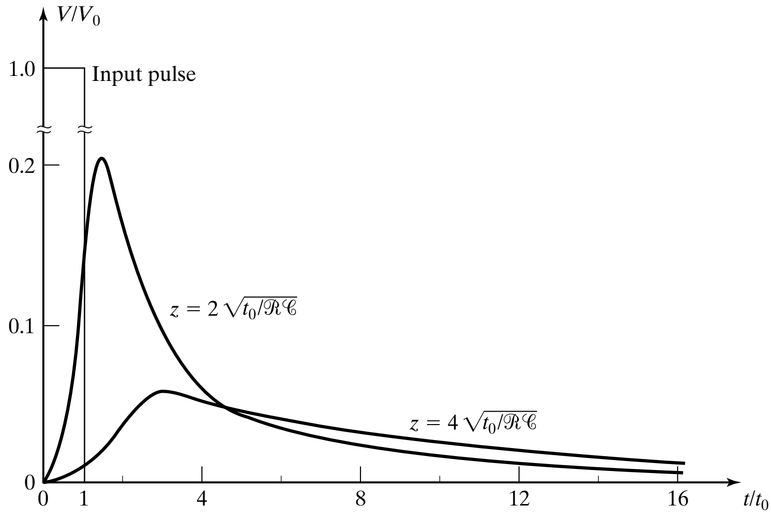

FIGURE 7.40

Line voltage variations with time for two values of distance for the system of Fig. 7.37, excited by a rectangular pulse of duration t_0, also shown in the figure.

Finally, let us consider the voltage source in the system of Fig. 7.40 to be a rectangular pulse of amplitude V_0 and duration t_0. Then according to superposition, the rectangular pulse is equivalent to the sum of two constant voltages, one of value V_0 connected at $t = 0$ and the second of value $-V_0$ connected at $t = t_0$. Applying the result of Fig. 7.39(b) to each constant voltage source and using superposition, we can find the response to the rectangular pulse. Thus, the time variations of the line voltage for the two values of z equal to $2\sqrt{t_0/\mathcal{RC}}$ and $4\sqrt{t_0/\mathcal{RC}}$ are as shown in Fig. 7.40. Note the difference in the vertical scales between Figs. 7.39(b) and 7.40. It can be seen from Fig. 7.40 that, as the value of z is increased, the attainment of the maximum of the pulse is delayed and the value of the maximum is reduced.

K7.7. Distortionless line; Noninductive, leakage-free cable; Diffusion

D7.16. Assume that the duration of the source voltage pulse in Example 7.9 is 1.0 μs, instead of 0.1 μs. Find the value of the line voltage at $t = 1.5$ μs, for each of the following values of z: **(a)** 50 m; **(b)** 75 m; and **(c)** 100 m.

Ans. **(a)** 34.05 V; **(b)** 28.39 V **(c)** 24.53 V

SUMMARY

In this chapter, we began our study of sinusoidal steady-state analysis of lossless transmission lines by expressing the general solutions for the phasor line voltage and line current in terms of the distance variable d, measured from the load toward the source. These solutions are

$$\bar{V}(d) = \bar{V}^{+}e^{j\beta d} + \bar{V}^{-}e^{-j\beta d}$$

$$\bar{I}(d) = \frac{1}{Z_0}(\bar{V}^{+}e^{j\beta d} - \bar{V}^{-}e^{-j\beta d})$$

By applying these general solutions to the case of a line short-circuited at the far end and obtaining the particular solutions for that case, we discussed the standing-wave phenomenon resulting from the complete reflection of waves by the short circuit. We introduced the concept of a standing-wave pattern and discussed the phenomenon of natural oscillations. We examined the frequency behavior of the input impedance of a short-circuited line of length l, given by

$$\bar{Z}_{\text{in}} = jZ_0 \tan \beta l$$

and illustrated (1) its application in a technique for locating a short circuit in a line and (2) the computation of resonant frequencies for a system formed by connecting together short-circuited line sections.

Next we considered the general case of a line terminated by an arbitrary load $\bar{Z}_R$ and introduced the concept of the generalized voltage reflection coefficient, as the ratio of the phasor reflected wave voltage at any value of d to the phasor incident wave voltage at that value of d. It is given by

$$\bar{\Gamma}(d) = \bar{\Gamma}_R e^{-j2\beta d}$$

where

$$\bar{\Gamma}_R = |\bar{\Gamma}_R|e^{j\theta} = \frac{\bar{Z}_R - Z_0}{\bar{Z}_R + Z_0}$$

is the voltage reflection coefficient at the load. We then expressed the solutions for the line voltage and line current in terms of $\bar{\Gamma}(d)$ and discussed the construction of standing-wave patterns from the solutions. We learned that together with the property that distance between successive voltage minima of the standing-wave patterns is $\lambda/2$, the quantities

$$\text{SWR} = \frac{1 + |\bar{\Gamma}_R|}{1 - |\bar{\Gamma}_R|}$$

and

$$d_{\text{min}} = \frac{\lambda}{4\pi}(\theta + \pi)$$

constitute an important set of parameters associated with the standing waves. The SWR, which is the ratio of the maximum voltage amplitude to the minimum voltage amplitude in the standing-wave pattern, and d_{min}, which is the distance of the first voltage minimum of the standing-wave pattern from the load, are easily measurable quantities. We then defined the ratio of the complex line voltage to the complex line current at a given value of d to be the line impedance $\bar{Z}(d)$, given by

$$\bar{Z}(d) = Z_0 \frac{1 + \bar{\Gamma}(d)}{1 - \bar{\Gamma}(d)}$$

and discussed its several properties as well as the computation of power flow along the line from considerations of input impedance of the line.

We then turned our attention to the topic of transmission-line matching, which consists of eliminating standing waves by connecting a matching device near the load such that the line views an effective impedance equal to its own characteristic impedance, on the generator side of the matching device. We discussed the need for matching and three techniques of matching: (1) quarter-wave transformer, (2) single stub, and (3) double stub. The quarter-wave transformer technique is based on a property of the line impedance that

$$[\bar{Z}(d)]\left[\bar{Z}\left(d + \frac{\lambda}{4}\right)\right] = Z_0^2$$

whereas the stub-matching techniques make use of the property that the input impedance of a lossless line short-circuited (or open-circuited) at the far end is purely reactive. We also discussed the departure of SWR from unity as the frequency is varied from that at which the match is achieved, and we illustrated a procedure for computation of the SWR versus frequency.

Next we introduced the Smith chart, a popular graphical aid in the solution of transmission-line problems. We learned that the Smith chart is based on the transformation from the $\bar{z}$-plane to the $\bar{\Gamma}$-plane in accordance with the relationship

$$\bar{\Gamma}(d) = \frac{\bar{z}(d) - 1}{\bar{z}(d) + 1}$$

where

$$\bar{z}(d) = \frac{\bar{Z}(d)}{Z_0}$$

is the normalized line impedance. We discussed the construction of the Smith chart, some basic procedures, and the solution of transmission-line matching problems. We also discussed a useful property associated with the transformation of the reflection coefficient across a discontinuity and illustrated its application by means of an example.

