CHAPTER 4

Fields and Waves In Material Media

Thus far in our study of fields and waves, we have considered the medium to be free space. In this chapter, we extend our study to material media. Materials contain charged particles that respond to applied electric and magnetic fields to produce secondary fields. We will learn that there are three basic phenomena resulting from the interaction of the charged particles with the electric and magnetic fields. These are conduction, polarization, and magnetization. Although a given material may exhibit all three properties, it is classified as a conductor (including semiconductor), a dielectric, or a magnetic material, depending on whether conduction, polarization, or magnetization is the predominant phenomenon. Thus, we introduce these materials one at a time and develop a set of constitutive relations for the material media that enable us to avoid the necessity of explicitly taking into account the interaction of the charged particles with the fields.

We shall then use the constitutive relations together with Maxwell's equations to extend our study of uniform plane waves to material media, first for the general case and then for several special cases. To study problems involving two or more different media, we shall then derive *boundary conditions*, which are a set of conditions for the fields to satisfy at the boundaries between the different media. Finally, we shall use the boundary conditions to study the reflection and transmission of uniform plane waves at plane boundaries.

4.1 CONDUCTORS AND SEMICONDUCTORS

Conduction

Depending on their response to an applied electric field, materials may be classified as conductors, semiconductors, or dielectrics. According to the classical model, an atom consists of a tightly bound, positively charged nucleus surrounded by a diffuse electron could having an equal and opposite charge to the nucleus. While the electrons for the most part are less tightly bound, the majority of them are associated with the nucleus and are known as *bound* electrons. These bound electrons can be displaced, but not removed from the influence of the nucleus upon the application of an electric field. Not taking part in this bonding mechanism are the *free*, or *conduction*, electrons. These electrons are constantly under thermal agitation, being released from the parent atom at one point and recaptured at another point. In the absence of an applied electric field, their motion is completely random; that is, the average thermal velocity on a macroscopic scale is zero, so that there is no net current and the electron cloud maintains a fixed position. When an electric field is applied, an additional velocity is superimposed on the random velocities, thereby causing a *drift* of the average position of the electrons along the direction opposite to that of the electric field. This process is known as *conduction*. In certain materials, a large number of electrons may take part in this process. These materials are known as *conductors*. In certain other materials, only very few or a negligible number of electrons may participate in conduction. These materials are known as *dielectrics*, or insulators. A class of materials for which conduction occurs not only by electrons but also by another type of carriers known as *holes*—vacancies created by detachment of electrons due to breaking of covalent bonds with other atoms—is intermediate to that of conductors and dielectrics. These materials are called *semiconductors*.

The quantum theory describes the motion of the current carriers in terms of energy levels. According to this theory, the electrons in an atom can have associated with them only certain discrete values of energy. When a large number of atoms are packed together, as in a crystalline solid, each energy level in the individual atom splits into a number of levels with slightly different energies, with the degree of splitting governed by the interatomic spacing, thereby giving rise to allowed bands of energy levels that may be widely separated, may be close together, or may even overlap. Four possible energy band diagrams are shown in Fig. 4.1, in which a forbidden band consists of energy levels that no electron in any atom of the solid can occupy. For case (a), the lower allowed band is only partially filled at the temperature of absolute zero. At higher temperatures, the

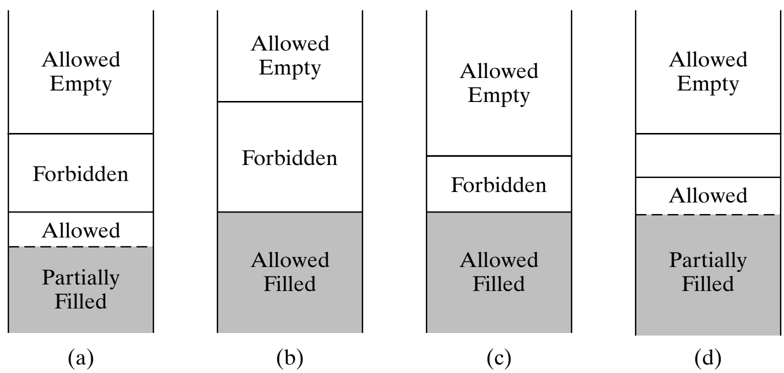


FIGURE 4.1

Energy band diagrams for different cases: (a) and (d) conductor; (b) dielectric; and (c) semiconductor.

electron population in the band spreads out somewhat, but only very few electrons reach higher energy levels. Thus, since there are many unfilled levels in the same band, it is possible to increase the energy of the system by moving the electrons to these unoccupied levels very easily by the application of an electric field, thereby resulting in drift of the electrons. The material is then classified as a conductor. For cases (b) and (c), the lower band is completely filled, whereas the next-higher band is completely empty at the temperature of absolute zero. If the width of the forbidden band is very large as in (b), the situation at normal temperatures is essentially the same as at absolute zero, and, hence, there are no neighboring empty energy levels for the electrons to move. The only way for conduction to take place is for the electrons in the filled band to get excited and move to the next higher band. But this is very difficult to achieve with reasonable electric fields, and the material is then classified as a dielectric. Only by supplying a very large amount of energy can an electron be excited to move from the lower band to the higher band, where it has neighboring empty levels available for causing conduction. The dielectric is said to break down under such conditions. If, on the other hand, the width of the forbidden band in which the Fermi level lies is not too large, as in (c), some of the electrons in the lower band move into the upper band at normal temperatures, so that conduction can take place under the influence of an electric field, not only in the upper band, but also in the lower band because of the vacancies (holes) left by the electrons that moved into the upper band. The material is then classified as a semiconductor. A semiconductor crystal in pure form is known as an intrinsic semiconductor. The properties of an intrinsic crystal can be altered by introducing impurities into it. The crystal is then said to be an extrinsic semiconductor. For case (d), two allowed bands overlap; one or both of the bands is only partially filled and the situation corresponds to a conductor.

In the foregoing discussion, we classified materials on the basis of their ability to permit conduction of electrons under the application of an external electric field. For conductors, we are interested in knowing about the relationship between the *drift velocity* of the electrons and the applied electric field, since the predominant process is conduction. But for collisions with the atomic lattice, the electric field continuously accelerates the electrons in the direction opposite to it as they move about at random. Collisions with the atomic lattice, however, provide the frictional mechanism by means of which the electrons lose some of the momentum gained between collisions. The net effect is as though the electrons drift with an average drift velocity $\mathbf{v}_d$, under the influence of the force exerted by the applied electric field and an opposing force due to the frictional mechanism. This opposing force is proportional to the momentum of the electron and inversely proportional to the average time τ between collisions. Thus, the equation of motion of an electron is given by

$$m\frac{d\mathbf{v}_d}{dt} = e\mathbf{E} - \frac{m\mathbf{v}_d}{\tau} \tag{4.1}$$

where e and m are the charge and mass of an electron.

Rearranging (4.1), we have

$$m\frac{d\mathbf{v}_d}{dt} + \frac{m}{\tau}\mathbf{v}_d = e\mathbf{E} \tag{4.2}$$

For the sudden application of a constant electric field $\mathbf{E}_0$ at $t = 0$, the solution for (4.2) is given by

$$\mathbf{v}_d = \frac{e\tau}{m}\mathbf{E}_0 - \frac{e\tau}{m}\mathbf{E}_0 e^{-t/\tau} \tag{4.3}$$

where we have evaluated the arbitrary constant of integration by using the initial condition that $\mathbf{v}_d = \mathbf{0}$ at $t = 0$. The values of τ for typical conductors such as copper are of the order of 10^{-14} s, so that the exponential term on the right side of (4.3) decays to a negligible value in a time much shorter than that of practical interest. Thus, neglecting this term, we have

$$\mathbf{v}_d = \frac{e\tau}{m}\mathbf{E}_0 \tag{4.4}$$

and the drift velocity is proportional in magnitude and opposite in direction to the applied electric field, since the value of e is negative.

In fact, since we can represent a time-varying field as a superposition of step functions starting at appropriate times, the exponential term in (4.3) may be neglected as long as the electric field varies slowly compared to τ. For fields varying sinusoidally with time, this means that as long as the period T of the sinusoidal variation is several times the value of τ, or the radian frequency $\omega \ll 2\pi/\tau$, the drift velocity follows the variations in the electric field. Since $1/\tau \approx 10^{14}$, this condition is satisfied even at frequencies up to several hundred gigahertz (a gigahertz is 10^9 Hz). Thus, for all practical purposes, we can assume that

$$\mathbf{v}_d = \frac{e\tau}{m}\mathbf{E} \tag{4.5}$$

Mobility

Now, we define the *mobility*, μ_e, of the electron as the ratio of the magnitudes of the drift velocity and the applied electric field. Then we have

$$\boxed{\mu_e = \frac{|\mathbf{v}_d|}{|\mathbf{E}|} = \frac{|e|\tau}{m}} \tag{4.6}$$

and

$$\mathbf{v}_d = -\mu_e\mathbf{E} \quad \text{for electrons} \tag{4.7a}$$

For values of τ typically of the order of 10^{-14} s, we note by substituting for $|e|$ and m on the right side of (4.6) that the electron mobilities are of the order of 10^{-3} C-s/kg. Alternative units for the mobility are square meters per volt-second. In semiconductors, conduction is due not only to the movement of electrons, but also to the movement of holes. We can define the mobility μ_h of a hole similarly to μ_e as the ratio of the drift velocity of the hole to the applied electric field. Thus, we have

$$\mathbf{v}_d = \mu_h \mathbf{E} \quad \text{for holes} \tag{4.7b}$$

Note from (4.7b) that conduction of a hole takes place along the direction of the applied electric field, since a hole is a vacancy created by the removal of an electron and, hence, a hole movement is equivalent to the movement of a positive charge of value equal to the magnitude of the charge of an electron. In general, the mobility of holes is lower than the mobility of electrons for a particular semiconductor. For example, for silicon, the values of μ_e and μ_h are 0.135 m^2/V-s and 0.048 m^2/V-s, respectively. Semiconductors are denoted *n*-type or *p*-type, depending on whether the conduction is predominantly due to the movement of electrons or holes.

Conduction current

The drift of electrons in a conductor and that of electrons and holes in a semiconductor is equivalent to a current flow. This current is known as the *conduction current*. The conduction current density may be obtained in the following manner. If there are N_e free electrons per cubic meter of the material, then the amount of charge ΔQ passing through an infinitesimal area ΔS normal to the drift velocity at a point in the material in a time Δt is given by

$$\Delta Q = N_e e(\Delta S)(v_d \, \Delta t) \tag{4.8}$$

The current ΔI flowing across ΔS is given by

$$\Delta I = \frac{|\Delta Q|}{\Delta t} = N_e |e| v_d \, \Delta S \tag{4.9}$$

The magnitude of the current density at the point is the ratio of ΔI to ΔS in the limit ΔS tends to zero, and the direction is opposite to that of $\mathbf{v}_d$. Thus, the conduction current density $\mathbf{J}_c$ resulting from the drift of electrons in the conductor is given by

$$\mathbf{J}_c = -N_e |e| \mathbf{v}_d \tag{4.10}$$

Substituting for $\mathbf{v}_d$ from (4.7a), we have

$$\mathbf{J}_c = \mu_e N_e |e| \mathbf{E} \tag{4.11}$$

Conductivity

Defining a quantity σ as

$$\boxed{\sigma = \mu_e N_e |e|} \tag{4.12}$$

we obtain the simple and important relationship between $\mathbf{J}_c$ and $\mathbf{E}$:

$$\boxed{\mathbf{J}_c = \sigma \mathbf{E}} \tag{4.13}$$

The quantity σ is known as the electrical *conductivity* of the material, and (4.13) is known as Ohm's law valid at a point. We shall show later that the well-known

Ohm's law in circuit theory follows from it. In a semiconductor, the current density is the sum of the contributions due to the drifts of electrons and holes. If the densities of holes and electrons are N_h and N_e, respectively, the conduction current density is given by

$$\boxed{\mathbf{J}_c = (\mu_h N_h|e| + \mu_e N_e|e|)\mathbf{E}} \tag{4.14}$$

Thus, the conductivity of a semiconducting material is given by

$$\boxed{\sigma = \mu_h N_h|e| + \mu_e N_e|e|} \tag{4.15a}$$

For an intrinsic semiconductor, $N_h = N_e$, so that (4.15a) reduces to

$$\boxed{\sigma = (\mu_h + \mu_e)N_e|e|} \tag{4.15b}$$

The units of conductivity are (meter^2/volt-second)(coulomb/meter^3) or ampere/volt-meter, also commonly known as siemens per meter (S/m), where a siemen is an ampere per volt. The ranges of conductivities for conductors, semiconductors, and dielectrics are shown in Fig. 4.2. Values of conductivities for a few materials are listed in Table 4.1. The constant values of conductivities do not imply that the conduction current density is proportional to the applied electric field intensity for all values of current density and field intensity. However, the range of current densities for which the material is linear, that is, for which the conductivity is a constant, is very large for conductors.

In considering electromagnetic wave propagation in conducting media, the conduction current density given by (4.13) must be employed for the current density term on the right side of Ampere's circuital law. Thus, Maxwell's curl equation for $\mathbf{H}$ for a conducting medium is given by

$$\nabla \times \mathbf{H} = \mathbf{J}_c + \frac{\partial \mathbf{D}}{\partial t} = \sigma \mathbf{E} + \frac{\partial \mathbf{D}}{\partial t} \tag{4.16}$$

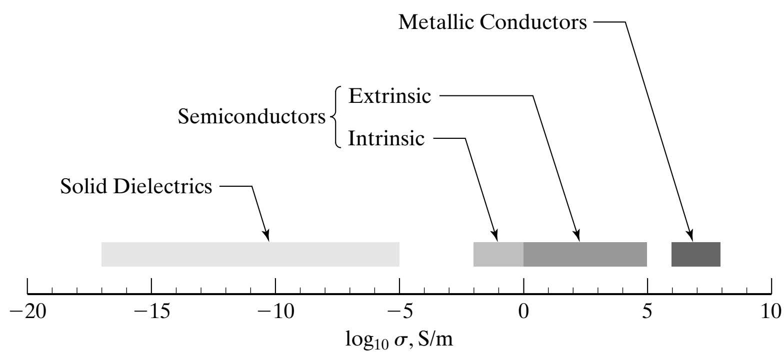


FIGURE 4.2

Ranges of conductivities for conductors, semiconductors, and dielectrics.

Table 4.1 Conductivities of Some Materials

Material	Conductivity (S/m)	Material	Conductivity (S/m)
Silver	6.1×10^7	Seawater	4
Copper	5.8×10^7	Intrinsic germanium	2.2
Gold	4.1×10^7	Intrinsic silicon	1.6×10^{-3}
Aluminum	3.5×10^7	Fresh water	10^{-3}
Tungsten	1.8×10^7	Distilled water	2×10^{-4}
Brass	1.5×10^7	Dry earth	10^{-5}
Solder	7.0×10^6	Bakelite	10^{-9}
Lead	4.8×10^6	Glass	$10^{-10} - 10^{-14}$
Constantin	2.0×10^6	Mica	$10^{-11} - 10^{-15}$
Mercury	1.0×10^6	Fused quartz	0.4×10^{-17}

We shall use this equation in Sec. 4.4 to obtain the solution for sinusoidally time-varying uniform plane waves in a material medium.

Conductor in a static electric field

Let us now consider a conductor placed in a static electric field, as shown in Fig. 4.3(a). The free electrons in the conductor move opposite to the direction lines of the electric field. If there is a way in which the flow of electrons can be continued to form a closed circuit, then a continuous flow of current takes place. Since the conductor is bounded by free space, the electrons are held at the boundary from moving further. Thus, a negative surface charge forms on the boundary, accompanied by an equal amount of positive surface charge, as shown in Fig. 4.3(b), since the conductor as a whole is neutral. The surface charge distribution formed in this manner produces a secondary electric field which, together with the applied electric field, makes the field inside the conductor zero. We shall illustrate the computation of the surface charge densities by means of a simple example.

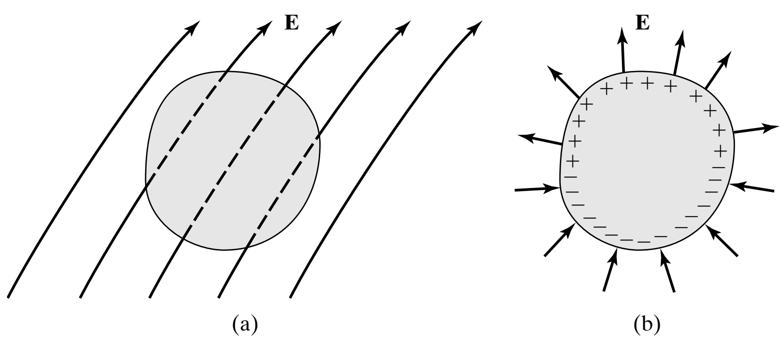


FIGURE 4.3

For illustrating the surface charge formation at the boundary of a conductor placed in a static electric field.

Example 4.1 Plane conducting slab in a uniform static electric field

Let us consider an infinite plane conducting slab of thickness d occupying the region between $z = 0$ and $z = d$ and in a uniform electric field $\mathbf{E} = E_0\mathbf{a}_z$ produced by two infinite plane sheets of equal and opposite uniform charge densities on either side of the slab, as shown in Fig. 4.4(a). We wish to find the charge densities induced on the surfaces of the slab.

Since the applied electric field is uniform and is directed along the z-direction, a negative charge of uniform density forms on the surface $z = 0$ due to the accumulation of free electrons at that surface. A positive charge of equal and opposite uniform density forms on the surface $z = d$ due to a deficiency of electrons at that surface. Let these surface charge densities be $-\rho_{S0}$ and ρ_{S0}, respectively. To satisfy the property that the field in the interior of the conductor is zero, the secondary field produced by the surface charges must be equal and opposite to the applied field; that is, it must be equal to $-E_0\mathbf{a}_z$. Now, each surface charge produces a field intensity directed normally from it and having a magnitude $1/2\varepsilon_0$ times the charge density so that the field due to the two surface charges together is equal to $-(\rho_{S0}/\varepsilon_0)\mathbf{a}_z$ inside the conductor and zero outside the conductor, as shown in Fig. 4.4(b). Thus, for zero field inside the conductor,

$$-\frac{\rho_{S0}}{\varepsilon_0}\mathbf{a}_z = -E_0\mathbf{a}_z$$

or

$$\rho_{S0} = \varepsilon_0 E_0$$

The field outside the conductor remains the same as the applied field since the secondary field in that region due to the surface charges is zero. The induced surface charge

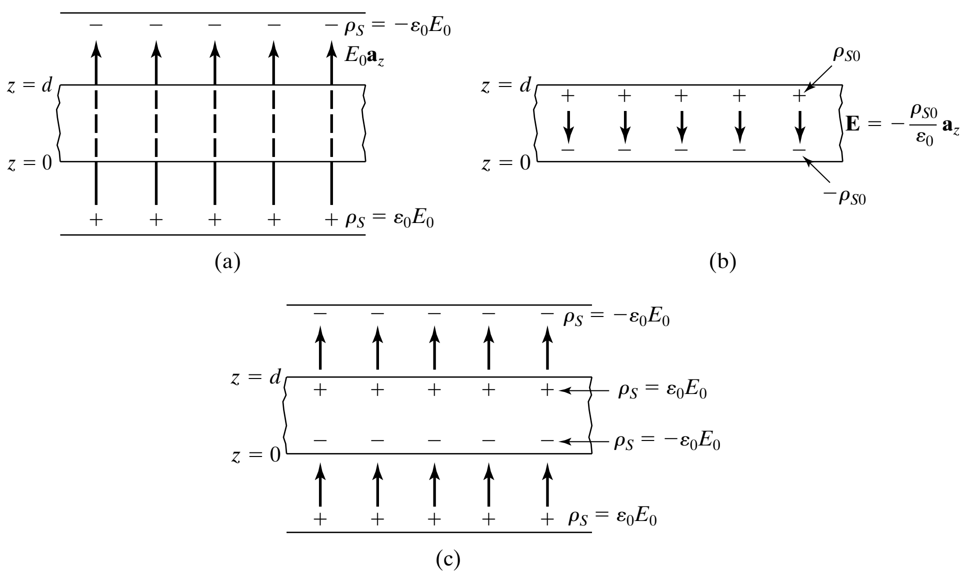


FIGURE 4.4

(a) Infinite plane slab conductor in a uniform applied field. (b) Induced surface charge at the boundaries of the conductor and the secondary field. (c) Sum of the applied and the secondary fields.

distribution and the fields inside and outside the conductor are shown in Fig. 4.4(c). In the general case, the induced surface charge produces a secondary field outside the conductor also, thereby changing the applied field.

Ohm's law, resistance

Returning now to (4.13), we shall show that the well-known Ohm's law in circuit theory follows from it. To do this, let us consider a bar of conducting material of conductivity σ, length l, and uniform cross-sectional area A, between the ends of which a voltage V is applied, as shown in Fig. 4.5. The voltage sets up an electric field directed along the length of the conductor, thereby giving rise to conduction current. Assuming, for simplicity, uniformity of the electric field, the voltage between the two ends of the conductor is given by the electric field intensity times the length of the conductor, that is,

$$V = El \tag{4.17}$$

Then from (4.13) and (4.17), the conduction current density magnitude is given by

$$J_c = \sigma E = \frac{\sigma V}{l} \tag{4.18}$$

Assuming uniformity of the field and hence of the conduction current density in the cross-sectional area of the conductor, we then obtain the conduction current to be

$$I = J_c A = \frac{\sigma A}{l} V \tag{4.19}$$

Upon rearrangement, we get

$$V = I \frac{l}{\sigma A} \tag{4.20}$$

which is in the form of the familiar Ohm's law,

$$\boxed{V = IR} \tag{4.21}$$

From (4.20), the resistance R of the conducting bar can now be identified as

$$\boxed{R = \frac{l}{\sigma A}} \tag{4.22}$$

the units of R being ohms.

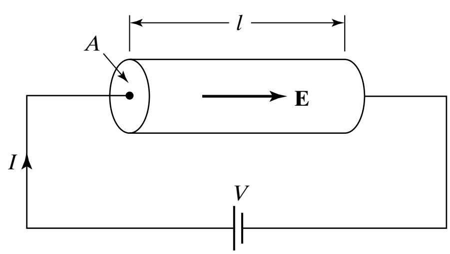


FIGURE 4.5

For the derivation of Ohm's law in circuit theory.

Hall effect

We shall conclude this section with a discussion of the Hall effect, an important phenomenon employed in the determination of charge densities in conducting and semiconducting materials, as well as in other techniques such as the measurement of fluid flow using electromagnetic flow meters. Let us consider the p-type semiconducting material in the form of a rectangular bar shown in Fig. 4.6, in which holes drift in the x-direction with a velocity $\mathbf{v} = v_x\mathbf{a}_x$ due to an applied voltage between the two ends of the bar. If a magnetic field $\mathbf{B} = B_z\mathbf{a}_z$ is applied in a perpendicular direction, then the drifting holes will experience a magnetic force $\mathbf{F}_m$ that deflects them in the $\mathbf{a}_x \times \mathbf{a}_z$ or $-\mathbf{a}_y$-direction. This deflection of holes toward the $-y$-direction establishes an electric field $\mathbf{E}_H = E_y\mathbf{a}_y$ in the material, resulting in the development of a voltage between the two sides of the bar. This phenomenon is known as the *Hall effect*, and the voltage developed is known as the *Hall voltage*. Were it not for the establishment of the Hall electric field, the holes would continually deflect toward the $-y$-direction as they drift in the x-direction. The Hall electric field exerts force $\mathbf{F}_H$ on the holes in the $+y$-direction, which in the steady-state balances exactly the magnetic force $\mathbf{F}_m$ in the $-y$-direction so that the net y-directed force is zero. According to the Lorentz force equation (1.89), the Hall electric field that achieves this balance is given by

$$\begin{aligned} q(\mathbf{E}_H + \mathbf{v} \times \mathbf{B}) &= q(E_y\mathbf{a}_y + v_x\mathbf{a}_x \times B_z\mathbf{a}_z) \\ &= q(E_y - v_xB_z)\mathbf{a}_y = \mathbf{0} \end{aligned} \tag{4.23}$$

or $E_y = v_xB_z$. By using this result, the hole density can be computed from a measurement of the Hall voltage for known values of the magnetic field B_z, the current I, and the cross-sectional dimensions of the bar. If the material is n-type instead of p-type, then the charge carriers are electrons, and $\mathbf{v}$ would be in the $-x$-direction. The deflection of the charge carriers will still be toward the $-y$-direction since the charge is negative. This results in an electric field in the

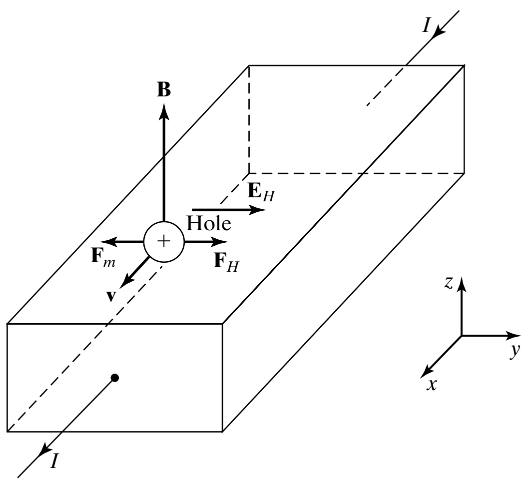


FIGURE 4.6

For illustrating the Hall effect phenomenon.

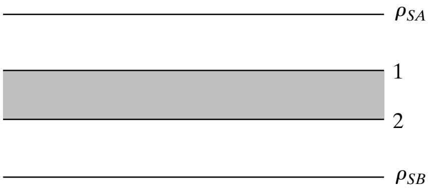


FIGURE 4.7
For Problem D4.2.

$-y$-direction and hence in a Hall voltage of opposite polarity to that in the case of the p-type material. Thus, the polarity of the Hall voltage can be used to determine if the charge carriers are holes or electrons.

K4.1. Conduction; Conduction current density; Conductivity; Ohm's law; Conductor in a static electric field; Resistance; Hall effect.

D4.1. Find the magnitude of the electric field intensity required to establish the flow of a conduction current of 0.1 A across an area of 1 cm^2 normal to the field for each of the following cases: **(a)** in copper; **(b)** in an intrinsic semiconductor material with electron and hole mobilities of 3600 cm^2/V-s and 1700 cm^2/V-s, respectively, and electron and hole densities of 2.5×10^{13} cm^{-3}; and **(c)** in a metallic wire of circular cross section of radius 1 mm, length 1 m, and resistance 1 ohm.

Ans. **(a)** 17.24 μV/m; **(b)** 471.1 V/m; **(c)** 3.14 mV/m.

D4.2. An infinite plane conducting slab lies between, and parallel to, two infinite plane sheets of charge of uniform surface charge densities ρ_{SA} and ρ_{SB}, as shown by the cross-sectional view in Fig. 4.7. Find the surface charge densities on the two surfaces of the slab: **(a)** ρ_{S1} and **(b)** ρ_{S2}.

Ans. **(a)** $\frac{1}{2}(\rho_{SB} - \rho_{SA})$; **(b)** $\frac{1}{2}(\rho_{SA} - \rho_{SB})$.

