CHAPTER 3

Maxwell's Equations in Differential Form, and Uniform Plane Waves in Free Space

In Chapter 2, we introduced Maxwell's equations in integral form. We learned that the quantities involved in the formulation of these equations are the scalar quantities, electromotive force, magnetomotive force, magnetic flux, displacement flux, charge, and current, which are related to the field vectors and source densities through line, surface, and volume integrals. Thus, the integral forms of Maxwell's equations, while containing all the information pertinent to the interdependence of the field and source quantities over a given region in space, do not permit us to study directly the interaction between the field vectors and their relationships with the source densities at individual points. It is our goal in this chapter to derive the differential forms of Maxwell's equations that apply directly to the field vectors and source densities at a given point.

We shall derive Maxwell's equations in differential form by applying Maxwell's equations in integral form to infinitesimal closed paths, surfaces, and volumes, in the limit that they shrink to points. We will find that the differential equations relate the spatial variations of the field vectors at a given point to their temporal variations and to the charge and current densities at that point. Using Maxwell's equations in differential form, we introduce the important topic of uniform plane waves and the associated concepts, fundamental to gaining an understanding of the basic principles of electromagnetic wave propagation.

3.1 FARADAY'S LAW AND AMPÈRE'S CIRCUITAL LAW

Faraday's law, special case

We recall from Chapter 2 that Faraday's law is given in integral form by

$$\boxed{\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S}} \tag{3.1}$$

where S is any surface bounded by the closed path C. In the most general case, the electric and magnetic fields have all three components (x, y, and z) and are dependent on all three coordinates (x, y, and z) in addition to time (t). For simplicity, we shall, however, first consider the case in which the electric field has an x component only, which is dependent only on the z coordinate, in addition to time. Thus,

$$\boxed{\mathbf{E} = E_x(z, t)\mathbf{a}_x} \tag{3.2}$$

In other words, this simple form of time-varying electric field is everywhere directed in the x-direction and it is uniform in planes parallel to the xy-plane.

Let us now consider a rectangular path C of infinitesimal size lying in a plane parallel to the xz-plane and defined by the points (x, z), $(x, z + \Delta z)$, $(x + \Delta x, z + \Delta z)$, and $(x + \Delta x, z)$, as shown in Fig. 3.1. According to Faraday's law, the emf around the closed path C is equal to the negative of the time rate of change of the magnetic flux enclosed by C. The emf is given by the line integral of $\mathbf{E}$ around C. Thus, evaluating the line integrals of $\mathbf{E}$ along the four sides of the rectangular path, we obtain

$$\int_{(x,\, z)}^{(x,\, z+\Delta z)} \mathbf{E} \cdot d\mathbf{l} = 0 \qquad \text{since } E_z = 0 \tag{3.3a}$$

$$\int_{(x,\, z+\Delta z)}^{(x+\Delta x,\, z+\Delta z)} \mathbf{E} \cdot d\mathbf{l} = [E_x]_{z+\Delta z}\, \Delta x \tag{3.3b}$$

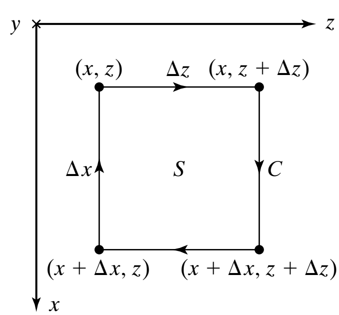

FIGURE 3.1

Infinitesimal rectangular path lying in a plane parallel to the xz-plane.

$$\int_{(x+\Delta x,\, z+\Delta z)}^{(x+\Delta x,\, z)} \mathbf{E} \cdot d\mathbf{l} = 0 \qquad \text{since } E_z = 0 \tag{3.3c}$$

$$\int_{(x+\Delta x,\, z)}^{(x,\, z)} \mathbf{E} \cdot d\mathbf{l} = -[E_x]_z \, \Delta x \tag{3.3d}$$

Adding up (3.3a)–(3.3d), we obtain

$$\begin{aligned} \oint_C \mathbf{E} \cdot d\mathbf{l} &= [E_x]_{z+\Delta z} \, \Delta x - [E_x]_z \, \Delta x \\ &= \{[E_x]_{z+\Delta z} - [E_x]_z\} \, \Delta x \end{aligned} \tag{3.4}$$

In (3.3a)–(3.3d) and (3.4), $[E_x]_z$ and $[E_x]_{z+\Delta z}$ denote values of E_x evaluated along the sides of the path for which $z = z$ and $z = z + \Delta z$, respectively.

To find the magnetic flux enclosed by C, let us consider the plane surface S bounded by C. According to the right-hand screw rule, we must use the magnetic flux crossing S toward the positive y-direction, that is, into the page, since the path C is traversed in the clockwise sense. The only component of $\mathbf{B}$ normal to the area S is the y-component. Also since the area is infinitesimal in size, we can assume B_y to be uniform over the area and equal to its value at (x, z). The required magnetic flux is then given by

$$\int_S \mathbf{B} \cdot d\mathbf{S} = [B_y]_{(x,\, z)} \, \Delta x \, \Delta z \tag{3.5}$$

Substituting (3.4) and (3.5) into (3.1) to apply Faraday's law to the rectangular path C under consideration, we get

$$\{[E_x]_{z+\Delta z} - [E_x]_z\} \, \Delta x = -\frac{d}{dt}\{[B_y]_{(x,\, z)} \, \Delta x \, \Delta z\}$$

or

$$\frac{[E_x]_{z+\Delta z} - [E_x]_z}{\Delta z} = -\frac{\partial [B_y]_{(x,\, z)}}{\partial t} \tag{3.6}$$

If we now let the rectangular path shrink to the point (x, z) by letting Δx and Δz tend to zero, we obtain

$$\lim_{\substack{\Delta x \to 0 \\ \Delta z \to 0}} \frac{[E_x]_{z+\Delta z} - [E_x]_z}{\Delta z} = -\lim_{\substack{\Delta x \to 0 \\ \Delta z \to 0}} \frac{\partial [B_y]_{(x,\, z)}}{\partial t}$$

or

$$\boxed{\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t}} \tag{3.7}$$

Equation (3.7) is Faraday's law in differential form for the simple case of **E** given by (3.2). It relates the variation of E_x with z (space) at a point to the variation of B_y with t (time) at that point. Since this derivation can be carried out for any arbitrary point (x, y, z), it is valid for all points. It tells us in particular that an E_x associated with a time-varying B_y has a differential in the z-direction. This is to be expected since if this is not the case, $\oint \mathbf{E} \cdot d\mathbf{l}$ around the infinitesimal rectangular path would be zero.

Example 3.1 Finding B for a given E

Given $\mathbf{E} = 10 \cos (6\pi \times 10^8 t - 2\pi z)\, \mathbf{a}_x$ V/m, let us find **B** that satisfies (3.7).

From (3.7), we have

$$
\begin{aligned}
\frac{\partial B_y}{\partial t} &= -\frac{\partial E_x}{\partial z} \\
&= -\frac{\partial}{\partial z}[10 \cos (6\pi \times 10^8 t - 2\pi z)] \\
&= -20\pi \sin (6\pi \times 10^8 t - 2\pi z) \\
B_y &= \frac{10^{-7}}{3} \cos (6\pi \times 10^8 t - 2\pi z) \\
\mathbf{B} &= \frac{10^{-7}}{3} \cos (6\pi \times 10^8 t - 2\pi z)\, \mathbf{a}_y
\end{aligned}
$$

Faraday's law, general case

We shall now proceed to derive the differential form of (3.1) for the general case of the electric field having all three components (x, y, z), each of them depending on all three coordinates (x, y, and z), in addition to time (t); that is,

$$\boxed{\mathbf{E} = E_x(x, y, z, t)\mathbf{a}_x + E_y(x, y, z, t)\mathbf{a}_y + E_z(x, y, z, t)\mathbf{a}_z} \tag{3.8}$$

To do this, let us consider the three infinitesimal rectangular paths in planes parallel to the three mutually orthogonal planes of the Cartesian coordinate system, as shown in Fig. 3.2. Evaluating $\oint \mathbf{E} \cdot d\mathbf{l}$ around the closed paths *abcda, adefa*, and *afgba*, we get

$$
\begin{aligned}
\oint_{abcda} \mathbf{E} \cdot d\mathbf{l} = {} & [E_y]_{(x,\, z)}\, \Delta y + [E_z]_{(x, y+\Delta y)}\, \Delta z \\
& -[E_y]_{(x,\, z+\Delta z)}\, \Delta y - [E_z]_{(x,\, y)}\, \Delta z
\end{aligned} \tag{3.9a}
$$

$$
\begin{aligned}
\oint_{adefa} \mathbf{E} \cdot d\mathbf{l} = {} & [E_z]_{(x,\, y)}\, \Delta z + [E_x]_{(y,\, z+\Delta z)}\, \Delta x \\
& -[E_z]_{(x+\Delta x,\, y)}\, \Delta z - [E_x]_{(y,\, z)}\, \Delta x
\end{aligned} \tag{3.9b}
$$

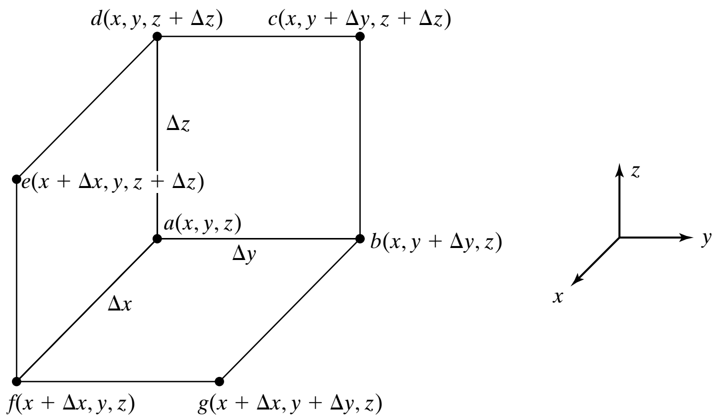

FIGURE 3.2

Infinitesimal rectangular paths in three mutually orthogonal planes.

$$\oint_{afgba} \mathbf{E} \cdot d\mathbf{l} = [E_x]_{(y,\, z)}\,\Delta x + [E_y]_{(x+\Delta x,\, z)}\,\Delta y - [E_x]_{(y+\Delta y,\, z)}\,\Delta x - [E_y]_{(x,\, z)}\,\Delta y \tag{3.9c}$$

In (3.9a)–(3.9c), the subscripts associated with the field components in the various terms on the right sides of the equations denote the values of the coordinates that remain constant along the sides of the closed paths corresponding to the terms. Now, evaluating $\int \mathbf{B} \cdot d\mathbf{S}$ over the surfaces *abcd, adef*, and *afgb*, keeping in mind the right-hand screw rule, we have

$$\int_{abcd} \mathbf{B} \cdot d\mathbf{S} = [B_x]_{(x,\, y,\, z)}\,\Delta y\,\Delta z \tag{3.10a}$$

$$\int_{adef} \mathbf{B} \cdot d\mathbf{S} = [B_y]_{(x,\, y,\, z)}\,\Delta z\,\Delta x \tag{3.10b}$$

$$\int_{afgb} \mathbf{B} \cdot d\mathbf{S} = [B_z]_{(x,\, y,\, z)}\,\Delta x\,\Delta y \tag{3.10c}$$

Applying Faraday's law to each of the three paths by making use of (3.9a)–(3.9c) and (3.10a)–(3.10c) and simplifying, we obtain

$$\frac{[E_z]_{(x,\, y+\Delta y)} - [E_z]_{(x,\, y)}}{\Delta y} - \frac{[E_y]_{(x,\, z+\Delta z)} - [E_y]_{(x,\, z)}}{\Delta z} = -\frac{\partial [B_x]_{(x,\, y,\, z)}}{\partial t} \tag{3.11a}$$

$$\frac{[E_x]_{(y,\, z+\Delta z)} - [E_x]_{(y,\, z)}}{\Delta z} - \frac{[E_z]_{(x+\Delta x,\, y)} - [E_z]_{(x,\, y)}}{\Delta x} = -\frac{\partial [B_y]_{(x,\, y,\, z)}}{\partial t} \tag{3.11b}$$

$$\frac{[E_y]_{(x+\Delta x,\, z)} - [E_y]_{(x,\, z)}}{\Delta x} - \frac{[E_x]_{(y+\Delta y,\, z)} - [E_x]_{(y,\, z)}}{\Delta y} = -\frac{\partial [B_z]_{(x,\, y,\, z)}}{\partial t} \tag{3.11c}$$