Finally, we extended our analysis of lossless lines briefly to lossy lines, with the discussion of (1) the distributed equivalent circuit, (2) computation of characteristic impedance and propagation constant from input impedance measurements, (3) computation of power flow at the generator and load ends of the line, and power dissipated on the line, and (4) two special cases, distortionless propagation and diffusion, of pulses on lossy lines.

REVIEW QUESTIONS

Q7.1. Discuss the general solutions for the line voltage and line current in terms of the distance variable d in the sinusoidal steady state.

Q7.2. State the boundary condition at a short circuit on a line. For an open-circuited line, what is the boundary condition to be satisfied at the open circuit?

Q7.3. What is a standing wave? How do complete standing waves arise? Discuss their characteristics.

Q7.4. What is a standing-wave pattern? Discuss the voltage and current standing-wave patterns for a short-circuited line.

Q7.5. Explain the phenomenon of natural oscillations and the determination of natural frequencies of oscillation by means of an example.

Q7.6. Discuss the variation with frequency of the input reactance of a short-circuited line and its application in the determination of the location of a short circuit.

Q7.7. Outline the method of computation of resonant frequencies of a system formed by connecting together two short-circuited line sections.

Q7.8. How is the generalized voltage reflection coefficient defined? Discuss its variation along the line.

Q7.9. Discuss the sketching of standing-wave patterns for line voltage and current on a line terminated by an arbitrary load.

Q7.10. Define the standing-wave ratio (SWR). What are the standing-wave ratios for **(a)** a semi-infinitely long line; **(b)** a short-circuited line; **(c)** an open-circuited line; and **(d)** a line terminated by its characteristic impedance?

Q7.11. Discuss the slotted-line technique for performing standing-wave measurements on a line and the determination of an unknown load impedance from the standing-wave measurements.

Q7.12. How is line impedance defined? Summarize the several properties of line impedance.

Q7.13. Outline the procedure for the determination of time-average power flow down a line from input impedance considerations.

Q7.14. Define normalized line impedance and normalized line admittance. How are they related to the voltage reflection coefficient?

Q7.15. Discuss the reasons for transmission-line matching and the principle behind matching.

Q7.16. Which property of line impedance forms the basis for the quarter-wave transformer (QWT) technique of transmission-line matching? Outline the solution for the QWT matching problem.

Q7.17. What is a stub? Outline the solution for the single-stub matching problem.

Q7.18. Outline the solution for the double-stub matching problem.

Q7.19. Discuss the bandwidth associated with a transmission-line matched system and the procedure for obtaining the SWR in the main line versus frequency.

Q7.20. What is the basis behind the construction of the Smith chart? Briefly discuss the mapping of the normalized line impedances onto the $\overline{\Gamma}$-plane.

Q7.21. Why is a circle with its center at the center of the Smith chart known as a constant SWR circle? Where on the circle is the corresponding SWR value marked?

Q7.22. Using the Smith chart, how do you find the normalized line admittance at a point on the line, given the normalized line impedance at that point?

Q7.23. Briefly describe the solution to the single-stub matching problem by using the Smith chart.

Q7.24. Briefly describe the solution to the double-stub matching problem by using the Smith chart.

Q7.25. Discuss the forbidden region of $\overline{y}_1'$ for possible match associated with the double-stub matching technique.

Q7.26. Discuss the transformation of the reflection coefficient from one side of a transmission-line discontinuity to the other side of the discontinuity and an application of the property associated with this transformation.

Q7.27. Discuss the modification of the distributed equivalent circuit for the lossless line case to the lossy line case.

Q7.28. What are the conditions under which a lossy line can be classified as a low-loss line? Compare the propagation parameters of the low-loss line with those for the lossless line.

Q7.29. Discuss the computation of $\bar{Z}_0$ and $\overline{\gamma}$ for a lossy line from a knowledge of the input impedances of the line with short-circuit and open-circuit terminations.

Q7.30. Briefly outline the procedure for the computation of time-average power flow at the input and the load ends of a lossy line and, hence, the time-average power dissipated in the line.

Q7.31. State and explain the condition for distortionless transmission along a lossy line. Discuss the propagation of a pulse along the distortionless line by means of an example.

Q7.32. Discuss the phenomenon of diffusion along a lossy line with reference to the special case of the noninductive, leakage-free cable.

PROBLEMS

Section 7.1

P7.1. Solutions for line voltage and current for an open-circuited line. For a line open-circuited at the far end, as shown in Fig. 7.41, obtain the solutions for the phasor line voltage and current, and sketch the voltage and current standing-wave patterns, as in Fig. 7.4.

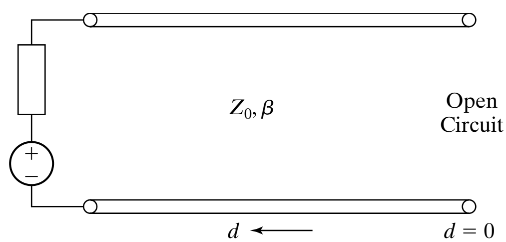

FIGURE 7.41
For Problem P7.1.

P7.2. Open-circuited line excited by a source of two harmonically related frequencies. In the system shown in Fig. 7.42, the line is open-circuited at the far end, the source voltage is

$$V_g(t) = V_0 \cos \pi f_0 t \cos 3\pi f_0 t$$

and $l = \lambda/4$ at $f = f_0$. Find the root-mean-square (rms) values of the line voltage and line current at values of d/l equal to 0, $\frac{1}{3}$, $\frac{1}{2}$, and 1. (*Note:* The rms value of the sum of the voltages of two harmonically related frequencies is equal to the square root of the sum of the squares of the rms values of the individual voltages.)

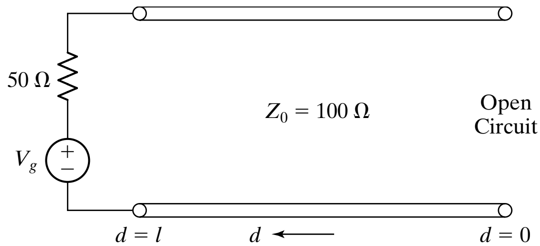

FIGURE 7.42
For Problem P7.2.

P7.3. Short-circuited line excited by a nonsinusiodal periodic source. In the system shown in Fig. 7.43(a), the source voltage is periodic, as shown in Fig. 7.43(b). Find the reading of the ammeter A if it reads root-mean-square values.