4.2 DIELECTRICS

Polarization, electric dipole

In the preceding section, we learned that conductors are characterized by an abundance of *conduction*, or *free*, electrons that give rise to conduction current under the influence of an applied electric field. In this section, we turn our attention to dielectric materials in which the *bound* electrons are predominant. Under the application of an external electric field, the bound electrons of an atom are displaced such that the centroid of the electron cloud is separated from the centroid of the nucleus. The atom is then said to be *polarized*, thereby creating an *electric dipole*, as shown in Fig. 4.8(a). This kind of polarization is called *electronic polarization*. The schematic representation of an electric dipole is shown in Fig. 4.8(b). The strength of the dipole is defined by the electric dipole moment $\mathbf{p}$ given by

$$\boxed{\mathbf{p} = Q\mathbf{d}} \tag{4.24}$$

where $\mathbf{d}$ is the vector displacement between the centroids of the positive and negative charges, each of magnitude Q coulombs.

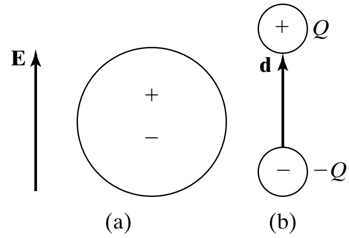


FIGURE 4.8

(a) Electric dipole. (b) Schematic representation of an electric dipole.

In certain dielectric materials, polarization may exist in the molecular structure of the material even under the application of no external electric field. The polarization of individual atoms and molecules, however, is randomly oriented, and hence the net polarization on a *macroscopic* scale is zero. The application of an external field results in torques acting on the *microscopic* dipoles, as shown in Fig. 4.9, to convert the initially random polarization into a partially coherent one along the field, on a macroscopic scale. This kind of polarization is known as *orientational polarization*. A third kind of polarization, known as *ionic polarization*, results from the separation of positive and negative ions in molecules formed by the transfer of electrons from one atom to another in the molecule. Certain materials exhibit permanent polarization, that is, polarization even in the absence of an applied electric field. Electrets, when allowed to solidify in the applied electric field, become permanently polarized, and ferroelectric materials exhibit spontaneous, permanent polarization.

On a macroscopic scale, we define a vector **P**, called the *polarization vector*, as the *electric dipole moment per unit volume*. Thus, if N denotes the number of molecules per unit volume of the material, then there are $N\,\Delta v$ molecules in a volume Δv and

$$\boxed{\mathbf{P} = \frac{1}{\Delta v}\sum_{j=1}^{N\,\Delta v}\mathbf{p}_j = N\mathbf{p}} \tag{4.25}$$

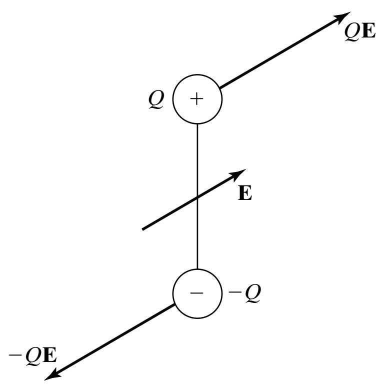


FIGURE 4.9

Torque acting on an electric dipole in an external electric field.

where $\mathbf{p}$ is the average dipole moment per molecule. The units of $\mathbf{P}$ are coulomb-meter/meter^3 or coulombs per square meter. It is found that for many dielectric materials, the polarization vector is related to the electric field $\mathbf{E}$ in the dielectric in the simple manner given by

$$\boxed{\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}} \tag{4.26}$$

where χ_e, a dimensionless parameter, is known as the *electric susceptibility*. The quantity χ_e is a measure of the ability of the material to become polarized and differs from one dielectric to another.

Dielectric in an electric field

When a dielectric material is placed in an electric field, the induced dipoles produce a secondary electric field such that the resultant field, that is, the sum of the originally applied field and the secondary field, and the polarization vector satisfy (4.26). We shall illustrate this by means of a simple example.

Example 4.2 Plane dielectric slab in a uniform static electric field

Let us consider an infinite plane dielectric slab of thickness d sandwiched between two infinite plane sheets of equal and opposite uniform charge densities ρ_{S0} and $-\rho_{S0}$ in the $z = 0$ and $z = d$ planes, respectively, as shown in Fig. 4.10(a). We wish to investigate the effect of polarization in the dielectric.

In the absence of the dielectric, the electric field between the sheets of charge is given by

$$\mathbf{E}_a = \frac{\rho_{S0}}{\varepsilon_0}\mathbf{a}_z$$

In the presence of the dielectric, this field acts as the applied electric field, inducing dipole moments in the dielectric with the negative charges separated from the positive charges and pulled away from the direction of the field. Since the electric field and the electric susceptibility are uniform, the density of the induced dipole moments, that is, the polarization vector $\mathbf{P}$, is uniform, as shown in Fig. 4.10(b). Such a distribution results in exact neutralization of all the charges except at the boundaries of the dielectric since, for each positive (or negative) charge not on the surface, there is the same amount of negative (or positive) charge associated with the dipole adjacent to it, thereby canceling its effect. Thus, the net result is the formation of a positive surface charge at the boundary $z = d$ and a negative surface charge at the boundary $z = 0$, as shown in Fig. 4.10(c). These surface charges are known as polarization surface charges since they are due to the polarization in the dielectric. In view of the uniform density of the dipole moments, the surface charge densities are uniform. Also, in the absence of a net charge in the interior of the dielectric, the surface charge densities must be equal in magnitude to preserve the charge neutrality of the dielectric.

Let us therefore denote the surface charge densities as

$$\rho_{pS} = \begin{cases} \rho_{pS0} & \text{for} \quad z = d \\ -\rho_{pS0} & \text{for} \quad z = 0 \end{cases}$$

where the subscript p in addition to the other subscripts stands for polarization. If we now consider a vertical column of infinitesimal rectangular cross-sectional area ΔS cut

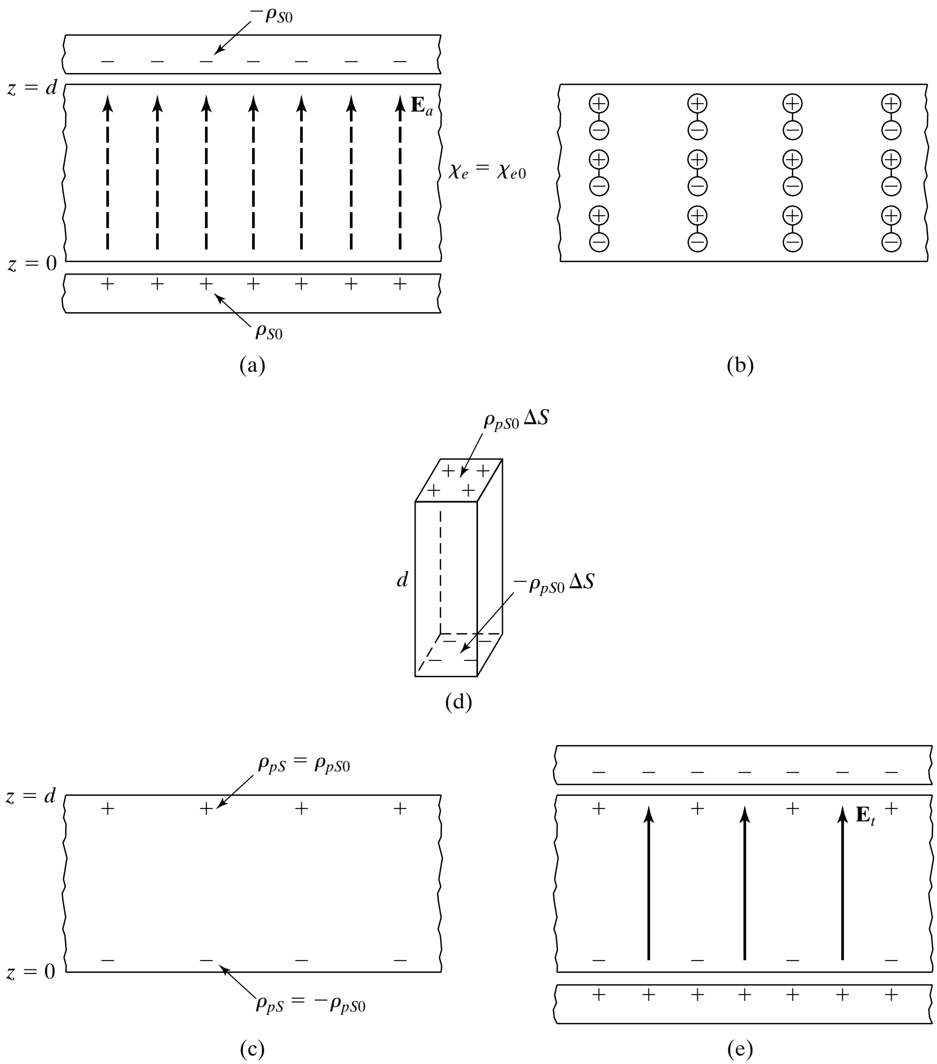


FIGURE 4.10

For investigating the effect of polarization induced in a dielectric material sandwiched between two infinite plane sheets of charge.

out from the dielectric, as shown in Fig. 4.10(d), the equal and opposite surface charges make the column appear as a dipole of moment $(\rho_{pS0}\,\Delta S)d\mathbf{a}_z$. On the other hand, writing

$$\mathbf{P} = P_0\mathbf{a}_z \tag{4.27}$$

where P_0 is a constant in view of the uniformity of the induced polarization, the dipole moment of the column is equal to $\mathbf{P}$ times the volume of the column, or $P_0(d\,\Delta S)\mathbf{a}_z$. Equating the dipole moments computed in the two different ways, we have

$$\rho_{pS0} = P_0$$

Thus, we have related the surface charge density to the magnitude of the polarization vector. Now, the surface charge distribution produces a secondary field $\mathbf{E}_s$ given by

$$\mathbf{E}_s = \begin{cases} -\dfrac{\rho_{pS0}}{\varepsilon_0}\mathbf{a}_z = -\dfrac{P_0}{\varepsilon_0}\mathbf{a}_z & \text{for} \quad 0 < z < d \\ \mathbf{0} & \text{otherwise} \end{cases}$$

Denoting the total field in the dielectric to be $\mathbf{E}_t$, we have

$$\mathbf{E}_t = \mathbf{E}_a + \mathbf{E}_s = \frac{\rho_{S0}}{\varepsilon_0}\mathbf{a}_z - \frac{P_0}{\varepsilon_0}\mathbf{a}_z \tag{4.28}$$

But from (4.26),

$$\mathbf{P} = \varepsilon_0 \chi_{e0} \mathbf{E}_t \tag{4.29}$$

Substituting (4.27) and (4.28) into (4.29), we obtain

$$P_0 = \chi_{e0}(\rho_{S0} - P_0)$$

or

$$P_0 = \frac{\chi_{e0}\rho_{S0}}{1 + \chi_{e0}} \tag{4.30}$$

Thus, the polarization surface charge densities are given by

$$\rho_{pS} = \begin{cases} \dfrac{\chi_{e0}\rho_{S0}}{1 + \chi_{e0}} & \text{for} \quad z = d \\ -\dfrac{\chi_{e0}\rho_{S0}}{1 + \chi_{e0}} & \text{for} \quad z = 0 \end{cases} \tag{4.31}$$

and the electric field intensity in the dielectric is

$$\mathbf{E}_t = \frac{\rho_{S0}}{\varepsilon_0(1 + \chi_{e0})}\mathbf{a}_z \tag{4.32}$$

as shown in Fig. 4.10(e).

Let us now consider the case of the infinite plane current sheet of Fig. 3.14, radiating uniform plane waves, except that now the space on either side of the current sheet is a dielectric material instead of free space. The electric field in the medium induces polarization. The polarization in turn acts together with other factors to govern the behavior of the electromagnetic field. For the case under consideration, the electric field is entirely in the x-direction and uniform in x and y. Thus the induced dipoles are all oriented in the x-direction, on a macroscopic scale, with the dipole moment per unit volume given by

$$\mathbf{P} = P_x\mathbf{a}_x = \varepsilon_0 \chi_e E_x \mathbf{a}_x \tag{4.33}$$

where E_x is understood to be a function of z and t.

If we now consider an infinitesimal surface of area $\Delta y\, \Delta z$ parallel to the yz plane, we can write E_x associated with that infinitesimal area to be equal to $E_0 \cos \omega t$, where E_0 is a constant. The time history of the induced dipoles associated with that area can be sketched for one complete period of the current source, as shown in Fig. 4.11. In view of the cosinusoidal variation of the electric field with time, the dipole moment of the individual dipoles varies in a cosinusoidal manner with maximum strength in the positive x direction at $t = 0$, decreasing sinusoidally to zero strength at $t = \pi/2\omega$ and then reversing to the negative x direction, increasing to maximum strength in that direction at $t = \pi/\omega$, and so on.

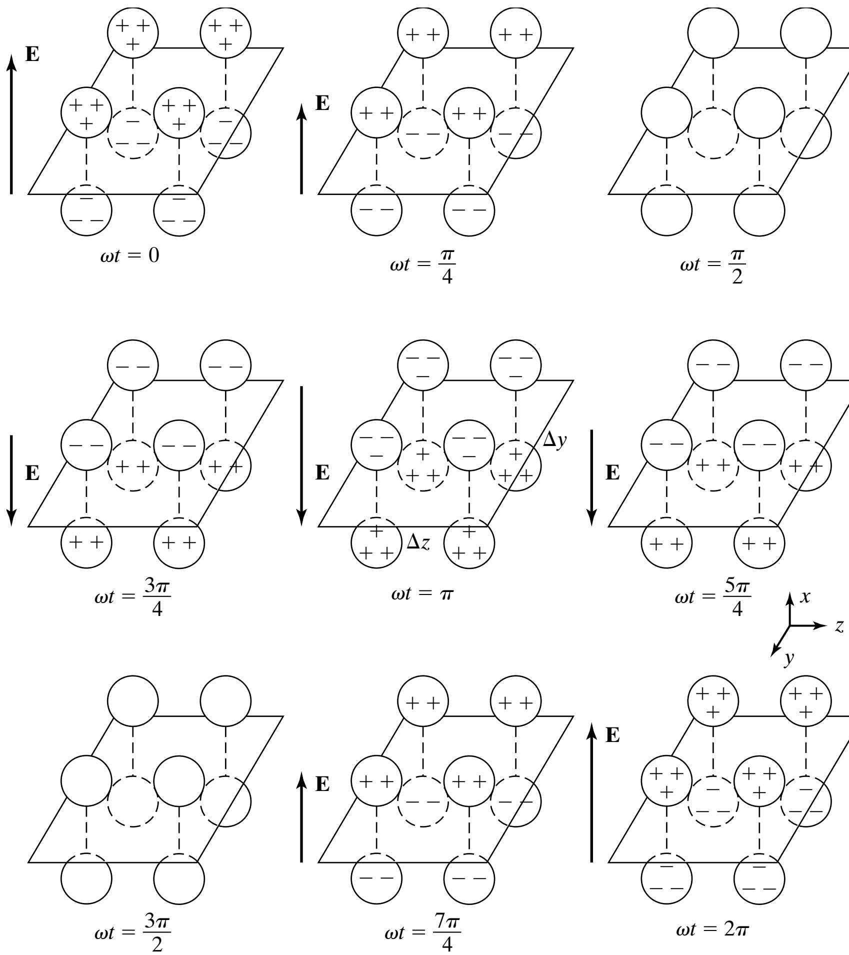


FIGURE 4.11

Time history of induced electric dipoles in a dielectric material under the influence of a sinusoidally time-varying electric field.

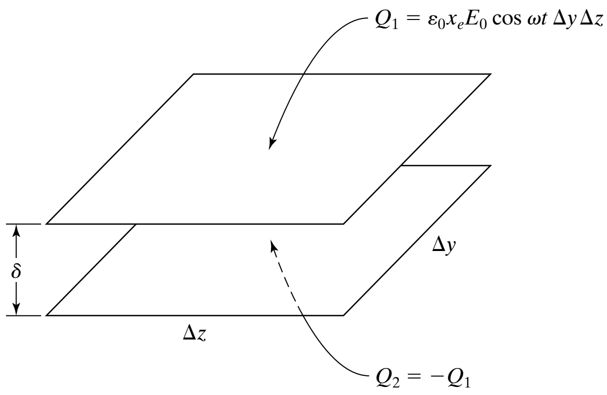


FIGURE 4.12

Two plane sheets of equal and opposite time-varying charges equivalent to the phenomenon depicted in Fig. 4.11.

The arrangement can be considered as two plane sheets of equal and opposite time-varying charges displaced by the amount δ in the x direction, as shown in Fig. 4.12. To find the magnitude of either charge, we note that the dipole moment per unit volume is

$$P_x = \varepsilon_0 \chi_e E_0 \cos \omega t \tag{4.34}$$

Since the total volume occupied by the dipoles is $\delta \, \Delta y \, \Delta z$, the total dipole moment associated with the dipoles is $\varepsilon_0 \chi_e E_0 \cos \omega t(\delta \, \Delta y \, \Delta z)$. The dipole moment associated with two equal and opposite sheet charges is equal to the magnitude of either sheet charge multiplied by the displacement between the two sheets. Hence we obtain the magnitude of either sheet charge to be $\varepsilon_0 \chi_e E_0 \cos \omega t \, \Delta y \, \Delta z$. Thus we have a situation in which a sheet charge $Q_1 = \varepsilon_0 \chi_e E_0 \cos \omega t \, \Delta y \, \Delta z$ is above the surface and a sheet charge $Q_2 = -Q_1 = -\varepsilon_0 \chi_e E_0 \cos \omega t \, \Delta y \, \Delta z$ is below the surface. This is equivalent to a current flowing across the surface, since the charges are varying with time.

We call this current the "polarization current" since it results from the time variation of the electric dipole moments induced in the dielectric due to polarization. The polarization current crossing the surface in the positive x direction, that is, from below to above, is

$$I_{px} = \frac{dQ_1}{dt} = -\varepsilon_0 \chi_e E_0 \omega \sin \omega t \, \Delta y \, \Delta z \tag{4.35}$$

where the subscript p denotes polarization. By dividing I_{px} by $\Delta y \, \Delta z$ and letting the area tend to zero, we obtain the polarization current density associated with the points on the surface as

$$\begin{aligned} J_{px} &= \lim_{\substack{\Delta y \to 0 \\ \Delta z \to 0}} \frac{I_{px}}{\Delta y \, \Delta z} = -\varepsilon_0 \chi_e E_0 \omega \sin \omega t \\ &= \frac{\partial}{\partial t}(\varepsilon_0 \chi_e E_0 \cos \omega t) = \frac{\partial P_x}{\partial t} \end{aligned}$$

or

$$\boxed{\mathbf{J}_p = \frac{\partial \mathbf{P}}{\partial t}} \tag{4.36}$$

Although we have deduced this result by considering the special case of the infinite plane current sheet, it is valid in general.

In considering electromagnetic wave propagation in a dielectric medium, the polarization current density given by (4.36) must be included with the current density term on the right side of Ampere's circuital law. Thus considering Ampere's circuital law in differential form for the general case given by (3.21), we have

$$\begin{aligned}\nabla \times \mathbf{H} &= \mathbf{J} + \mathbf{J}_p + \frac{\partial}{\partial t}(\varepsilon_0 \mathbf{E}) \\ &= \mathbf{J} + \frac{\partial \mathbf{P}}{\partial t} + \frac{\partial}{\partial t}(\varepsilon_0 \mathbf{E}) \\ &= \mathbf{J} + \frac{\partial}{\partial t}(\varepsilon_0 \mathbf{E} + \mathbf{P})\end{aligned} \tag{4.37}$$

In order to make (4.37) consistent with the corresponding equation for free space given by (3.21), we now revise the definition of the displacement vector **D** to read as

$$\boxed{\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}} \tag{4.38}$$

Substituting for **P** by using (4.26), we obtain

$$\begin{aligned}\mathbf{D} &= \varepsilon_0 \mathbf{E} + \varepsilon_0 \chi_e \mathbf{E} \\ &= \varepsilon_0 (1 + \chi_e) \mathbf{E} \\ &= \varepsilon_0 \varepsilon_r \mathbf{E}\end{aligned} \tag{4.39}$$

or

$$\boxed{\mathbf{D} = \varepsilon \mathbf{E}} \tag{4.40}$$

where we define

$$\boxed{\varepsilon_r = 1 + \chi_e} \tag{4.41}$$

and

$$\boxed{\varepsilon = \varepsilon_0 \varepsilon_r} \tag{4.42}$$

The quantity ε_r is known as the *relative permittivity* or *dielectric constant* of the dielectric, and ε is the *permittivity* of the dielectric. The permittivity ε takes into account the effects of polarization, and there is no need to consider them when we use for ε for ε_0! The relative permittivity is an experimentally measurable parameter. Its values for several dielectric materials are listed in Table 4.2.

Table 4.2 Relative Permittivities of Some Materials

Material	Relative Permittivity	Material	Relative Permittivity
Air	1.0006	Dry earth	5
Paper	2.0-3.0	Mica	6
Teflon	2.1	Neoprene	6.7
Polystyrene	2.56	Wet earth	10
Plexiglass	2.6-3.5	Ethyl alcohol	24.3
Nylon	3.5	Glycerol	42.5
Fused quartz	3.8	Distilled water	81
Bakelite	4.9	Titanium dioxide	100

Returning now to Example 4.2, we observe that in the absence of the dielectric between the sheets of charge,

$$\mathbf{E} = \mathbf{E}_a = \frac{\rho_{S0}}{\varepsilon_0}\mathbf{a}_z \tag{4.43a}$$

$$\mathbf{D} = \varepsilon_0\mathbf{E}_a = \rho_{S0}\mathbf{a}_z \tag{4.43b}$$

since $\mathbf{P}$ is equal to zero. In the presence of the dielectric between the sheets of charge,

$$\mathbf{E} = \mathbf{E}_t = \frac{\rho_{S0}}{\varepsilon_0(1 + \chi_{e0})}\mathbf{a}_z = \frac{\rho_{S0}}{\varepsilon}\mathbf{a}_z \tag{4.44a}$$

$$\mathbf{D} = \varepsilon\mathbf{E} = \rho_{S0}\mathbf{a}_z \tag{4.44b}$$

Thus, the $\mathbf{D}$ fields are the same in both cases, independent of the permittivity of the medium, whereas the expressions for the $\mathbf{E}$ fields differ in the permittivities, that is, with ε_0 replaced by ε. The situation in general is, however, not so simple because the dielectric alters the original field distribution. In the case of Example 4.2, the geometry is such that the original field distribution is not altered by the dielectric. Also in the general case, the situation is equivalent to having a polarization volume charge inside the dielectric in addition to polarization surface charges on its boundaries.

Anisotropic dielectric materials

The nature of (4.13), which is characteristic of conductors, and of (4.40), which is characteristic of dielectrics, implies that $\mathbf{J}_c$ in the case of conductors and $\mathbf{D}$ in the case of dielectrics are in the same direction as that of $\mathbf{E}$. Such materials are said to be *isotropic* materials. For *anisotropic* materials, this is not necessarily the case. To explain, we shall consider *anisotropic dielectric materials*. Then $\mathbf{D}$ is not in general in the same direction as that of $\mathbf{E}$. This arises because the induced polarization is such that the polarization vector $\mathbf{P}$ is not necessarily in the same direction as that of $\mathbf{E}$. In fact, the angle between the directions of the applied $\mathbf{E}$ and the resulting $\mathbf{P}$ depends on the direction of $\mathbf{E}$. The relationship between $\mathbf{D}$ and $\mathbf{E}$ is then expressed in the form of a matrix equation as

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \tag{4.45}$$

Thus, each component of $\mathbf{D}$ is in general dependent on each component of $\mathbf{E}$. The square matrix in (4.45) is known as the *permittivity tensor* of the anisotropic dielectric.

Although $\mathbf{D}$ is not in general parallel to $\mathbf{E}$ for anisotropic dielectrics, there are certain polarizations of $\mathbf{E}$ for which $\mathbf{D}$ is parallel to $\mathbf{E}$. These are said to correspond to the characteristic polarizations, where the word *polarization* here refers to the direction of the field, not to the creation of electric dipoles. We shall consider an example to investigate the characteristic polarizations.

Example 4.3 Characteristics of an anisotropic dielectric material

An anisotropic dielectric material is characterized by the permittivity tensor

$$[\varepsilon] = \varepsilon_0 \begin{bmatrix} 7 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Let us find $\mathbf{D}$ for several cases of $\mathbf{E}$.

Substituting the given permittivity matrix into (4.45), we obtain

$$\begin{aligned} D_x &= 7\varepsilon_0 E_x + 2\varepsilon_0 E_y \\ D_y &= 2\varepsilon_0 E_x + 4\varepsilon_0 E_y \\ D_z &= 3\varepsilon_0 E_z \end{aligned}$$

For $\mathbf{E} = E_0\mathbf{a}_z$, $\mathbf{D} = 3\varepsilon_0 E_0 \mathbf{a}_z = 3\varepsilon_0\mathbf{E}$; $\mathbf{D}$ is parallel to $\mathbf{E}$.
For $\mathbf{E} = E_0\mathbf{a}_x$, $\mathbf{D} = 7\varepsilon_0 E_0\mathbf{a}_x + 2\varepsilon_0 E_0 \mathbf{a}_y$; $\mathbf{D}$ is not parallel to $\mathbf{E}$.
For $\mathbf{E} = E_0\mathbf{a}_y$, $\mathbf{D} = 2\varepsilon_0 E_0\mathbf{a}_x + 4\varepsilon_0 E_0 \mathbf{a}_y$; $\mathbf{D}$ is not parallel to $\mathbf{E}$.
For $\mathbf{E} = E_0(\mathbf{a}_x + 2\mathbf{a}_y)$, $\mathbf{D} = 11\varepsilon_0 E_0\mathbf{a}_x + 10\varepsilon_0 E_0 \mathbf{a}_y$; $\mathbf{D}$ is not parallel to $\mathbf{E}$.
For $\mathbf{E} = E_0(2\mathbf{a}_x + \mathbf{a}_y)$, $\mathbf{D} = 16\varepsilon_0 E_0\mathbf{a}_x + 8\varepsilon_0 E_0 \mathbf{a}_y = 8\varepsilon_0 E_0(2\mathbf{a}_x + \mathbf{a}_y) = 8\varepsilon_0\mathbf{E}$; $\mathbf{D}$ is parallel of $\mathbf{E}$.

When $\mathbf{D}$ is parallel to $\mathbf{E}$, that is, for the characteristic polarizations of $\mathbf{E}$, one can define an *effective permittivity* as the ratio of $\mathbf{D}$ to $\mathbf{E}$. Thus, for the case of $\mathbf{E} = E_0\mathbf{a}_z$, the effective permittivity is $3\varepsilon_0$, and for the case of $\mathbf{E} = E_0(2\mathbf{a}_x + \mathbf{a}_y)$, the effective permittivity is $8\varepsilon_0$. For the characteristic polarizations, the anisotropic material behaves effectively as an isotropic dielectric having the permittivity equal to the corresponding effective permittivity.

K4.2. Polarization; Electric dipole; Polarization vector; Polarization charge; Polarization current; Permittivity; Relative permittivity; Anisotropic dielectric; Characteristic polarizations; Effective permittivity.