If we now let all three paths shrink to the point a by letting Δx, Δy, and Δz tend to zero, (3.11a)–(3.11c) reduce to

$$\boxed{\begin{aligned} \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} &= -\frac{\partial B_x}{\partial t} \quad (3.12a) \\ \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} &= -\frac{\partial B_y}{\partial t} \quad (3.12b) \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} &= -\frac{\partial B_z}{\partial t} \quad (3.12c) \end{aligned}}$$

Equations (3.12a)–(3.12c) are the differential equations governing the relationships between the space variations of the electric field components and the time variations of the magnetic field components at a point. In particular, we note that the space derivatives are all lateral derivatives, that is, derivatives evaluated along directions lateral to the directions of the field components and not along the directions of the field components. An examination of one of the three equations is sufficient to reveal the physical meaning of these relationships. For example, (3.12a) tells us that a time-varying B_x at a point results in an electric field at that point having y- and z-components such that their net right-lateral differential normal to the x-direction is nonzero. The right-lateral differential of E_y normal to the x-direction is its derivative in the $\mathbf{a}_y \times \mathbf{a}_x$, or $-\mathbf{a}_z$-direction, that is, $\partial E_y/\partial(-z)$ or $-\partial E_y/\partial z$. The right-lateral differential of E_z normal to the x-direction is its derivative in the $\mathbf{a}_z \times \mathbf{a}_x$, or $\mathbf{a}_y$-direction, that is, $\partial E_z/\partial y$. Thus, the net right-lateral differential of the y- and z-components of the electric field normal to the x-direction is $(-\partial E_y/\partial z) + (\partial E_z/\partial y)$, or $(\partial E_z/\partial y - \partial E_y/\partial z)$. Figure 3.3(a) shows an example in which the net right-lateral differential is zero although the individual derivatives are nonzero. This is because $\partial E_z/\partial y$ and $\partial E_y/\partial z$ are both positive and equal so that their difference is zero. On the other hand, for the example in Fig. 3.3(b), $\partial E_z/\partial y$ is positive and $\partial E_y/\partial z$ is negative so that their difference, that is, the net right-lateral differential, is nonzero.

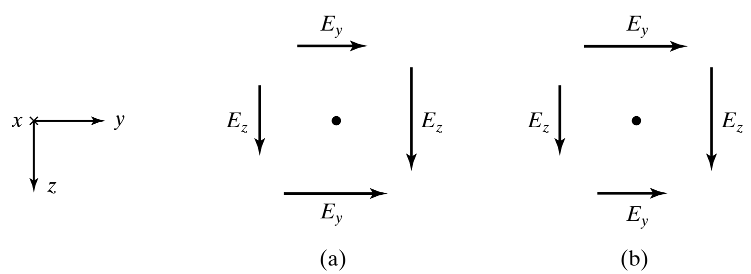

FIGURE 3.3

For illustrating (a) zero and (b) nonzero net right-lateral differential of E_y and E_z normal to the x-direction.

Equations (3.12a)–(3.12c) can be combined into a single vector equation as given by

Curl (del cross)

$$\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\mathbf{a}_z$$
$$= -\frac{\partial B_x}{\partial t}\mathbf{a}_x - \frac{\partial B_y}{\partial t}\mathbf{a}_y - \frac{\partial B_z}{\partial t}\mathbf{a}_z \tag{3.13}$$

This can be expressed in determinant form as

$$\boxed{\begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = -\frac{\partial \mathbf{B}}{\partial t}} \tag{3.14}$$

or as

$$\left(\mathbf{a}_x\frac{\partial}{\partial x} + \mathbf{a}_y\frac{\partial}{\partial y} + \mathbf{a}_z\frac{\partial}{\partial z}\right) \times (E_x\mathbf{a}_x + E_y\mathbf{a}_y + E_z\mathbf{a}_z) = -\frac{\partial \mathbf{B}}{\partial t} \tag{3.15}$$

The left side of (3.14) or (3.15) is known as the *curl of* **E**, denoted as $\nabla \times \mathbf{E}$ (del cross **E**), where ∇ (del) is the vector operator given by

$$\boxed{\nabla = \mathbf{a}_x\frac{\partial}{\partial x} + \mathbf{a}_y\frac{\partial}{\partial y} + \mathbf{a}_z\frac{\partial}{\partial z}} \tag{3.16}$$

Thus, we have

$$\boxed{\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}} \tag{3.17}$$

Equation (3.17) is Maxwell's equation in differential form corresponding to Faraday's law. It tells us that at a point in an electromagnetic field, the curl of the electric field intensity is equal to the time rate of decrease of the magnetic flux density. We shall discuss curl further in Section 3.3, but note that for static fields, $\nabla \times \mathbf{E}$ is equal to the null vector. Thus, for a static vector field to be realized as an electric field, the components of its curl must all be zero.

Although we have deduced (3.17) from (3.1) by considering the Cartesian coordinate system, it is independent of the coordinate system since (3.1) is independent of the coordinate system. The expressions for the curl of a vector in cylindrical and spherical coordinate systems are derived in Appendix B. They are reproduced here together with that in (3.14) for the Cartesian coordinate system.

CARTESIAN

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \tag{3.18a}$$

CYLINDRICAL

$$\nabla \times \mathbf{A} = \begin{vmatrix} \dfrac{\mathbf{a}_r}{r} & \mathbf{a}_\phi & \dfrac{\mathbf{a}_z}{r} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix} \tag{3.18b}$$

SPHERICAL

$$\nabla \times \mathbf{A} = \begin{vmatrix} \dfrac{\mathbf{a}_r}{r^2 \sin\theta} & \dfrac{\mathbf{a}_\theta}{r\sin\theta} & \dfrac{\mathbf{a}_\phi}{r} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix} \tag{3.18c}$$

Example 3.2 Evaluating curls of vector fields

Find the curls of the following vector fields: **(a)** $y\mathbf{a}_x - x\mathbf{a}_y$ and **(b)** $\mathbf{a}_\phi$ in cylindrical coordinates.

(a) Using (3.18a), we have

$$\begin{aligned} \nabla \times (y\mathbf{a}_x - x\mathbf{a}_y) &= \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} \\ &= \mathbf{a}_x\left[-\frac{\partial}{\partial z}(-x)\right] + \mathbf{a}_y\left[\frac{\partial}{\partial z}(y)\right] + \mathbf{a}_z\left[\frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial y}(y)\right] \\ &= -2\mathbf{a}_z \end{aligned}$$

(b) Using (3.18b), we obtain

$$\nabla \times \mathbf{a}_\phi = \begin{vmatrix} \dfrac{\mathbf{a}_r}{r} & \mathbf{a}_\phi & \dfrac{\mathbf{a}_z}{r} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ 0 & r & 0 \end{vmatrix} = \frac{\mathbf{a}_r}{r}\left[-\frac{\partial}{\partial z}(r)\right] + \frac{\mathbf{a}_z}{r}\left[\frac{\partial}{\partial r}(r)\right] = \frac{1}{r}\mathbf{a}_z$$

Ampère's circuital law, general case

We shall now consider the derivation of the differential form of Ampère's circuital law given in integral form by

$$\boxed{\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt}\int_S \mathbf{D} \cdot d\mathbf{S}} \tag{3.19}$$

where S is any surface bounded by the closed path C. To do this, we need not repeat the procedure employed in the case of Faraday's law. Instead, we note from (3.1) and (3.17) that in converting to the differential form from integral form, the line integral of $\mathbf{E}$ around the closed path C is replaced by the curl of $\mathbf{E}$, the surface integral of $\mathbf{B}$ over the surface S bounded by C is replaced by $\mathbf{B}$ itself, and the total time derivative is replaced by partial derivative, as shown:

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S}$$

$$\overbrace{\nabla \times}\, \mathbf{E} = -\frac{\partial}{\partial t}(\mathbf{B})$$

Then using the analogy between Ampère's circuital law and Faraday's law, we can write the following:

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt}\int_S \mathbf{D} \cdot d\mathbf{S}$$

$$\overbrace{\nabla \times}\, \mathbf{H} = \mathbf{J} + \frac{\partial}{\partial t}(\mathbf{D})$$

Thus, for the general case of the magnetic field having all three components (x, y, and z), each of them depending on all three coordinates (x, y, and z), in addition to time (t), that is, for

$$\boxed{\mathbf{H} = H_x(x, y, z, t)\mathbf{a}_x + H_y(x, y, z, t)\mathbf{a}_y + H_z(x, y, z, t)\mathbf{a}_z} \tag{3.20}$$

the differential form of Ampère's circuital law is given by

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}} \tag{3.21}$$

The quantity $\partial \mathbf{D}/\partial t$ is known as the *displacement current density*. Equation (3.21) tells us that at a point in an electromagnetic field, the curl of the magnetic field intensity is equal to the sum of the current density due to flow of charges and the displacement current density. In Cartesian coordinates, (3.21) becomes

$$\begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ H_z & H_y & H_z \end{vmatrix} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{3.22}$$

This is equivalent to three scalar equations relating the lateral space derivatives of the components of **H** to the components of the current density and the time derivatives of the electric field components. These scalar equations can be interpreted in a manner similar to the interpretation of (3.12a)–(3.12c) in the case of Faraday's law. Also, expressions similar to (3.22) can be written in the cylindrical and spherical coordinate systems by using the determinant expansions for the curl in those coordinate systems, given by (3.18b) and (3.18c), respectively.

Ampère's circuital law, special case

Having obtained the differential form of Ampère's circuital law for the general case, we can now simplify it for any particular case. Let us consider the particular case of

$$\boxed{\mathbf{H} = H_y(z, t)\mathbf{a}_y} \tag{3.23}$$

that is, a magnetic field directed everywhere in the y-direction and uniform in planes parallel to the xy-plane. Then since **H** does not depend on x and y, we can replace $\partial/\partial x$ and $\partial/\partial y$ in the determinant expansion for $\nabla \times \mathbf{H}$ by zeros. In addition, setting $H_x = H_z = 0$, we have

$$\begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 0 & 0 & \dfrac{\partial}{\partial z} \\ 0 & H_y & 0 \end{vmatrix} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{3.24}$$

Equating like components on the two sides and noting that the y- and z-components on the left side are zero, we obtain

$$-\frac{\partial H_y}{\partial z} = J_x + \frac{\partial D_x}{\partial t}$$

or

$$\boxed{\frac{\partial H_y}{\partial z} = -J_x - \frac{\partial D_x}{\partial t}} \tag{3.25}$$

Equation (3.25) is Ampère's circuital law in differential form for the simple case of **H** given by (3.23). It relates the variation of H_y with z (space) at a point to the current density J_x and to the variation of D_x with t (time) at that point. It tells us in particular that an H_y associated with a current density J_x or a time-varying D_x, or a nonzero combination of the two quantities, has a differential in the z-direction.

Example 3.3 Simultaneous satisfaction of Faraday's and Ampere's circuital laws by E and B

Given $\mathbf{E} = E_0 z^2 e^{-t}\mathbf{a}_x$ in free space ($\mathbf{J} = \mathbf{0}$). We wish to determine if there exists a magnetic field such that both Faraday's law and Ampère's circuital law are satisfied simultaneously.

Using Faraday's law and Ampère's circuital law in succession, we have

$$\begin{aligned}
\frac{\partial B_y}{\partial t} &= -\frac{\partial E_x}{\partial z} = -2E_0 z e^{-t} \\
B_y &= 2E_0 z e^{-t} \\
H_y &= \frac{2E_0}{\mu_0} z e^{-t} \\
\frac{\partial D_x}{\partial t} &= -\frac{\partial H_y}{\partial z} = -\frac{2E_0}{\mu_0} e^{-t} \\
D_x &= \frac{2E_0}{\mu_0} e^{-t} \\
E_x &= \frac{2E_0}{\mu_0 \varepsilon_0} e^{-t} \\
\mathbf{E} &= \frac{2E_0}{\mu_0 \varepsilon_0} e^{-t}\mathbf{a}_x
\end{aligned}$$

which is not the same as the original $\mathbf{E}$. Hence, a magnetic field does not exist which together with the given $\mathbf{E}$ satisfies both laws simultaneously. The pair of fields $\mathbf{E} = E_0 z^2 e^{-t}\mathbf{a}_x$ and $\mathbf{B} = 2E_0 z e^{-t}\mathbf{a}_y$ satisfies only Faraday's law, whereas the pair of fields $\mathbf{B} = 2E_0 z e^{-t}\mathbf{a}_y$ and $\mathbf{E} = (2E_0/\mu_0\varepsilon_0)e^{-t}\mathbf{a}_x$ satisfies only Ampère's circuital law.