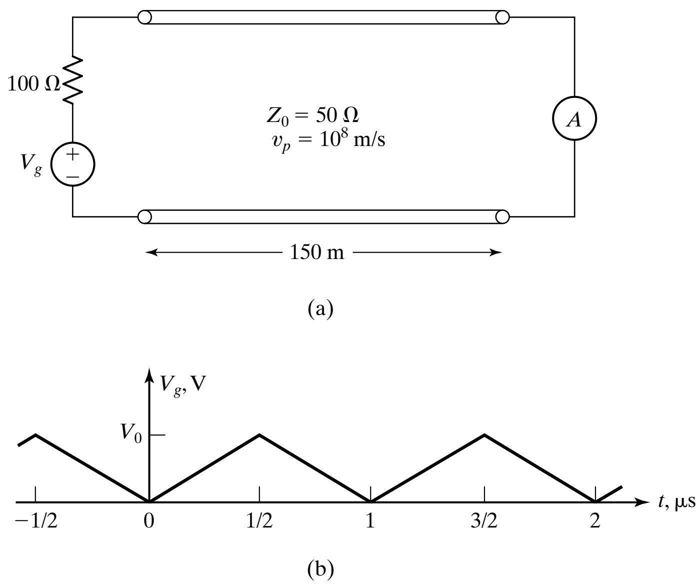

FIGURE 7.43
For Problem P7.3.

P7.4. A parallel-plate resonator. In the system shown in Fig. 7.44, a nonmagnetic ($\mu = \mu_0$), lossless material medium is sandwiched between two parallel, perfect conductors. For uniform plane waves bouncing back and forth normal to the conductors, find the following: **(a)** the minimum value of l for which the natural frequency of oscillation is 2.5 GHz if the medium is a perfect dielectric of permittivity $2.25\varepsilon_0$, and **(b)** the expression for the lowest natural frequency of oscillation if the medium is a plasma, which can be thought of as equivalent to a perfect dielectric of permittivity $\varepsilon_0(1 - f_N^2/f^2)$, where f_N, known as the plasma frequency, is a constant and f is the wave frequency.

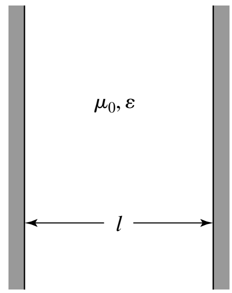

FIGURE 7.44
For Problem P7.4.

P7.5. Natural frequencies of oscillation for a ring transmission line. A ring transmission line is formed as shown in Fig. 7.45(a) by connecting the ends a and a' of the conductors of a line of length l [shown in Fig. 7.45(b)] to the ends b and b', respectively, of the same conductors. Find the natural frequencies of oscillation of the system.

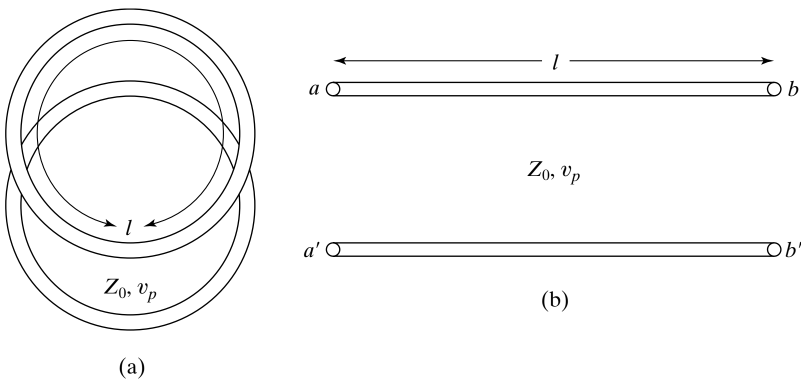

FIGURE 7.45
For Problem P7.5.

P7.6. Natural frequencies of oscillation for a twisted-ring transmission line. Repeat Problem P7.5 for a twisted-ring transmission line formed by connecting the ends a and a' of the conductors [see Fig. 7.45(b)] to the ends b' and b, respectively.

P7.7. Input impedance of a line at low frequencies. Show that, for $f \ll v_p/2\pi l$, the input impedance of a short-circuited line of length l and phase velocity v_p is essentially that of a single inductor of value $\mathscr{L}l$, where $\mathscr{L}$ is the inductance per unit length of the line. Assuming that the criterion $f \ll v_p/2\pi l$ is satisfied for frequencies $f \leq 0.1v_p/2\pi l$, compute the maximum length of an air-dielectric short-circuited line for which the input impedance is approximately that of an inductance equal to the total inductance of the line for $f = 100$ MHz.

P7.8. Location of a short circuit in a line. In the example involving the location of a short circuit in a line, solve for the distance of the short circuit from the generator by considering the standing-wave patterns for the two frequencies of interest and deducing the number of wavelengths at one of the two frequencies.

P7.9. Finding the resonant frequencies for a transmission-line resonant system. A transmission line of characteristic impedance $Z_0 = 100\ \Omega$, phase velocity $v_p = 2 \times 10^8$ m/s, and length $l = 20$ cm is short-circuited at one end and terminated by an inductor of value 0.1 μH at the other end. Find the three lowest resonant frequencies of the system.

P7.10. Finding the resonant frequencies for a parallel-plate resonator with two dielectrics. The arrangement shown in Fig. 7.46 is that of a parallel-plate resonator made up of two dielectric slabs sandwiched between perfect conductors and in which uniform plane waves bounce back and forth normal to the conductors. **(a)** Show that the resonant frequencies of the system are given by the roots of the characteristic equation

$$\tan \omega\sqrt{\mu_0\varepsilon_1}t + \sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\tan \omega\sqrt{\mu_0\varepsilon_2}(l - t) = 0$$

(b) Find the five lowest resonant frequencies if $t = l/2$, $l = 5$ cm, $\varepsilon_1 = 4\varepsilon_0$, and $\varepsilon_2 = 16\varepsilon_0$.

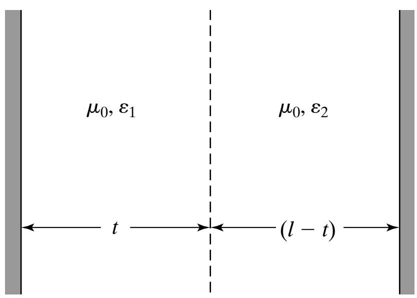

FIGURE 7.46

For Problem P7.10.

Section 7.2

P7.11. Standing-wave parameters for a line terminated by a reactive load. For a line of characteristic impedance Z_0 terminated by a purely reactive load jX, show that the SWR is equal to infinity and the value of d_{min} is $(\lambda/2\pi)[\pi - \tan^{-1}(X/Z_0)]$ for $X > 0$ and $(\lambda/2\pi)\tan^{-1}(|X|/Z_0)$ for $X < 0$.

P7.12. Finding the load impedance from standing-wave measurements. A slotted coaxial line of characteristic impedance 75 Ω was used to measure an unknown load impedance. First, the receiving end of the line was short-circuited. The voltage minima were found to be 20 cm apart. One of the minima was marked as the reference point. Next, the unknown impedance was connected to the receiving end of the line. The SWR was found to be 3.0 and a voltage minimum was found to be 6 cm from the reference point toward the load. Find the value of the unknown load impedance.