D4.3. Infinite plane sheets of uniform charge densities 1 μC/m^2 and -1 μC/m^2 occupy the planes $z = 0$ and $z = d$, respectively. The region $0 < z < d$ is a dielectric of permittivity $4\varepsilon_0$. Find the values of **(a)** $\mathbf{D}$, **(b)** $\mathbf{E}$, and **(c)** $\mathbf{P}$ in the region $0 < z < d$.

Ans. **(a)** $10^{-6}\,\mathbf{a}_z$ C/m^2; **(b)** $9000\pi\mathbf{a}_z$ V/m; **(c)** $0.75 \times 10^{-6}\,\mathbf{a}_z$ C/m^2.

D4.4. For an anisotropic dielectric material characterized by the **D** to **E** relationship

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \varepsilon_0 \begin{bmatrix} 8 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

find the value of the effective relative permittivity for each of the following electric field intensities corresponding to the characteristic polarizations: **(a)** $\mathbf{E} = E_0\mathbf{a}_z$; **(b)** $\mathbf{E} = E_0(\mathbf{a}_x - 2\mathbf{a}_y)$; and **(c)** $\mathbf{E} = E_0(2\mathbf{a}_x + \mathbf{a}_y)$.

Ans. **(a)** 9; **(b)** 4; **(c)** 9.

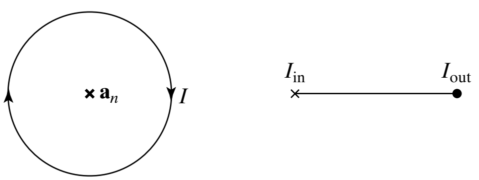
4.3 MAGNETIC MATERIALS

Magnetization, magnetic dipole

In the preceding two sections, we have been concerned with the response of materials to electric fields. We now turn our attention to materials known as magnetic materials, which, as the name implies, are classified according to their magnetic behavior. According to a simplified atomic model, the electrons associated with a particular nucleus orbit around the nucleus in circular paths while spinning about themselves. In addition, the nucleus itself has a spin motion associated with it. Since the movement of charge constitutes a current, these orbital and spin motions are equivalent to current loops of atomic dimensions. A current loop is the magnetic analog of the electric dipole. Thus, each atom can be characterized by a superposition of magnetic dipole moments corresponding to the electron orbital motions, electron spin motions, and the nuclear spin. However, owing to the heavy mass of the nucleus, the angular velocity of the nuclear spin is much smaller than that of an electron spin, and hence the equivalent current associated with the nuclear spin is much smaller than the equivalent current associated with an electron spin. The dipole moment due to the nuclear spin can therefore be neglected in comparison with the other two effects. The schematic representations of a magnetic dipole as seen from along its axis and from a point in its plane are shown in Figs. 4.13(a) and (b), respectively. The strength of the dipole is defined by the magnetic dipole moment **m** given by

$$\boxed{\mathbf{m} = IA\mathbf{a}_n} \tag{4.46}$$

where A is the area enclosed by the current loop, and $\mathbf{a}_n$ is the unit vector normal to the plane of the loop and directed in the right-hand sense.



(a) (b)

FIGURE 4.13

Schematic representation of a magnetic dipole as seen from (a) along its axis and (b) a point in its plane.

In many materials, the net magnetic moment of each atom is zero; that is, on the average, the magnetic dipole moments corresponding to the various electronic orbital and spin motions add up to zero. An external magnetic field has the effect of inducing a net dipole moment by changing the angular velocities of the electronic orbits, thereby magnetizing the material. This kind of magnetization, known as *diamagnetism*, is in fact prevalent in all materials. In certain materials known as *paramagnetic materials*, the individual atoms possess net nonzero magnetic moments even in the absence of an external magnetic field. These *permanent* magnetic moments of the individual atoms are, however, randomly oriented so that the net magnetization on a macroscopic scale is zero. An applied magnetic field has the effect of exerting torques on the individual permanent dipoles, as shown in Fig. 4.14, that convert, on a macroscopic scale, the initially random alignment into a partially coherent one along the magnetic field, that is, with the normal to the current loop directed along the magnetic field. This kind of magnetization is known as *paramagnetism*. Certain materials known as *ferromagnetic, antiferromagnetic*, and *ferrimagnetic* materials exhibit permanent magnetization, that is, magnetization even in the absence of an applied magnetic field.

On a macroscopic scale, we define a vector **M**, called the *magnetization vector*, as the *magnetic dipole moment per unit volume*. Thus, if N denotes the number of molecules per unit volume of the material, then there are $N\,\Delta v$ molecules in a volume Δv and

$$\boxed{\mathbf{M} = \frac{1}{\Delta v}\sum_{j=1}^{N\,\Delta v}\mathbf{m}_j = N\mathbf{m}} \tag{4.47}$$

where **m** is the average dipole moment per molecule. The units of **M** are ampere-meter^2/meter^3 or amperes per meter. It is found that for many magnetic materials, the magnetization vector is related to the magnetic field **B** in the material in the simple manner given by

$$\boxed{\mathbf{M} = \frac{\chi_m}{1+\chi_m}\frac{\mathbf{B}}{\mu_0}} \tag{4.48}$$

where χ_m, a dimensionless parameter, is known as the *magnetic susceptibility*. The quantity χ_m is a measure of the ability of the material to become magnetized and differs from one magnetic material to another.

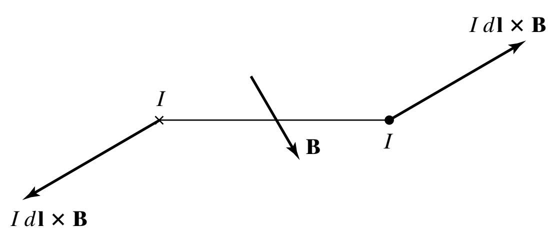


FIGURE 4.14

Torque acting on a magnetic dipole in an external magnetic field.

Magnetic material in a magnetic field

When a magnetic material is placed in a magnetic field, the induced dipoles produce a secondary magnetic field such that the resultant field, that is, the sum of the originally applied field and the secondary field, and the magnetization vector satisfy (4.48). We shall illustrate this by means of an example.

Example 4.4 Plane magnetic material slab in a uniform static magnetic field

Let us consider an infinite plane magnetic material slab of thickness d sandwiched between two infinite plane sheets of equal and opposite uniform current densities $J_{S0}\mathbf{a}_y$ and $-J_{S0}\mathbf{a}_y$ in the $z = 0$ and $z = d$ planes, respectively, as shown in Fig. 4.15(a). We wish to investigate the effect of magnetization in the magnetic material.

In the absence of the magnetic material, the magnetic field between the sheets of current is given by

$$\begin{aligned}\mathbf{B}_a &= \mu_0 J_{S0}\mathbf{a}_y \times \mathbf{a}_z \\ &= \mu_0 J_{S0}\mathbf{a}_x\end{aligned}$$

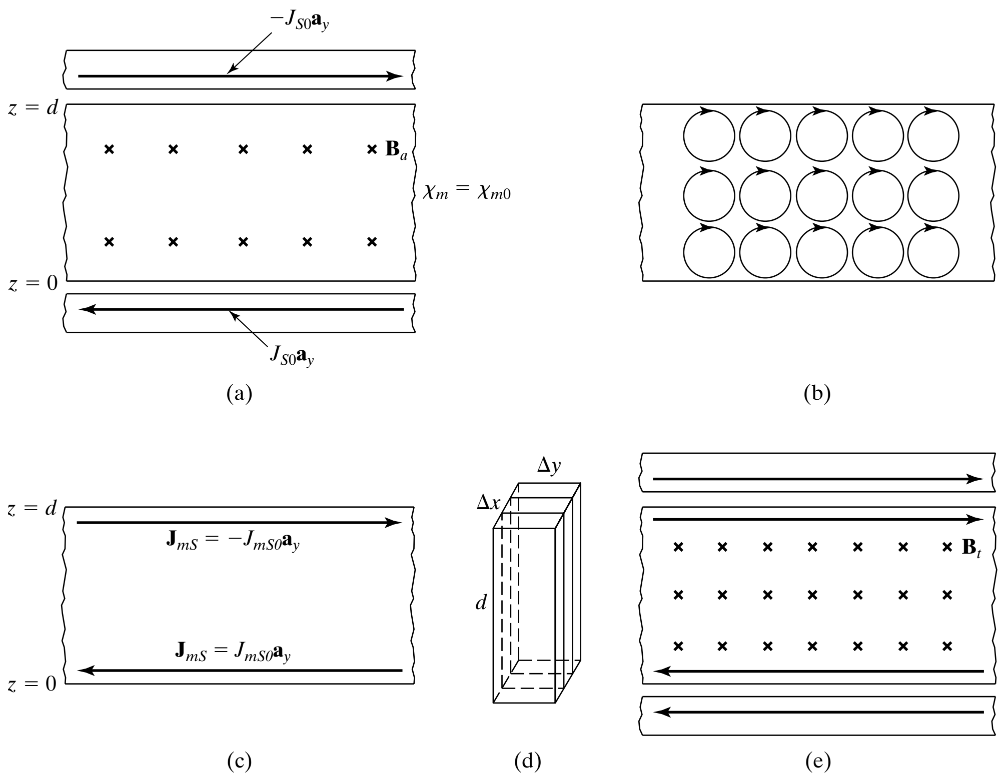


FIGURE 4.15

For investigating the effect of magnetization induced in a magnetic material sandwiched between two infinite plane sheets of current.

In the presence of the magnetic material, this field acts as the applied magnetic field resulting in magnetic dipole moments in the material that are oriented along the field. Since the magnetic field and the magnetic susceptibility are uniform, the density of the dipole moments, that is, the magnetization vector **M**, is uniform as shown in Fig. 4.15(b). Such a distribution results in exact cancelation of currents everywhere except at the boundaries of the material since, for each current segment not on the surface, there is a current segment associated with the dipole adjacent to it and carrying the same amount of current in the opposite direction, thereby canceling its effect. Thus, the net result is the formation of a negative y-directed surface current at the boundary $z = d$ and a positive y-directed surface current at the boundary $z = 0$, as shown in Fig. 4.15(c). These surface currents are known as magnetization surface currents, since they are due to the magnetization in the material. In view of the uniform density of the dipole moments, the surface current densities are uniform. Also, in the absence of a net current in the interior of the magnetic material, the surface current densities must be equal in magnitude so that whatever current flows on one surface returns via the other surface.

Let us therefore denote the surface current densities as

$$\mathbf{J}_{mS} = \begin{cases} J_{mS0}\mathbf{a}_y & \text{for} \quad z = 0 \\ -J_{mS0}\mathbf{a}_y & \text{for} \quad z = d \end{cases}$$

where the subscript m in addition to the other subscripts stands for magnetization. If we now consider a vertical column of infinitesimal rectangular cross-sectional area $\Delta S = (\Delta x)(\Delta y)$ cut out from the magnetic material, as shown in Fig. 4.15(d), the rectangular current loop of width Δx makes the column appear as a dipole of moment $(J_{mS0}\,\Delta x)(d\,\Delta y)\mathbf{a}_x$. On the other hand, writing

$$\mathbf{M} = M_0\mathbf{a}_x \tag{4.49}$$

where M_0 is a constant in view of the uniformity of the magnetization, the dipole moment of the column is equal to **M** times the volume of the column, or $M_0(d\,\Delta x\,\Delta y)\mathbf{a}_x$. Equating the dipole moments computed in the two different ways, we have

$$J_{mS0} = M_0$$

Thus, we have related the surface current density to the magnitude of the magnetization vector. Now, the surface current distribution produces a secondary field $\mathbf{B}_s$ given by

$$\mathbf{B}_s = \begin{cases} \mu_0 J_{mS0}\mathbf{a}_x = \mu_0 M_0\mathbf{a}_x & \text{for} \quad 0 < z < d \\ \mathbf{0} & \text{otherwise} \end{cases}$$

Denoting the total field inside the magnetic material to be $\mathbf{B}_t$, we have

$$\begin{aligned} \mathbf{B}_t &= \mathbf{B}_a + \mathbf{B}_s = \mu_0 J_{S0}\mathbf{a}_x + \mu_0 M_0\mathbf{a}_x \\ &= \mu_0(J_{S0} + M_0)\,\mathbf{a}_x \end{aligned} \tag{4.50}$$

But, from (4.48),

$$\mathbf{M} = \frac{\chi_{m0}}{1 + \chi_{m0}}\frac{\mathbf{B}_t}{\mu_0} \tag{4.51}$$

Substituting (4.49) and (4.50) into (4.51), we have

$$M_0 = \frac{\chi_{m0}}{1 + \chi_{m0}}(J_{S0} + M_0)$$

or

$$M_0 = \chi_{m0} J_{S0} \tag{4.52}$$

Thus, the magnetization surface current densities are given by

$$\mathbf{J}_{mS} = \begin{cases} \chi_{m0} J_{S0} \mathbf{a}_y & \text{for} \quad z = 0 \\ -\chi_{m0} J_{S0} \mathbf{a}_y & \text{for} \quad z = d \end{cases} \tag{4.53}$$

and the magnetic flux density in the magnetic material is

$$\mathbf{B}_t = \mu_0 (1 + \chi_{m0}) J_{S0} \mathbf{a}_x \tag{4.54}$$

as shown in Fig. 4.15(e).

Let us now consider the case of the infinite plane current sheet of Fig. 3.14, radiating uniform plane waves, except that now the space on either side of the current sheet possesses magnetic material properties in addition to dielectric properties. The magnetic field in the medium induces magnetization. The magnetization in turn acts together with other factors to govern the behavior of the electromagnetic field. For the case under consideration, the magnetic field is entirely in the y-direction and uniform in x and y. Thus the induced dipoles are all oriented with their axes in the y-direction, on a macroscopic scale, with the dipole moment per unit volume given by

$$\mathbf{M} = M_x \mathbf{a}_y = \frac{\chi_m}{1 + \chi_m} \frac{B_y}{\mu_0} \mathbf{a}_y \tag{4.55}$$

where B_y is understood to be a function of z and t.

Let us now consider an infinitesimal surface of area $\Delta y \, \Delta z$ parallel to the yz plane and the magnetic dipoles associated with the two areas $\Delta y \, \Delta z$ to the left and to the right of the center of this area as shown in Fig. 4.16(a). Since B_y is a function of z, we can assume the dipoles in the left area to have a different moment than the dipoles in the right area for any given time. If the dimension of an individual dipole is δ in the x direction, then the total dipole moment associated with the dipoles in the left area is $[M_y]_{z-\Delta z/2} \, \delta \, \Delta y \, \Delta z$ and the total dipole moment associated with the dipoles in the right area is $[M_y]_{z+\Delta z/2} \, \delta \, \Delta y \, \Delta z$.

The arrangement of dipoles can be considered to be equivalent to two rectangular surface current loops as shown in Fig. 4.16 (b) with the left side current loop having a dipole moment $[M_y]_{z-\Delta z/2} \, \delta \, \Delta y \, \Delta z$ and the right side current loop having a dipole moment $[M_y]_{z+\Delta z/2} \, \delta \, \Delta y \, \Delta z$. Since the magnetic dipole moment of a rectangular surface current loop is simply equal to the product of the surface current and the cross-sectional area of the loop, the surface current associated with the left loop is $[M_y]_{z-\Delta z/2} \, \Delta y$ and the surface current associated with the right loop is $[M_y]_{z+\Delta z/2} \, \Delta y$. Thus we have a situation in which a current equal to $[M_y]_{z-\Delta z/2} \, \Delta y$ is crossing the area $\Delta y \, \Delta z$ in the positive x direction, and a current equal to $[M_y]_{z+\Delta z/2} \, \Delta y$ is crossing the same area in the negative x direction. This is equivalent to a net current flowing across the surface.

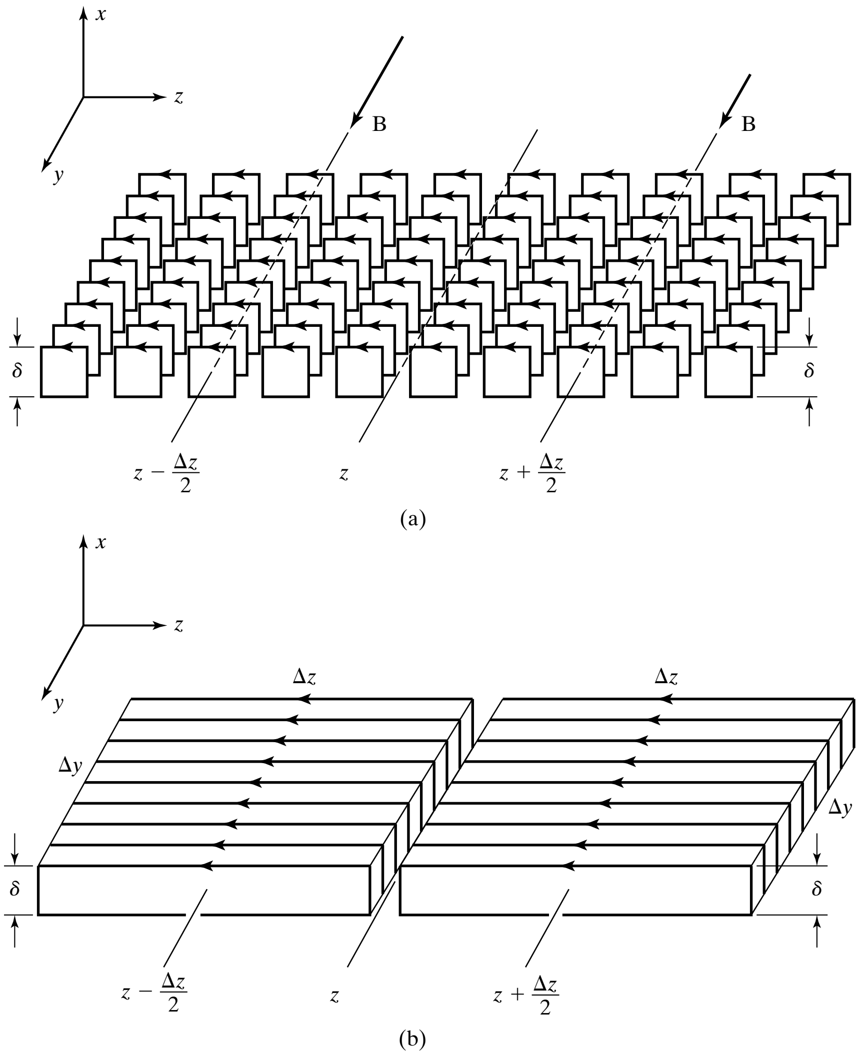


FIGURE 4.16

(a) Induced magnetic dipoles in a magnetic material. (b) Equivalent surface current loops.

We call this current the "magnetization current," since it results from the space variation of the magnetic dipole moments induced in the magnetic material due to magnetization. The net magnetization current crossing the surface in the positive x direction is

$$I_{mx} = [M_y]_{z-\Delta z/2}\,\Delta y - [M_y]_{z+\Delta z/2}\,\Delta y \tag{4.56}$$

where the subscript m denotes magnetization. By dividing I_{mx} by $\Delta y\,\Delta z$ and letting the area tend to zero, we obtain the magnetization current density associated with the points on the surface as

$$J_{mx} = \operatorname*{Lim}_{\substack{\Delta y \to 0 \\ \Delta z \to 0}} \frac{I_{mx}}{\Delta y \, \Delta z} = \operatorname*{Lim}_{\Delta z \to 0} \frac{[M_y]_{z-\Delta z/2} - [M_y]_{z+\Delta z/2}}{\Delta z}$$

$$= -\frac{\partial M_y}{\partial z}$$

or

$$J_{mx}\mathbf{a}_x = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & M_y & 0 \end{vmatrix}$$

or

$$\boxed{\mathbf{J}_m = \boldsymbol{\nabla} \times \mathbf{M}} \tag{4.57}$$

Although we have deduced this result by considering the special case of the infinite plane current sheet, it is valid in general.

In considering electromagnetic wave propagation in a magnetic material medium, the magnetization current density given by (4.57) must be included with the current density term on the right side of Ampere's circuital law. Thus considering Ampere's circuital law in differential form for the general case given by (3.21), we have

$$\begin{aligned} \boldsymbol{\nabla} \times \frac{\mathbf{B}}{\mu_0} &= \mathbf{J} + \mathbf{J}_m + \frac{\partial \mathbf{D}}{\partial t} \qquad (4.58) \\ &= \mathbf{J} + \boldsymbol{\nabla} \times \mathbf{M} + \frac{\partial \mathbf{D}}{\partial t} \end{aligned}$$

or

$$\boldsymbol{\nabla} \times \left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M}\right) = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{4.59}$$

In order to make (4.59) consistent with the corresponding equation for free space given by (3.21), we now revise the definition of the magnetic field intensity vector **H** to read as

$$\boxed{\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}} \tag{4.60}$$

Substituting for **M** by using (4.48), we obtain

$$\begin{aligned}\mathbf{H} &= \frac{\mathbf{B}}{\mu_0} - \frac{\chi_m}{1+\chi_m}\frac{\mathbf{B}}{\mu_0}\\ &= \frac{\mathbf{B}}{\mu_0(1+\chi_m)}\\ &= \frac{\mathbf{B}}{\mu_0\mu_r}\end{aligned} \tag{4.61}$$

or

$$\boxed{\mathbf{H} = \frac{\mathbf{B}}{\mu}} \tag{4.62}$$

where we define

$$\boxed{\mu_r = 1 + \chi_m} \tag{4.63}$$

and

$$\boxed{\mu = \mu_0\mu_r} \tag{4.64}$$

The quantity μ_r is known as the *relative permeability* of the magnetic material and μ is the *permeability* of the magnetic material. The permeability μ takes into account the effects of magnetization, and there is no need to consider them when we use μ for μ_0!

Returning now to Example 4.4, we observe that in the absence of the magnetic material between the sheets of current,

$$\mathbf{B} = \mathbf{B}_a = \mu_0 J_{S0}\mathbf{a}_x \tag{4.65a}$$

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} = J_{S0}\mathbf{a}_x \tag{4.65b}$$

since **M** is equal to zero. In the presence of the magnetic material between the sheets of current,

$$\mathbf{B} = \mathbf{B}_t = \mu_0(1+\chi_m)J_{S0}\mathbf{a}_x = \mu J_{S0}\mathbf{a}_x \tag{4.66a}$$

$$\mathbf{H} = \frac{\mathbf{B}}{\mu} = J_{S0}\mathbf{a}_x \tag{4.66b}$$

Thus, the **H** fields are the same in both cases, independent of the permeability of the medium, whereas the expressions for the **B** fields differ in the permeabilities, that is, with μ_0 replaced by μ. The situation in general is, however, not so simple because the magnetic material alters the original field distribution. In the case of Example 4.4, the geometry is such that the original field distribution is not altered by the magnetic material. Also, in the general case, the situation is equivalent to having a magnetization volume current inside the material in addition to the surface current at the boundaries. For anisotropic magnetic materials, **H** is not in general parallel to **B** and the relationship between the two quantities is expressed in the form of a matrix equation, as given by

$$\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} \mu_{xx} & \mu_{xy} & \mu_{xz} \\ \mu_{yx} & \mu_{yy} & \mu_{yz} \\ \mu_{zx} & \mu_{zy} & \mu_{zz} \end{bmatrix} \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} \tag{4.67}$$

just as in the case of the relationship between **D** and **E** for anistropic dielectric materials.

Ferromagnetic materials

For many materials for which the relationship between **H** and **B** is linear, the relative permeability does not differ appreciably from unity, unlike the case of linear dielectric materials, for which the relative permittivity can be very large, as shown in Table 4.2. In fact, for diamagnetic materials, the magnetic susceptibility χ_m is a small negative number of the order -10^{-4} to -10^{-8}, whereas for paramagnetic materials, χ_m is a small positive number of the order 10^{-3} to 10^{-7}. Ferromagnetic materials, however, possess large values of relative permeability on the order of several hundreds, thousands, or more. The relationship between **B** and **H** for these materials is nonlinear, resulting in a non-unique value of μ_r, for a given material. In fact, these materials are characterized by hysteresis, that is, the relationship between **B** and **H** dependent on the past history of the material.

Ferromagnetic materials possess strong dipole moments, owing to the predominance of the electron spin moments over the electron orbital moments. The theory of ferromagnetism is based on the concept of magnetic *domains*, as formulated by Weiss in 1907. A magnetic domain is a small region in the material in which the atomic dipole moments are all aligned in one direction, due to strong interaction fields arising from the neighboring dipoles. In the absence of an external magnetic field, although each domain is magnetized to saturation, the magnetizations in various domains are randomly oriented, as shown in Fig. 4.17(a) for a single crystal specimen. The random orientation results from minimization of the

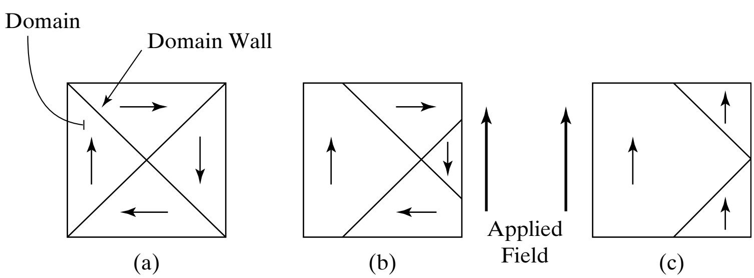


FIGURE 4.17

For illustrating the different steps in the magnetization of a ferromagnetic specimen: (a) unmagnetized state; (b) domain wall motion; and (c) domain rotation.

associated energy. The net magnetization is therefore zero on a macroscopic scale. With the application of a weak external magnetic field, the volumes of the domains in which the original magnetizations are favorably oriented relative to the applied field grow at the expense of the volumes of the other domains, as shown in Fig. 4.17(b). This feature is known as domain wall motion. Upon removal of the applied field, the domain wall motion reverses, bringing the material close to its original state of magnetization. With the application of stronger external fields, the domain wall motion continues to such an extent that it becomes irreversible; that is, the material does not return to its original unmagnetized state on a macroscopic scale upon removal of the field. With the application of still stronger fields, the domain wall motion is accompanied by domain rotation, that is, alignment of the magnetizations in the individual domains with the applied field, as shown in Fig. 4.17(c), thereby magnetizing the material to saturation. The material retains some magnetization along the direction of the applied field even after removal of the field. In fact, an external field opposite to the original direction has to be applied to bring the net magnetization back to zero.