Lumped-circuit theory approximations

To generalize the observation made in the example just discussed, there are certain pairs of time-varying electric and magnetic fields that satisfy only Faraday's law as given by (3.17) and certain other pairs that satisfy only Ampère's circuital law as given by (3.21). In the strictest sense, every physically realizable pair of time-varying electric and magnetic fields must satisfy simultaneously both laws as given by (3.17) and (3.21). However, under the low-frequency approximation, it is valid for the fields to satisfy the laws with certain terms neglected in one or both laws. Lumped-circuit theory is based on such approximations. Thus, the terminal voltage-to-current relationship $V(t) = d[LI(t)]/dt$ for an inductor is obtained by ignoring the effect of the time-varying electric field, that is, $\partial\mathbf{D}/\partial t$ term in Ampère's circuital law. The terminal current-to-voltage relationship $I(t) = d[CV(t)]/dt$ for a capacitor is obtained by ignoring the effect of the time-varying magnetic field, that is, $\partial\mathbf{B}/\partial t$ term in Faraday's law. The terminal voltage-to-current relationship $V(t) = RI(t)$ for a resistor is obtained by ignoring the effects of both time-varying electric field and time-varying magnetic field, that is, both $\partial\mathbf{D}/\partial t$ term in Ampère's circuital law and $\partial\mathbf{B}/\partial t$ term in Faraday's law. In contrast to these approximations, electromagnetic wave propagation phenomena and transmission-line (distributed circuit) theory are based on the simultaneous application of the two laws with all terms included, that is, as given by (3.17) and (3.21).

We shall conclude this section with an example involving no time variations.

Example 3.4 Magnetic field of a current distribution from Ampere's circuital law in differential form

Let us consider the current distribution given by

$$\mathbf{J} = J_0\mathbf{a}_x \qquad \text{for } -a < z < a$$

as shown in Fig. 3.4(a), where J_0 is a constant, and find the magnetic field everywhere.

Since the current density is independent of x and y, the field is also independent of x and y. Also, since the current density is not a function of time, the field is static. Hence $(\partial D_x/\partial t) = 0$, and we have

$$\frac{\partial H_y}{\partial z} = -J_x$$

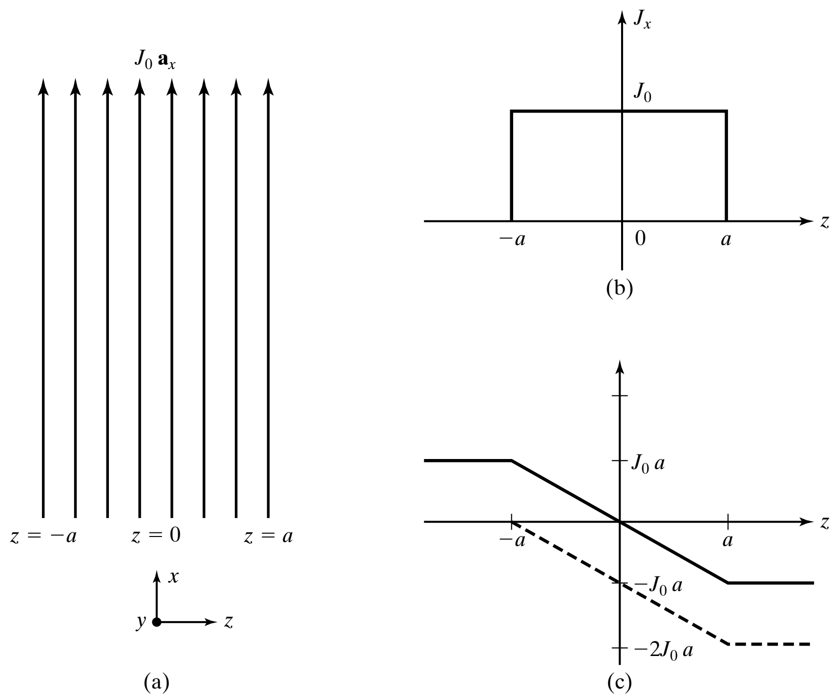

FIGURE 3.4

The determination of magnetic field due to a current distribution.

Integrating both sides with respect to z, we obtain

$$H_y = -\int_{-\infty}^{z} J_x \, dz + C$$

where C is the constant of integration.

The variation of J_x with z is shown in Fig. 3.4(b). Integrating $-J_x$ with respect to z, that is, finding the area under the curve of Fig. 3.4(b) as a function of z, and taking its negative, we obtain the result shown by the dashed curve in Fig. 3.4(c) for $-\int_{-\infty}^{z} J_x \, dz$. From symmetry considerations, the field must be equal and opposite on either side of the current region $-a < z < a$. Hence, we choose the constant of integration C to be equal to $J_0 a$, thereby obtaining the final result for H_y as shown by the solid curve in Fig. 3.4(c). Thus, the magnetic field intensity due to the current distribution is given by

$$\mathbf{H} = \begin{cases} J_0 a \mathbf{a}_y & \text{for } z < -a \\ -J_0 z \mathbf{a}_y & \text{for } -a < z < a \\ -J_0 a \mathbf{a}_y & \text{for } z > a \end{cases}$$

The magnetic flux density, $\mathbf{B}$, is equal to $\mu_0 \mathbf{H}$.

K3.1. Faraday's law in differential form; Ampere's circuital law in differential form; Curl of a vector; Lumped circuit theory approximations.

D3.1. Given $\mathbf{E} = E_0 \cos(6\pi \times 10^8 t - 2\pi z)\, \mathbf{a}_x$ V/m, find the time rate of increase of B_y at $t = 10^{-8}$ s for each of the following values of z: **(a)** 0; **(b)** $\frac{1}{4}$ m; and **(c)** $\frac{2}{3}$ m.
Ans. **(a)** 0; **(b)** $2\pi E_0$; **(c)** $-\sqrt{3}\pi E_0$.

D3.2. For the vector field $\mathbf{A} = xy^2 \mathbf{a}_x + xz \mathbf{a}_y + x^2 yz \mathbf{a}_z$, find the following: **(a)** the net right-lateral differential of A_x and A_y normal to the z-direction at the point $(1, 1, 1)$; **(b)** the net right-lateral differential of A_y and A_z normal to the x-direction at the point $(1, 2, 1)$; and **(c)** the net right-lateral differential of A_z and A_x normal to the y-direction at the point $(1, 1, -1)$.
Ans. **(a)** -1; **(b)** 0; **(c)** 2.

D3.3. Given $\mathbf{J} = \mathbf{0}$ and $\mathbf{H} = H_0 e^{-(3\times 10^8 t - z)^2} \mathbf{a}_y$ A/m, find the time rate of increase of D_x for each of the following cases: **(a)** $z = 2$ m, $t = 10^{-8}$ s; **(b)** $z = 3$ m, $t = \frac{1}{3} \times 10^{-8}$ s; and **(c)** $z = 3$ m, $t = 10^{-8}$ s.
Ans. **(a)** $-0.7358 H_0$; **(b)** $0.0733 H_0$; **(c)** 0.

3.2 GAUSS' LAWS AND THE CONTINUITY EQUATION

Thus far, we have derived Maxwell's equations in differential form corresponding to the two Maxwell's equations in integral form involving the line integrals of $\mathbf{E}$ and $\mathbf{H}$ around the closed path, that is, Faraday's law and Ampère's circuital law, respectively. The remaining two Maxwell's equations in integral form, namely, Gauss' law for the electric field and Gauss' law for the magnetic field, are concerned with the closed surface integrals of $\mathbf{D}$ and $\mathbf{B}$, respectively. In this section, we shall derive the differential forms of these two equations.

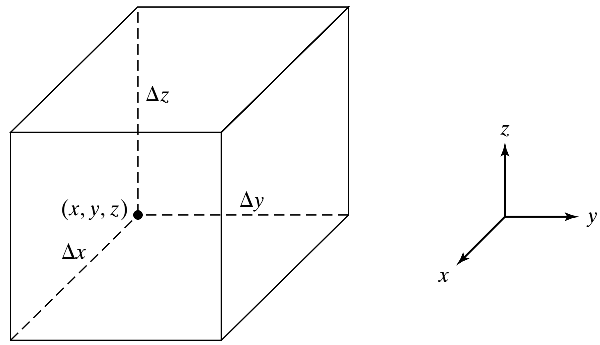

FIGURE 3.5

Infinitesimal rectangular box.

Gauss' law for the electric field

We recall from Section 2.5 that Gauss' law for the electric field is given by

$$\boxed{\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho \, dv} \tag{3.26}$$

where V is the volume enclosed by the closed surface S. To derive the differential form of this equation, let us consider a rectangular box of edges of infinitesimal lengths Δx, Δy, and Δz and defined by the six surfaces $x = x$, $x = x + \Delta x$, $y = y$, $y = y + \Delta y$, $z = z$, and $z = z + \Delta z$, as shown in Fig. 3.5, in a region of electric field

$$\boxed{\mathbf{D} = \mathbf{D}_x(x, y, z, t)\mathbf{a}_x + D_y(x, y, z, t)\mathbf{a}_y + D_z(x, y, z, t)\mathbf{a}_z} \tag{3.27}$$

and charge of density $\rho(x, y, z, t)$. According to Gauss' law for the electric field, the displacement flux emanating from the box is equal to the charge enclosed by the box. The displacement flux is given by the surface integral of $\mathbf{D}$ over the surface of the box, which comprises six plane surfaces. Thus, evaluating the displacement flux emanating from the box through each of the six plane surfaces of the box, we have

$$\int \mathbf{D} \cdot d\mathbf{S} = -[D_x]_x \, \Delta y \, \Delta z \qquad \text{for the surface } x = x \tag{3.28a}$$

$$\int \mathbf{D} \cdot d\mathbf{S} = [D_x]_{x+\Delta x} \, \Delta y \, \Delta z \qquad \text{for the surface } x = x + \Delta x \tag{3.28b}$$

$$\int \mathbf{D} \cdot d\mathbf{S} = -[D_y]_y \, \Delta z \, \Delta x \qquad \text{for the surface } y = y \tag{3.28c}$$

$$\int \mathbf{D} \cdot d\mathbf{S} = [D_y]_{y+\Delta y} \, \Delta z \, \Delta x \qquad \text{for the surface } y = y + \Delta y \tag{3.28d}$$

$$\int \mathbf{D} \cdot d\mathbf{S} = -[D_z]_z \, \Delta x \, \Delta y \qquad \text{for the surface } z = z \tag{3.28e}$$

$$\int \mathbf{D} \cdot d\mathbf{S} = [D_z]_{z+\Delta z} \, \Delta x \, \Delta y \qquad \text{for the surface } z = z + \Delta z \tag{3.28f}$$

Adding up (3.28a)–(3.28f), we obtain the total displacement flux emanating from the box to be

$$\begin{aligned}\oint_S \mathbf{D}\cdot d\mathbf{S} = &\{[D_x]_{x+\Delta x} - [D_x]_x\}\,\Delta y\,\Delta z \\ &+ \{[D_y]_{y+\Delta y} - [D_y]_y\}\,\Delta z\,\Delta x \\ &+ \{[D_z]_{z+\Delta z} - [D_z]_z\}\,\Delta x\,\Delta y\end{aligned} \tag{3.29}$$

Now the charge enclosed by the rectangular box is given by

$$\int_V \rho\, dv = \rho(x, y, z, t)\cdot\Delta x\,\Delta y\,\Delta z = \rho\Delta x\,\Delta y\,\Delta z \tag{3.30}$$

where we have assumed ρ to be uniform throughout the volume of the box and equal to its value at (x, y, z), since the box is infinitesimal in volume.