P7.13. Normal incidence of uniform plane waves onto a dielectric slab. In the system shown in Fig. 7.47, assume uniform plane waves of frequency f incident normally onto the interface from medium 1. **(a)** Find the SWR in medium 1 for $f = 10^9$ Hz if $l = 5$ cm. **(b)** Find the three lowest values of f for which complete transmission occurs if $l = 5$ cm. **(c)** Find the three lowest values of l for which complete transmission occurs for $f = 10^9$ Hz.

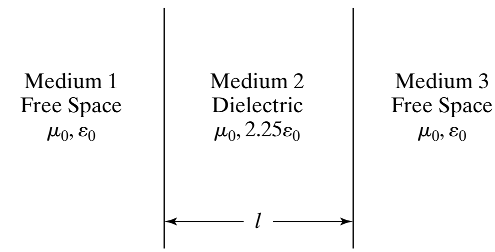

FIGURE 7.47

For Problem P7.13.

P7.14. Complete transmission of uniform plane waves through a dielectric slab. For the system shown in Fig. 7.48, find the lowest value of l for which no reflection occurs for a uniform plane wave having the electric field

$$\mathbf{E} = E_0 \cos 4\pi \times 10^9 t \cos \pi \times 10^9 t \, \mathbf{a}_x$$

at $z = 0$ normally incident on the interface from medium 1.

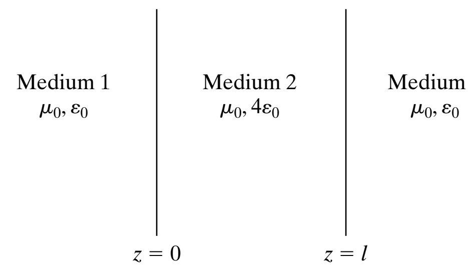

FIGURE 7.48

For Problem P7.14.

P7.15. Uniform plane-wave transmission for three media in cascade. For uniform plane waves of frequency f incident normally onto the interface from medium 1 in the system shown in Fig. 7.49, find the fraction of the incident power transmitted into medium 3 for each of the following values of f: **(a)** 3000 MHz; **(b)** 6000 MHz; and **(c)** 1500 MHz.

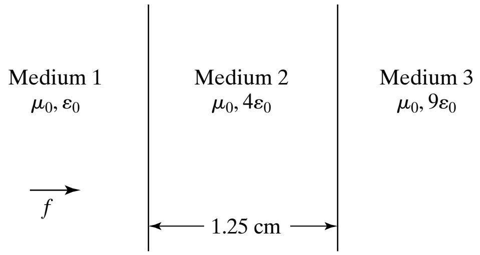

FIGURE 7.49
For Problem P7.15.

P7.16. Parallel-plate resonator with the plates coated with a dielectric. The arrangement shown in Fig. 7.50 is that of a parallel-plate resonator made up of two plane perfect conductors coated with a dielectric and in which uniform plane waves bounce back and forth normal to the plates. **(a)** Show that the characteristic equation for the resonant frequency is given by

$$\tan \omega\sqrt{\mu_0\varepsilon_0}(l - t) \tan \omega\sqrt{\mu_0\varepsilon_0}t = \sqrt{\frac{\varepsilon_d}{\varepsilon_0}}$$

(b) Find the value of the lowest resonant frequency for $t = l/2 = 5$ cm and $\varepsilon_d = 4\varepsilon_0$.

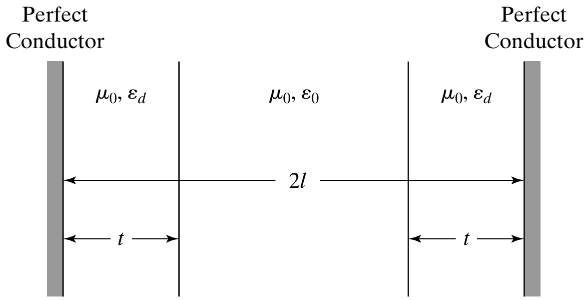

FIGURE 7.50
For Problem P7.16.

P7.17. Finding the power delivered to the load from considerations of line input impedance. For the system shown in Fig. 7.51, find the input impedance of the line and the time-average power delivered to the load.

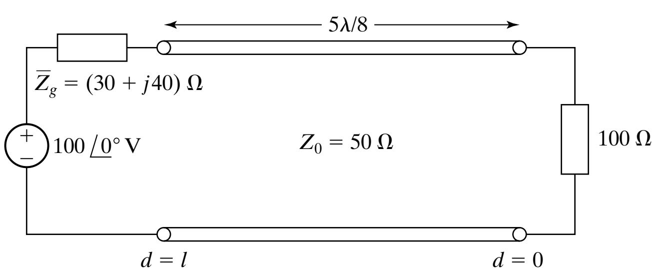

FIGURE 7.51
For Problem P7.17.

P7.18. Application of maximum power transfer theorem for a transmission-line system. In the system in Fig. 7.52, find: **(a)** the value of the load impedance $\bar{Z}_R$ that enables maximum power transfer from the generator to the load and **(b)** the power transferred to the load for the value found in (a). (*Hint:* Apply maximum power transfer theorem at $d = l$.)

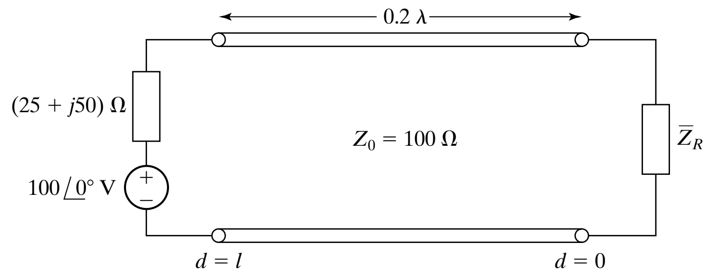

FIGURE 7.52

For Problem P7.18.

P7.19. Finding the input impedance of a ring transmission-line system. The ring transmission line of Fig. 7.45(a) is excited by connecting a voltage source $V_g(t) = V_0 \cos \omega t$ across the conductors at some location on the line [such as aa' (or bb') in Fig. 7.45(b)]. Find the impedance viewed by the voltage source.

P7.20. Application of maximum power transfer theorem for a transmission-line system. In the system shown in Fig. 7.53, find the values of the reactance X and the characteristic impedance Z_{02} of line 2 for which the power delivered to the load $\bar{Z}_R$ is a maximum.

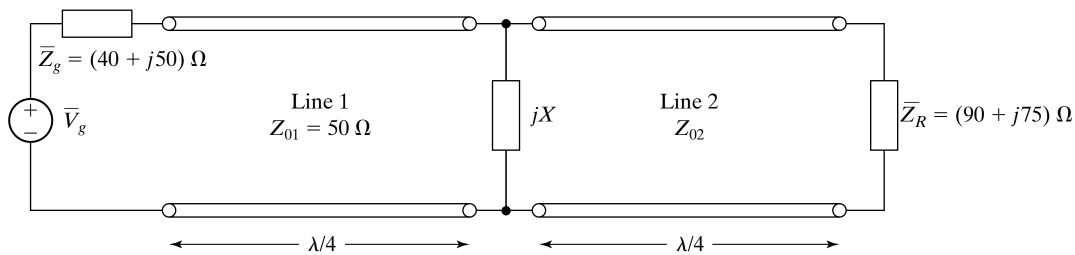

FIGURE 7.53

For Problem P7.20.