Hysteresis curve

We may now discuss the relationship between **B** and **H** for a ferromagnetic material, which is depicted graphically as shown by a typical curve in Fig. 4.17. This curve is known as the hysteresis curve, or the *B–H* curve. To trace the development of the hysteresis effect, we start with an unmagnetized sample of ferromagnetic material in which both **B** and **H** are initially zero, corresponding to point *a* on the curve. As *H* is increased, the magnetization builds up, thereby increasing *B* gradually along the curve *ab* and finally to saturation at *b*, according to the following sequence of events as discussed earlier: (1) reversible motion of domain walls, (2) irreversible motion of domain walls, and (3) domain rotation. The regions corresponding to these events along the curve *ab* as well as other portions of the hysteresis curve are shown marked 1, 2, and 3, respectively, in Fig. 4.18. If the value of *H* is now decreased to zero, the value of *B*

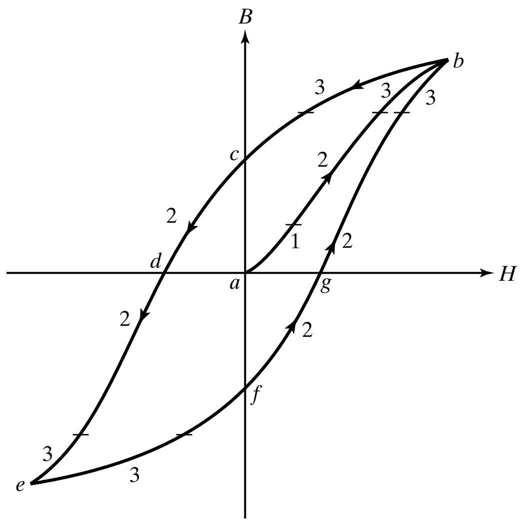


FIGURE 4.18

Hysteresis curve for a ferromagnetic material.

does not retrace the curve *ab* backward, but instead follows the curve *bc*, which indicates that a certain amount of magnetization remains in the material even after the magnetizing field is completely removed. In fact, it requires a magnetic field intensity in the opposite direction to bring B back to zero, as shown by the portion *cd* of the curve. The value of B at the point c is known as the *remanence*, or *retentivity*, whereas the value of H at d is known as the *coercivity* of the material. Further increase in $\mathbf{H}$ in this direction results in the saturation of $\mathbf{B}$ in the direction opposite to that corresponding to b, as shown by the portion *de* of the curve. If $\mathbf{H}$ is now decreased to zero, reversed in direction, and increased, the resulting variation of $\mathbf{B}$ occurs in accordance with the curve *efgb*, thereby completing the hysteresis loop.

The nature of the hysteresis curve suggests that the hysteresis phenomenon can be used to distinguish between two states, for example, "1" and "0" in a binary number magnetic memory system. There are several kinds of magnetic memories. Although differing in details, all these are based on the principles of storing and retrieving information in regions on a magnetic medium. In disk, drum, and tape memories, the magnetic medium moves, whereas in bubble and core memories, the medium is stationary. We shall briefly discuss here only the floppy disk, or diskette, used as secondary memory in personal computers.[1]

Floppy disk

The floppy disk consists of a coating of ferrite material applied over a thin flexible nonmagnetic substrate for physical support. Ferrites are a class of magnetic materials characterized by almost rectangular-shaped hysteresis loops so that the two remanent states are well-defined. The disk is divided into many circular tracks, and each track is subdivided into regions called sectors, as shown in Fig. 4.19. To access a sector, an electromagnetic read/write head moves across the spinning disk to the appropriate track and waits for the correct sector to rotate beneath it. The head consists of a ferrite core around which a coil is wound and with a gap at the bottom, as shown in Fig. 4.20. Writing data on the disk is done by passing current through the coil. The current generates a magnetic field that in the core confines essentially to the material, but in the air gap spreads out into the magnetic medium below it, thereby magnetizing the region to represent the 0 state. To store the 1 state in a region, the current in the coil is reversed to magnetize the medium in the reverse direction. Reading of data from the disk is accomplished by the changing magnetic field from the magnetized

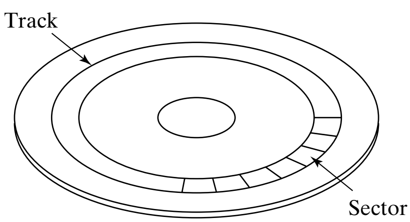


FIGURE 4.19

Arrangement of sectors on a floppy disk.

[1]See, for example, Robert M. White, "Disk-Storage Technology," *Scientific American*, August 1980, pp. 138–148.

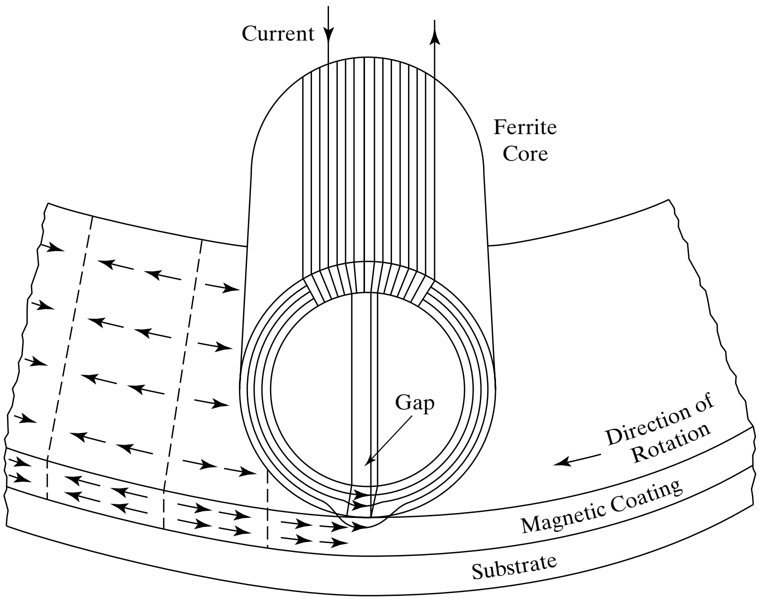


FIGURE 4.20

Writing of data on a floppy disk.

regions on the disk inducing a voltage in the coil of the head as the disk rotates under the head. The voltage is induced in accordance with Faraday's law (which we covered in Section 2.3) whenever there is a change in magnetic flux linked by the coil. We have here only discussed the basic principles behind storing data on the disk and retrieving data from it. There are a number of ways in which bits can be encoded on the disk. We shall, however, not pursue the topic here.

K4.3. Magnetization; Magnetic dipole; Magnetization vector; Magnetization current; Permeability; Relative permeability; Ferromagnetic materials; Hysteresis.

D4.5. Find the magnetic dipole moment for each of the following cases: **(a)** 1 μC of charge in a circular orbit of radius $1/\sqrt{\pi}$ mm in the xy-plane around the z-axis in the sense of increasing ϕ with angular velocity of 1 revolution per millisecond; **(b)** a square current loop having the vertices at the points $A(10^{-3}, 0, 0)$, $B(0, 10^{-3}, 0)$, $C(-10^{-3}, 0, 0)$, and $D(0, -10^{-3}, 0)$ with current 0.1 A flowing in the sense $ABCDA$; and **(c)** an equilateral triangular current loop having vertices at the points $A(10^{-3}, 0, 0)$, $B(0, 10^{-3}, 0)$, and $C(0, 0, 10^{-3})$ with current 0.1 A flowing in the sense $ABCA$.

Ans. **(a)** $10^{-9}\,\mathbf{a}_z$ A-m^2; **(b)** $2\times10^{-7}\,\mathbf{a}_z$ A-m^2; **(c)** $5\times10^{-8}(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z)$ A-m^2.

D4.6. Infinite plane sheets of current densities $0.1\mathbf{a}_y$ A/m and $-0.1\mathbf{a}_y$ A/m occupy the planes $z = 0$ and $z = d$, respectively. The region $0 < z < d$ is a magnetic material of permeability $100\mu_0$. Find **(a) H**, **(b) B**, and **(c) M** in the region $0 < z < d$.

Ans. **(a)** $0.1\mathbf{a}_x$ A/m; **(b)** $4\pi \times 10^{-6}\,\mathbf{a}_x$ Wb/m^2; **(c)** $9.9\mathbf{a}_x$ A/m.

4.4 WAVE EQUATION AND SOLUTION FOR MATERIAL MEDIUM

In the previous three sections, we introduced conductors, dielectrics, and magnetic materials, and developed the relationships (4.13), (4.40) and (4.62), which take into account the phenomena of conduction, polarization, and magnetization, respectively. In this section, we make use of these relationships, in conjunction with Maxwell's curl equations, to extend our discussion of uniform plane wave propagation in free space in Sections 3.4 and 3.5 to a material medium. These relationships, known as the *constitutive relations*, are given by

$$\boxed{\begin{aligned} \mathbf{J}_c &= \sigma\mathbf{E} \quad &(4.68a)\\ \mathbf{D} &= \varepsilon\mathbf{E} \quad &(4.68b)\\ \mathbf{H} &= \frac{\mathbf{B}}{\mu} \quad &(4.68c)\end{aligned}}$$

so that the Maxwell's equations for the material medium are

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\mu\frac{\partial \mathbf{H}}{\partial t} \tag{4.69a}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}_c + \frac{\partial \mathbf{D}}{\partial t} = \sigma\mathbf{E} + \varepsilon\frac{\partial \mathbf{E}}{\partial t} \tag{4.69b}$$

To discuss electromagnetic wave propagation in the material medium, let us consider the infinite plane current sheet of Fig. 3.14, except that now the medium on either side of the sheet is a material instead of free space, as shown in Fig. 4.21.

The electric and magnetic fields for the simple case of the infinite plane current sheet in the $z = 0$ plane and carrying uniformly distributed current in the negative x-direction as given by

$$\mathbf{J}_S = -J_{S0}\cos\omega t\,\mathbf{a}_x \tag{4.70}$$

are of the form

$$\mathbf{E} = E_x(z, t)\mathbf{a}_x \tag{4.71a}$$

$$\mathbf{H} = H_y(z, t)\mathbf{a}_y \tag{4.71b}$$

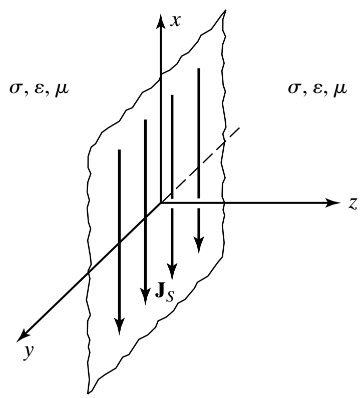


FIGURE 4.21

Infinite plane current sheet embedded in a material medium.

The corresponding simplified forms of the Maxwell's curl equations are

$$\boxed{\begin{aligned}\frac{\partial E_x}{\partial z} &= -\mu\frac{\partial H_y}{\partial t} \\ \frac{\partial H_y}{\partial z} &= -\sigma E_x - \varepsilon\frac{\partial E_x}{\partial t}\end{aligned}} \qquad \begin{aligned}&(4.72a)\\&(4.72b)\end{aligned}$$

Without the σE_x term on the right side of (4.72b), these two equations would be the same as (3.72a) and (3.72b) with μ_0 replaced by μ and ε_0 replaced by ε. The addition of the σE_x term complicates the solution in time domain. Hence, it is convenient to consider the solution for the sinusoidally time-varying case by using the phasor technique. See Appendix A for phasor technique.

Wave equation

Thus, letting

$$E_x(z,t) = \text{Re}[\bar{E}_x(z)e^{j\omega t}] \tag{4.73a}$$

$$H_y(z,t) = \text{Re}[\bar{H}_y(z)e^{j\omega t}] \tag{4.73b}$$

and replacing E_x and H_y in (4.72a) and (4.72b) by their phasors $\bar{E}_x$ and $\bar{H}_y$, respectively, and $\partial/\partial t$ by $j\omega$, we obtain the corresponding differential equations for the phasors $\bar{E}_x$ and $\bar{H}_y$ as

$$\boxed{\begin{aligned}\frac{\partial \bar{E}_x}{\partial z} &= -j\omega\mu\bar{H}_y \\ \frac{\partial \bar{H}_y}{\partial z} &= -\sigma\bar{E}_x - j\omega\varepsilon\bar{E}_x = -(\sigma + j\omega\varepsilon)\bar{E}_x\end{aligned}} \qquad \begin{aligned}&(4.74a)\\&(4.74b)\end{aligned}$$

Differentiating (4.74a) with respect to z and using (4.74b), we obtain

$$\frac{\partial^2 \bar{E}_x}{\partial z^2} = -j\omega\mu\frac{\partial \bar{H}_y}{\partial z} = j\omega\mu(\sigma + j\omega\varepsilon)\bar{E}_x \tag{4.75}$$

Defining

$$\boxed{\bar{\gamma} = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}} \tag{4.76}$$

and substituting in (4.75), we have

$$\boxed{\frac{\partial^2 \bar{E}_x}{\partial z^2} = \bar{\gamma}^2\bar{E}_x} \tag{4.77}$$

which is the wave equation for $\bar{E}_x$ in the material medium.

The solution to the wave equation (4.77) is given by

$$\bar{E}_x(z) = \bar{A}e^{-\bar{\gamma}z} + \bar{B}e^{\bar{\gamma}z} \tag{4.78}$$

where $\bar{A}$ and $\bar{B}$ are arbitrary constants. Noting that $\bar{\gamma}$ is a complex number and, hence, can be written as

$$\boxed{\bar{\gamma} = \alpha + j\beta} \tag{4.79}$$

and also writing $\bar{A}$ and $\bar{B}$ in exponential form as $Ae^{j\theta}$ and $Be^{j\phi}$, respectively, we have

$$\bar{E}_x(z) = Ae^{j\theta}e^{-\alpha z}e^{-j\beta z} + Be^{j\phi}e^{\alpha z}e^{j\beta z}$$

or

$$\begin{aligned} E_x(z, t) &= \text{Re}[\bar{E}_x(z)e^{j\omega t}] \\ &= \text{Re}[Ae^{j\theta}e^{-\alpha z}e^{-j\beta z}e^{j\omega t} + Be^{j\phi}e^{\alpha z}e^{j\beta z}e^{j\omega t}] \\ &= Ae^{-\alpha z}\cos(\omega t - \beta z + \theta) + Be^{\alpha z}\cos(\omega t + \beta z + \phi) \end{aligned} \tag{4.80}$$

We now recognize the two terms on the right side of (4.80) as representing uniform plane waves propagating in the positive z- and negative z-directions, respectively, with phase constant β, in view of the factors $\cos(\omega t - \beta z + \theta)$ and $\cos(\omega t + \beta z + \phi)$, respectively. They are, however, multiplied by the factors $e^{-\alpha z}$ and $e^{\alpha z}$, respectively. Hence, the amplitude of the field differs from one constant phase surface to another. Since there cannot be a $(+)$ wave in the region $z < 0$, that is, to the left of the current sheet, and since there cannot be a $(-)$ wave in the region $z > 0$, that is, to the right of the current sheet, the solution for the electric field is given by

$$\mathbf{E}(z, t) = \begin{cases} Ae^{-\alpha z}\cos(\omega t - \beta z + \theta)\,\mathbf{a}_x & \text{for} \quad z > 0 \\ Be^{\alpha z}\cos(\omega t + \beta z + \phi)\,\mathbf{a}_x & \text{for} \quad z < 0 \end{cases} \tag{4.81}$$

Attenuation constant

To discuss how the amplitude of E_x varies with z on either side of the current sheet, we note that since σ, ε, and μ are all positive, the phase angle of $j\omega\mu(\sigma + j\omega\varepsilon)$ lies between 90° and 180°, and hence the phase angle of $\bar{\gamma}$ lies between 45° and 90°, making α and β positive quantities. This means that $e^{-\alpha z}$ decreases with increasing value of z, that is, in the positive z-direction, and $e^{\alpha z}$ decreases with decreasing value of z, that is, in the negative z-direction. Thus, the exponential factors $e^{-\alpha z}$ and $e^{\alpha z}$ associated with the solutions for E_x in (4.81) have the effect of decreasing the amplitude of the field, that is, attenuating it as it propagates away from the sheet to either side of it. For this reason, the quantity α is known as the *attenuation constant*. The attenuation per unit length is equal to e^{α}. In terms of decibels, this is equal to $20 \log_{10} e^{\alpha}$, or 8.686α dB. The units of α are nepers per meter. The quantity $\bar{\gamma}$ is known as the *propagation constant*, since its real and imaginary parts, α and β, together determine the propagation characteristics, that is, attenuation and phase shift of the wave.

Having found the solution for the electric field of the wave and discussed its general properties, we now turn to the solution for the corresponding magnetic field by substituting for $\bar{E}_x$ in (4.74a). Thus,

$$\begin{aligned}\bar{H}_y &= -\frac{1}{j\omega\mu}\frac{\partial \bar{E}_x}{\partial z} = \frac{\bar{\gamma}}{j\omega\mu}(\bar{A}e^{-\bar{\gamma}z} - \bar{B}e^{\bar{\gamma}z}) \\ &= \sqrt{\frac{\sigma + j\omega\varepsilon}{j\omega\mu}}(\bar{A}e^{-\bar{\gamma}z} - \bar{B}e^{\bar{\gamma}z}) \\ &= \frac{1}{\bar{\eta}}(\bar{A}e^{-\bar{\gamma}z} - \bar{B}e^{\bar{\gamma}z})\end{aligned} \tag{4.82}$$

where

$$\boxed{\bar{\eta} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}} \tag{4.83}$$

is the intrinsic impedance of the medium, which is now complex. Writing

$$\boxed{\bar{\eta} = |\bar{\eta}|\, e^{j\tau}} \tag{4.84}$$

we obtain the solution for $H_y(z, t)$ as

$$\begin{aligned}H_y(z,t) &= \mathrm{Re}[\bar{H}_y(z)e^{j\omega t}] \\ &= \mathrm{Re}\left[\frac{1}{|\bar{\eta}|e^{j\tau}}Ae^{j\theta}e^{-\alpha z}e^{-j\beta z}e^{j\omega t} - \frac{1}{|\bar{\eta}|e^{j\tau}}Be^{j\phi}e^{\alpha z}e^{j\beta z}e^{j\omega t}\right] \\ &= \frac{A}{|\bar{\eta}|}e^{-\alpha z}\cos(\omega t - \beta z + \theta - \tau) - \frac{B}{|\bar{\eta}|}e^{\alpha z}\cos(\omega t + \beta z + \phi - \tau)\end{aligned} \tag{4.85}$$

Remembering that the first and second terms on the right side of (4.85) correspond to $(+)$ and $(-)$ waves, respectively, and, hence, represent the solutions for the magnetic field in the regions $z > 0$ and $z < 0$, respectively, we write

$$\mathbf{H}(z,t) = \begin{cases} \dfrac{A}{|\bar{\eta}|}e^{-\alpha z}\cos(\omega t - \beta z + \theta - \tau)\,\mathbf{a}_y & \text{for} \quad z > 0 \qquad (4.86a) \\ -\dfrac{B}{|\bar{\eta}|}e^{\alpha z}\cos(\omega t + \beta z + \phi - \tau)\,\mathbf{a}_y & \text{for} \quad z < 0 \qquad (4.86b) \end{cases}$$

Electromagnetic field due to the current sheet

To complete the solution for the electromagnetic field due to the current sheet embedded in the material medium, we need to find the values of the constants A, B, θ, and ϕ. To do this, we proceed in the same manner as in Sec. 3.4, using Fig. 3.17, except with a material medium on either side of the current sheet. Thus, applying Faraday's law in integral form to the rectangular closed path *abcda* in the limit that the sides *bc* and *da* $\rightarrow 0$, with the sides *ab* and *dc* remaining on either side of the current sheet, we have

$$(ab)[E_x]_{z=0+} - (dc)[E_x]_{z=0-} = 0 \tag{4.87}$$

or $A \cos(\omega t + \theta) - B \cos(\omega t + \phi) = 0$, giving us $A = B$ and $\theta = \phi$. The solutions for **E** and **H** reduce to

$$\mathbf{E}(z, t) = Ae^{\mp\alpha z} \cos(\omega t \mp \beta z + \theta)\,\mathbf{a}_x \qquad \text{for} \quad z \gtrless 0 \tag{4.88a}$$

$$\mathbf{H}(z, t) = \pm\frac{A}{|\bar{\eta}|}e^{\mp\alpha z} \cos(\omega t \mp \beta z + \theta - \tau)\,\mathbf{a}_y \quad \text{for} \quad z \gtrless 0 \tag{4.88b}$$

Now, applying Ampere's circuital law in integral form to the rectangular closed path *efghe* in Fig. 3.17, but with a material medium on either side of the current sheet, in the limit that the sides *fg* and *he* $\rightarrow 0$, with the sides *ef* and *hg* remaining on either side of the current sheet, we have

$$(ef)[H_y]_{z=0+} - (hg)[H_y]_{z=0-} = (ef)J_{S0} \cos \omega t \tag{4.89}$$

or

$$\frac{2A}{|\bar{\eta}|} \cos(\omega t + \theta - \tau) = J_{S0} \cos \omega t$$

$$A = \frac{|\bar{\eta}|J_{S0}}{2} \qquad \text{and} \qquad \theta = \tau$$

Thus, the electromagnetic field due to the infinite plane current sheet of surface current density

$$\boxed{\mathbf{J}_S = -J_{S0} \cos \omega t\,\mathbf{a}_x \qquad \text{for} \quad z = 0} \tag{4.90}$$

and with a material medium characterized by σ, ε, and μ on either side of it is given by

$$\mathbf{E}(z, t) = \frac{|\bar{\eta}|J_{S0}}{2}e^{\mp\alpha z} \cos(\omega t \mp \beta z + \tau)\,\mathbf{a}_x \quad \text{for} \quad z \gtrless 0 \tag{4.91a}$$

$$\mathbf{H}(z, t) = \pm\frac{J_{S0}}{2}e^{\mp\alpha z} \cos(\omega t \mp \beta z)\,\mathbf{a}_y \qquad \text{for} \quad z \gtrless 0 \tag{4.91b}$$

Propagation characteristics

As we have already discussed, (4.91a) and (4.91b) represent sinusoidally time-varying uniform plane waves, getting attenuated as they propagate away from the current sheet. The phenomenon is illustrated in Fig. 4.22, which shows sketches of current density on the sheet and the distance variation of the electric and magnetic fields on either side of the current sheet for three values of t. As in Fig. 3.22, it should be understood that in these sketches, the field variations depicted along the z-axis hold also for any other line parallel to the z-axis. We shall now discuss further the propagation characteristics associated with these waves:

1. From (4.76) and (4.79), we have

$$\bar{\gamma}^2 = (\alpha + j\beta)^2 = j\omega\mu(\sigma + j\omega\varepsilon)$$

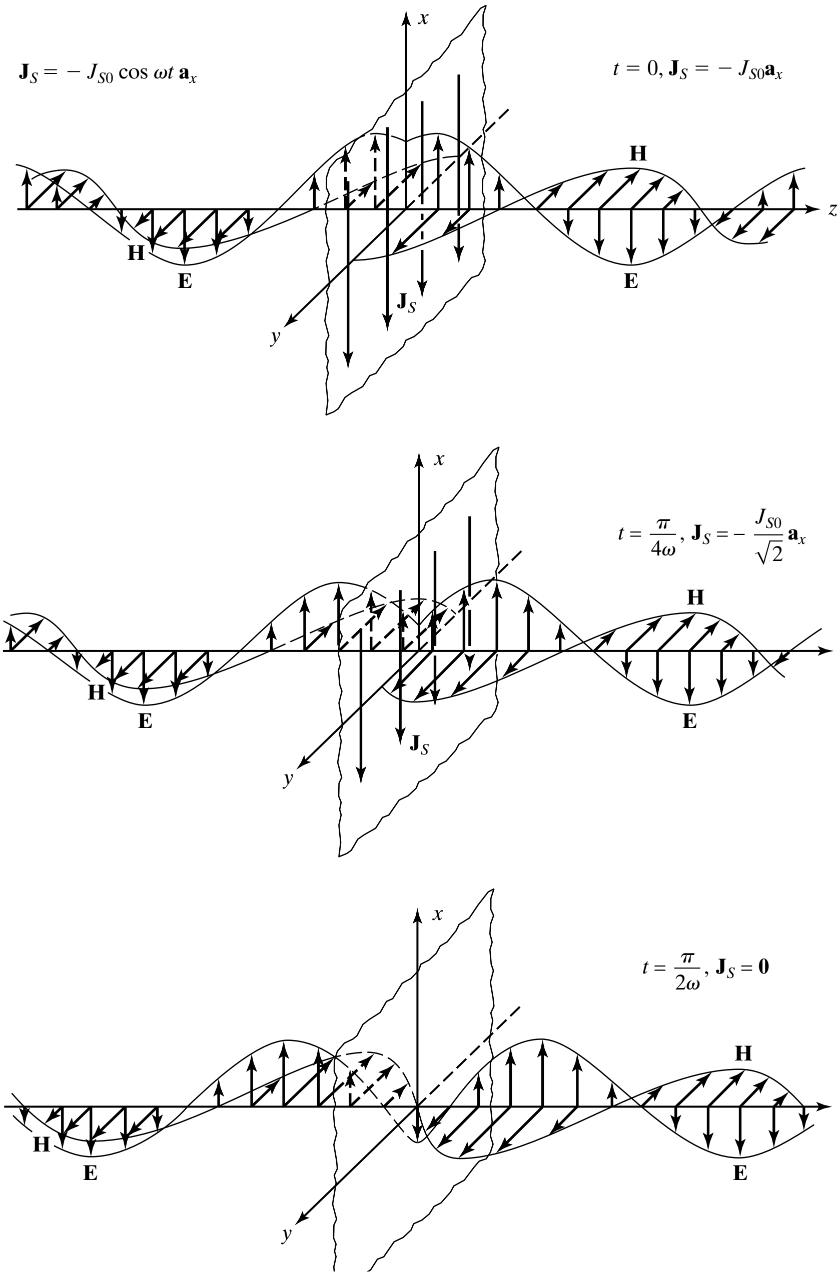


FIGURE 4.22

Time history of uniform plane electromagnetic wave radiating away from an infinite plane current sheet embedded in a material medium.

or

$$\alpha^2 - \beta^2 = -\omega^2\mu\varepsilon \tag{4.92a}$$
$$2\alpha\beta = \omega\mu\sigma \tag{4.92b}$$

Squaring (4.92a) and (4.92b) and adding and then taking the square root, we obtain

$$\alpha^2 + \beta^2 = \omega^2\mu\varepsilon\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} \tag{4.93}$$

From (4.92a) and (4.93), we then have

$$\alpha^2 = \frac{1}{2}\left[-\omega^2\mu\varepsilon + \omega^2\mu\varepsilon\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2}\right]$$
$$\beta^2 = \frac{1}{2}\left[\omega^2\mu\varepsilon + \omega^2\mu\varepsilon\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2}\right]$$

Since α and β are both positive, we finally get

$$\alpha = \frac{\omega\sqrt{\mu\varepsilon}}{\sqrt{2}}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]^{1/2} \tag{4.94}$$
$$\beta = \frac{\omega\sqrt{\mu\varepsilon}}{\sqrt{2}}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right]^{1/2} \tag{4.95}$$

We note from (4.94) and (4.95) that α and β are both dependent on σ through the factor $\sigma/\omega\varepsilon$. This factor, known as the *loss tangent*, is the ratio of the magnitude of the conduction current density $\sigma\bar{E}_x$ to the magnitude of the displacement current density $j\omega\varepsilon\bar{E}_x$ in the material medium. In practice, the loss tangent is, however, not simply inversely proportional to ω, since both σ and ε are generally functions of frequency. In fact, for many materials, the dependence of $\sigma/\omega\varepsilon$ on ω is more toward constant over wide frequency ranges.