Substituting (3.29) and (3.30) into (3.26), we get

$$\begin{aligned}\{[D_x]_{x+\Delta x} - [D_x]_x\}\,\Delta y\,\Delta z + \{[D_y]_{y+\Delta y} - [D_y]_y\}\,\Delta z\,\Delta x \\ + \{[D_z]_{z+\Delta z} - [D_z]_z\}\,\Delta x\,\Delta y = \rho\Delta x\,\Delta y\,\Delta z\end{aligned}$$

or, dividing throughout by the volume Δv,

$$\frac{[D_x]_{x+\Delta x} - [D_x]_x}{\Delta x} + \frac{[D_y]_{y+\Delta y} - [D_y]_y}{\Delta y} + \frac{[D_z]_{z+\Delta z} - [D_z]_z}{\Delta z} = \rho \tag{3.31}$$

If we now let the box shrink to the point (x, y, z) by letting Δx, Δy, and Δz tend to zero, we obtain

$$\begin{aligned}\lim_{\Delta x\to 0}\frac{[D_x]_{x+\Delta x} - [D_x]_x}{\Delta x} + \lim_{\Delta y\to 0}\frac{[D_y]_{y+\Delta y} - [D_y]_y}{\Delta y} \\ + \lim_{\Delta z\to 0}\frac{[D_z]_{z+\Delta z} - [D_z]_z}{\Delta z} = \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0\\ \Delta z\to 0}} \rho\end{aligned}$$

or

$$\boxed{\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \rho} \tag{3.32}$$

Equation (3.32) is the differential equation governing the relationship between the space variations of the components of **D** to the charge density. In particular, we note that the derivatives are all longitudinal derivatives, that is, derivatives evaluated along the directions of the field components, in contrast to the lateral derivatives encountered in Section 3.1. Thus, (3.32) tells us that the net longitudinal differential, that is, the algebraic sum of the longitudinal

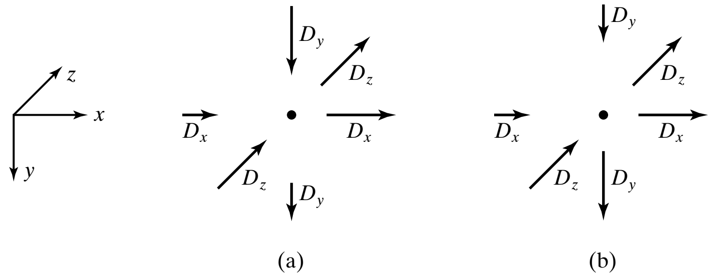

FIGURE 3.6

For illustrating (a) zero and (b) nonzero net longitudinal differential of the components of **D**.

derivatives, of the components of **D** at a point in space is equal to the charge density at that point. Conversely, a charge density at a point results in an electric field having components of **D** such that their net longitudinal differential is nonzero. Figure 3.6(a) shows an example in which the net longitudinal differential is zero. This is because $\partial D_x/\partial x$ and $\partial D_y/\partial y$ are equal in magnitude but opposite in sign, whereas $\partial D_z/\partial z$ is zero. On the other hand, for the example in Fig. 3.6(b), both $\partial D_x/\partial x$ and $\partial D_y/\partial y$ are positive and $\partial D_z/\partial z$ is zero, so that the net longitudinal differential is nonzero.

Divergence (del dot)

Equation (3.32) can be written in vector notation as

$$\left(\mathbf{a}_x \frac{\partial}{\partial x} + \mathbf{a}_y \frac{\partial}{\partial y} + \mathbf{a}_z \frac{\partial}{\partial z}\right) \cdot (D_x\mathbf{a}_x + D_y\mathbf{a}_y + D_z\mathbf{a}_z) = \rho \tag{3.33}$$

The left side of (3.33) is known as the *divergence of* **D**, denoted as $\nabla \cdot \mathbf{D}$ (del dot **D**). Thus, we have

$$\boxed{\nabla \cdot \mathbf{D} = \rho} \tag{3.34}$$

Equation (3.34) is Maxwell's equation in differential form corresponding to Gauss' law for the electric field. It tells us that the divergence of the displacement flux density at a point is equal to the charge density at that point. We shall discuss divergence further in Section 3.3.

Example 3.5 Electric field of a charge distribution from Gauss' law in differential form

Let us consider the charge distribution given by

$$\rho = \begin{cases} -\rho_0 & \text{for } -a < x < 0 \\ \rho_0 & \text{for } 0 < x < a \end{cases}$$

as shown in Fig. 3.7(a), where ρ_0 is a constant, and find the electric field everywhere.

Since the charge density is independent of y and z, the field is also independent of y and z, thereby giving us $\partial D_y/\partial y = \partial D_z/\partial z = 0$ and reducing Gauss' law for the electric field to

$$\frac{\partial D_x}{\partial x} = \rho$$

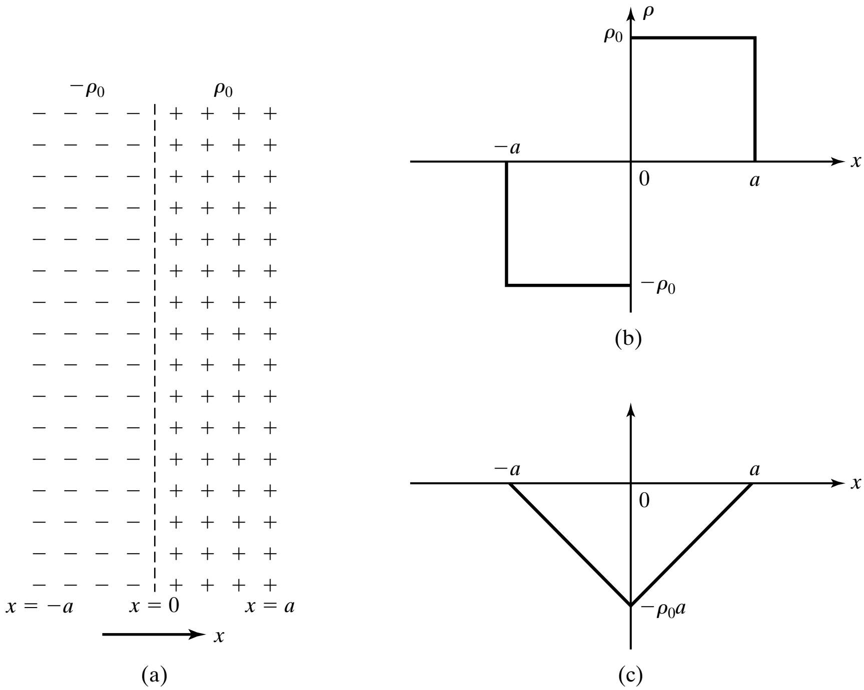

FIGURE 3.7

The determination of electric field due to a charge distribution.

Integrating both sides with respect to x, we obtain

$$D_x = \int_{-\infty}^{x} \rho \, dx + C$$

where C is the constant of integration.

The variation of ρ with x is shown in Fig. 3.7(b). Integrating ρ with respect to x, that is, finding the area under the curve of Fig. 3.7(b) as a function of x, we obtain the result shown in Fig. 3.7(c) for $\int_{-\infty}^{x} \rho \, dx$. The constant of integration C is zero since the symmetry of the field required by the symmetry of the charge distribution is already satisfied by the curve of of Fig. 3.7(c). Alternatively, it can be seen that any nonzero value of C would remain even if the charge distribution is allowed to disappear, and hence it is not attributable to the given charge distribution. Thus, the displacement flux density due to the charge distribution is given by

$$\mathbf{D} = \begin{cases} 0 & \text{for } x < -a \\ -\rho_0(x + a)\mathbf{a}_x & \text{for } -a < x < 0 \\ \rho_0(x - a)\mathbf{a}_x & \text{for } 0 < x < a \\ 0 & \text{for } x > a \end{cases}$$

The electric field intensity, $\mathbf{E}$, is equal to $\mathbf{D}/\varepsilon_0$.

Although we have deduced (3.34) from (3.26) by considering the Cartesian coordinate system, it is independent of the coordinate system since (3.26) is independent of the coordinate system. The expressions for the divergence of a vector in cylindrical and spherical coordinate systems are derived in Appendix B. They are reproduced here together with that in (3.32) for the Cartesian coordinate system.

CARTESIAN

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \tag{3.35a}$$

CYLINDRICAL

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \tag{3.35b}$$

SPHERICAL

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r \sin\theta}\frac{\partial A_\phi}{\partial \phi} \tag{3.35c}$$

Example 3.6 Evaluating divergences of vector fields

Find the divergences of the following vector fields: **(a)** $3x\mathbf{a}_x + (y - 3)\mathbf{a}_y + (2 - z)\mathbf{a}_z$ and **(b)** $r^2 \sin\theta\, \mathbf{a}_\theta$ in spherical coordinates.

(a) Using (3.35a), we have

$$\begin{aligned}\nabla \cdot [3x\mathbf{a}_x + (y - 3)\mathbf{a}_y + (2 - z)\mathbf{a}_z] &= \frac{\partial}{\partial x}(3x) + \frac{\partial}{\partial y}(y - 3) + \frac{\partial}{\partial z}(2 - z) \\ &= 3 + 1 - 1 = 3\end{aligned}$$

(b) Using (3.35c), we obtain

$$\begin{aligned}\nabla \cdot r^2 \sin\theta\, \mathbf{a}_\theta &= \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(r^2 \sin^2\theta) \\ &= \frac{1}{r \sin\theta}(2r^2 \sin\theta \cos\theta) \\ &= 2r\cos\theta\end{aligned}$$

Gauss' law for the magnetic field

We shall now consider the derivation of the differential form of Gauss' law for the magnetic field given in integral form by

$$\boxed{\oint_S \mathbf{B} \cdot d\mathbf{S} = 0} \tag{3.36}$$

where S is any closed surface. To do this, we need not repeat the procedure employed in the case of Gauss' law for the electric field. Instead, we note from (3.26) and (3.34) that in converting to the differential form from integral form, the surface integral of **D** over the closed surface S is replaced by the divergence of **D** and the volume integral of ρ is replaced by ρ itself, as shown:

$$\boxed{\begin{array}{c} \oint_S \mathbf{D}\cdot d\mathbf{S} = \int_V \rho\, dv \\ \downarrow \\ \nabla\cdot\mathbf{D} = \rho \end{array}}$$

Then using the analogy between the two Gauss' laws, we can write the following:

$$\boxed{\begin{array}{c} \oint_S \mathbf{B}\cdot d\mathbf{S} = 0 = \int_V 0\, dv \\ \downarrow \\ \nabla\cdot\mathbf{B} = 0 \end{array}}$$

Thus, Gauss' law in differential form for the magnetic field

$$\mathbf{B} = B_x(x, y, z, t)\mathbf{a}_x + B_y(x, y, z, t)\mathbf{a}_y + B_z(x, y, z, t)\mathbf{a}_z \tag{3.37}$$

is given by

$$\boxed{\nabla\cdot\mathbf{B} = 0} \tag{3.38}$$

which tells us that the divergence of the magnetic flux density at a point is equal to zero. Conversely, for a vector field to be realized as a magnetic field, its divergence must be zero. In Cartesian coordinates, (3.38) becomes

$$\boxed{\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0} \tag{3.39}$$

pointing out that the net longitudinal differential of the components of **B** is zero. Also, expressions similar to (3.39) can be written in cylindrical and spherical coordinate systems by using the expressions for the divergence in those coordinate systems, given by (3.35b) and (3.35c), respectively.

Example 3.7 Realizability of a vector field as a magnetic field

Determine if the vector $\mathbf{A} = (1/r^2)(\cos\phi\, \mathbf{a}_r + \sin\phi\, \mathbf{a}_\phi)$ in cylindrical coordinates can represent a magnetic field **B**.

Noting that

$$\begin{aligned}\nabla \cdot \mathbf{A} &= \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\cos\phi}{r}\right) + \frac{1}{r}\frac{\partial}{\partial \phi}\left(\frac{\sin\phi}{r^2}\right)\\ &= -\frac{\cos\phi}{r^3} + \frac{\cos\phi}{r^3} = 0\end{aligned}$$

we conclude that the given vector can represent a **B**.

Continuity equation

We shall conclude this section by deriving the differential form of the law of conservation of charge given in integral form by

$$\boxed{\oint_S \mathbf{J}\cdot d\mathbf{S} = -\frac{d}{dt}\int_V \rho\, dv} \tag{3.40}$$

Using analogy with Gauss' law for the electric field, we can write the following:

$$\boxed{\begin{gathered}\oint_S \mathbf{J}\cdot d\mathbf{S} = -\frac{d}{dt}\int_V \rho\, dv\\ \overbrace{\nabla\cdot}\ \mathbf{J} = -\frac{\partial}{\partial t}(\rho)\end{gathered}}$$

Thus, the differential form of the law of conservation of charge is given by

$$\boxed{\nabla\cdot\mathbf{J} = -\frac{\partial\rho}{\partial t}} \tag{3.41}$$

Equation (3.41) is familiarly known as the *continuity equation*. It tells us that the divergence of the current density due to flow of charges at a point is equal to the time rate of decrease of the charge density at that point. It can be expanded in a given coordinate system by using the expression for the divergence in that coordinate system.