Section 7.3

P7.21. Eliminating reflections by using a quarter-wave dielectric coating. In the arrangement shown in Fig. 7.54, a quarter-wave dielectric coating is employed to eliminate reflections of uniform plane waves of frequency 1000 MHz incident normally from free space onto a dielectric of permittivity $4\varepsilon_0$. Assuming that $\mu = \mu_0$, find the thickness in centimeters and the permittivity of the dielectric coating.

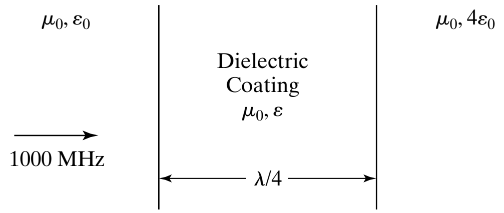

FIGURE 7.54

For Problem P7.21.

P7.22. Minimizing the standing-wave ratio in a line with a quarter-wave section. In the system shown in Fig. 7.55, the $\lambda/4$ section of characteristic impedance 50 Ω is used to minimize the SWR to the left of the section. Find analytically the minimum value of d_1 that minimizes the SWR and the minimum value of the SWR.

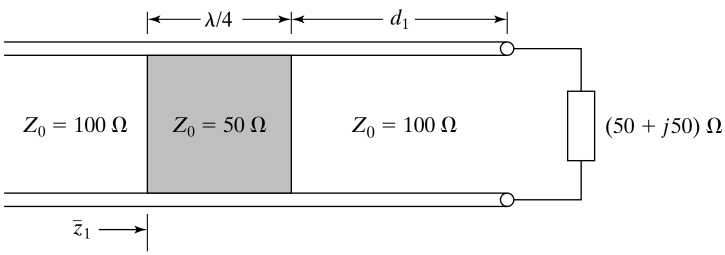

FIGURE 7.55

For Problem P7.22.

P7.23. Alternated-line transformer matching arrangement. Figure 7.56 shows an arrangement, known as the alternated-line transformer, for achieving a matched interconnection between two lines of different characteristic impedances Z_{01} and Z_{02}. It consists of two sections of the same characteristic impedances as those of the lines to be matched but alternated, as shown in the figure. The electrical lengths of the two sections are equal. Show that to achieve a match, the required

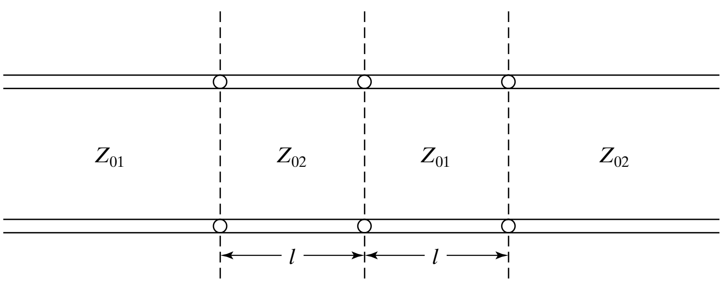

FIGURE 7.56

For Problem P7.23.

electrical length of each section is

$$\frac{l}{\lambda} = \frac{1}{2\pi} \tan^{-1} \sqrt{\frac{n}{n^2 + n + 1}}$$

where $n = Z_{02}/Z_{01}$.

P7.24. Resolving the problem of nonexistence of solution in double-stub matching technique. We learned that in the double-stub matching technique, a solution does not exist for b_2 if $g' > 1/\sin^2 \beta d_{12}$—see (7.52) and associated discussion. Show that one means of resolving this problem is by increasing d_1 by $\lambda/4$.

P7.25. Hybrid matching system using parallel and series stubs. Figure 7.57 shows a hybrid arrangement of a parallel short-circuited stub and a series short-circuited stub connected at a fixed distance d_1 from the load in order to achieve a match between the line and the load. With reference to the notation shown in the figure, show that in order to achieve a match, the required input reactance of the series stub x_2 is given by

$$x_2 = \sqrt{\frac{1 - g'}{g'}}$$

and the required input susceptance of the parallel stub b_1 is given by

$$b_1 = \frac{x_2}{1 + x_2^2} - b'$$

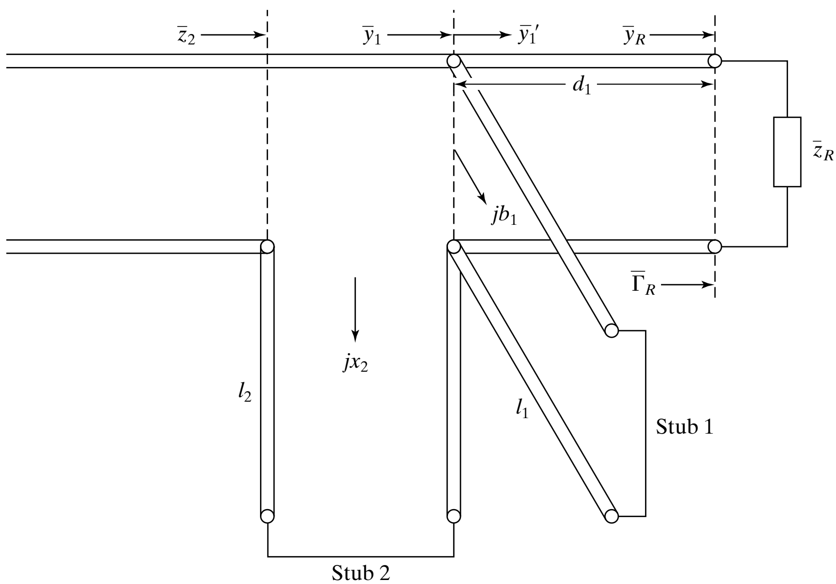

FIGURE 7.57

For Problem P7.25.

where $(g' + jb')$ is equal to $\bar{y}'_1$. Discuss the condition for which a solution does not exist for a fixed value of d_1, and a remedy to get around the problem.

P7.26. Finding the bandwidth of a quarter-wave transformer matched system. In the arrangement shown in Fig. 7.58, a quarter-wave transformer is employed to eliminate reflections of uniform plane waves of frequency 2500 MHz incident normally from the free-space side. **(a)** Find analytically the bandwidth between frequencies on either side of 2500 MHz at which the SWR in free space is 2.0. **(b)** What is the maximum SWR in free space as the frequency is varied on either side of 2500 MHz?

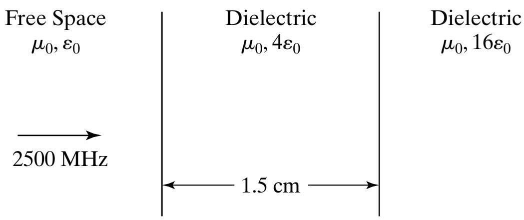

FIGURE 7.58
For Problem P7.26.