2. The phase velocity of the wave along the direction of propagation is given by

$$v_p = \frac{\omega}{\beta} = \frac{\sqrt{2}}{\sqrt{\mu\varepsilon}}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right]^{-1/2} \tag{4.96}$$

We note that the phase velocity is dependent on the frequency of the wave. Thus, waves of different frequencies travel with different phase velocities. Consequently, for a signal comprising a band of frequencies, the different frequency components do not maintain the same phase relationships as they propagate in the medium. This phenomenon is known as *dispersion*. We shall discuss dispersion in detail in Chapter 8.

3. The wavelength in the medium is given by

$$\lambda = \frac{2\pi}{\beta} = \frac{\sqrt{2}}{f\sqrt{\mu\varepsilon}}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right]^{-1/2} \tag{4.97}$$

In view of the attenuation of the wave with distance, the field variation with distance is not sinusoidal. Hence, the wavelength is not exactly equal to the distance between two consecutive positive maxima as in Fig. 3.23. It is, however, still exactly equal to the distance between two alternate zero crossings.

4. The ratio of the amplitude of the electric field to the amplitude of the magnetic field is equal to $|\overline{\eta}|$, the magnitude of the complex intrinsic impedance of the medium. The electric and magnetic fields are out of phase by τ, the phase angle of the intrinsic impedance. In terms of the phasor or complex field components, we have

$$\frac{\overline{E}_x}{\overline{H}_y} = \begin{cases} \overline{\eta} & \text{for the } (+) \text{ wave} \\ -\overline{\eta} & \text{for the } (-) \text{ wave} \end{cases} \tag{4.98}$$

5. From (4.76) and (4.83), we note that

$$\overline{\gamma}\,\overline{\eta} = j\omega\mu \tag{4.99a}$$

$$\frac{\overline{\gamma}}{\overline{\eta}} = \sigma + j\omega\varepsilon \tag{4.99b}$$

so that

$$\boxed{\begin{aligned} \sigma &= \text{Re}\left(\frac{\overline{\gamma}}{\overline{\eta}}\right) && (4.100a) \\ \varepsilon &= \frac{1}{\omega}\text{Im}\left(\frac{\overline{\gamma}}{\overline{\eta}}\right) && (4.100b) \\ \mu &= \frac{1}{j\omega\overline{\gamma}\,\overline{\eta}} && (4.100c) \end{aligned}}$$

Using (4.100a)–(4.100c), we can compute the material parameters σ, ε, and μ from a knowledge of the propagation parameters $\overline{\gamma}$ and $\overline{\eta}$ at the frequency of interest.

6. To obtain the electromagnetic field due to a nonsinusoidal source, it is necessary to consider its frequency components and apply superposition, since waves of different frequencies are attenuated by different amounts and travel with different phase velocities. The nonsinusoidal signal changes shape as it propagates in the material medium, unlike in the case of free space.

We shall now consider an example of the computation of $\overline{\gamma}$ and $\overline{\eta}$ given σ, ε, μ, and f.

Example 4.5 Finding propagation parameters of a material medium from its material parameters

The material parameters of a certain food item are given by $\sigma = 2.17$ S/m, $\varepsilon = 47\varepsilon_0$, and $\mu = \mu_0$ at the operating frequency $f = 2.45$ GHz of a microwave oven. We wish to find the propagation parameters α, β, λ, v_p, and $\bar{\eta}$.

Although explicit expressions for α and β in terms of ω, σ, ε, and μ are given by (4.94) and (4.95), it is instructive to compute their values by using complex algrebra in conjunction with the expression for $\bar{\gamma}$ given by (4.76). Thus, we have

$$\begin{aligned}
\bar{\gamma} &= \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)} \\
&= \sqrt{j\omega\mu \cdot j\omega\varepsilon\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)} \\
&= j\frac{\omega\sqrt{\varepsilon_r}}{c}\sqrt{1 - j\frac{\sigma}{\omega\varepsilon_r\varepsilon_0}} \\
&= j\frac{2\pi \times 2.45 \times 10^9 \times \sqrt{47}}{3 \times 10^8}\sqrt{1 - j\frac{2.17 \times 36\pi}{2\pi \times 2.45 \times 10^9 \times 47 \times 10^{-9}}} \\
&= j351.782\sqrt{1 - j0.3392} \\
&= j351.782\sqrt{1.0560\angle{-18.7369^\circ}} \\
&= 351.782\angle{90^\circ} \times 1.0276\angle{-9.3685^\circ} \\
&= 361.4912\angle{80.6315^\circ} \\
&= 58.85 + j356.67
\end{aligned}$$

so that

$$\begin{aligned}
\alpha &= 58.85 \text{ Np/m} \\
\beta &= 356.67 \text{ rad/m} \\
\lambda &= \frac{2\pi}{\beta} = 0.0176 \text{ m} \\
v_p &= \frac{\omega}{\beta} = 0.4316 \times 10^8 \text{ m/s}
\end{aligned}$$

Proceeding in a similar manner with (4.83), we obtain

$$\begin{aligned}
\bar{\eta} &= \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} \\
&= \sqrt{\frac{j\omega\mu}{j\omega\varepsilon[1 - j(\sigma/\omega\varepsilon)]}} \\
&= \frac{\eta_0}{\sqrt{\varepsilon_r}}\frac{1}{\sqrt{1 - j(\sigma/\omega\varepsilon)}} \\
&= \frac{120\pi}{\sqrt{47}}\frac{1}{\sqrt{1 - j0.3392}} \\
&= \frac{54.9898}{1.0276\angle{-9.3685^\circ}} \\
&= 53.51\angle{9.37^\circ}\,\Omega
\end{aligned}$$

Poynting's theorem for a material medium

We shall conclude this section by generalizing the Poynting's theorem (3.118), derived in Sec. 3.7, to a material medium. Thus, substituting $\mathbf{J} = \mathbf{J}_c = \sigma\mathbf{E}$ so that $\mathbf{E} \cdot \mathbf{J} = \mathbf{E} \cdot \sigma\mathbf{E} = \sigma E^2$, and replacing ε_0 by ε and μ_0 by μ, in (3.118), we obtain

$$\boxed{\oint_S \mathbf{P} \cdot d\mathbf{S} = -\int_V (\sigma E^2)\, dv - \frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\varepsilon E^2\right) dv - \frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\mu H^2\right) dv} \tag{4.101}$$

where $\mathbf{P}$ is the instantaneous Poynting vector given by

$$\boxed{\mathbf{P} = \mathbf{E} \times \mathbf{H}} \tag{4.102}$$

We also recall that the time-average Poynting vector, $\langle \mathbf{P} \rangle$, is given by

$$\langle \mathbf{P} \rangle = \frac{1}{2}\text{Re}\,[\overline{\mathbf{E}} \times \overline{\mathbf{H}}^*] \tag{4.103}$$

In (4.101), the quantity σE^2 is the power density associated with the work done by the field, having to do with the conduction current in the material. Since power is dissipated in causing the conduction current to flow, it is the power dissipation density. Thus, it follows that the power dissipation density, the electric stored energy density, and the magnetic stored energy density, associated with electric and magnetic fields in a material medium are given, respectively, by

$$p_d = \sigma E^2 \tag{4.104a}$$

$$w_e = \frac{1}{2}\varepsilon E^2 \tag{4.104b}$$

$$w_m = \frac{1}{2}\mu H^2 \tag{4.104c}$$

Example 4.6 Power flow for a uniform plane wave in seawater

Let us consider the electric field of a uniform plane wave propagating in seawater ($\sigma = 4$ S/m, $\varepsilon = 80\varepsilon_0$, and $\mu = \mu_0$) in the positive z-direction and having the electric field

$$\mathbf{E} = 1 \cos 5 \times 10^4\, \pi t\, \mathbf{a}_x \text{ V/m}$$

at $z = 0$. We wish to find the instantaneous power flow per unit area normal to the z-direction as a function of z and the time-average power flow per unit area normal to the z-direction as a function of z.

From the expression for $\mathbf{E}$, we note that the frequency of the wave is 25 kHz. At this frequency in seawater, the propagation parameters can be computed to be $\alpha = \beta \approx 0.628$ and $\overline{\eta} = 0.222\underline{/45^\circ}$. The expressions for the instantaneous electric and magnetic fields are therefore given by

$$\mathbf{E} = 1e^{-0.628z} \cos\,(5 \times 10^4\pi t - 0.628z)\, \mathbf{a}_x \text{ V/m}$$
$$\mathbf{H} = 4.502e^{-0.628z} \cos\,(5 \times 10^4\pi t - 0.628z - \pi/4)\, \mathbf{a}_y \text{ A/m}$$

The instantaneous Poynting vector is then given by

$$\begin{aligned}\mathbf{P} &= \mathbf{E} \times \mathbf{H}\\ &= 4.502e^{-1.256z} \cos (5 \times 10^4 \pi t - 0.628z)\\ &\quad \cdot \cos (5 \times 10^4 \pi t - 0.628z - \pi/4)\, \mathbf{a}_z \text{ W/m}^2\end{aligned}$$

Thus, the instantaneous power flow per unit area normal to the z-direction, which is simply the z-component of the instantaneous Poynting vector, is

$$P_z = 2.251e^{-1.256z} [\cos \pi/4 + \cos (10^5 \pi t - 1.256z - \pi/4)] \text{ W/m}^2$$

Finally, the time-average power flow per unit area normal to the z-direction is

$$\begin{aligned}\langle P_z \rangle &= 2.251e^{-1.256z} \cos \pi/4\\ &= 1.592e^{-1.256z} \text{ W/m}^2\end{aligned}$$

K4.4. Material medium; Sinusoidal waves; Material parameters; Propagation parameters; Attenuation and phase constants; Complex propagation constant; Complex intrinsic impedance; Poynting's theorem for material medium; Power dissipation density; Electric stored energy density; Magnetic stored energy density.

D4.7. Compute the propagation constant and intrinsic impedance for the following cases: **(a)** $\sigma = 10^{-5}$ S/m, $\varepsilon = 5\varepsilon_0$, $\mu = \mu_0$, and $f = 10^5$ Hz; and **(b)** $\sigma = 4$ S/m, $\varepsilon = 80\varepsilon_0$, $\mu = \mu_0$, and $f = 10^9$ Hz.

Ans. **(a)** $(0.00083 + j0.00476)$ m^{-1}, $163.54\angle 9.9^\circ\,\Omega$; **(b)** $(77.84 + j202.86)$ m^{-1}, $36.34\angle 20.99^\circ\,\Omega$.

D4.8. For a uniform plane wave of frequency 10^6 Hz propagating in a nonmagnetic ($\mu = \mu_0$) material medium, the propagation constant is known to be $(0.05 + j0.1)$ m^{-1}. Find the following: **(a)** the distance in which the fields are attenuated by e^{-1}; **(b)** the distance in which the fields undergo a change of phase by 1 rad; **(c)** the distance that a constant phase of the wave travels in 1 μs; **(d)** the ratio of the amplitudes of the electric and magnetic fields; and **(e)** the phase difference between the electric and magnetic fields.

Ans. **(a)** 20 m; **(b)** 10 m; **(c)** 62.83 m; **(d)** 70.62 Ω; **(e)** 0.1476π.

D4.9. The magnetic field associated with a uniform plane wave propagating in the $+z$-direction in a nonmagnetic ($\mu = \mu_0$) material medium is given by

$$\mathbf{H} = H_0 e^{-z} \cos (6\pi \times 10^7 t - \sqrt{3}z)\, \mathbf{a}_y \text{ A/m}$$

Find the following: **(a)** the instantaneous power flow across a surface of area 1 m^2 in the $z = 0$ plane at $t = 0$; **(b)** the time-average power flow across a surface of area 1 m^2 in the $z = 0$ plane; and **(c)** the time-average power flow across a surface of area 1 m^2 in the $z = 1$ m plane.

Ans. **(a)** $102.57H_0^2$ W; **(b)** $51.28H_0^2$ W; **(c)** $6.94H_0^2$ W.

4.5 UNIFORM PLANE WAVES IN DIELECTRICS AND CONDUCTORS

In the preceding section, we discussed uniform plane electromagnetic wave propagation in a material medium for the general case. In this section, we consider special cases as follows:

Case 1: Perfect dielectrics. Perfect dielectrics are characterized by $\sigma = 0$. Then

$$\bar{\gamma} = \sqrt{j\omega\mu \cdot j\omega\varepsilon} = j\omega\sqrt{\mu\varepsilon} \tag{4.105}$$

is purely imaginary, so that

$$\alpha = 0 \tag{4.106a}$$
$$\beta = \omega\sqrt{\mu\varepsilon} \tag{4.106b}$$
$$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\varepsilon}} \tag{4.106c}$$
$$\lambda = \frac{2\pi}{\beta} = \frac{1}{f\sqrt{\mu\varepsilon}} \tag{4.106d}$$

Further,

$$\bar{\eta} = \sqrt{\frac{j\omega\mu}{j\omega\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon}} \tag{4.107}$$

is purely real. Thus, the waves propagate without attenuation and with the electric and magnetic fields in phase, as in free space but with ε_0 replaced by ε and μ_0 replaced by μ. In terms of the relative permittivity ε_r and the relative permeability μ_r of the perfect dielectric medium, the propagation parameters are

$$\boxed{\beta = \beta_0\sqrt{\mu_r\varepsilon_r}} \tag{4.108a}$$
$$\boxed{v_p = \frac{c}{\sqrt{\mu_r\varepsilon_r}}} \tag{4.108b}$$
$$\boxed{\lambda = \frac{\lambda_0}{\sqrt{\mu_r\varepsilon_r}}} \tag{4.108c}$$
$$\boxed{\eta = \eta_0\sqrt{\frac{\mu_r}{\varepsilon_r}}} \tag{4.108d}$$

where the quantities with subscripts "0" refer to free space.

Case 2: Imperfect dielectrics. Imperfect dielectrics are characterized by $\sigma \neq 0$, but $\sigma/\omega\varepsilon \ll 1$. Recalling that $\sigma\bar{E}_x$ is the conduction current density and $\omega\varepsilon\bar{E}_x$ is the displacement current density, we note that this condition is equivalent to stating that the magnitude of the conduction current density is small compared to the magnitude of the displacement current density. Using the binomial expansion

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots$$

we can then write

$$\begin{aligned}\bar{\gamma} &= \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)} \\ &= \sqrt{j\omega\mu \cdot j\omega\varepsilon\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)} = j\omega\sqrt{\mu\varepsilon}\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)^{1/2} \\ &\approx \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}\left(1 - \frac{\sigma^2}{8\omega^2\varepsilon^2}\right) + j\omega\sqrt{\mu\varepsilon}\left(1 + \frac{\sigma^2}{8\omega^2\varepsilon^2}\right)\end{aligned} \tag{4.109}$$

so that

$$\alpha \approx \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}\left(1 - \frac{\sigma^2}{8\omega^2\varepsilon^2}\right) \tag{4.110a}$$

$$\beta \approx \omega\sqrt{\mu\varepsilon}\left(1 + \frac{\sigma^2}{8\omega^2\varepsilon^2}\right) \tag{4.110b}$$

$$v_p = \frac{\omega}{\beta} \approx \frac{1}{\sqrt{\mu\varepsilon}}\left(1 - \frac{\sigma^2}{8\omega^2\varepsilon^2}\right) \tag{4.110c}$$

$$\lambda = \frac{2\pi}{\beta} \approx \frac{1}{f\sqrt{\mu\varepsilon}}\left(1 - \frac{\sigma^2}{8\omega^2\varepsilon^2}\right) \tag{4.110d}$$

Further,

$$\begin{aligned}\bar{\eta} &= \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} \\ &= \sqrt{\frac{j\omega\mu}{j\omega\varepsilon}}\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)^{-1/2}\end{aligned}$$

so that

$$\bar{\eta} \approx \sqrt{\frac{\mu}{\varepsilon}}\left[\left(1 - \frac{3}{8}\frac{\sigma^3}{\omega^2\varepsilon^2}\right) + j\frac{\sigma}{2\omega\varepsilon}\right] \tag{4.111}$$

In (4.109)–(4.111), we have retained all terms up to and including the second power in $\sigma/\omega\varepsilon$ and have neglected all higher-order terms, since $\sigma/\omega\varepsilon \ll 1$. For a value of $\sigma/\omega\varepsilon$ equal to 0.1, the quantities β, v_p, and λ are different from those for the corresponding perfect dielectric case by a factor of only 1/800, whereas the intrinsic impedance has a real part differing from the intrinsic impedance of the perfect dielectric medium by a factor of 3/800 and an imaginary part, which is 1/20 of the intrinsic impedance of the perfect dielectric medium. Thus, for all practical purposes, the only significant feature different from the perfect dielectric case is the attenuation.

Case 3: Good conductors. Good conductors are characterized by $\sigma/\omega\varepsilon \gg 1$, just the opposite of imperfect dielectrics. This condition is equivalent to stating that the magnitude of the conduction current density is large compared to the magnitude of the displacement current density. Then

$$\begin{aligned}\bar{\gamma} &= \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)} \\ &\approx \sqrt{j\omega\mu\sigma} \\ &= \sqrt{\omega\mu\sigma}\, e^{j\pi/4} \\ &= \sqrt{\pi f\mu\sigma}(1 + j)\end{aligned} \tag{4.112}$$

so that

$$\boxed{\begin{aligned}\alpha &\approx \sqrt{\pi f\mu\sigma} && (4.113a)\\ \beta &\approx \sqrt{\pi f\mu\sigma} && (4.113b)\\ v_p = \frac{\omega}{\beta} &\approx \sqrt{\frac{4\pi f}{\mu\sigma}} && (4.113c)\\ \lambda = \frac{2\pi}{\beta} &\approx \sqrt{\frac{4\pi}{f\mu\sigma}} && (4.113d)\end{aligned}}$$

Further,

$$\begin{aligned}\bar{\eta} &= \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} \\ &\approx \sqrt{\frac{j\omega\mu}{\sigma}}\end{aligned}$$

or

$$\boxed{\begin{aligned}\bar{\eta} &\approx \sqrt{\frac{\omega\mu}{\sigma}}\, e^{j\pi/4} \\ &= \sqrt{\frac{\pi f\mu}{\sigma}}(1 + j)\end{aligned}} \tag{4.114}$$

We note that α, β, v_p, and $\bar{\eta}$ are proportional to $\sqrt{f}$, provided that σ and μ are constants. This behavior is much different from the imperfect dielectric case.

Skin effect

To discuss the propagation characteristics of a wave inside a good conductor, let us consider the case of copper. The constants for copper are $\sigma = 5.80 \times 10^7$ S/m, $\varepsilon = \varepsilon_0$, and $\mu = \mu_0$. Hence, the frequency at which σ is equal to $\omega\varepsilon$ for copper is equal to $5.8 \times 10^7/2\pi\varepsilon_0$, or 1.04×10^{18} Hz. Thus, at frequencies of even several gigahertz, copper behaves like an excellent conductor. To obtain an idea of the attenuation of the wave inside the conductor, we note that the attenuation undergone in a distance of one wavelength is equal to $e^{-\alpha\lambda}$ or $e^{-2\pi}$. In terms of decibels, this is equal to $20 \log_{10} e^{2\pi} = 54.58$ dB. In fact, the field is attenuated by a factor e^{-1}, or 0.368, in a distance equal to $1/\alpha$. This distance is known as the *skin depth* and is denoted by the symbol δ. From (4.113a), we obtain

$$\boxed{\delta = \frac{1}{\sqrt{\pi f\mu\sigma}}} \tag{4.115}$$

The skin depth for copper is equal to

$$\frac{1}{\sqrt{\pi f \times 4\pi \times 10^{-7} \times 5.8 \times 10^{7}}} = \frac{0.066}{\sqrt{f}}\text{ m}$$

Thus, in copper, the fields are attenuated by a factor e^{-1} in a distance of 0.066 mm even at the low frequency of 1 MHz, thereby resulting in the concentration of the fields near to the skin of the conductor. This phenomenon is known as the *skin effect*. It also explains *shielding* by conductors.

Underwater communication

To discuss further the characteristics of wave propagation in a good conductor, we note that the ratio of the wavelength in the conducting medium to the wavelength in a dielectric medium having the same ε and μ as those of the conductor is given by

$$\frac{\lambda_{\text{conductor}}}{\lambda_{\text{dielectric}}} \approx \frac{\sqrt{4\pi/f\mu\sigma}}{1/f\sqrt{\mu\varepsilon}} = \sqrt{\frac{4\pi f\varepsilon}{\sigma}} = \sqrt{\frac{2\omega\varepsilon}{\sigma}} \tag{4.116}$$

Since $\sigma/\omega\varepsilon \gg 1$, $\lambda_{\text{conductor}} \ll \lambda_{\text{dielectric}}$. For example, for seawater, $\sigma = 4$ S/m, $\varepsilon = 80\varepsilon_0$, and $\mu = \mu_0$, so that the ratio of the two wavelengths for $f = 25$ kHz ($\sigma/\omega\varepsilon = 36{,}000$) is equal to 0.00745. Thus, for $f = 25$ kHz, the wavelength in seawater is 1/134 of the wavelength in a dielectric having the same ε and μ as those of seawater and a still smaller fraction of the wavelength in free space. Furthermore, the lower the frequency, the smaller is this fraction. Since it is the electrical length (i.e., the length in terms of the wavelength) instead of the physical length that determines the radiation characteristics of an antenna, this means that antennas of much shorter length can be used in seawater than in free space. Together with the property that $\alpha \propto \sqrt{f}$, this illustrates that the lower the frequency, the more suitable it is for underwater communication.

For a given frequency, the higher the value of σ, the greater is the value of the attenuation constant, the smaller is the value of the skin depth, and hence the less deep the waves can penetrate. For example, in the heating of malignant tissues (hyperthermia) by RF (radio-frequency) radiation, the waves penetrate much deeper into fat (low water content) than into muscle (high water content).[2]

Equation (4.114) tells us that the intrinsic impedance of a good conductor has a phase angle of 45°. Hence, the electric and magnetic fields in the medium are out of phase by 45°. The magnitude of the intrinsic impedance is given by

$$|\bar{\eta}| = \left|(1 + j)\sqrt{\frac{\pi f\mu}{\sigma}}\right| = \sqrt{\frac{2\pi f\mu}{\sigma}} \tag{4.117}$$

As a numerical example, for copper, this quantity is equal to

$$\sqrt{\frac{2\pi f \times 4\pi \times 10^{-7}}{5.8 \times 10^{7}}} = 3.69 \times 10^{-7}\sqrt{f}\ \Omega$$

[2] F. Sterzer et al., "RF Therapy for Malignancy," *IEEE Spectrum*, December 1980, pp. 32–37.

Thus, the intrinsic impedance of copper has as low a magnitude as 0.369 Ω even at a frequency of 10^{12} Hz. In fact, by recognizing that

$$|\bar{\eta}| = \sqrt{\frac{2\pi f\mu}{\sigma}} = \sqrt{\frac{\omega\varepsilon}{\sigma}}\sqrt{\frac{\mu}{\varepsilon}} \tag{4.118}$$

we note that the magnitude of the intrinsic impedance of a good conductor medium is a small fraction of the intrinsic impedance of a dielectric medium having the same ε and μ. It follows that for the same electric field, the magnetic field inside a good conductor is much larger than the magnetic field inside a dielectric having the same ε and μ as those of the conductor.

Case 4: Perfect conductors. Perfect conductors are idealizations of good conductors in the limit that $\sigma \to \infty$. From (4.115), we note that the skin depth is equal to zero, and, hence, there is no penetration of fields into the material. Thus, no time-varying fields can exist inside a perfect conductor.

Summarizing the discussion of the special cases, we observe that as σ varies from 0 to ∞, a material is classified as a perfect dielectric for $\sigma = 0$, an imperfect dielectric for $\sigma \neq 0$ but $\ll \omega\varepsilon$, a good conductor for $\sigma \gg \omega\varepsilon$, and finally a perfect conductor in the limit that $\sigma \to \infty$. This implies that a material of nonzero σ behaves as an imperfect dielectric for $f \gg f_q$ but as a good conductor for $f \ll f_q$, where f_q, the transition frequency, is equal to $\sigma/2\pi\varepsilon$. In practice, however, the situation is not so simple because, as was already mentioned in Section 4.4, σ and ε are in general functions of frequency.

K4.5. Perfect dielectric; Imperfect dielectric; Good conductor; Conduction current versus displacement current; Skin effect; Perfect conductor.

D4.10. For a nonmagnetic ($\mu = \mu_0$) perfect dielectric material, find the relative permittivity for each of the following cases: **(a)** the phase velocity in the dielectric is one-third of its value in free space; **(b)** the rate of change of phase with distance at a fixed time in the dielectric for a wave of frequency f_0 is the same as the rate of change of phase with distance at a fixed time in free space for a wave of frequency $2f_0$; **(c)** for the same frequency, the wavelength in the dielectric is two-thirds of its value in free space; and **(d)** for the same electric-field amplitude, the magnetic-field amplitude in the dielectric is four times its value in free space.

Ans. **(a)** 9; **(b)** 4; **(c)** 2.25; **(d)** 16.

D4.11. For a uniform plane wave of frequency $f = 10^5$ Hz propagating in a good conductor medium, the fields undergo attenuation by the factor $e^{-\pi}$ in a distance of 2.5 m. Find the following: **(a)** the distance in which the fields undergo a change of phase by 2π rad for $f = 10^5$ Hz; **(b)** the distance by which a constant phase travels in 1 μs for $f = 10^5$ Hz; and **(c)** the distance by which a constant phase travels in 1 μs for $f = 10^4$ Hz, assuming the material parameters to be the same as at $f = 10^5$ Hz.

Ans. **(a)** 5 m; **(b)** 0.5 m; **(c)** 0.1581 m.

D4.12. The electric fields of uniform plane waves of the same frequency propagating in three different materials 1, 2, and 3 are given, respectively, by

(a) $\mathbf{E}_1 = E_0 e^{-0.4\pi z} \cos(2\pi \times 10^5 t - 0.4\pi z)\,\mathbf{a}_x$

(b) $\mathbf{E}_2 = E_0 e^{-2\pi \times 10^{-5} z} \cos(2\pi \times 10^5 t - 2\pi \times 10^{-3} z)\, \mathbf{a}_x$
(c) $\mathbf{E}_3 = E_0 e^{-0.004z} \cos(2\pi \times 10^5 t - 0.01z)\, \mathbf{a}_x$

For each material, determine if at the frequency of operation, it can be classified as an imperfect dielectric or a good conductor or neither of the two.

Ans. **(a)** Good conductor; **(b)** Imperfect dielectric; **(c)** Neither.

4.6 BOUNDARY CONDITIONS

Boundary condition explained

In our study of electromagnetics, we will be considering many problems involving more than one medium. Examples are reflections of waves at an air–dielectric interface, determination of capacitance for a multiple-dielectric capacitor, and guiding of waves in a metallic waveguide. To solve a problem involving a boundary surface between different media, we need to know the conditions satisfied by the field components at the boundary. These are known as the *boundary conditions*. They are a set of relationships relating the field components at a point adjacent to and on one side of the boundary, to the field components at a corresponding point adjacent to and on the other side of the boundary. These relationships arise from the fact that Maxwell's equations in integral form involve closed paths and surfaces and they must be satisfied for all possible closed paths and surfaces, whether they lie entirely in one medium or encompass a portion of the boundary between two different media. In the latter case, Maxwell's equations in integral form must be satisfied collectively by the fields on either side of the boundary, thereby resulting in the boundary conditions.