K3.2. Gauss' law for the electric field in differential form; Gauss' law for the magnetic field in differential form; Divergence of a vector; Continuity equation.

D3.4. For the vector field $\mathbf{A} = yz\mathbf{a}_x + xy\mathbf{a}_y + xyz^2\mathbf{a}_z$, find the net longitudinal differential of the components of **A** at the following points: **(a)** $(1, 1, -1)$; **(b)** $(1, 1, -\frac{1}{2})$; and **(c)** $(1, 1, 1)$.

Ans. **(a)** -1; **(b)** 0; **(c)** 3.

D3.5. The following hold at a point in a charge-free region: (i) the sum of the longitudinal differentials of D_x and D_y is D_0 and (ii) the longitudinal differential of D_y

is three times the longitudinal differential of D_z. Find: **(a)** $\partial D_x/\partial x$; **(b)** $\partial D_y/\partial y$; and **(c)** $\partial D_z/\partial z$.

Ans. **(a)** $4D_0$; **(b)** $-3D_0$; **(c)** $-D_0$.

D3.6. In a small region around the origin, the current density due to flow of charges is given by $\mathbf{J} = J_0(x^2\mathbf{a}_x + y^2\mathbf{a}_y + z^2\mathbf{a}_z)$ A/m^2, where J_0 is a constant. Find the time rate of increase of the charge density at each of the following points: **(a)** $(0.02, 0.01, 0.01)$; **(b)** $(0.02, -0.01, -0.01)$; and **(c)** $(-0.02, -0.01, 0.01)$.

Ans. **(a)** $-0.08J_0$ (C/m^3)/s; **(b)** 0; **(c)** $0.04J_0$ (C/m^3)/s.

3.3 CURL AND DIVERGENCE

In Sections 3.1 and 3.2, we derived the differential forms of Maxwell's equations and the law of conservation of charge from their integral forms. Maxwell's equations are given by

$$\boxed{\begin{aligned} \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} && (3.42a)\\ \nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} && (3.42b)\\ \nabla \cdot \mathbf{D} &= \rho && (3.42c)\\ \nabla \cdot \mathbf{B} &= 0 && (3.42d) \end{aligned}}$$

whereas the continuity equation is given by

$$\boxed{\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}} \tag{3.43}$$

These equations contain two new vector (differential) operations, namely, the curl and the divergence. The curl of a vector is a vector quantity, whereas the divergence of a vector is a scalar quantity. In this section, we shall introduce the basic definitions of curl and divergence and then discuss physical interpretations of these quantities. We shall also derive two associated theorems.

A. Curl

Curl, basic definition

To discuss curl first, let us consider Ampère's circuital law without the displacement current density term; that is,

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{3.44}$$

We wish to express $\nabla \times \mathbf{H}$ at a point in the current region in terms of $\mathbf{H}$ at that point. If we consider an infinitesimal surface $\Delta\mathbf{S}$ at the point and take the dot product of both sides of (3.44) with $\Delta\mathbf{S}$, we get

$$(\nabla \times \mathbf{H}) \cdot \Delta\mathbf{S} = \mathbf{J} \cdot \Delta\mathbf{S} \tag{3.45}$$

But $\mathbf{J} \cdot \Delta\mathbf{S}$ is simply the current crossing the surface $\Delta\mathbf{S}$, and according to Ampère's circuital law in integral form without the displacement current term,

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \mathbf{J} \cdot \Delta\mathbf{S} \tag{3.46}$$

where C is the closed path bounding $\Delta\mathbf{S}$. Comparing (3.45) and (3.46), we have

$$(\nabla \times \mathbf{H}) \cdot \Delta\mathbf{S} = \oint_C \mathbf{H} \cdot d\mathbf{l}$$

or

$$(\nabla \times \mathbf{H}) \cdot \Delta S\, \mathbf{a}_n = \oint_C \mathbf{H} \cdot d\mathbf{l} \tag{3.47}$$

where $\mathbf{a}_n$ is the unit vector normal to ΔS and directed toward the side of advance of a right-hand screw as it is turned around C. Dividing both sides of (3.47) by ΔS, we obtain

$$(\nabla \times \mathbf{H}) \cdot \mathbf{a}_n = \frac{\oint_C \mathbf{H} \cdot d\mathbf{l}}{\Delta S} \tag{3.48}$$

The maximum value of $(\nabla \times \mathbf{H}) \cdot \mathbf{a}_n$, and hence that of the right side of (3.48), occurs when $\mathbf{a}_n$ is oriented parallel to $\nabla \times \mathbf{H}$, that is, when the surface ΔS is oriented normal to the current density vector $\mathbf{J}$. This maximum value is simply $|\nabla \times \mathbf{H}|$. Thus,

$$|\nabla \times \mathbf{H}| = \left[\frac{\oint_C \mathbf{H} \cdot d\mathbf{l}}{\Delta S}\right]_{\max}$$

Since the direction of $\nabla \times \mathbf{H}$ is the direction of $\mathbf{J}$, or that of the unit vector normal to ΔS, we can then write

$$\nabla \times \mathbf{H} = \left[\frac{\oint_C \mathbf{H} \cdot d\mathbf{l}}{\Delta S}\right]_{\max} \mathbf{a}_n$$

This result is, however, approximate, since (3.47) is exact only in the limit that ΔS tends to zero. Thus,

$$\nabla \times \mathbf{H} = \lim_{\Delta S \to 0} \left[\frac{\oint_C \mathbf{H} \cdot d\mathbf{l}}{\Delta S}\right]_{\max} \mathbf{a}_n \tag{3.49}$$

which is the expression for $\nabla \times \mathbf{H}$ at a point in terms of $\mathbf{H}$ at that point. Although we have derived this for the $\mathbf{H}$ vector, it is a general result and, in fact, is often the starting point for the introduction of curl. Thus, for any vector field $\mathbf{A}$,

$$\boxed{\nabla \times \mathbf{A} = \lim_{\Delta S \to 0} \left[\frac{\oint_C \mathbf{A} \cdot d\mathbf{l}}{\Delta S} \right]_{\max} \mathbf{a}_n} \tag{3.50}$$

Equation (3.50) tells us that to find the curl of a vector at a point in that vector field, we first consider an infinitesimal surface at that point and compute the closed line integral or circulation of the vector around the periphery of this surface by orienting the surface such that the circulation is maximum. We then divide the circulation by the area of the surface to obtain the maximum value of the circulation per unit area. Since we need this maximum value of the circulation per unit area in the limit that the area tends to zero, we do this by gradually shrinking the area and making sure that each time we compute the circulation per unit area, an orientation for the area that maximizes this quantity is maintained. The limiting value to which the maximum circulation per unit area approaches is the *magnitude* of the curl. The limiting direction to which the normal vector to the surface approaches is the *direction* of the curl. The task of computing the curl is simplified if we consider one component at a time and compute that component, since then it is sufficient if we always maintain the orientation of the surface normal to that component axis. In fact, this is what we did in Section 3.1, which led us to the determinant expression for the curl in Cartesian coordinates, by choosing for convenience rectangular surfaces whose sides are all parallel to the coordinate planes.

Physical interpretation of curl

We are now ready to discuss the physical interpretation of the curl. We do this with the aid of a simple device known as the *curl meter*, which responds to the circulation of the vector field. Although the curl meter may take several forms, we shall consider one consisting of a circular disk that floats in water with a paddle wheel attached to the bottom of the disk, as shown in Fig. 3.8. A dot at the periphery on top of the disk serves to indicate any rotational motion of the curl meter about its axis (i.e., the axis of the paddle wheel). Let us now consider a stream of rectangular cross section carrying water in the z-direction, as shown in Fig. 3.8(a). Let us assume the velocity $\mathbf{v}$ of the water to be independent of height but increasing sinusoidally from a value of zero at the banks to a maximum value v_0 at the center, as shown in Fig. 3.8(b), and investigate the behavior of the curl meter when it is placed vertically at different points in the stream. We assume that the size of the curl meter is vanishingly small so that it does not disturb the flow of water as we probe its behavior at different points.

Since exactly in midstream the blades of the paddle wheel lying on either side of the centerline are hit by the same velocities, the paddle wheel does not rotate. The curl meter simply slides down the stream without any rotational motion, that is, with the dot on top of the disk maintaining the same position relative to the center of the disk, as shown in Fig. 3.8(c). At a point to the left of the midstream, the blades of the paddle wheel are hit by a greater velocity on the right side than on the left side so that the paddle wheel rotates in the counterclockwise

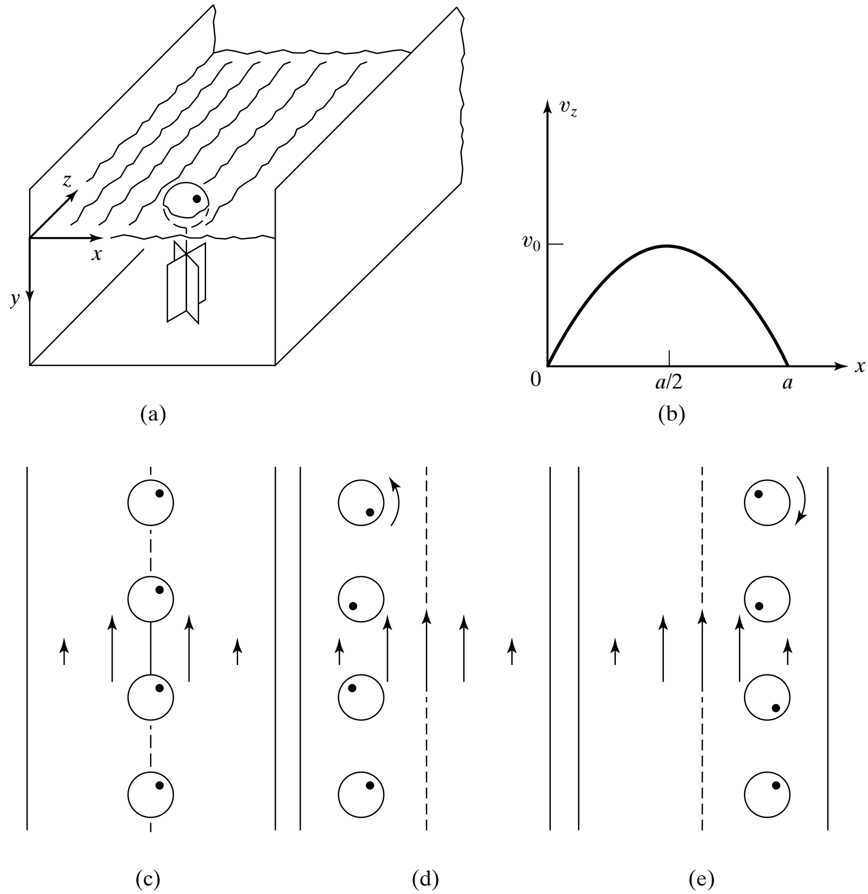

FIGURE 3.8

For explaining the physical interpretation of curl using the curl meter.

sense, as seen looking along the positive y-axis. The curl meter rotates in the counterclockwise direction about its axis as it slides down the stream, as indicated by the changing position of the dot on top of the disk relative to the center of the disk, as shown in Fig. 3.8(d). At a point to the right of midstream, the blades of the paddle wheel are hit by a greater velocity on the left side than on the right side so that the paddle wheel rotates in the clockwise sense, as seen looking along the positive y-axis. The curl meter rotates in the clockwise direction about its axis as it slides down the stream, as indicated by the changing position of the dot on top of the disk relative to the center of the disk, as shown in Fig 3.8(e).

If we now pick up the curl meter and insert it in the water with its axis parallel to the x-axis, the curl meter does not rotate because its blades are hit with the same force above and below its axis. If the curl meter is inserted in the water with its axis parallel to the z-axis, it does not rotate since the water flow is then parallel to the blades.