Section 7.4

P7.27. Property of the transformation forming the basis for the Smith chart. The transformation

$$\bar{\Gamma} = \frac{\bar{z} - 1}{\bar{z} + 1}$$

which forms the basis for the construction of the Smith chart maps circles in the complex $\bar{z}$-plane onto circles in the complex $\bar{\Gamma}$-plane. For the circle in the $\bar{z}$-plane given by $(r - 2)^2 + x^2 = 1$, find the equation for the circle in the $\bar{\Gamma}$-plane. (*Hint:* Consider three points on the circle in the $\bar{z}$-plane, find the corresponding three points in the $\bar{\Gamma}$-plane, and then find the equation.)

P7.28. Property of the transformation forming the basis for the Smith chart. Using the inverse of the procedure suggested in Problem P7.27, find the equation of the circle in the $\bar{z}$-plane that maps onto the circle in the $\bar{\Gamma}$-plane given by $(\text{Re}\,\bar{\Gamma} - 0.25)^2 + (\text{Im}\,\bar{\Gamma})^2 = 0.0625$.

P7.29. Several basic procedures using the Smith chart. For a transmission line of characteristic impedance 50 Ω, terminated by a load impedance $(100 + j50)$ Ω, find the following quantities by using the Smith chart: **(a)** reflection coefficient at the load; **(b)** SWR on the line; **(c)** the distance of the first voltage minimum of the standing-wave pattern from the load; **(d)** the line impedance at $d = 0.15\lambda$; **(e)** the line admittance at $d = 0.15\lambda$; and **(f)** the location nearest to the load at which the real part of the line admittance is equal to the line characteristic admittance.

P7.30. Finding SWR by using the Smith chart for an arrangement involving several media. In the arrangement shown in Fig. 7.59, uniform plane waves of frequency 2500 MHz are incident normally from medium 1 onto the interface

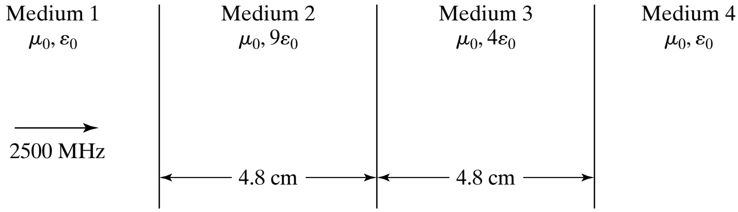

FIGURE 7.59

For Problem P7.30.

between medium 1 and medium 2. Using the Smith chart, find the SWR in: **(a)** medium 3; **(b)** medium 2; and **(c)** medium 1.

Section 7.5

P7.31. Finding load impedance from standing-wave measurements by using the Smith chart. Standing-wave measurements on a line of characteristic impedance 60 Ω indicate an SWR of 4.0 and a voltage minimum at a distance of 0.4λ from the load. Determine the value of the load impedance by using the Smith chart.

P7.32. Solution of a single-stub matching problem by using the Smith chart. A transmission line of characteristic impedance 60 Ω is terminated by a certain load impedance. It is found that the SWR on the line is equal to 4.0 and that the first voltage minimum of the standing-wave pattern is located to be at 0.2λ from the load. Using the Smith chart, determine the location nearest to the load and the length of a short-circuited stub of characteristic impedance 60 Ω connected in parallel with the line required to achieve a match between the line and the load.

P7.33. Solution of a double-stub matching problem by using the Smith chart. Standing-wave measurements on a line of characteristic impedance 60 Ω indicate SWR on the line to be 4.0 and the location of the first voltage minimum of the standing-wave pattern to be 0.1λ from the load. Assuming that $d_1 = 0.05\lambda$ and $d_{12} = 0.375\lambda$ and using the Smith chart, find the lengths of the two short-circuited stubs of characteristic impedance 60 Ω required to achieve a match between the line and the load.

P7.34. Limits for the nonexistence of a solution in double-stub matching technique. It is proposed to match a transmission line of characteristic impedance 100 Ω to a load impedance $(20 - j100)$ Ω by using a double-stub arrangement with spacing between stubs, d_{12}, equal to $5\lambda/8$. Determine the forbidden range of values of d_1 within the first half-wavelength to achieve the match using the Smith chart.

P7.35. Minimizing the SWR in a line with a quarter-wave section by using the Smith chart. Solve Problem P7.22 by using the Smith chart.

P7.36. Solution of matching with two quarter-wave sections by using the Smith chart. In the system shown in Fig. 7.60, two line sections, each of length $\lambda/4$ and characteristic impedance 50 Ω, are employed. By using the Smith chart, find the locations of the two $\lambda/4$ sections, that is, the values of l_1 and l_2 to achieve a match between the 100-Ω line and the load. Use the notation shown in the figure.

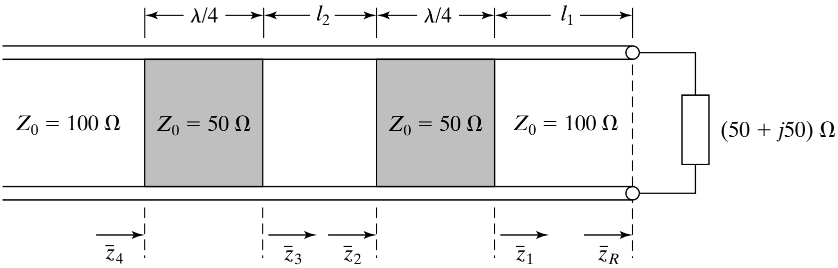

FIGURE 7.60
For Problem P7.36.

P7.37. Bandwidth of a quarter-wave transformer matched system by using the Smith chart. Solve Problem P7.26 by using the Smith chart.

P7.38. Solution of alternated-line transformer matching problem by using the Smith chart. Consider $Z_{01} = 50\ \Omega$ and $Z_{02} = 100\ \Omega$ for the alternated-line transformer arrangement of Problem P7.23. By using the Smith chart, obtain the minimum value of l/λ for achieving the match and show that it agrees with the solution given in Problem P7.23.

P7.39. Solution of hybrid parallel-series matching problem by using the Smith chart. For the hybrid parallel-series stub matching arrangement of Problem P7.25, illustrate the solution with the use of the Smith chart by considering a line of characteristic impedance 50 Ω terminated by a load of $(40 + j40)\ \Omega$. Assume $d_1 = \lambda/8$ and the characteristic impedance of the stubs to be 50 Ω.

P7.40. Investigation of maximum power transfer achievement problem by using the Smith chart. In the system shown in Fig. 7.61, it is desired to transfer the maximum possible power from the source to the load. By using the Smith chart, find, if possible, the location and the length of a short-circuited stub of characteristic impedance 100 Ω connected in parallel with the line that will enable this to be achieved.