We shall derive the boundary conditions by considering the Maxwell's equations

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S} \tag{4.119a}$$

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt}\int_S \mathbf{D} \cdot d\mathbf{S} \tag{4.119b}$$

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho\, dv \tag{4.119c}$$

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0 \tag{4.119d}$$

and applying them one at a time to a closed path or a closed surface encompassing the boundary, and in the limit that the area enclosed by the closed path or the volume bounded by the closed surface goes to zero. Thus, let us consider two semi-infinite media separated by a plane boundary, as shown in Fig. 4.23. Let us denote the quantities pertinent to medium 1 by subscript 1 and the quantities pertinent to medium 2 by subscript 2. Let $\mathbf{a}_n$ be the unit normal vector to the surface and directed into medium 1, as shown in Fig. 4.23, and let all normal components of fields at the boundary in both media denoted by an additional subscript n be directed along $\mathbf{a}_n$. Let the surface charge density (C/m^2) and the

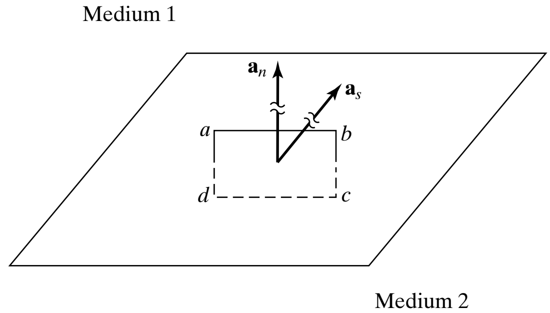


FIGURE 4.23

For deriving the boundary conditions resulting from Faraday's law and Ampère's circuital law.

surface current density (A/m) on the boundary be ρ_S and $\mathbf{J}_S$, respectively. Note that, in general, the fields at the boundary in both media and the surface charge and current densities are functions of position on the boundary.

Boundary condition for $E_{tangential}$

First, we consider a rectangular closed path *abcda* of infinitesimal area in the plane normal to the boundary and with its sides *ab* and *cd* parallel to and on either side of the boundary, as shown in Fig. 4.23. Applying Faraday's law (4.119a) to this path in the limit that *ad* and *bc* → 0 by making the area *abcd* tend to zero, but with *ab* and *cd* remaining on either side of the boundary, we have

$$\lim_{\substack{ad\to 0\\ bc\to 0}} \oint_{abcda} \mathbf{E}\cdot d\mathbf{l} = -\lim_{\substack{ad\to 0\\ bc\to 0}} \frac{d}{dt}\int_{\substack{\text{area}\\ abcd}} \mathbf{B}\cdot d\mathbf{S} \tag{4.120}$$

In this limit, the contributions from *ad* and *bc* to the integral on the left side of (4.120) approach zero. Since *ab* and *cd* are infinitesimal, the sum of the contributions from *ab* and *cd* becomes $[E_{ab}(ab) + E_{cd}(cd)]$, where E_{ab} and E_{cd} are the components of $\mathbf{E}_1$ and $\mathbf{E}_2$ along *ab* and *cd*, respectively. The right side of (4.120) is equal to zero, since the magnetic flux crossing the area *abcd* approaches zero as the area *abcd* tends to zero. Thus, (4.120) gives

$$E_{ab}(ab) + E_{cd}(cd) = 0$$

or, since *ab* and *cd* are equal and $E_{dc} = -E_{cd}$,

$$E_{ab} - E_{dc} = 0 \tag{4.121}$$

Let us now define $\mathbf{a}_s$ to be the unit vector normal to the area *abcd* and in the direction of advance of a right-hand screw as it is turned in the sense of the closed path *abcda*. Noting then that $\mathbf{a}_s \times \mathbf{a}_n$ is the unit vector along *ab*, we can write (4.121) as

$$\mathbf{a}_s \times \mathbf{a}_n \cdot (\mathbf{E}_1 - \mathbf{E}_2) = 0$$

Rearranging the order of the scalar triple product, we obtain

$$\mathbf{a}_s \cdot \mathbf{a}_n \times (\mathbf{E}_1 - \mathbf{E}_2) = 0 \tag{4.122}$$

Since we can choose the rectangle *abcd* to be in any plane normal to the boundary, (4.122) must be true for all orientations of $\mathbf{a}_s$. It then follows that

$$\boxed{\mathbf{a}_n \times (\mathbf{E}_1 - \mathbf{E}_2) = \mathbf{0}} \tag{4.123a}$$

or, in scalar form,

$$\boxed{E_{t1} - E_{t2} = 0} \tag{4.123b}$$

where E_{t1} and E_{t2} are the components of $\mathbf{E}_1$ and $\mathbf{E}_2$, respectively, tangential to the boundary. In words, (4.123a) and (4.123b) state that *at any point on the boundary, the components of* $\mathbf{E}_1$ *and* $\mathbf{E}_2$ *tangential to the boundary are equal.*

Boundary condition for $\mathrm{H}_{tangential}$

Similarly, applying Ampère's circuital law (4.119b) to the closed path in the limit that *ad* and $bc \rightarrow 0$, we have

$$\lim_{\substack{ad\to 0\\ bc\to 0}} \oint_{abcda} \mathbf{H}\cdot d\mathbf{l} = \lim_{\substack{ad\to 0\\ bc\to 0}} \int_{\substack{\text{area}\\ abcd}} \mathbf{J}\cdot d\mathbf{S} + \lim_{\substack{ad\to 0\\ bc\to 0}} \frac{d}{dt} \int_{\substack{\text{area}\\ abcd}} \mathbf{D}\cdot d\mathbf{S} \tag{4.124}$$

Using the same argument as for the left side of (4.120), we obtain the quantity on the left side of (4.124) to be equal to $[H_{ab}(ab) + H_{cd}(cd)]$, where H_{ab} and H_{cd} are the components of $\mathbf{H}_1$ and $\mathbf{H}_2$ along *ab* and *cd*, respectively. The second integral on the right side of (4.124) is zero since the displacement flux crossing the area *abcd* approaches zero as the area *abcd* tends to zero. The first integral on the right side of (4.124) would also be equal to zero but for a contribution from the surface current on the boundary, because letting the area *abcd* tend to zero with *ab* and *cd* on either side of the boundary reduces only the volume current, if any, enclosed by it to zero, keeping the surface current still enclosed by it. This contribution is the surface current flowing normal to the line that *abcd* approaches as it tends to zero, that is, $[\mathbf{J}_S \cdot \mathbf{a}_s](ab)$. Thus, (4.124) gives

$$H_{ab}(ab) + H_{cd}(cd) = (\mathbf{J}_S \cdot \mathbf{a}_s)(ab)$$

or, since *ab* and *cd* are equal and $H_{dc} = -H_{cd}$,

$$H_{ab} - H_{dc} = \mathbf{J}_S \cdot \mathbf{a}_s \tag{4.125}$$

In terms of $\mathbf{H}_1$ and $\mathbf{H}_2$, we have

$$\mathbf{a}_s \times \mathbf{a}_n \cdot (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_S \cdot \mathbf{a}_s$$

or

$$\mathbf{a}_s \cdot \mathbf{a}_n \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{a}_s \cdot \mathbf{J}_S \tag{4.126}$$

Since (4.126) must be true for all orientations of $\mathbf{a}_s$, that is, for a rectangle *abcd* in any plane normal to the boundary, it follows that

$$\boxed{\mathbf{a}_n \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_S} \tag{4.127a}$$

or, in scalar form,

$$\boxed{H_{t1} - H_{t2} = J_S} \tag{4.127b}$$

where H_{t1} and H_{t2} are the components of $\mathbf{H}_1$ and $\mathbf{H}_2$, respectively, tangential to the boundary. In words, (4.127a) and (4.127b) state that *at any point on the boundary, the components of* $\mathbf{H}_1$ *and* $\mathbf{H}_2$ *tangential to the boundary are discontinuous by the amount equal to the surface current density at that point.* It should be noted that the information concerning the direction of $\mathbf{J}_S$ relative to that of $(\mathbf{H}_1 - \mathbf{H}_2)$, which is contained in (4.127a), is not present in (4.127b). Thus, in general, (4.127b) is not sufficient, and it is necessary to use (4.127a).

Boundary condition for $\mathbf{D}_{normal}$

Now, we consider a rectangular box *abcdefgh* of infinitesimal volume enclosing an infinitesimal area of the boundary and parallel to it, as shown in Fig. 4.24. Applying Gauss' law for the electric field (4.119c) to this box in the limit that the side surfaces (abbreviated *ss*) tend to zero by making the volume of the box tend to zero but with the sides *abcd* and *efgh* remaining on either side of the boundary, we have

$$\lim_{ss\to 0} \oint_{\substack{\text{surface}\\ \text{of the box}}} \mathbf{D}\cdot d\mathbf{S} = \lim_{ss\to 0} \int_{\substack{\text{volume}\\ \text{of the box}}} \rho\, dv \tag{4.128}$$

In this limit, the contributions from the side surfaces to the integral on the left side of (4.128) approach zero. The sum of the contributions from the top and bottom surfaces becomes $[D_{n1}(abcd) - D_{n2}(efgh)]$ since *abcd* and *efgh* are infinitesimal. The quantity on the right side of (4.128) would be zero but for the surface charge on the boundary, since letting the volume of the box tend to zero with the sides *abcd* and *efgh* on either side of it reduces only the volume charge, if any, enclosed by it to zero, keeping the surface charge still enclosed by it. This surface charge is equal to $\rho_S(abcd)$. Thus, (4.128) gives

$$D_{n1}(abcd) - D_{n2}(efgh) = \rho_S(abcd)$$

or, since *abcd* and *efgh* are equal,

$$\boxed{D_{n1} - D_{n2} = \rho_S} \tag{4.129a}$$

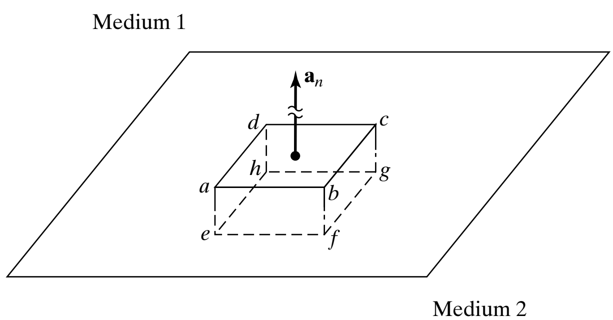


FIGURE 4.24

For deriving the boundary conditions resulting from the two Gauss' laws.

In terms of $\mathbf{D}_1$ and $\mathbf{D}_2$, (4.129a) is given by

$$\boxed{\mathbf{a}_n \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_S} \tag{4.129b}$$

In words, (4.129a) and (4.129b) state that *at any point on the boundary, the components of* $\mathbf{D}_1$ and $\mathbf{D}_2$ *normal to the boundary are discontinuous by the amount of the surface charge density at that point.*

Boundary condition for B_{normal}

Similarly, applying Gauss' law for the magnetic field (4.119d) to the box *abcdefgh* in the limit that the side surfaces tend to zero, we have

$$\lim_{ss \to 0} \oint_{\substack{\text{surface} \\ \text{of the box}}} \mathbf{B} \cdot d\mathbf{S} = 0 \tag{4.130}$$

Using the same argument as for the left side of (4.128), we obtain the quantity on the left side of (4.130) to be equal to $[B_{n1}(abcd) - B_{n2}(efgh)]$. Thus, (4.130) gives

$$B_{n1}(abcd) - B_{n2}(efgh) = 0$$

or, since *abcd* and *efgh* are equal

$$\boxed{B_{n1} - B_{n2} = 0} \tag{4.131a}$$

In terms of $\mathbf{B}_1$ and $\mathbf{B}_2$, (4.131a) is given by

$$\mathbf{a}_n \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0 \tag{4.131b}$$

In words, (4.131a) and (4.131b) state that *at any point on the boundary, the components of* $\mathbf{B}_1$ *and* $\mathbf{B}_2$ *normal to the boundary are equal.*

Summarizing the boundary conditions, we have

$$\mathbf{a}_n \times (\mathbf{E}_1 - \mathbf{E}_2) = \mathbf{0} \tag{4.132a}$$
$$\mathbf{a}_n \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_S \tag{4.132b}$$
$$\mathbf{a}_n \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_S \tag{4.132c}$$
$$\mathbf{a}_n \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0 \tag{4.132d}$$

or, in scalar form,

$$E_{t1} - E_{t2} = 0 \tag{4.133a}$$
$$H_{t1} - H_{t2} = J_S \tag{4.133b}$$
$$D_{n1} - D_{n2} = \rho_S \tag{4.133c}$$
$$B_{n1} - B_{n2} = 0 \tag{4.133d}$$

as illustrated in Fig. 4.25. Although we have derived these boundary conditions by considering a plane interface between the two media, it should be obvious that we can consider any arbitrary-shaped boundary and obtain the same results by letting the sides *ab* and *cd* of the rectangle and the top and bottom surfaces of

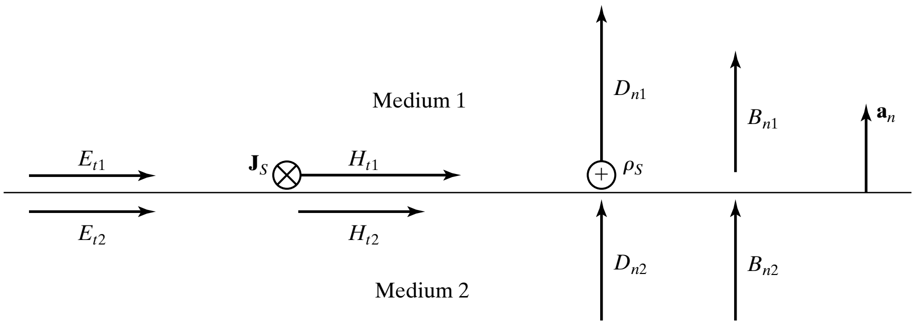


FIGURE 4.25

For illustrating the boundary conditions at an interface between two different media.

the box tend to zero, in addition to the limits that the sides *ad* and *bc* of the rectangle and the side surfaces of the box tend to zero.

The boundary conditions given by (4.132a) – (4.132d) are general. When they are applied to particular cases, the special properties of the pertinent media come into play. Two such cases are important to be considered. They are as follows.

Boundary conditions at interface between perfect dielectrics

Interface between two perfect dielectric media: Since for a perfect dielectric, $\sigma = 0$, $\mathbf{J}_c = \sigma\mathbf{E} = \mathbf{0}$. Thus, there cannot be any conduction current in a perfect dielectric, which in turn rules out any accumulation of free charge on the surface of a perfect dielectric. Hence, in applying the boundary conditions (4.132a)–(4.132d) to an interface between two perfect dielectric media, we set ρ_S and $\mathbf{J}_S$ equal to zero, thereby obtaining

$$\boxed{\begin{aligned} \mathbf{a}_n \times (\mathbf{E}_1 - \mathbf{E}_2) &= \mathbf{0} \\ \mathbf{a}_n \times (\mathbf{H}_1 - \mathbf{H}_2) &= \mathbf{0} \\ \mathbf{a}_n \cdot (\mathbf{D}_1 - \mathbf{D}_2) &= 0 \\ \mathbf{a}_n \cdot (\mathbf{B}_1 - \mathbf{B}_2) &= 0 \end{aligned}} \qquad \begin{aligned} &(4.134\text{a}) \\ &(4.134\text{b}) \\ &(4.134\text{c}) \\ &(4.134\text{d}) \end{aligned}$$

These boundary conditions tell us that the tangential components of **E** and **H** and the normal components of **D** and **B** are continuous at the boundary.

Boundary conditions on a perfect conductor surface

Surface of a perfect conductor: No time-varying fields can exist in a perfect conductor. In view of this, the boundary conditions on a perfect conductor surface are obtained by setting the fields with subscript 2 in (4.132a) – (4.132d) equal to zero. Thus, we obtain

$$\boxed{\begin{aligned} \mathbf{a}_n \times \mathbf{E} &= \mathbf{0} \\ \mathbf{a}_n \times \mathbf{H} &= \mathbf{J}_S \\ \mathbf{a}_n \cdot \mathbf{D} &= \rho_S \\ \mathbf{a}_n \cdot \mathbf{B} &= 0 \end{aligned}} \qquad \begin{aligned} &(4.135\text{a}) \\ &(4.135\text{b}) \\ &(4.135\text{c}) \\ &(4.135\text{d}) \end{aligned}$$

where we have also omitted subscripts 1, so that **E**, **H**, **D**, and **B** are the fields on the perfect conductor surface. The boundary conditions (4.135a) and (4.135d) tell us that on a perfect conductor surface, the tangential component of the electric field intensity and the normal component of the magnetic field intensity are zero. Hence, the electric field must be completely normal, and the magnetic field must be completely tangential to the surface. The remaining two boundary conditions (4.135c) and (4.135b) tell us that the (normal) displacement flux density is equal to the surface charge density and the (tangential) magnetic field intensity is equal in magnitude to the surface current density.

Example 4.7 Application of boundary conditions

In Fig. 4.26, the region $x < 0$ is a perfect conductor, the region $0 < x < d$ is a perfect dielectric of $\varepsilon = 2\varepsilon_0$ and $\mu = \mu_0$, and the region $x > d$ is free space. The electric and magnetic fields in the region $0 < x < d$ are given at a particular instant of time by

$$\begin{aligned}\mathbf{E} &= E_1 \cos \pi x \sin 2\pi z \, \mathbf{a}_x + E_2 \sin \pi x \cos 2\pi z \, \mathbf{a}_z \\ \mathbf{H} &= H_1 \cos \pi x \sin 2\pi z \, \mathbf{a}_y\end{aligned}$$

We wish to find (a) ρ_S and $\mathbf{J}_S$ on the surface $x = 0$ and (b) **E** and **H** for $x = d+$, that is, immediately adjacent to the $x = d$ plane and on the free-space side, at that instant of time.

(a) Denoting the perfect dielectric medium $(0 < x < d)$ to be medium 1 and the perfect conductor medium $(x < 0)$ to be medium 2, we have $\mathbf{a}_n = \mathbf{a}_x$, and all fields with subscript 2 are equal to zero. Then from (4.132c) and (4.132b), we obtain

$$\begin{aligned}[\rho_S]_{x=0} &= \mathbf{a}_n \cdot [\mathbf{D}_1]_{x=0} = \mathbf{a}_x \cdot 2\varepsilon_0 E_1 \sin 2\pi z \, \mathbf{a}_x \\ &= 2\varepsilon_0 E_1 \sin 2\pi z \\ [\mathbf{J}_S]_{x=0} &= \mathbf{a}_n \times [\mathbf{H}_1]_{x=0} = \mathbf{a}_x \times H_1 \sin 2\pi z \, \mathbf{a}_y \\ &= H_1 \sin 2\pi z \, \mathbf{a}_z\end{aligned}$$

Note that the remaining two boundary conditions (4.132a) and (4.132d) are already satisfied by the given fields, since E_y and B_x do not exist and for $x = 0$, $E_z = 0$. Also note that what we have done here is equivalent to using (4.135a) – (4.135d), since the boundary is the surface of a perfect conductor.

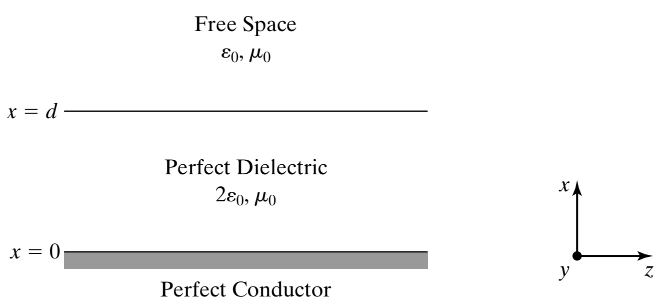


FIGURE 4.26

For illustrating the application of boundary conditions.

(b) Denoting the perfect dielectric medium $(0 < x < d)$ to be medium 1 and the free-space medium $(x > d)$ to be medium 2 and setting $\rho_S = 0$, we obtain from (4.133a) and (4.133c)

$$\begin{aligned}
[E_y]_{x=d+} &= [E_y]_{x=d-} = 0 \\
[E_z]_{x=d+} &= [E_z]_{x=d-} = E_2 \sin \pi d \cos 2\pi z \\
[D_x]_{x=d+} &= [D_x]_{x=d-} = 2\varepsilon_0 [E_x]_{x=d-} \\
&= 2\varepsilon_0 E_1 \cos \pi d \sin 2\pi z \\
[E_x]_{x=d+} &= \frac{1}{\varepsilon_0}[D_x]_{x=d+} \\
&= 2E_1 \cos \pi d \sin 2\pi z
\end{aligned}$$

Thus

$$[\mathbf{E}]_{x=d+} = 2E_1 \cos \pi d \sin 2\pi z \, \mathbf{a}_x + E_2 \sin \pi d \cos 2\pi z \, \mathbf{a}_z$$

Setting $\mathbf{J}_S = \mathbf{0}$ and using (4.133b) and (4.133d), we obtain

$$\begin{aligned}
[H_y]_{x=d+} &= [H_y]_{x=d-} = H_1 \cos \pi d \sin 2\pi z \\
[H_z]_{x=d+} &= [H_z]_{x=d-} = 0 \\
[B_x]_{x=d+} &= [B_x]_{x=d-} = 0
\end{aligned}$$

Thus,

$$[\mathbf{H}]_{x=d+} = H_1 \cos \pi d \sin 2\pi z \, \mathbf{a}_y$$

Note that what we have done here is equivalent to using (4.134a) – (4.134d), since the boundary is the interface between two perfect dielectrics.

K4.6. Boundary conditions; Tangential component of **E**; Tangential component of **H**; Normal component of **D**; Normal component of **B**.

D4.13. For each of the following values of the displacement flux density at a point on the surface of a perfect conductor (no electric field inside and hence $E_t = 0$ on the surface), find the surface charge density at that point: **(a)** $\mathbf{D} = D_0(\mathbf{a}_x - 2\mathbf{a}_y + 2\mathbf{a}_z)$ and pointing away from the surface; **(b)** $\mathbf{D} = D_0(\mathbf{a}_x + \sqrt{3}\mathbf{a}_z)$ and pointing toward the surface; and **(c)** $\mathbf{D} = D_0(0.8\mathbf{a}_x + 0.6\mathbf{a}_y)$ and pointing away from the surface. Assume D_0 to be positive for all cases.

Ans. **(a)** $3D_0$; **(b)** $-2D_0$; **(c)** D_0.

D4.14. The region $x > 0$ is a perfect dielectric of permittivity $2\varepsilon_0$ and the region $x < 0$ is a perfect dielectric of permittivity $3\varepsilon_0$. Consider the field components at point 1 on the $+x$-side of the boundary to be denoted by subscript 1 and the field components at the adjacent point 2 on the $-x$-side of the boundary to be denoted by subscript 2. If $\mathbf{E}_1 = E_0(2\mathbf{a}_x + \mathbf{a}_y)$, find the following: **(a)** E_{x1}/E_{x2}; **(b)** E_1/E_2; and **(c)** D_1/D_2.

Ans. **(a)** 1.5; **(b)** $3/\sqrt{5}$; **(c)** $2/\sqrt{5}$.

D4.15. The plane $z = 0$ forms the boundary between free space $(z > 0)$ and another medium. Find the following: **(a)** $\mathbf{J}_S(0, 0, 0)$ at $t = 0$ if $z < 0$ is a perfect conductor and $\mathbf{H}(0, 0, 0+) = H_0(3\mathbf{a}_x - 4\mathbf{a}_y) \cos \omega t$; **(b)** $\mathbf{H}(0, 0, 0+)$ if $z < 0$ is a magnetic material of $\mu = 20\mu_0$ and $\mathbf{H}(0, 0, 0-) = H_0(10\mathbf{a}_x + \mathbf{a}_z)$; and **(c)** the ratio of $B(0, 0, 0-)$ to $B(0, 0, 0+)$ for the case of **(b)**.

Ans. **(a)** $H_0(4\mathbf{a}_x + 3\mathbf{a}_y)$; **(b)** $10H_0(\mathbf{a}_x + 2\mathbf{a}_z)$; **(c)** 8.989.

4.7 REFLECTION AND TRANSMISSION OF UNIFORM PLANE WAVES

Thus far, we have considered uniform plane wave propagation in unbounded media. Practical situations are characterized by propagation involving several different media. When a wave is incident on a boundary between two different media, a reflected wave is produced. In addition, if the second medium is not a perfect conductor, a transmitted wave is set up. Together, these waves satisfy the boundary conditions at the interface between the two media. In this section, we shall consider these phenomena for waves incident normally on plane boundaries.

Normal incidence on a plane interface

To do this, let us consider the situation shown in Fig. 4.27 in which steady-state conditions are established by uniform plane waves of radian frequency ω propagating normal to the plane interface $z = 0$ between two media characterized by two different sets of values of σ, ε, and μ, where $\sigma \neq \infty$. We shall assume that a $(+)$ wave is incident from medium 1 $(z < 0)$ onto the interface, thereby setting up a reflected $(-)$ wave in that medium, and a transmitted $(+)$ wave in medium 2 $(z > 0)$. For convenience, we shall work with the phasor or complex field components. Thus, considering the electric fields to be in the x-direction and the magnetic fields to be in the y-direction, we can write the solution for the complex field components in medium 1 to be

$$\bar{E}_{1x}(z) = \bar{E}_1^+ e^{-\bar{\gamma}_1 z} + \bar{E}_1^- e^{\bar{\gamma}_1 z} \tag{4.136a}$$

$$\begin{aligned} \bar{H}_{1y}(z) &= \bar{H}_1^+ e^{-\bar{\gamma}_1 z} + \bar{H}_1^- e^{\bar{\gamma}_1 z} \\ &= \frac{1}{\bar{\eta}_1}(\bar{E}_1^+ e^{-\bar{\gamma}_1 z} - \bar{E}_1^- e^{\bar{\gamma}_1 z}) \end{aligned} \tag{4.136b}$$

where $\bar{E}_1^+, \bar{E}_1^-, \bar{H}_1^+$, and $\bar{H}_1^-$ are the incident and reflected wave electric and magnetic field components, respectively, at $z = 0-$ in medium 1 and

$$\bar{\gamma}_1 = \sqrt{j\omega\mu_1(\sigma_1 + j\omega\varepsilon_1)} \tag{4.137a}$$

$$\bar{\eta}_1 = \sqrt{\frac{j\omega\mu_1}{\sigma_1 + j\omega\varepsilon_1}} \tag{4.137b}$$

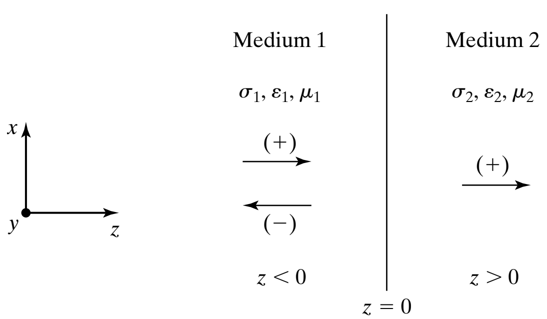


FIGURE 4.27

Normal incidence of uniform plane waves on a plane interface between two different media.