To relate the behavior of the curl meter with the curl of the velocity vector field of the water flow, we note that since the velocity vector is given by

$$\mathbf{v} = v_z(x)\mathbf{a}_z = v_0 \sin \frac{\pi x}{a} \mathbf{a}_z$$

its curl is given by

$$\begin{aligned}\nabla \times \mathbf{v} &= \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & 0 & v_z \end{vmatrix} \\ &= -\frac{\partial v_z}{\partial x}\mathbf{a}_y \\ &= -\frac{\pi v_0}{a} \cos \frac{\pi x}{a} \mathbf{a}_y\end{aligned}$$

Therefore, the x- and z-components of the curl are zero, whereas the y-component is nonzero varying with x in a cosinusoidal manner, from negative values left of midstream, to zero at midstream, to positive values right of midstream. Thus, no rotation of the curl meter corresponds to zero value for the component of the curl along its axis. Rotation of the curl meter in the counterclockwise or left-hand sense as seen looking along its axis corresponds to a nonzero negative value, and rotation in the clockwise or right-hand sense corresponds to a nonzero positive value for the component of the curl. It can further be visualized that the rate of rotation of the curl meter is a measure of the magnitude of the pertinent nonzero component of the curl.

The foregoing illustration of the physical interpretation of the curl of a vector field can be used to visualize the behavior of electric and magnetic fields. Thus, from

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

we know that at a point in an electromagnetic field, the circulation of the electric field per unit area in a given plane is equal to the component of $-\partial\mathbf{B}/\partial t$ along the unit vector normal to that plane and directed in the right-hand sense. Similarly, from

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

we know that at a point in an electromagnetic field, the circulation of the magnetic field per unit area in a given plane is equal to the component of $\mathbf{J} + \partial\mathbf{D}/\partial t$ along the unit vector normal to that plane and directed in the right-hand sense.

B. Divergence

Divergence, basic definition

Turning now to the discussion of divergence, let us consider Gauss' law for the electric field in differential form; that is,

$$\nabla \cdot \mathbf{D} = \rho \tag{3.51}$$

We wish to express $\nabla \cdot \mathbf{D}$ at a point in the charge region in terms of $\mathbf{D}$ at that point. If we consider an infinitesimal volume Δv at that point and multiply both sides of (3.51) by Δv, we get

$$(\nabla \cdot \mathbf{D})\, \Delta v = \rho\, \Delta v \tag{3.52}$$

But $\rho\, \Delta v$ is simply the charge contained in the volume Δv, and according to Gauss' law for the electric field in integral form,

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \rho\, \Delta v \tag{3.53}$$

where S is the closed surface bounding Δv. Comparing (3.52) and (3.53), we have

$$(\nabla \cdot \mathbf{D})\, \Delta v = \oint_S \mathbf{D} \cdot d\mathbf{S} \tag{3.54}$$

Dividing both sides of (3.54) by Δv, we obtain

$$\nabla \cdot \mathbf{D} = \frac{\oint_S \mathbf{D} \cdot d\mathbf{S}}{\Delta v} \tag{3.55}$$

This result is however approximate since (3.54) is exact only in the limit that Δv tends to zero. Thus,

$$\nabla \cdot \mathbf{D} = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{D} \cdot d\mathbf{S}}{\Delta v} \tag{3.56}$$

which is the expression for $\nabla \cdot \mathbf{D}$ at a point in terms of $\mathbf{D}$ at that point. Although we have derived this for the $\mathbf{D}$ vector, it is a general result and, in fact, is often the starting point for the introduction of divergence. Thus, for any vector field, $\mathbf{A}$,

$$\boxed{\nabla \cdot \mathbf{A} = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta v}} \tag{3.57}$$

Equation (3.57) tells us that to find the divergence of a vector at a point in that vector field, we first consider an infinitesimal volume at that point and compute the surface integral of the vector over the surface bounding that volume, that is, the outward flux of the vector field from that volume. We then divide the flux by the volume to obtain the flux per unit volume. Since we need this flux per unit volume in the limit that the volume tends to zero, we do this by gradually shrinking the volume. The limiting value to which the flux per unit volume approaches is the value of the divergence of the vector field at the point to which the volume is shrunk. In fact, this is what we did in Section 3.2, which led to the expression for the divergence in Cartesian coordinates, by choosing for convenience the volume of a rectangular box whose surfaces are parallel to the coordinate planes.

Physical interpretation of divergence

We are now ready to discuss the physical interpretation of the divergence. To simplify this task, we shall consider the continuity equation given by

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \tag{3.58}$$

Let us investigate three different cases: (1) positive value, (2) negative value, and (3) zero value of the time rate of decrease of the charge density at a point, that is, the divergence of the current density vector at that point. We shall do this with the aid of a simple device, which we shall call the *divergence meter*. The divergence meter can be imagined to be a tiny elastic balloon that encloses the point and that expands when hit by charges streaming outward from the point and contracts when acted on by charges streaming inward toward the point. For case 1, that is, when the time rate of decrease of the charge density at the point is positive, there is a net amount of charge streaming out of the point in a given time, resulting in a net current flow outward from the point that will make the imaginary balloon expand. For case 2, that is, when the time rate of decrease of the charge density at the point is negative or the time rate of increase of the charge density is positive, there is a net amount of charge streaming toward the point in a given time, resulting in a net current flow toward the point that will make the imaginary balloon contract. For case 3, that is, when the time rate of decrease of the charge density at the point is zero, the balloon will remain unaffected, since the charge is streaming out of the point at exactly the same rate as it is streaming into the point. The situation corresponding to case 1 is illustrated in Figs. 3.9(a) and (b), whereas that corresponding to case 2 is illustrated in Figs. 3.9(c) and (d), and that corresponding to case 3 is illustrated in Fig. 3.9(e). Note that in Figs. 3.9(a), (c), and (e), the imaginary balloon slides along the lines of current flow while responding to the divergence by expanding, contracting, or remaining unaffected.

Generalizing the foregoing discussion to the physical interpretation of the divergence of any vector field at a point, we can imagine the vector field to be a velocity field of streaming charges acting on the divergence meter and obtain in most cases a qualitative picture of the divergence of the vector field. If the divergence meter expands, the divergence is positive and a source of the flux of the vector field exists at that point. If the divergence meter contracts, the divergence is negative and a sink of the flux of the vector field exists at that point. It can be further visualized that the rate of expansion or contraction of the divergence meter is a measure of the magnitude of the divergence. If the divergence meter remains unaffected, the divergence is zero, and neither a source nor a sink of the flux of the vector field exists at that point; alternatively, there can exist at the point pairs of sources and sinks of equal strengths.

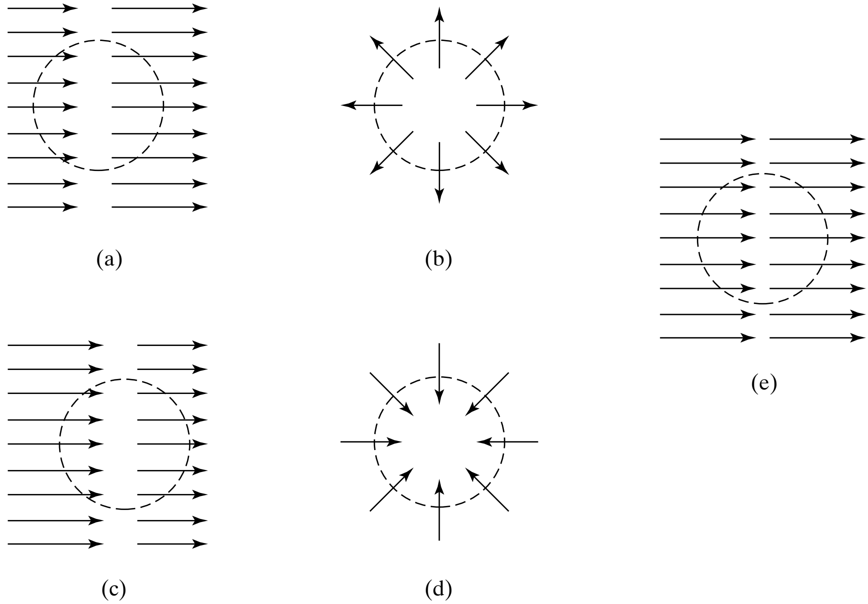

FIGURE 3.9

For explaining the physical interpretation of divergence using the divergence meter.

C. Stokes' and Divergence Theorems

We shall now derive two useful theorems in vector calculus, *Stokes' theorem* and the *divergence theorem*. Stokes' theorem relates the closed line integral of a vector field to the surface integral of the curl of that vector field, whereas the divergence theorem relates the closed surface integral of a vector field to the volume integral of the divergence of that vector field.

Stokes' theorem

To derive Stokes' theorem, let us consider an arbitrary surface S in a magnetic field region and divide this surface into a number of infinitesimal surfaces $\Delta S_1, \Delta S_2, \Delta S_3, \ldots,$ bounded by the contours $C_1, C_2, C_3, \ldots,$ respectively. Then, applying (3.45) to each one of these infinitesimal surfaces and adding up, we get

$$\sum_j (\nabla \times \mathbf{H})_j \cdot \Delta S_j \, \mathbf{a}_{nj} = \oint_{C_1} \mathbf{H} \cdot d\mathbf{l} + \oint_{C_2} \mathbf{H} \cdot d\mathbf{l} + \cdots \tag{3.59}$$

where $\mathbf{a}_{nj}$ are unit vectors normal to the surfaces ΔS_j chosen in accordance with the right-hand screw rule. In the limit that the number of infinitesimal surfaces tends to infinity, the left side of (3.59) approaches to the surface integral of $\nabla \times \mathbf{H}$ over the surface S. The right side of (3.59) is simply the closed line integral of $\mathbf{H}$ around the contour C, since the contributions to the line integrals from the portions of the contours interior to C cancel, as shown in Fig. 3.10. Thus, we get

$$\int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S} = \oint_C \mathbf{H} \cdot d\mathbf{l} \tag{3.60}$$

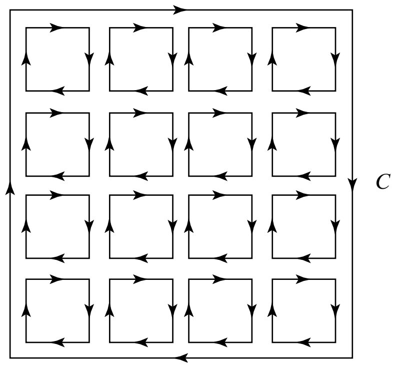

FIGURE 3.10
For deriving Stokes' theorem.

Equation (3.60) is Stokes' theorem. Although we have derived it by considering the **H** field, it is general and can be derived from the application of (3.50) to a geometry such as that in Fig. 3.10. Thus, for any vector field **A**,

$$\oint_C \mathbf{A} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} \tag{3.61}$$

where S is any surface bounded by C.

Example 3.8 Evaluation of line integral around a closed path using Stokes' theorem

Let us evaluate the line integral of Example 2.1 by using Stokes' theorem.

For $\mathbf{F} = x\mathbf{a}_y$,

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & x & 0 \end{vmatrix} = \mathbf{a}_z$$

With reference to Fig. 2.4, we then have

$$\begin{aligned} \oint_{ABCDA} \mathbf{F} \cdot d\mathbf{l} &= \int_{\substack{\text{area} \\ ABCDA}} (\nabla \times \mathbf{F}) \cdot d\mathbf{S} \\ &= \int_{\substack{\text{area} \\ ABCDA}} \mathbf{a}_z \cdot dx\, dy\, \mathbf{a}_z \\ &= \int_{\substack{\text{area} \\ ABCDA}} dx\, dy \\ &= \text{area } ABCDA \\ &= 6 \end{aligned}$$

which agrees with the result obtained in Example 2.1.

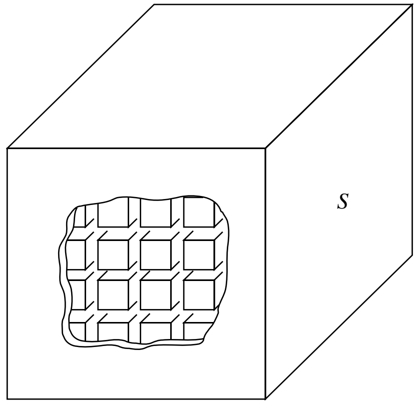

FIGURE 3.11

For deriving the divergence theorem.