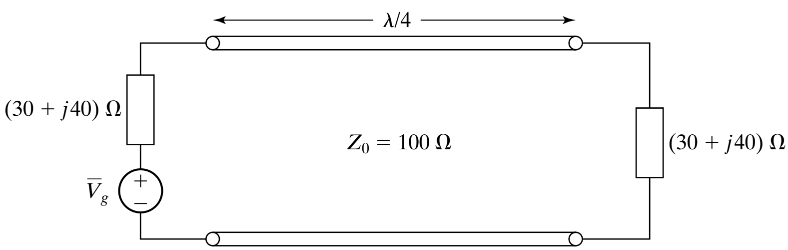

FIGURE 7.61
For Problem P7.40.

Section 7.6

P7.41. Computation of propagation parameters from line parameters for a lossy line. For a lossy line having the parameters $\mathcal{R} = 0.03\ \Omega/\text{m}$, $\mathcal{L} = 1.0\ \mu\text{H/m}$, $\mathcal{G} = 3 \times 10^{-9}$ S/m, and $\mathcal{C} = 50$ pF/m, compute the values of $\bar{Z}_0$ and $\bar{\gamma}$ for $f = 10$ kHz.

P7.42. Propagation parameters for a lossy line from input impedance measurements. The input impedance of a lossy line of length 50 m is measured at a frequency of 100 MHz for two cases: with the output short-circuited, it is $(10 + j49)\ \Omega$, and with the output open circuited, it is $(10 - j49)\ \Omega$. Find: **(a)** the characteristic impedance of the line; **(b)** the attenuation constant of the line; and **(c)** the phase velocity in the line, assuming its approximate value to be 1.75×10^8 m/s.

P7.43. Computation of power flow and power dissipated for a lossy line. For the lossy transmission-line system shown in Fig. 7.62, find: **(a)** the time-average power flow at the input end of the line; **(b)** the time-average power delivered to the load; and **(c)** the time-average power dissipated in the line.

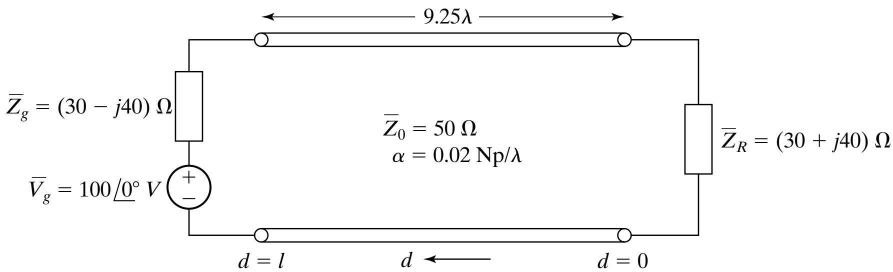

FIGURE 7.62
For Problem P7.43.

P7.44. An arrangement for eliminating reflections from a perfect conductor. In the arrangement shown in Fig. 7.63, uniform plane waves are incident normally onto a coating of good conductor material of conductivity σ and thickness l on a perfect dielectric slab of thickness $\lambda/4$ and backed by a perfect conductor. Show that no reflection occurs from the coating if $|\overline{\gamma}_c l| \ll 1$ and $\sigma = 1/\eta_0 l$, where $\overline{\gamma}_c$ is the propagation constant in the good conductor material.

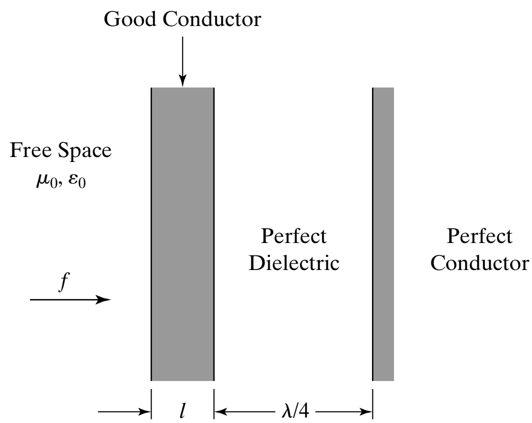

FIGURE 7.63
For Problem P7.44.

Section 7.7

P7.45. Energy storage and dissipation in a distortionless line. For a (+) wave alone or a (−) wave alone on a distortionless transmission line, show that the energy is stored equally in the inductance and capacitance of the line, and that the energy is dissipated equally in the resistance and conductance of the line.

P7.46. Pulse propagation along a lossy, but distortionless, transmission-line system. For the distortionless line system of Fig. 7.34, assume that the source voltage V_g is the triangular pulse of duration 1 μs shown in Fig. 7.64, instead of the rectangular pulse of duration 0.1 μs shown in Fig. 7.34. Find and sketch **(a)** the voltage V_R across the load resistor versus t, **(b)** the line voltage versus z for $t = 0.5\ \mu$s, and **(c)** the line voltage versus z for $t = 1.5\ \mu$s.

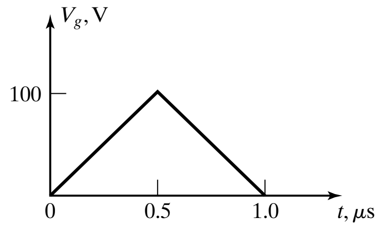

FIGURE 7.64
For Problem P7.46.

P7.47. Maximum value of current and its timing on the noninductive, leakage-free cable.

Show that the time-variation of the current on the noninductive, leakage-free cable, given by (7.93), is characterized by a maximum value of $\sqrt{\dfrac{2}{\pi e}}\dfrac{V_0}{\mathcal{R}z}$ at $t = \dfrac{1}{2}\mathcal{R}\mathcal{C}z^2$.

P7.48. Diffusion of fields in a highly conducting medium. Show that the time-domain behavior of electromagnetic fields in a highly conducting medium (displacement current density negligible) is characterized by diffusion. Consider for simplicity the case of $\mathbf{E} = E_x(z, t)\mathbf{a}_x$ and $\mathbf{H} = H_y(z, t)\mathbf{a}_y$.

REVIEW PROBLEMS

R7.1. Short-circuited line excited by a nonsinusiodal periodic source. In the system shown in Fig. 7.65(a), the source current is periodic, as shown in Fig. 7.65(b). Find the rms value of the current through the short-circuit.

R7.2. A transmission-line resonant system. **(a)** For the system shown in Fig. 7.66, find the value of L for which the system is resonant at $f = 10^9$ Hz. **(b)** What is the next resonant frequency of the system, greater than 10^9 Hz? **(c)** If L is increased by a small amount δ%, by what percent should l be changed such that the system remains resonant at $f = 10^9$ Hz? **(d)** Repeat part (c) for the resonant frequency found in part (b).