Recall that the real field corresponding to a complex field component is obtained by multiplying the complex field component by $e^{j\omega t}$ and taking the real part of the product. The complex field components in medium 2 are given by

$$\bar{E}_{2x}(z) = \bar{E}_2^+ e^{-\bar{\gamma}_2 z} \tag{4.138a}$$

$$\begin{aligned} \bar{H}_{2y}(z) &= \bar{H}_2^+ e^{-\bar{\gamma}_2 z} \\ &= \frac{\bar{E}_2^+}{\bar{\eta}_2} e^{-\bar{\gamma}_2 z} \end{aligned} \tag{4.138b}$$

where $\bar{E}_2^+$ and $\bar{H}_2^+$ are the transmitted wave electric- and magnetic-field components at $z = 0+$ in medium 2 and

$$\bar{\gamma}_2 = \sqrt{j\omega\mu_2(\sigma_2 + j\omega\varepsilon_2)} \tag{4.139a}$$

$$\bar{\eta}_2 = \sqrt{\frac{j\omega\mu_2}{\sigma_2 + j\omega\varepsilon_2}} \tag{4.139b}$$

Reflection and transmission coefficients

To satisfy the boundary conditions at $z = 0$, we note that (1) the components of both electric and magnetic fields are entirely tangential to the interface and (2) in view of the finite conductivities of the media, no surface current exists on the interface (currents flow in the volumes of the media). Hence, from the phasor forms of the boundary conditions (4.133a) and (4.133b), we have

$$[\bar{E}_{1x}]_{z=0} = [\bar{E}_{2x}]_{z=0} \tag{4.140a}$$

$$[\bar{H}_{1y}]_{z=0} = [\bar{H}_{2y}]_{z=0} \tag{4.140b}$$

Applying these to the solution pairs given by (4.136a, b) and (4.138a, b), we have

$$\bar{E}_1^+ + \bar{E}_1^- = \bar{E}_2^+ \tag{4.141a}$$

$$\frac{1}{\bar{\eta}_1}(\bar{E}_1^+ - \bar{E}_1^-) = \frac{1}{\bar{\eta}_2}\bar{E}_2^+ \tag{4.141b}$$

We now define the *reflection coefficient* at the boundary, denoted by the symbol $\bar{\Gamma}$, to be the ratio of the reflected wave electric field at the boundary to the incident wave electric field at the boundary. From (4.141a) and (4.141b), we obtain

$$\boxed{\bar{\Gamma} = \frac{\bar{E}_1^-}{\bar{E}_1^+} = \frac{\bar{\eta}_2 - \bar{\eta}_1}{\bar{\eta}_2 + \bar{\eta}_1}} \tag{4.142}$$

Note that the ratio of the reflected wave magnetic field at the boundary to the incident wave magnetic field at the boundary is given by

$$\frac{\bar{H}_1^-}{\bar{H}_1^+} = \frac{-\bar{E}_1^-/\bar{\eta}_1}{\bar{E}_1^+/\bar{\eta}_1} = -\frac{\bar{E}_1^-}{\bar{E}_1^+} = -\bar{\Gamma} \tag{4.143}$$

The ratio of the transmitted wave electric field at the boundary to the incident wave electric field at the boundary, known as the *transmission coefficient* and denoted by the symbol $\bar{\tau}$, is given by

$$\boxed{\bar{\tau} = \frac{\bar{E}_2^+}{\bar{E}_1^+} = \frac{\bar{E}_1^+ + \bar{E}_1^-}{\bar{E}_1^+} = 1 + \bar{\Gamma}} \tag{4.144}$$

where we have used (4.141a). The ratio of the transmitted wave magnetic field at the boundary to the incident wave magnetic field at the boundary is given by

$$\frac{\bar{H}_2^+}{\bar{H}_1^+} = \frac{\bar{H}_1^+ + \bar{H}_1^-}{\bar{H}_1^+} = 1 - \bar{\Gamma} \tag{4.145}$$

The reflection and transmission coefficients given by (4.142) and (4.144), respectively, enable us to find the reflected and transmitted wave fields for a given incident wave field. We observe the following properties of $\bar{\Gamma}$ and $\bar{\tau}$:

1. For $\bar{\eta}_2 = \bar{\eta}_1$, $\bar{\Gamma} = 0$ and $\bar{\tau} = 1$. The incident wave is entirely transmitted. The situation then corresponds to a "matched" condition. A trivial case occurs when the two media have identical values of the material parameters.
2. For $\sigma_1 = \sigma_2 = 0$, that is, when both media are perfect dielectrics, $\bar{\eta}_1$ and $\bar{\eta}_2$ are real. Hence, $\bar{\Gamma}$ and $\bar{\tau}$ are real. In particular, if the two media have the same permeability μ but different permittivities ε_1 and ε_2, then

$$\begin{aligned}\bar{\Gamma} &= \frac{\sqrt{\mu/\varepsilon_2} - \sqrt{\mu/\varepsilon_1}}{\sqrt{\mu/\varepsilon_2} + \sqrt{\mu/\varepsilon_1}} \\ &= \frac{1 - \sqrt{\varepsilon_2/\varepsilon_1}}{1 + \sqrt{\varepsilon_2/\varepsilon_1}}\end{aligned} \tag{4.146}$$

$$\bar{\tau} = \frac{2}{1 + \sqrt{\varepsilon_2/\varepsilon_1}} \tag{4.147}$$

3. For $\sigma_2 \to \infty$, $\bar{\eta}_2 \to 0$, $\bar{\Gamma} \to -1$, and $\bar{\tau} \to 0$. Thus, if medium 2 is a perfect conductor, the incident wave is entirely reflected, as it should be since there cannot be any time-varying fields inside a perfect conductor. The superposition of the reflected and incident waves would then give rise to the so-called complete standing waves in medium 1. We shall discuss complete standing waves as well as partial standing waves when we study the topic of sinusoidal steady-state analysis of waves on transmission lines in Chapter 7.

Example 4.8 Normal incidence of a uniform plane wave onto a material medium

Region 1 ($z < 0$) is free space, whereas region 2 ($z > 0$) is a material medium characterized by $\sigma = 10^{-4}$ S/m, $\varepsilon = 5\varepsilon_0$, and $\mu = \mu_0$. For a uniform plane wave having the electric field

$$\mathbf{E}_i = E_0 \cos\,(3\pi \times 10^5 t - 10^{-3}\pi z)\,\mathbf{a}_x \text{ V/m}$$

incident on the interface $z = 0$ from region 1, we wish to obtain the expressions for the reflected and transmitted wave electric and magnetic fields.

From computation as in Example 4.5 for $\sigma = 10^{-4}$ S/m, $\varepsilon = 5\varepsilon_0$, $\mu = \mu_0$, and $f = (3\pi \times 10^5)/2\pi = 1.5 \times 10^5$ Hz,

$$\bar{\gamma} = (6.283 + j9.425) \times 10^{-3}$$
$$\bar{\eta} = 104.559\underline{/33.69^\circ} = 104.559\underline{/0.1872\pi}$$

Then

$$\begin{aligned}\bar{\Gamma} &= \frac{\bar{\eta} - \eta_0}{\bar{\eta} + \eta_0} = \frac{104.559\underline{/33.69^\circ} - 377}{104.559\underline{/33.69^\circ} + 377}\\ &= 0.6325\underline{/161.565^\circ} = 0.6325\underline{/0.8976\pi}\\ \bar{\tau} &= 1 + \bar{\Gamma} = 1 + 0.6325\underline{/161.565^\circ}\\ &= 0.4472\underline{/26.565^\circ} = 0.4472\underline{/0.1476\pi}\end{aligned}$$

Thus, the reflected and transmitted wave electric and magnetic fields are given by

$$\begin{aligned}\mathbf{E}_r &= 0.6325E_0 \cos\,(3\pi \times 10^5 t + 10^{-3}\pi z + 0.8976\pi)\,\mathbf{a}_x \text{ V/m}\\ \mathbf{H}_r &= -\frac{0.6325E_0}{377} \cos\,(3\pi \times 10^5 t + 10^{-3}\pi z + 0.8976\pi)\,\mathbf{a}_y \text{ A/m}\\ &= -1.678 \times 10^{-3}E_0 \cos\,(3\pi \times 10^5 t + 10^{-3}\pi z + 0.8976\pi)\,\mathbf{a}_y \text{ A/m}\\ \mathbf{E}_t &= 0.4472E_0 e^{-6.283\times10^{-3}z}\\ &\quad\cdot \cos\,(3\pi \times 10^5 t - 9.425 \times 10^{-3}z + 0.1476\pi)\,\mathbf{a}_x \text{ V/m}\\ \mathbf{H}_t &= \frac{0.4472E_0}{104.559} e^{-6.283\times10^{-3}z}\\ &\quad\cdot \cos\,(3\pi \times 10^5 t - 9.425 \times 10^{-3}z + 0.1476\pi - 0.1872\pi)\,\mathbf{a}_y \text{ A/m}\\ &= 4.277 \times 10^{-3}E_0 e^{-6.283\times10^{-3}z}\\ &\quad\cdot \cos\,(3\pi \times 10^5 t - 9.425 \times 10^{-3}z - 0.0396\pi)\,\mathbf{a}_y \text{ A/m}\end{aligned}$$

Note that at $z = 0$, the boundary conditions of $\mathbf{E}_i + \mathbf{E}_r = \mathbf{E}_t$ and $\mathbf{H}_i + \mathbf{H}_r = \mathbf{H}_t$ are satisfied, since

$$E_0 + 0.6325E_0 \cos 0.8976\pi = 0.4472E_0 \cos 0.1476\pi$$

and

$$\frac{E_0}{377} - 1.678 \times 10^3 E_0 \cos 0.8976\pi = 4.277 \times 10^3 E_0 \cos\,(-0.0396\pi)$$

K4.7. Plane interface between two material media; Normal incidence of uniform plane waves; Reflection; Transmission; Reflection and transmission coefficients.

D4.16. For each of the following cases of uniform plane waves of frequency $f = 1$ MHz incident normally from medium 1 $(z < 0)$ onto the interface $(z = 0)$ with medium 2 $(z > 0)$, find the values of $\bar{\Gamma}$ and $\bar{\tau}$: **(a)** Medium 1 is free space and the parameters of medium 2 are $\sigma = 10^{-3}$ S/m, $\varepsilon = 6\varepsilon_0$, and $\mu = \mu_0$; and **(b)** the parameters of medium 1 are $\sigma = 4$ S/m, $\varepsilon = 80\varepsilon_0$, and $\mu = \mu_0$, and the parameters of medium 2 are $\sigma = 10^{-3}$ S/m, $\varepsilon = 80\varepsilon_0$, and $\mu = \mu_0$.

Ans. **(a)** $0.6909\underline{/64.177^\circ}$, $0.3846\underline{/29.331^\circ}$; **(b)** $0.9486\underline{/-2.4155^\circ}$, $1.948\underline{/-1.177^\circ}$.

D4.17. The regions $z < 0$ and $z > 0$ are nonmagnetic ($\mu = \mu_0$) perfect dielectrics of permittivities ε_1 and ε_2, respectively. For a uniform plane wave incident from the region $z < 0$ normally onto the boundary $z = 0$, find $\varepsilon_2/\varepsilon_1$ for each of the following to hold at $z = 0$: **(a)** the electric field of the reflected wave is $-1/3$ times the electric field of the incident wave; **(b)** the electric field of the transmitted wave is 0.4 times the electric field of the incident wave; and **(c)** the electric field of the transmitted wave is six times the electric field of the reflected wave.

Ans. **(a)** 4; **(b)** 16; **(c)** 4/9.

SUMMARY

In this Chapter, we introduced materials. We learned that materials can be classified as (1) conductors, (2) semiconductors, (3) dielectrics, and (4) magnetic materials, depending on the nature of the response of the charged particles in the materials to applied fields. Conductors are characterized by conduction, which is the phenomenon of steady drift of free electrons under the influence of an applied electric field, thereby resulting in a conduction current. In semiconductors, also characterized by conduction, the charge carriers are not only electrons, but also holes. We learned that the conduction current density is related to the electric field intensity in the manner

$$\mathbf{J}_c = \sigma \mathbf{E} \tag{4.148}$$

where σ is the conductivity of the material. We discussed (1) the formation of surface charge at the boundaries of a conductor placed in a static electric field, (2) the derivation of Ohm's law in circuit theory, and (3) the Hall effect.

Dielectrics are characterized by polarization, which is the phenomenon of the creation and net alignment of electric dipoles, formed by the displacement of the centroids of the electron clouds from the centroids of the nucleii of the atoms, along the direction of an applied electric field. Magnetic materials are characterized by magnetization, which is the phenomenon of net alignment of the axes of the magnetic dipoles, formed by the electron orbital and spin motion around the nucleii of the atoms, along the direction of an applied magnetic field. To eliminate the need for explicitly taking into account the effects of polarization and magnetization, we revised the definitions of the displacement flux density vector and the magnetic field intensity vector, introduced in Sec. 2.3 for free space, to be applicable for a material medium. The revised definitions are

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$$
$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}$$

respectively, where **P** is the polarization vector, and **M** is the magnetization vector. We learned that for isotropic materials, these expressions simplify to

$$\mathbf{D} = \varepsilon \mathbf{E} \tag{4.149}$$

$$\mathbf{H} = \frac{\mathbf{B}}{\mu} \tag{4.150}$$

where

$$\varepsilon = \varepsilon_0 \varepsilon_r$$
$$\mu = \mu_0 \mu_r$$

are the permittivity and the permeability, respectively, of the material and the quantities ε_r and μ_r are the relative permittivity and the relative permeability, respectively, which take into account implicitly the effects of polarization and magnetization, respectively. Equations (4.148), (4.149), and (4.150) are known as the constitutive relations. We also discussed the hysteresis phenomenon associated with ferromagnetic materials and discussed an application based on the use of the hysteresis curve.

Next, we extended the treatment of uniform plane waves to a material medium. Starting with Maxwell's equations for a material medium given by

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\mu \frac{\partial \mathbf{H}}{\partial t}$$
$$\nabla \times \mathbf{H} = \mathbf{J}_c + \frac{\partial \mathbf{D}}{\partial t} = \sigma \mathbf{E} + \varepsilon \frac{\partial \mathbf{E}}{\partial t}$$

and using the phasor technique, we considered the infinite plane current sheet of uniform surface current density

$$\mathbf{J}_S = -J_{S0} \cos \omega t \, \mathbf{a}_x \text{ A/m}$$

in the xy-plane and embedded in the material medium, and obtained the electromagnetic field due to it to be

$$\mathbf{E} = \frac{|\overline{\eta}| J_{S0}}{2} e^{\mp \alpha z} \cos(\omega t \mp \beta z + \tau) \, \mathbf{a}_x \quad \text{for} \quad z \gtrless 0 \tag{4.151a}$$

$$\mathbf{H} = \pm \frac{J_{S0}}{2} e^{\mp \alpha z} \cos(\omega t \mp \beta z) \, \mathbf{a}_y \quad \text{for} \quad z \gtrless 0 \tag{4.151b}$$

In (4.151a, b), α and β are the attenuation and phase constants given, respectively, by the real and imaginary parts of the propagation constant, $\overline{\gamma}$. Thus,

$$\overline{\gamma} = \alpha + j\beta = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}$$

The quantities $|\overline{\eta}|$ and τ are the magnitude and phase angle, respectively, of the intrinsic impedance, $\overline{\eta}$, of the medium. Thus,

$$\overline{\eta} = |\overline{\eta}| e^{j\tau} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}$$

The solution given by (4.151a) and (4.151b) tells us that the wave propagation in the material medium is characterized by attenuation, as indicated by $e^{\mp\alpha z}$ and a phase difference between **E** and **H** in the amount τ. We also learned that these properties as well as the phase velocity are frequency-dependent.

We also generalized the Poynting's theorem, introduced in Sec. 3.7 for free space, to a material medium and learned that the power dissipation density associated with the phenomenon of conduction, and the electric and magnetic stored energy densities are given, respectively, by

$$p_d = \sigma E^2$$
$$w_e = \tfrac{1}{2}\varepsilon E^2$$
$$w_m = \tfrac{1}{2}\mu H^2$$

The power flow out of a closed surface S, as given by the surface integral of the Poynting vector, $\mathbf{P}$, over S, plus the power dissipated in the volume V bounded by S, is always equal to the sum of the time rates of decrease of the electric and magnetic stored energies in the volume V, as given by the Poynting's theorem

$$\oint_S \mathbf{P} \cdot d\mathbf{S} = -\int_V \sigma E^2\, dv - \frac{\partial}{\partial t}\int_v \frac{1}{2}\varepsilon E^2\, dv - \frac{\partial}{\partial t}\int_v \frac{1}{2}\mu H^2\, dv$$

Having discussed uniform plane wave propagation for the general case of a medium characterized by σ, ε, and μ, we then considered several special cases. These are summarized in the following:

Perfect dielectrics. For these materials, $\sigma = 0$. Wave propagation occurs without attenuation as in free space but with the propagation parameters governed by ε and μ instead of ε_0 and μ_0, respectively.

Imperfect dielectrics. A material is classified as an imperfect dielectric for $\sigma \ll \omega\varepsilon$, that is, conduction current density small in magnitude compared to the displacement current density. The only significant feature of wave propagation in an imperfect dielectric as compared to that in a perfect dielectric is the attenuation undergone by the wave.

Good conductors. A material is classified as a good conductor for $\sigma \gg \omega\varepsilon$, that is, conduction current density large in magnitude compared to the displacement current density. Wave propagation in a good conductor medium is characterized by attenuation and phase constants both equal to $\sqrt{\pi f \mu \sigma}$. Thus for large values of f and/or σ, the fields do not penetrate very deep into the conductor. This phenomenon is known as the skin effect. From considerations of the frequency dependence of the attenuation and wavelength for a fixed σ, we learned that low frequencies are more suitable for communication with underwater objects. We also learned that the intrinsic impedance of a good conductor medium is very low in magnitude compared to that of a dielectric medium having the same ε and μ.

Perfect conductors. These are idealizations of good conductors in the limit $\sigma \rightarrow \infty$. For $\sigma = \infty$, the skin depth, that is, the distance in which the fields inside a conductor are attenuated by a factor e^{-1}, is zero. Hence, there can be no penetration of fields into a perfect conductor.

As a prelude to the consideration of problems involving more than one medium, we derived the boundary conditions resulting from the application of Maxwell's equations in integral form to closed paths and closed surfaces encompassing the boundary between two media, and in the limits that the areas enclosed by the closed paths and the volumes bounded by the closed surfaces go to zero. These boundary conditions are given by

$$\begin{aligned}
\mathbf{a}_n \times (\mathbf{E}_1 - \mathbf{E}_2) &= \mathbf{0} \\
\mathbf{a}_n \times (\mathbf{H}_1 - \mathbf{H}_2) &= \mathbf{J}_S \\
\mathbf{a}_n \cdot (\mathbf{D}_1 - \mathbf{D}_2) &= \rho_S \\
\mathbf{a}_n \cdot (\mathbf{B}_1 - \mathbf{B}_2) &= 0
\end{aligned}$$

where the subscripts 1 and 2 refer to media 1 and 2, respectively, and $\mathbf{a}_n$ is unit vector normal to the boundary at the point under consideration and directed into medium 1. In words, the boundary conditions state that at a point on the boundary, the tangential components of **E** and the normal components of **B** are continuous, whereas the tangential components of **H** are discontinuous by the amount equal to J_S at that point, and the normal components of **D** are discontinuous by the amount equal to ρ_S at that point.

Two important special cases of boundary conditions are as follows: **(a)** At the boundary between two perfect dielectrics, the tangential components of **E** and **H** and the normal components of **D** and **B** are continuous. **(b)** On the surface of a perfect conductor, the tangential component of **E** and the normal component of **B** are zero, whereas the normal component of **D** is equal to the surface charge density, and the tangential component of **H** is equal in magnitude to the surface current density.

Finally, we considered uniform plane waves incident normally onto a plane boundary between two media, and we learned how to compute the reflected and transmitted wave fields for a given incident wave field.

REVIEW QUESTIONS

Q4.1. Distinguish between bound electrons and free electrons in an atom and briefly describe the phenomenon of conduction.

Q4.2. Discuss the classification of a material as a conductor, semiconductor, or dielectric with the aid of energy band diagrams.

Q4.3. What is mobility? Give typical values of mobilities for electrons and holes.

Q4.4. State Ohm's law valid at a point, defining the conductivities for conductors and semiconductors.

Q4.5. Explain how conduction current in a material is taken into consideration in Maxwell's equations.

Q4.6. Discuss the formation of surface charge at the boundaries of a conductor placed in a static electric field.

Q4.7. Discuss the derivation of Ohm's law in circuit theory from the Ohm's law valid at a point.

Q4.8. Discuss the Hall effect.

Q4.9. Briefly describe the phenomenon of polarization in a dielectric material. What are the different kinds of polarization?

Q4.10. What is an electric dipole? How is its strength defined?

Q4.11. What is a polarization vector? How is it related to the electric field intensity?

Q4.12. Discuss the effect of polarization in a dielectric material using the example of polarization surface charge.

Q4.13. Discuss how polarization current arises in a dielectric material. How is it taken into account in Maxwell's equations?

Q4.14. Discuss the revised definition of displacement flux density and the permittivity concept.

Q4.15. What is an anisotropic dielectric material? When can an effective permittivity be defined for an anisotropic dielectric material?

Q4.16. Briefly describe the phenomenon of magnetization in a magnetic material. What are the different kinds of magnetic materials?

Q4.17. What is a magnetic dipole? How is its strength defined?

Q4.18. What is a magnetization vector? How is it related to the magnetic flux density?

Q4.19. Discuss the effect of magnetization in a magnetic material using the example of magnetization surface current.

Q4.20. Discuss how magnetization current arises in a magnetic material. How is it taken into account in Maxwell's equations?

Q4.21. Discuss the revised definition of magnetic field intensity and the permeability concept.

Q4.22. Discuss the phenomenon of hysteresis associated with ferromagnetic materials.

Q4.23. Discuss the principles behind storing data on a floppy disk and retrieving the data from it.

Q4.24. State the constitutive relations for a material medium.

Q4.25. Discuss the determination of the electromagnetic field due to an infinite plane current sheet of sinusoidally time-varying current density embedded in a material medium, explaining how it is made convenient by using the phasor technique.

Q4.26. What is the propagation constant for a material medium? Discuss the significance of its real and imaginary parts.

Q4.27. What is the intrinsic impedance for a material medium? What is the consequence of its complex nature?

Q4.28. What is loss tangent? Discuss its significance.

Q4.29. Discuss the consequence of the frequency dependence of the phase velocity of a wave in a material medium.

Q4.30. How would you obtain the electromagnetic field due to a current sheet of nonsinusoidally time-varying current density embedded in a material medium?

Q4.31. State Poynting's theorem for a material medium.

Q4.32. What are the power dissipation density, the electric stored energy density, and the magnetic stored energy density associated with an electromagnetic field in a material medium?

Q4.33. What is the condition for a medium to be a perfect dielectric? How do the characteristics of wave propagation in a perfect dielectric medium differ from those of wave propagation in free space?

Q4.34. What is the criterion for a material to be an imperfect dielectric? What is the significant feature of wave propagation in an imperfect dielectric as compared to that in a perfect dielectric?

Q4.35. What is the criterion for a material to be a good conductor? Give two examples of materials that behave as good conductors for frequencies of up to several gigahertz.

Q4.36. What is skin effect? Discuss skin depth, giving some numerical values.

Q4.37. Why are low-frequency waves more suitable than high-frequency waves for communication with underwater objects?

Q4.38. Discuss the consequence of the low intrinsic impedance of a good conductor as compared to that of a dielectric medium having the same ε and μ.

Q4.39. Why can there be no fields inside a perfect conductor?

Q4.40. What is a boundary condition? How do boundary conditions arise and how are they derived?

Q4.41. Summarize the boundary conditions for the general case of a boundary between two arbitrary media, indicating correspondingly the Maxwell's equations in integral form from which they are derived.

Q4.42. Discuss the boundary conditions on the surface of a perfect conductor.

Q4.43. Discuss the boundary conditions at the interface between two perfect dielectric media.

Q4.44. Discuss the determination of the reflected and transitted wave fields from the fields of a wave incident normally onto a plane boundary between two material media.

Q4.45. What is the consequence of a wave incident on a perfect conductor?

PROBLEMS

Section 4.1

P4.1. Kinetic energy of electron motion under thermal agitation. Consider two electrons moving under thermal agitation with velocities equal in magnitude and opposite in direction. A uniform electric field is applied along the direction of motion of one of the electrons. Show that the gain in kinetic energy by the accelerating electron is greater than the loss in kinetic energy by the decelerating electron.

P4.2. Drift velocity of electron motion in a conductor for a sinusoidal electric field. **(a)** For a sinusoidally time-varying electric field $\mathbf{E} = \mathbf{E}_0 \cos \omega t$, where $\mathbf{E}_0$ is a constant, show that the steady-state solution to (4.2) is given by

$$\mathbf{v}_d = \frac{\tau e}{m\sqrt{1 + \omega^2\tau^2}} \mathbf{E}_0 \cos\,(\omega t - \tan^{-1} \omega\tau)$$

(b) Based on the assumption of one free electron per atom, the free electron density N_e in silver is $5.86 \times 10^{28}\ \text{m}^{-3}$. Using the conductivity for silver given in Table 4.1, find the frequency at which the drift velocity lags the applied field by

$\pi/4$. What is the ratio of the mobility at this frequency to the mobility at zero frequency?

P4.3. **Surface charge densities for plane conducting slabs with net surface charge densities.** **(a)** An infinite plane conducting slab carries uniformly distributed surface charges on both of its surfaces. If the net surface charge density, that is, the sum of the surface charge densities on the two surfaces, is ρ_{S0} C/m^2, find the surface charge densities on the two surfaces. **(b)** Two infinite plane parallel conducting slabs 1 and 2 carry uniformly distributed surface charges on all four of their surfaces. If the net surface charge densities are ρ_{S1} and ρ_{S2} C/m^2, respectively, for the slabs 1 and 2, find the surface charge densities on all four surfaces.

P4.4. **Line charge in the presence of a plane conductor.** The region $x < 0$ is occupied by a conductor. An infinitely long line charge of uniform density ρ_{L0} is situated along the line passing through $(d, 0, 0)$ and parallel to the z-axis, where $d > 0$. From the secondary field required to make the total electric field inside the conductor equal to zero and from symmetry considerations, as shown by the cross-sectional view in Fig. 4.28, show that the field outside the conductor is the same as the field due to the line charge passing through $(d, 0, 0)$ and a parallel "image" line charge of uniform density $-\rho_{L0}$ along the line passing through $(-d, 0, 0)$. Find the expression for the electric field outside the conductor. *Hint*: Use the expression for the electric field intensity due to an infinitely long line charge of uniform density ρ_{L0} along the z-axis given by $(\rho_{L0}/2\pi\varepsilon_0 r)\mathbf{a}_r$.