Divergence theorem

To derive the divergence theorem, let us consider an arbitrary volume V in an electric field region and divide this volume into a number of infinitesimal volumes $\Delta v_1, \Delta v_2, \Delta v_3, \ldots$, bounded by the surfaces $S_1, S_2, S_3, \ldots$, respectively. Then, applying (3.54) to each one of these infinitesimal volumes and adding up, we get

$$\sum_j (\nabla \cdot \mathbf{D})_j \, \Delta v_j = \oint_{S_1} \mathbf{D} \cdot d\mathbf{S} + \oint_{S_2} \mathbf{D} \cdot d\mathbf{S} + \cdots \tag{3.62}$$

In the limit that the number of the infinitesimal volumes tends to infinity, the left side of (3.62) approaches to the volume integral of $\nabla \cdot \mathbf{D}$ over the volume V. The right side of (3.62) is simply the closed surface integral of $\mathbf{D}$ over S since the contribution to the surface integrals from the portions of the surfaces interior to S cancel, as shown in Fig. 3.11. Thus, we get

$$\int_V (\nabla \cdot \mathbf{D}) \, dv = \oint_S \mathbf{D} \cdot d\mathbf{S} \tag{3.63}$$

Equation (3.63) is the divergence theorem. Although we have derived it by considering the $\mathbf{D}$ field, it is general and can be derived from the application of (3.57) to a geometry such as that in Fig. 3.11. Thus, for any vector field $\mathbf{A}$,

$$\boxed{\oint_S \mathbf{A} \cdot d\mathbf{S} = \int_V (\nabla \cdot \mathbf{A}) \, dv} \tag{3.64}$$

where V is the volume bounded by S.

Example 3.9 Showing that the divergence of the curl of a vector is zero

Divergence of the curl of a vector

By using the Stokes and divergence theorems, show that for any vector field $\mathbf{A}$, $\nabla \cdot \nabla \times \mathbf{A} = 0$.

Let us consider volume V bounded by the closed surface $S_1 + S_2$, where S_1 and S_2 are bounded by the closed paths C_1 and C_2, respectively, as shown in Fig. 3.12. Note that

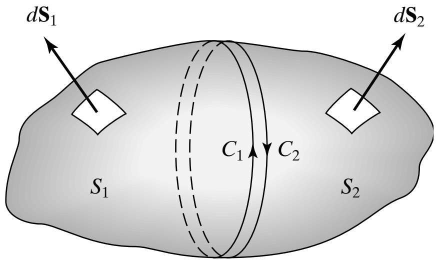

FIGURE 3.12
For proving the identity $\nabla \cdot \nabla \times \mathbf{A} = 0$.

C_1 and C_2 touch each other and are traversed in opposite senses and that $d\mathbf{S}_1$ and $d\mathbf{S}_2$ are directed in the right-hand sense relative to C_1 and C_2, respectively. Then, using divergence and Stokes' theorems in succession, we obtain

$$\begin{aligned}\int_V (\nabla \cdot \nabla \times \mathbf{A})\, dv &= \oint_{S_1+S_2} (\nabla \times \mathbf{A}) \cdot d\mathbf{S} \\ &= \int_{S_1} (\nabla \times \mathbf{A}) \cdot d\mathbf{S}_1 + \int_{S_2} (\nabla \times \mathbf{A}) \cdot d\mathbf{S}_2 \\ &= \oint_{C_1} \mathbf{A} \cdot d\mathbf{l} + \oint_{C_2} \mathbf{A} \cdot d\mathbf{l} \\ &= 0\end{aligned}$$

Since this result holds for any arbitrary volume V, it follows that

$$\boxed{\nabla \cdot \nabla \times \mathbf{A} = 0} \tag{3.65}$$

K3.3. Basic definition of curl; Physical interpretation of curl; Basic definition of divergence; Physical interpretation of divergence; Stokes' theorem; Divergence theorem; Divergence of the curl of a vector.

D3.7. With the aid of the curl meter, determine if the z-component of the curl of the vector field $\mathbf{A} = (x^2 - 4)\mathbf{a}_y$ is positive, zero, or negative at each of the following points: **(a)** $(2, -3, 1)$; **(b)** $(0, 2, 4)$; and **(c)** $(-1, 2, -1)$.

Ans. **(a)** positive; **(b)** zero; **(c)** negative.

D3.8. With the aid of the divergence meter, determine if the divergence of the vector field $\mathbf{A} = (x - 2)^2\mathbf{a}_x$ is positive, zero, or negative at each of the following points: **(a)** $(2, 4, 3)$; **(b)** $(1, 1, -1)$; and **(c)** $(3, -1, 4)$.

Ans. **(a)** zero; **(b)** negative; **(c)** positive.

D3.9. Using Stokes' theorem, find the absolute value of the line integral of the vector field $(x\mathbf{a}_y + \sqrt{3}y\mathbf{a}_z)$ around each of the following closed paths: **(a)** the perimeter of a square of sides 2 m lying in the xy-plane; **(b)** a circular path of radius $1/\sqrt{\pi}$ m lying in the xy-plane; and **(c)** the perimeter of an equilateral triangle of sides 2 m lying in the yz-plane.

Ans. **(a)** 4; **(b)** 1; **(c)** 3.

D3.10. Using the divergence theorem, find the surface integral of the vector field $(x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z)$ over each of the following closed surfaces: **(a)** the surface of a cube of sides 1 m; **(b)** the surface of a cylinder of radius $1/\sqrt{\pi}$ m and length 2 m; and **(c)** the surface of a sphere of radius $1/(\pi)^{1/3}$ m.

Ans. **(a)** 3; **(b)** 6; **(c)** 4.

3.4 UNIFORM PLANE WAVES IN TIME DOMAIN IN FREE SPACE

Uniform plane wave defined

In Section 3.1, we learned that the space variations of the electric- and magnetic-field components are related to the time variations of the magnetic- and electric-field components, respectively, through Maxwell's equations. This interdependence gives rise to the phenomenon of electromagnetic wave propagation. In the general case, electromagnetic wave propagation involves electric and magnetic fields having more than one component, each dependent on all three coordinates, in addition to time. However, a simple and very useful type of wave that serves as a building block in the study of electromagnetic waves consists of electric and magnetic fields that are perpendicular to each other and to the direction of propagation and are *uniform* in *planes* perpendicular to the direction of propagation. These waves are known as *uniform plane waves*. By orienting the coordinate axes such that the electric field is in the x-direction, the magnetic field is in the y-direction, and the direction of propagation is in the z-direction, as shown in Fig. 3.13, we have

$$\mathbf{E} = E_x(z, t)\mathbf{a}_x \tag{3.66a}$$

$$\mathbf{H} = H_y(z, t)\mathbf{a}_y \tag{3.66b}$$

Uniform plane waves do not exist in practice because they cannot be produced by finite-sized antennas. At large distances from physical antennas and ground, however, the waves can be approximated as uniform plane waves. Furthermore, the principles of guiding of electromagnetic waves along transmission lines and waveguides and the principles of many other wave phenomena can be studied basically in terms of uniform plane waves. Hence, it is very important that we understand the principles of uniform plane wave propagation.

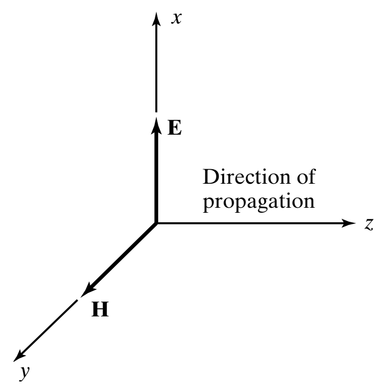

FIGURE 3.13

Directions of electric and magnetic fields and direction of propagation for a simple case of uniform plane wave.

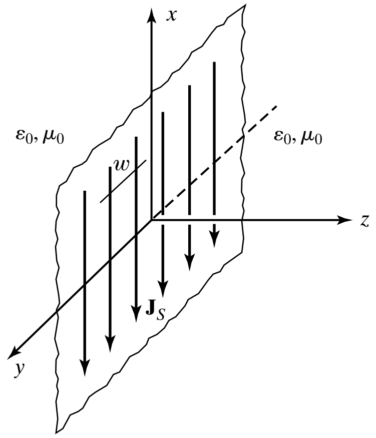

FIGURE 3.14

Infinite plane sheet in the xy-plane carrying surface current of uniform density.

Infinite plane current sheet source

To illustrate the phenomenon of interaction of electric and magnetic fields giving rise to uniform plane electromagnetic wave propagation and the principle of radiation of electromagnetic waves from an antenna, we shall consider a simple, idealized, hypothetical source. This source consists of an infinite sheet lying in the xy-plane, as shown in Fig. 3.14. On this infinite plane sheet, a uniformly distributed current flows in the negative x-direction, as given by

$$\boxed{\mathbf{J}_S = -J_S(t)\mathbf{a}_x \quad \text{for} \qquad z = 0} \tag{3.67}$$

where $J_S(t)$ is a given function of time. Because of the uniformity of the surface current density on the infinite sheet, if we consider any line of width w parallel to the y-axis, as shown in Fig. 3.14, the current crossing that line is simply given by w times the current density, that is, $wJ_S(t)$. If $J_S(t) = J_{S0} \cos \omega t$, then the current $wJ_{S0} \cos \omega t$, crossing the width w, actually alternates between negative x- and positive x-directions, that is, downward and upward. The time history of this current flow for one period of the sinusoidal variation is illustrated in Fig. 3.15, with the lengths of the lines indicating the magnitudes of the current. We shall consider the medium on either side of the current sheet to be free space.

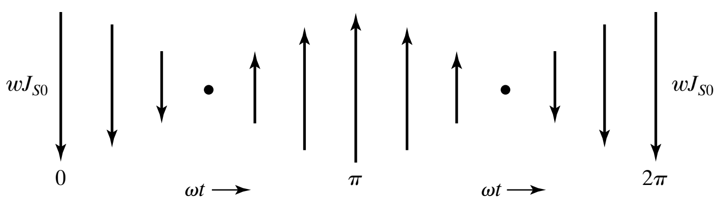

FIGURE 3.15

Time history of current flow across a line of width w parallel to the y-axis for the current sheet of Fig. 5.2, for $\mathbf{J}_S = -J_{S0} \cos \omega t \, \mathbf{a}_x$.

To find the electromagnetic field due to the time-varying current sheet, we shall begin with Faraday's law and Ampère's circuital law given, respectively, by

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{3.68a}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{3.68b}$$

and use a procedure that consists of the following steps:

1. Obtain the particular differential equations for the case under consideration.
2. Derive the general solution to the differential equations of step 1 without regard to the current on the sheet.
3. Show that the solution obtained in step 2 is a superposition of traveling waves propagating in the $+z$- and $-z$-directions.
4. Extend the general solution of step 2 to take into account the current on the sheet, thereby obtaining the required solution.

Although the procedure may be somewhat lengthy, we shall in the process learn several useful concepts and techniques.

1. To obtain the particular differential equations for the case under consideration, we first note that since (3.67) can be thought of as a current *distribution* having only an x-component of the current density that varies only with z, we can set J_y, J_z, and all derivatives with respect to x and y in (3.68a) and (3.68b) equal to zero. Hence, (3.68a) and (3.68b) reduce to

$$-\frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t} \tag{3.69a}$$

$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t} \tag{3.69b}$$

$$0 = -\frac{\partial B_z}{\partial t} \tag{3.69c}$$

$$-\frac{\partial H_y}{\partial z} = J_x + \frac{\partial D_x}{\partial t} \tag{3.70a}$$

$$\frac{\partial H_x}{\partial z} = \frac{\partial D_y}{\partial t} \tag{3.70b}$$

$$0 = \frac{\partial D_z}{\partial t} \tag{3.70c}$$

In these six equations, there are only two equations involving J_x and the pertinent electric- and magnetic-field components, namely, the simultaneous pair (3.69b) and (3.70a). Thus, the equations of interest are

$$\boxed{\begin{aligned} \frac{\partial E_x}{\partial z} &= -\frac{\partial B_y}{\partial t} \\ \frac{\partial H_y}{\partial z} &= -J_x - \frac{\partial D_x}{\partial t} \end{aligned}} \qquad \begin{aligned} &(3.71\text{a}) \\ &(3.71\text{b}) \end{aligned}$$

which are the same as (3.7) and (3.25), the simplified forms of Faraday's law and Ampère's circuital law, respectively, for the special case of electric and magnetic fields characterized by (3.66a) and (3.66b), respectively.