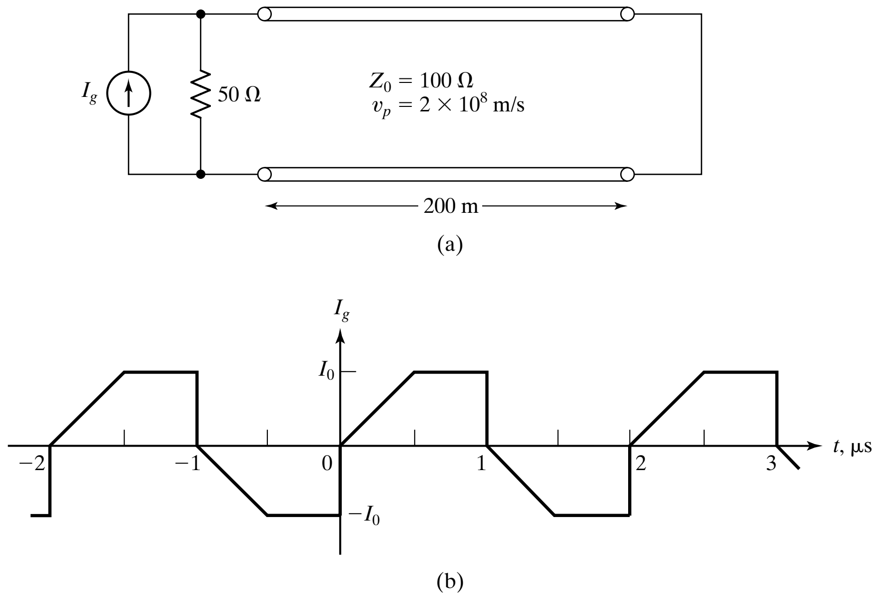

FIGURE 7.65
For Problem R7.1.

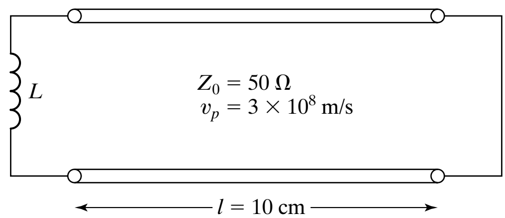

FIGURE 7.66
For Problem R7.2.

R7.3. Uniform plane-wave reflection and transmission for three media in cascade. In the arrangement shown in Fig. 7.67, a uniform plane wave having the electric field

$$\mathbf{E}_i = E_0 \cos(45\pi \times 10^8 t - 15\pi z) \cos(15\pi \times 10^8 t - 5\pi z)\,\mathbf{a}_x \text{ V/m}$$

is incident on the interface at $z = 0$. Find the fraction of the incident time-average power reflected back into medium 1 and the fraction transmitted into medium 3.

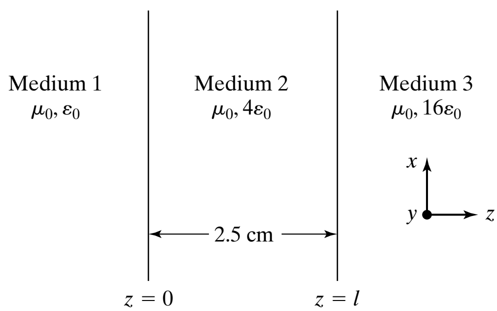

FIGURE 7.67
For Problem R7.3.

R7.4. Power flow in a system involving three transmission lines. In the system shown in Fig. 7.68, find **(a)** the time-average power delivered to the resistor R_2 and **(b)** the time-average power delivered to the resistor R_3.

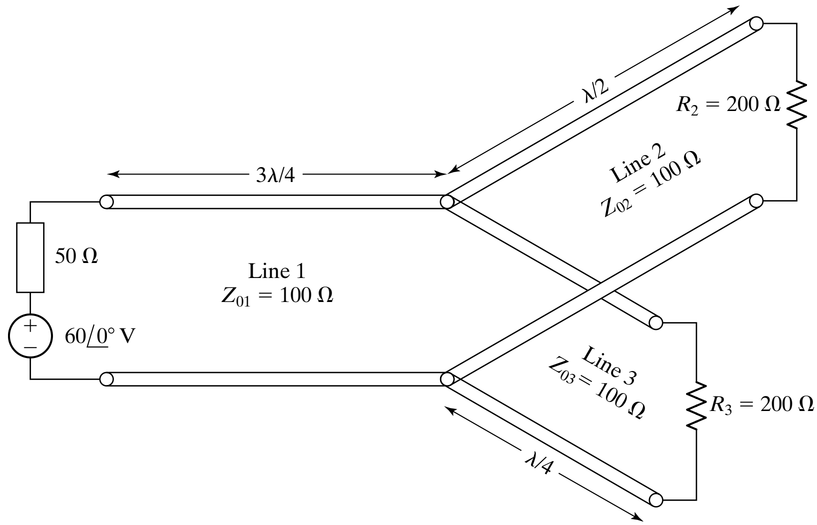

FIGURE 7.68
For Problem R7.4.

R7.5. Application of maximum power transfer theorem for a transmission-line system. In the system shown in Fig. 7.69, find the value of the reactance X and the minimum value of the line length l for which the time-average power delivered to the resistor R_L is a maximum. What are the values of this power and the SWR on the line?

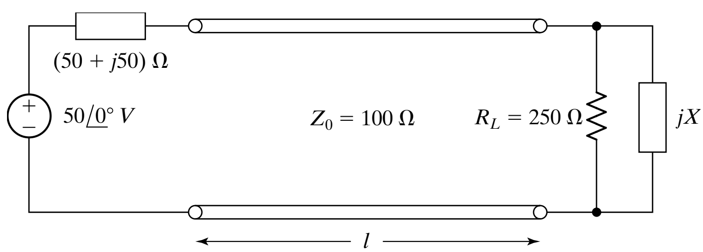

FIGURE 7.69
For Problem R7.5.

R7.6. Finding load impedance and minimizing SWR by using the Smith chart. Standing-wave measurements on a line of characteristic impedance 100 Ω indicate an SWR of 2.80 and a voltage minimum at a distance of 0.1λ from the load. By using the Smith chart, determine the value of the load impedance. It is desired to minimize the SWR on the line by connecting a line section at the load. By using the Smith chart, find the minimum required length of the line section and the minimum achievable SWR for each of the following line sections: **(a)** a short-circuited stub of characteristic impedance 100 Ω connected in parallel with the load; **(b)** a short-circuited stub of characteristic impedance 100 Ω connected in series with the load; and **(c)** a line section of characteristic impedance 50 Ω inserted between the main line and the load.

R7.7. Nonexistence of solution in the hybrid parallel-series stub matching technique. It is desired to achieve a match between a line of characteristic impedance 100 Ω to a load of $(120 - j160)\ \Omega$ by employing the hybrid parallel-series stub matching arrangement of Problem P7.25. Determine the forbidden range of values of d within the first half-wavelength from the load.

R7.8. Energy dissipation in a distortionless line system. For the distortionless transmission line system of Fig. 7.34, find the energy dissipated in the line from $t = 0$ to $t = \infty$. Repeat the solution for the case of a short circuit for the load and a value of 100 Ω for the resistor in series with the voltage source.