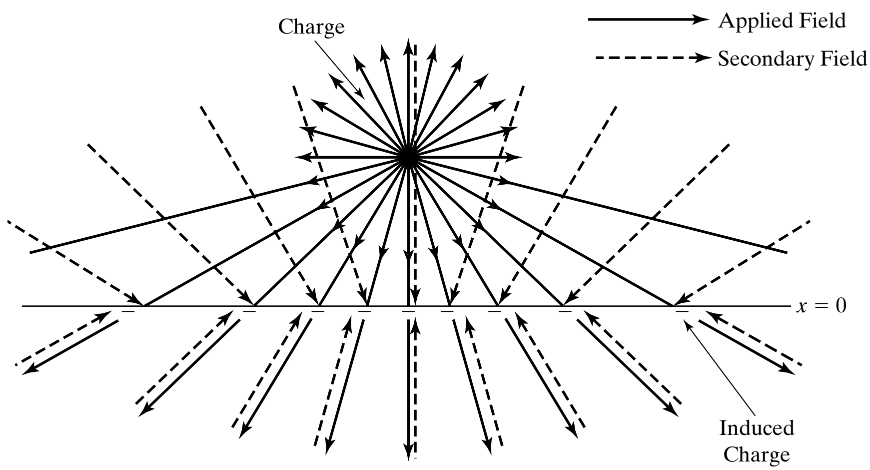


FIGURE 4.28

For Problem P4.4.

Section 4.2

P4.5. **Torque on an electric dipole in an applied electric field.** Show that the torque acting on an electric dipole of moment $\mathbf{p}$ due to an applied electric field $\mathbf{E}$ is $\mathbf{p} \times \mathbf{E}$. Compute the torque for a dipole consisting of 1 μC of charge at $(0, 0, 10^{-3})$ and $-1\ \mu$C of charge at $(0, 0, -10^{-3})$ in an electric field $\mathbf{E} = 10^3\,(2\mathbf{a}_x - \mathbf{a}_y + 2\mathbf{a}_z)$ V/m.

P4.6. Point charge surrounded by a spherical dielectric shell. A point charge Q is situated at the origin surrounded by a spherical dielectric shell of uniform permittivity $4\varepsilon_0$ and having inner and outer radii a and b, respectively. Find the following: **(a)** the **D** and **E** fields in the three regions $0 < r < a$, $a < r < b$, and $r > b$ and **(b)** the polarization vector inside the dielectric shell.

P4.7. Characteristics of an anisotropic dielectric material. An anisotropic dielectric material is characterized by the **D** to **E** relationship

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \varepsilon_0 \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

(a) Find **D** for $\mathbf{E} = E_0(\mathbf{a}_x + \mathbf{a}_y)$. **(b)** Find **D** for $\mathbf{E} = E_0(\mathbf{a}_x - \mathbf{a}_y)$. **(c)** Find **E** for $\mathbf{D} = D_0(\mathbf{a}_x + \mathbf{a}_y - 2\mathbf{a}_z)$. Comment on your result for each case.

P4.8. Characteristic polarizations and effective permittivities for an anisotropic dielectric. An anisotropic dielectric material is characterized by the **D** to **E** relationship

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & 0 \\ \varepsilon_{yx} & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

For $\mathbf{E} = E_x\mathbf{a}_x + E_y\mathbf{a}_y$, find the value(s) of E_y/E_x for which **D** is parallel to **E**. Find the effective permittivity for each case.

Section 4.3

P4.9. Magnetic dipole moment of a charged rotating disk of uniform charge density. Charge Q is distributed with uniform density on a circular disk of radius a lying in the xy-plane and rotating around the z-axis with angular velocity ω in the sense of increasing ϕ. Find the magnetic dipole moment of the rotating charge.

P4.10. Torque on a magnetic dipole in an applied magnetic field. Considering for simplicity a rectangular current loop in the xy-plane, show that the torque acting on a magnetic dipole of moment **m** due to an applied magnetic field **B** is $\mathbf{m} \times \mathbf{B}$. Then find the torque acting on a circular current loop of radius 1 mm, in the xy-plane, centered at the origin and with current 0.1 A flowing in the sense of increasing ϕ in a magnetic field $\mathbf{B} = 10^{-5}(2\mathbf{a}_x - 2\mathbf{a}_y + \mathbf{a}_z)$ Wb/m^2.

P4.11. Finding the parameters of a ferromagnetic material. A portion of the B–H curve for a ferromagnetic material can be approximated by the analytical expression

$$\mathbf{B} = \mu_0 k H \mathbf{H}$$

where k is a constant having units of meter per ampere. Find μ, μ_r, χ_m, and **M**.

P4.12. Finding effective permeability for an anisotropic magnetic material. An anisotropic magnetic material is characterized by the **B** to **H** relationship

$$\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = k\mu_0 \begin{bmatrix} 7 & 6 & 0 \\ 6 & 12 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix}$$

where k is a constant. Find the effective permeability for $\mathbf{H} = H_0(3\mathbf{a}_x - 2\mathbf{a}_y)$.

Section 4.4

P4.13. Finding fields for a plane-sheet sinusoidal current source in a material medium. An infinite plane sheet in the $z = 0$ plane carries a surface current of density

$$\mathbf{J}_s = -0.2 \cos 2\pi \times 10^6 t \, \mathbf{a}_x \text{ A/m}$$

The medium on either side of the sheet is characterized by $\sigma = 10^{-3}$ S/m, $\varepsilon = 6\varepsilon_0$, and $\mu = \mu_0$. Find **E** and **H** on either side of the current sheet.

P4.14. An array of two infinite plane current sheets in a material medium. Consider an array of two infinite plane, parallel, current sheets of uniform densities given by

$$\mathbf{J}_{S1} = -J_{S0} \cos 2\pi \times 10^6 t \, \mathbf{a}_x \quad \text{in the } z = 0 \text{ plane}$$
$$\mathbf{J}_{S2} = -kJ_{S0} \sin 2\pi \times 10^6 t \, \mathbf{a}_x \quad \text{in the } z = d \text{ plane}$$

situated in a medium characterized by $\sigma = 10^{-3}$ S/m, $\varepsilon = 6\varepsilon_0$, and $\mu = \mu_0$. **(a)** Find the minimum value of $d(>0)$ and the corresponding value of k for which the fields in the region $z < 0$ are zero. **(b)** For the values of d and k found in (a), obtain the electric-field intensity in the region $z > d$.

P4.15. Finding material parameters of a medium from propagation characteristics. A uniform plane wave of frequency 5×10^5 Hz propagating in a material medium has the following characteristics. **(i)** The fields are attenuated by the factor e^{-1} in a distance of 28.65 m. **(ii)** The fields undergo a change in phase by 2π in a distance of 111.2 m. **(iii)** The ratio of the amplitudes of the electric- and magnetic-field intensities at a point in the medium is 59.4. **(a)** What is the value of $\bar{\gamma}$? **(b)** What is the value of $\bar{\eta}$? **(c)** Find σ, ε, and μ of the medium.

P4.16. Finding fields for a plane-sheet nonsinusoidal current source in a material medium. Repeat Problem P4.13 for the surface current of density

$$\mathbf{J}_s = -0.2 \cos 2\pi \times 10^6 t \cos 4\pi \times 10^6 t \, \mathbf{a}_x \text{ A/m}$$

P4.17. Power flow and dissipation in a material medium. The magnetic field of a uniform plane wave propagating in a nonmagnetic ($\mu = \mu_0$) material medium is given by

$$\mathbf{H} = H_0 e^{-z} \cos (2\pi \times 10^6 t - 2z) \, \mathbf{a}_x \text{ A/m}$$

Find: **(a)** the time-average power flow per unit area normal to the z-direction and **(b)** the time-average power dissipated in the volume bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, and $z = 1$.

Section 4.5

P4.18. Finding parameters for a uniform plane-wave electric field in a perfect dielectric. The electric field of a uniform plane wave propagating in a perfect dielectric medium having $\mu = \mu_0$ is given by

$$\mathbf{E} = 10 \cos (3\pi \times 10^7 t - 0.2\pi x) \, \mathbf{a}_z$$

Find: **(a)** the frequency; **(b)** the wavelength; **(c)** the phase velocity; **(d)** the relative permittivity of the medium; and **(e)** the associated magnetic-field vector **H**.

P4.19. Plotting field variations for a nonsinusoidal current source in a perfect dielectric. An infinite plane sheet lying in the $z = 0$ plane carries a surface current of density $\mathbf{J}_S = -J_S(t)\mathbf{a}_x$ A/m, where $J_S(t)$ is as shown in Fig. 4.29. The medium on either side of the current sheet is a perfect dielectric of $\varepsilon = 2.25\varepsilon_0$ and $\mu = \mu_0$.

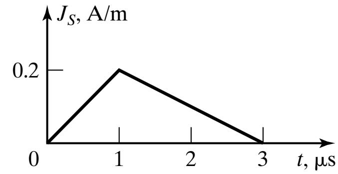


FIGURE 4.29
For Problem P4.19

Find and sketch **(a)** E_x versus t for $z = 200$ m; **(b)** H_y versus t for $z = -300$ m; **(c)** E_x versus z for $t = 2\ \mu$s; and **(d)** H_y versus z for $t = 3\ \mu$s.

P4.20. **Finding the parameters of a perfect dielectric from propagation characteristics.** For a uniform plane wave having $\mathbf{E} = E_x(z, t)\mathbf{a}_x$ and $\mathbf{H} = H_y(z, t)\mathbf{a}_y$ and propagating in the $+z$-direction in a perfect dielectric medium, the time variation of E_x in a constant z-plane and the distance variation of H_y for a fixed time are observed to be periodic, as shown in Figs. 4.30(a) and (b), respectively, for two complete cycles. Find the relative permittivity and the relative permeability of the medium.

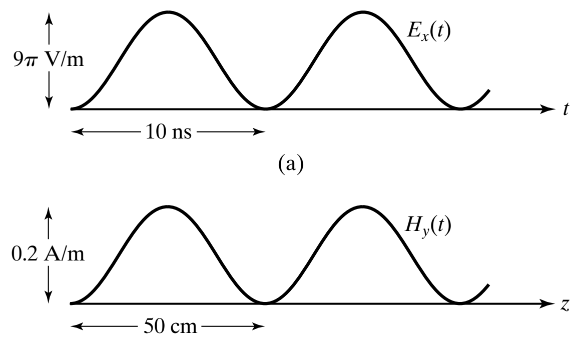


FIGURE 4.30
For Problem P4.20.

P4.21. **Computing propagation parameters for a uniform plane wave in ice.** For uniform plane wave propagation in ice ($\sigma \approx 10^{-6}$ S/m, $\varepsilon = 3\varepsilon_0$, and $\mu = \mu_0$), compute α, β, v_p, λ, and $\overline{\eta}$ for $f = 1$ MHz. What is the distance in which the fields are attenuated by the factor e^{-1}?

P4.22. **Computing propagation parameters for a uniform plane wave in seawater.** For uniform plane wave propagation in seawater ($\sigma = 4$ S/m, $\varepsilon = 80\varepsilon_0$, and $\mu = \mu_0$), compute α, δ, β, λ, v_p, and $\overline{\eta}$ for two frequencies: **(a)** $f = 10$ GHz and **(b)** $f = 100$ kHz.

P4.23. **Finding the electric field for a nonsinusoidal-wave magnetic field in a material medium.** For a uniform plane wave propagating in the $+z$-direction in a material medium, the magnetic field intensity in the $z = 0$ plane is given by

$$[\mathbf{H}]_{z=0} = 0.1 \cos^3 2\pi \times 10^8 t\ \mathbf{a}_y \text{ A/m}$$

Find $\mathbf{E}(z, t)$ for each of the following cases: **(a)** the medium is characterized by $\sigma = 0$, $\varepsilon = 9\varepsilon_0$, and $\mu = \mu_0$; **(b)** the medium is characterized by $\sigma = 10^{-3}$ S/m, $\varepsilon = 9\varepsilon_0$, and $\mu = \mu_0$; and **(c)** the medium is characterized by $\sigma = 10$ S/m, $\varepsilon = 9\varepsilon_0$, and $\mu = \mu_0$.

Section 4.6

P4.24. Verifying consistency of results with boundary conditions. Show that the results obtained for the electric field due to the sheet of charge in Example 1.9 and for the magnetic field due to the sheet of current in Example 1.12 are consistent with the boundary conditions.

P4.25. Applying boundary conditions at interface between dielectric and free space. Medium 1, consisting of the region $r < a$ in spherical coordinates, is a perfect dielectric of permittivity ε_1, whereas medium 2, consisting of the region $r > a$ in spherical coordinates, is free space. The electric field intensities in the two media are given by

$$\mathbf{E}_1 = E_{01}(\cos\theta\,\mathbf{a}_r - \sin\theta\,\mathbf{a}_\theta)$$
$$\mathbf{E}_2 = E_{02}\left[\left(1 + \frac{a^3}{2r^3}\right)\cos\theta\,\mathbf{a}_r - \left(1 - \frac{a^3}{4r^3}\right)\sin\theta\,\mathbf{a}_\theta\right]$$

respectively. Find ε_1.

P4.26. Applying boundary conditions at interface between dielectric and free space. A boundary separates free space from a perfect dielectric medium. At a point on the boundary, the electric field intensity on the free space side is $\mathbf{E}_1 = E_0(4\mathbf{a}_x + 2\mathbf{a}_y + 5\mathbf{a}_z)$, whereas on the dielectric side, it is $\mathbf{E}_2 = 3E_0(\mathbf{a}_x + \mathbf{a}_z)$, where E_0 is a constant. Find the permittivity of the dielectric medium.

P4.27. Applying boundary conditions at interface between magnetic material and free space. Medium 1, consisting of the region $r < a$ in spherical coordinates, is a magnetic material of permeability μ_1, whereas medium 2, consisting of the region $r > a$ in spherical coordinates, is free space. The magnetic flux densities in the two media are given by

$$\mathbf{B}_1 = B_{01}(\cos\theta\,\mathbf{a}_r - \sin\theta\,\mathbf{a}_\theta)$$
$$\mathbf{B}_2 = B_{02}\left[\left(1 + 1.94\frac{a^3}{r^3}\right)\cos\theta\,\mathbf{a}_r - \left(1 - 0.97\frac{a^3}{r^3}\right)\sin\theta\,\mathbf{a}_\theta\right]$$

respectively. Find μ_1.

P4.28. Verification and application of boundary conditions on a perfect conductor surface. In Problem P4.4, show that the applied and secondary fields together satisfy the boundary condition of zero tangential component of electric field on the conductor surface. From the boundary condition for the normal component of $\mathbf{D}$, find the charge density on the conductor surface and show that the total induced surface charge per unit width in the z-direction is $-\rho_{L0}$.

P4.29. Applying boundary conditions for a rectangular cavity resonator. The rectangular cavity resonator is a box consisting of the region $0 < x < a$, $0 < y < b$, and $0 < z < d$, and bounded by perfectly conducting walls on all of its six sides. The time-varying electric and magnetic fields inside the resonator are given by

$$\mathbf{E} = E_0 \sin\frac{\pi x}{a}\sin\frac{\pi z}{d}\cos\omega t\,\mathbf{a}_y$$
$$\mathbf{H} = H_{01}\sin\frac{\pi x}{a}\cos\frac{\pi z}{d}\sin\omega t\,\mathbf{a}_x - H_{02}\cos\frac{\pi x}{a}\sin\frac{\pi z}{d}\sin\omega t\,\mathbf{a}_z$$

where E_0, H_{01}, and H_{02} are constants. Find ρ_S and $\mathbf{J}_S$ on all six walls, assuming the medium inside the box to be a perfect dielectric of $\varepsilon = 4\varepsilon_0$.

P4.30. Finding fields for a plane-sheet current source with different media on either side. In Problem P4.13, assume that the region $z > 0$ is free space, whereas the region $z < 0$ is a material medium characterized by $\sigma = 10^{-3}$ S/m, $\varepsilon = 6\varepsilon_0$, and $\mu = \mu_0$. Find **E** and **H** on either side of the current sheet. (*Hint:* Make use of the complex electric and magnetic fields to satisfy the boundary conditions at $z = 0$.)

P4.31. Finding fields for a plane-sheet current source with different dielectrics on either side. An infinite plane sheet lying in the $z = 0$ plane carries a surface current of density

$$\mathbf{J}_s = -0.2 \cos 6\pi \times 10^8 t \, \mathbf{a}_x \text{ A/m}$$

The region $z > 0$ is a perfect dielectric of $\varepsilon = 2.25\varepsilon_0$ and $\mu = \mu_0$, whereas the region $z < 0$ is a perfect dielectric of $\varepsilon = 4\varepsilon_0$ and $\mu = \mu_0$. Find **E** and **H** on both sides of the sheet.

Section 4.7

P4.32. Normal incidence of a sinusoidal uniform plane wave onto a material medium. Region 1 ($z < 0$) is free space, whereas region 2 ($z > 0$) is a material medium characterized by $\sigma = 10^{-4}$ S/m, $\varepsilon = 5\varepsilon_0$, and $\mu = \mu_0$. For a uniform plane wave having the electric field

$$\mathbf{E}_i = E_0 \cos (3\pi \times 10^5 t - 10^{-3}\pi z) \, \mathbf{a}_x \text{ V/m}$$

incident on the interface $z = 0$ from region 1, obtain the expression for the reflected and transmitted wave electric fields.

P4.33. Normal incidence of a nonsinusoidal uniform plane wave onto a material medium. Repeat Problem P4.32 for the incident wave electric field given by

$$\mathbf{E}_i = E_0 \cos^3 (3\pi \times 10^5 t - 10^{-3}\pi z) \, \mathbf{a}_x \text{ V/m}$$

P4.34. Uniform plane wave reflection and transmission involving three media in cascade. In Fig. 4.31, medium 3 extends to infinity so that no reflected (−) wave exists in that medium. For a uniform plane wave having the electric field

$$\mathbf{E}_i = E_0 \cos (3 \times 10^8 \pi t - \pi z) \, \mathbf{a}_x \text{ V/m}$$

incident from medium 1 onto the interface $z = 0$, obtain the expressions for the phasor electric- and magnetic-field components in all three media.

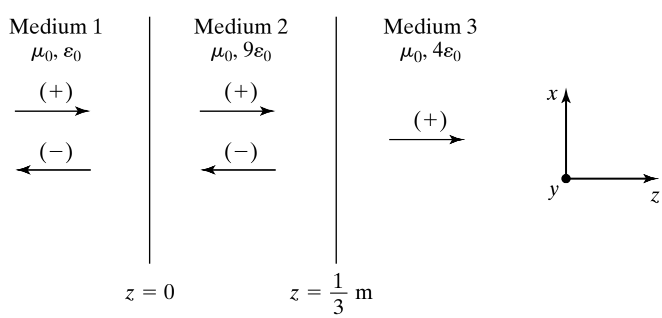


FIGURE 4.31
For Problem P4.34.

P4.35. Plotting field variations for a nonsinusoidal wave incident on a perfect dielectric. A uniform plane wave propagating in the $+z$-direction and having the electric field $\mathbf{E}_i = E_{xi}(t)\mathbf{a}_x$, where $E_{xi}(t)$ in the $z = 0$ plane is as shown in Fig. 4.32, is incident normally from free space ($z < 0$) onto a nonmagnetic ($\mu = \mu_0$), perfect dielectric ($z > 0$) of permittivity $4\varepsilon_0$. Find and sketch the following: **(a)** E_x versus z for $t = 1\ \mu$s and **(b)** H_y versus z for $t = 1\ \mu$s.

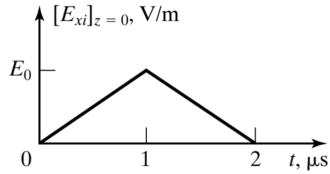


FIGURE 4.32
For Problem P4.35

P4.36. Normal incidence of a uniform plane wave on a perfect conductor surface. The region $z < 0$ is a perfect dielectric, whereas the region $z > 0$ is a perfect conductor, as shown in Fig. 4.33. For a uniform plane wave having the electric and magnetic fields

$$\mathbf{E}_i = E_0 \cos(\omega t - \beta z)\,\mathbf{a}_x$$
$$\mathbf{H}_i = \frac{E_0}{\eta} \cos(\omega t - \beta z)\,\mathbf{a}_y$$

where $\beta = \omega\sqrt{\mu\varepsilon}$ and $\eta = \sqrt{\mu/\varepsilon}$, obtain the expressions for the reflected wave electric and magnetic fields and hence the expressions for the total (incident + reflected) electric and magnetic fields in the dielectric, and the current density on the surface of the perfect conductor.

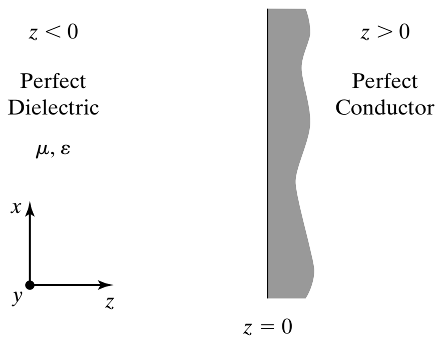


FIGURE 4.33
For Problem P4.36.

REVIEW PROBLEMS

R4.1. Finding surface charge densities for plane conducting slabs between two sheets of charge. Two infinite plane conducting slabs lie between and parallel to two infinite plane sheets of uniform surface charge densities ρ_{SA} and ρ_{SB}, as shown by the cross-sectional view in Fig. 4.34. Find the surface charge densities on all four surfaces of the slabs.

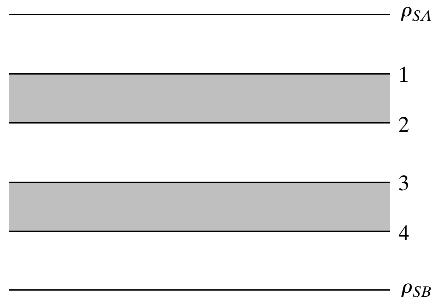


FIGURE 4.34

For Problem R4.1.

R4.2. Characteristic polarizations for an anisotropic dielectric. An anisotropic dielectric material is characterized by the **D** to **E** relationship

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \varepsilon_0 \begin{bmatrix} 6.5 & 1.5 & 0 \\ 1.5 & 2.5 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

Express $\mathbf{E} = E_0(\mathbf{a}_x - \mathbf{a}_y)$ as the linear combination of $\mathbf{E}_1$ and $\mathbf{E}_2$, which correspond to two of the characteristic polarizations of the material.

R4.3. Magnetic dipole moment of a charged rotating disk of nonuniform charge density. Charge Q is distributed with density proportional to r on a circular disk of radius a lying on the xy-plane with its center at the origin and rotating around the z-axis with angular velocity ω in the sense of increasing ϕ. Find the magnetic dipole moment.

R4.4. Finding H and the material parameters of a nonmagnetic medium from E in the medium. The electric field of a uniform plane wave propagating in the $+z$-direction in a nonmagnetic ($\mu = \mu_0$) material medium is given by

$$\mathbf{E} = 8.4e^{-0.0432z} \cos\,(4\pi \times 10^6 t - 0.1829z)\,\mathbf{a}_x \text{ V/m}$$

Find the magnetic field of the wave. Further, find the values of σ and ε of the medium.

R4.5. Infinite plane current sheet sandwiched between two different perfect dielectric media. An infinite plane current sheet of uniform density $\mathbf{J}_S = -J_S(t)\mathbf{a}_x$ is sandwiched between two perfect dielectric media, as shown in Fig. 4.35(a). If $J_S(t)$ is a triangular pulse of duration 3 μs, the plots of $E_x(t)$ at some value of z equal to z_0 (>0) and $H_y(z)$ for some value of t equal to t_0 (>0) are given by Figs. 4.35(b) and (c), respectively. If $J_S(t) = J_{S0} \cos 6\pi \times 10^8 t$ A/m, instead of being a pulse, find **E** and **H** on both sides of the sheet, and the time-average power radiated by the sheet for unit area of the sheet.

R4.6. Application of boundary conditions on a perfect conductor surface. The region $3x + 4y + 12z < 12$ is occupied by a perfect conductor. If at a point on the perfect conductor surface, the surface charge and current densities at a particular instant of time are ρ_{S0} C/m^2 and $J_{S0}(4\mathbf{a}_x - 3\mathbf{a}_y)$A/m, find **D** and **H** at that point at that instant of time.

R4.7. Application of boundary conditions at interface between dielectric and free space. Medium 1, consisting of the region $r < a$ in spherical coordinates, is a perfect dielectric of permittivity $\varepsilon_1 = 2\varepsilon_0$, whereas medium 2, consisting of the

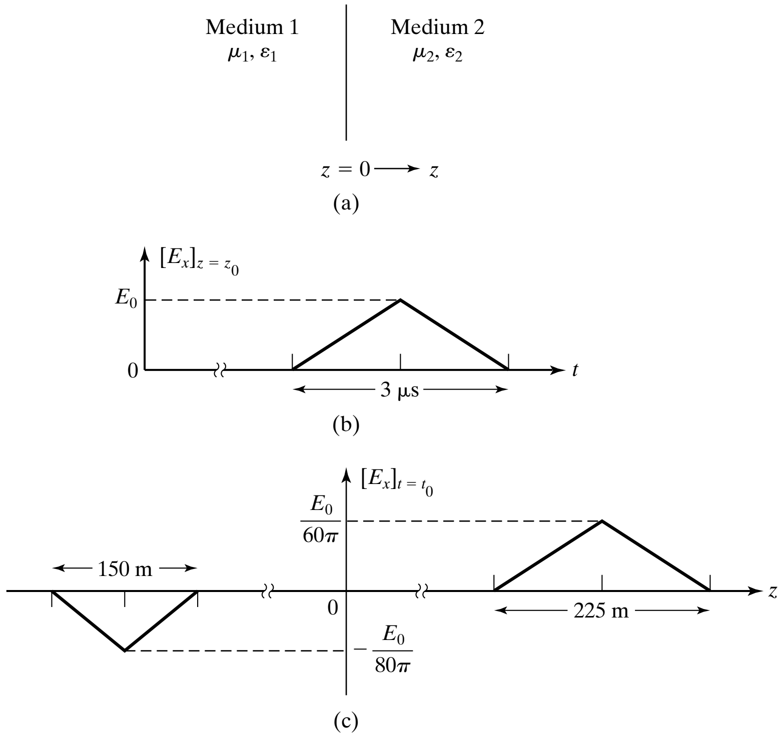


FIGURE 4.35

For Problem R4.5.

region $r > a$, is free space. The electric field intensity in medium 1 is given by $\mathbf{E}_1 = E_0\mathbf{a}_z$. Find the electric field intensity at the points **(a)** $(0, 0, a)$, **(b)** $(0, a, 0)$, and **(c)** $(0, a/\sqrt{2}, a/\sqrt{2})$, in Cartesian coordinates, in medium 2.

R4.8. Normal incidence of a uniform plane wave onto a slab of perfect dielectric. For a sinusoidally time-varying uniform plane wave incident normally from medium 1 on to the interface $z = 0$ in Fig. 4.36, show that there is a minimum value of the frequency for which a wave at that frequency or any integer multiple of that frequency undergoes no reflection at the interface. Further, find the maximum value of the period of a nonsinusoidal periodic wave for which no reflection occurs at the interface. Note that medium 1 and medium 3 are both free space.

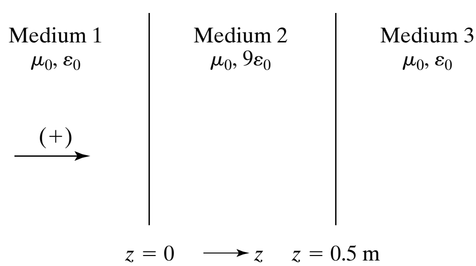


FIGURE 4.36

For Problem R4.8.