Derivation of wave equation

2. In applying (3.71a) and (3.71b) to (3.67), we note that J_x in (3.71b) is a volume current density, whereas (3.67) represents a surface current density. Hence, we shall solve (3.71a) and (3.71b) by setting $J_x = 0$ and then extend the solution to take into account the current on the sheet. For $J_x = 0$, (3.71a) and (3.71b) become

$$\boxed{\begin{aligned}\frac{\partial E_x}{\partial z} &= -\frac{\partial B_y}{\partial t} = -\mu_0\frac{\partial H_y}{\partial t} \\ \frac{\partial H_y}{\partial z} &= -\frac{\partial D_x}{\partial t} = -\varepsilon_0\frac{\partial E_x}{\partial t}\end{aligned}} \qquad \begin{aligned}&(3.72a)\\&(3.72b)\end{aligned}$$

Differentiating (3.72a) with respect to z and then substituting for $\partial H_y/\partial z$ from (3.72b), we obtain

$$\frac{\partial^2 E_x}{\partial z^2} = -\mu_0\frac{\partial}{\partial z}\left(\frac{\partial H_y}{\partial t}\right) = -\mu_0\frac{\partial}{\partial t}\left(\frac{\partial H_y}{\partial z}\right) = -\mu_0\frac{\partial}{\partial t}\left(-\varepsilon_0\frac{\partial E_x}{\partial t}\right)$$

or

$$\boxed{\frac{\partial^2 E_x}{\partial z^2} = \mu_0\varepsilon_0\frac{\partial^2 E_x}{\partial t^2}} \qquad (3.73)$$

We have thus eliminated H_y from (3.72a) and (3.72b) and obtained a single second-order partial differential equation involving E_x only. Equation (3.73) is known as the *wave equation*. In particular, it is a one-dimensional wave equation in time-domain form, that is, for arbitrary time dependence of E_x.

Solution of wave equation

To obtain the solution for (3.73), we introduce a change of variable by defining $\tau = z\sqrt{\mu_0\varepsilon_0}$. Substituting for z in (3.73) in terms of τ, we then have

$$\frac{\partial^2 E_x}{\partial \tau^2} = \frac{\partial^2 E_x}{\partial t^2} \qquad (3.74)$$

or

$$\begin{aligned}\frac{\partial^2 E_x}{\partial \tau^2} - \frac{\partial^2 E_x}{\partial t^2} &= 0 \\ \left(\frac{\partial}{\partial \tau} + \frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial \tau} - \frac{\partial}{\partial t}\right)E_x &= 0\end{aligned} \qquad (3.75)$$

where the quantities in parentheses are operators operating on one another and on E_x. Equation (3.75) is satisfied if

$$\left(\frac{\partial}{\partial \tau} \pm \frac{\partial}{\partial t}\right)E_x = 0$$

or

$$\frac{\partial E_x}{\partial \tau} = \mp \frac{\partial E_x}{\partial t} \tag{3.76}$$

Let us first consider the equation corresponding to the upper sign in (3.76); that is,

$$\frac{\partial E_x}{\partial \tau} = -\frac{\partial E_x}{\partial t}$$

This equation says that the partial derivative of $E_x(\tau, t)$ with respect to τ is equal to the negative of the partial derivative of $E_x(\tau, t)$ with respect to t. The simplest function that satisfies this requirement is the function $(t - \tau)$. It then follows that any arbitrary function of $(t - \tau)$, say, $f(t - \tau)$, satisfies the requirement since

$$\frac{\partial}{\partial t}[f(t - \tau)] = f'(t - \tau)\frac{\partial}{\partial t}(t - \tau) = f'(t - \tau)$$

and

$$\frac{\partial}{\partial \tau}[f(t - \tau)] = f'(t - \tau)\frac{\partial}{\partial \tau}(t - \tau) = -f'(t - \tau) = -\frac{\partial}{\partial t}[f(t - \tau)]$$

where the prime associated with $f'(t - \tau)$ denotes differentiation of f with respect to $(t - \tau)$. In a similar manner, the solution for the equation corresponding to the lower sign in (3.76), that is, for

$$\frac{\partial E_x}{\partial \tau} = \frac{\partial E_x}{\partial t}$$

can be seen to be any arbitrary function of $(t + \tau)$, say, $g(t + \tau)$. Combining the two solutions, we write the solution for (3.76) to be

$$E_x(\tau, t) = Af(t - \tau) + Bg(t + \tau) \tag{3.77}$$

where A and B are arbitrary constants.

Substituting now for τ in (3.77) in terms of z, we obtain the solution for (3.73) to be

$$\boxed{E_x(z, t) = Af(t - z\sqrt{\mu_0\varepsilon_0}) + Bg(t + z\sqrt{\mu_0\varepsilon_0})} \tag{3.78}$$

The corresponding solution for $H_y(z, t)$ can be obtained by substituting (3.78) into (3.72a) or (3.72b). Thus, using (3.72a),

$$\frac{\partial H_y}{\partial t} = \sqrt{\frac{\varepsilon_0}{\mu_0}}[Af'(t - z\sqrt{\mu_0\varepsilon_0}) - Bg'(t + z\sqrt{\mu_0\varepsilon_0})]$$

$$\boxed{H_y(z, t) = \frac{1}{\sqrt{\mu_0/\varepsilon_0}}[Af(t - z\sqrt{\mu_0\varepsilon_0}) - Bg(t + z\sqrt{\mu_0\varepsilon_0})]} \tag{3.79}$$

The fields given by (3.78) and (3.79) are the general solutions to the differential equations (3.72a) and (3.72b).

Traveling wave functions explained

3. To proceed further, we need to know the meanings of the functions f and g in (3.78) and (3.79). To discuss the meaning of f, let us consider a specific example

$$f(t - z\sqrt{\mu_0\varepsilon_0}) = (t - z\sqrt{\mu_0\varepsilon_0})^2$$

Plots of this function versus z for two values of t, $t = 0$ and $t = \sqrt{\mu_0\varepsilon_0}$, are shown in Fig. 3.16(a). An examination of these plots reveals that as time increases from 0 to $\sqrt{\mu_0\varepsilon_0}$, every point on the plot for $t = 0$ moves by one unit in the $+z$-direction,

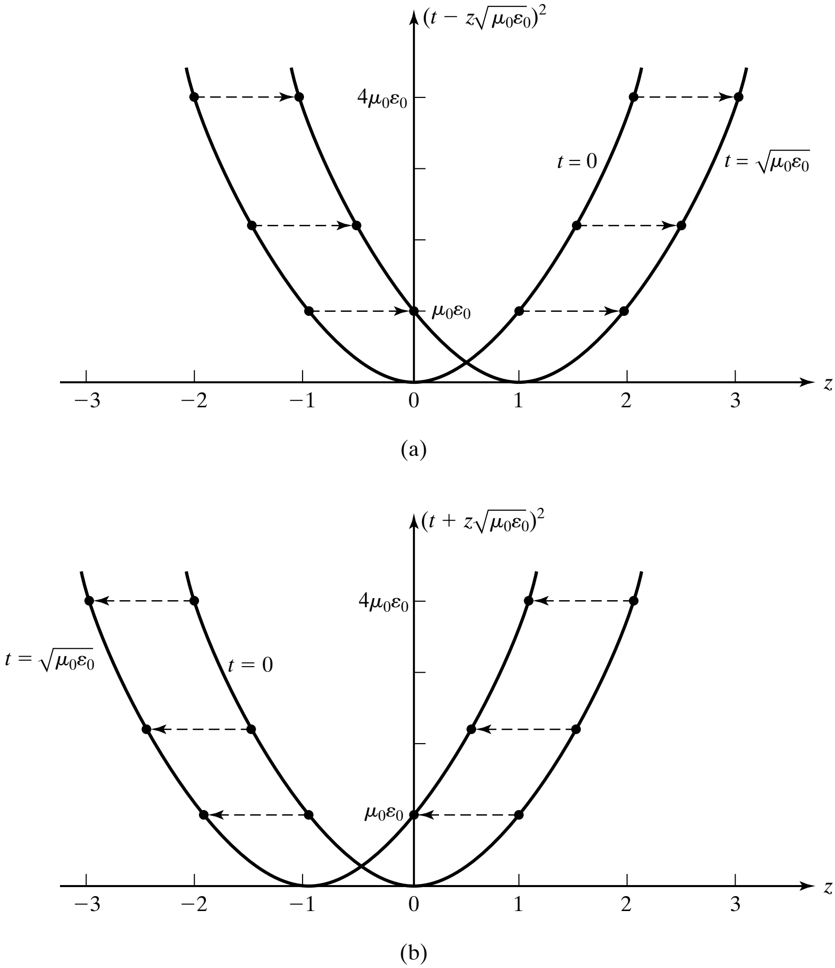

FIGURE 3.16

(a) Plots of the function $(t - z\sqrt{\mu_0\varepsilon_0})^2$ versus z for $t = 0$ and $t = \sqrt{\mu_0\varepsilon_0}$. (b) Plots of the function $(t + z\sqrt{\mu_0\varepsilon_0})^2$ versus z for $t = 0$ and $t = \sqrt{\mu_0\varepsilon_0}$.

thereby making the plot for $t = \sqrt{\mu_0\varepsilon_0}$ an exact replica of the plot for $t = 0$, except displaced by one unit in the $+z$-direction. The function f is therefore said to represent a *traveling wave propagating in the $+z$-direction*, or simply a $(+)$ *wave*. In particular, it is a *uniform plane wave* since its value does not vary with position in a given constant z-plane. By dividing the distance traveled by the time taken, the velocity of propagation of the wave can be obtained to be

$$\boxed{v_p = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = 3 \times 10^8 \text{ m/s}} \tag{3.80}$$

which is equal to c, the velocity of light in free space. Similarly, to discuss the meaning of g, we shall consider

$$g(t + z\sqrt{\mu_0\varepsilon_0}) = (t + z\sqrt{\mu_0\varepsilon_0})^2$$

Then plotting the function versus z for $t = 0$ and $t = \sqrt{\mu_0\varepsilon_0}$, as shown in Fig. 3.16(b), we can see that the plot for $t = \sqrt{\mu_0\varepsilon_0}$ is an exact replica of the plot for $t = 0$, except displaced by one unit in the $-z$-direction. The function g is therefore said to represent a *traveling wave propagating in the $-z$-direction*, or simply a $(-)$ *wave*. Once again, it is a uniform plane wave with the velocity of propagation equal to $1/\sqrt{\mu_0\varepsilon_0}$.

To generalize the foregoing discussion of the functions f and g, let us consider two pairs of t and z, say, t_1 and z_1, and $t_1 + \Delta t$ and $z_1 + \Delta z$. Then for the function f to maintain the same value for these two pairs of z and t, we must have

$$t_1 - z_1\sqrt{\mu_0\varepsilon_0} = (t_1 + \Delta t) - (z_1 + \Delta z)\sqrt{\mu_0\varepsilon_0}$$

or

$$\Delta z = \frac{1}{\sqrt{\mu_0\varepsilon_0}}\,\Delta t$$

Since $\sqrt{\mu_0\varepsilon_0}$ is a positive quantity, this indicates that as time progresses, a given value of the function moves forward in z with the velocity $1/\sqrt{\mu_0\varepsilon_0}$, thereby giving the characteristic of a $(+)$ wave for f. Similarly, for the function g to maintain the same value for the two pairs of t and z, we must have

$$t_1 + z_1\sqrt{\mu_0\varepsilon_0} = (t_1 + \Delta t) + (z_1 + \Delta z)\sqrt{\mu_0\varepsilon_0}$$

or

$$\Delta z = -\frac{1}{\sqrt{\mu_0\varepsilon_0}}\,\Delta t$$

The minus sign associated with $1/\sqrt{\mu_0\varepsilon_0}$ indicates that as time progresses, a given value of the function moves backward in z with the velocity $1/\sqrt{\mu_0\varepsilon_0}$, giving the characteristic of a $(-)$ wave for g.

We shall now define the intrinsic impedance of free space, η_0, to be

$$\boxed{\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \approx 120\pi\ \Omega = 377\ \Omega} \tag{3.81}$$

From (3.78) and (3.79), we see that η_0 is the ratio of E_x to H_y for the $(+)$ wave or the negative of the same ratio for the $(-)$ wave. Since the units of E_x are volts per meter and the units of H_y are amperes per meter, the units of E_x/H_y are volts per ampere or ohms, thereby giving the character of impedance for η_0. Replacing $\sqrt{\mu_0/\varepsilon_0}$ in (3.79) by η_0 and substituting v_p for $1/\sqrt{\mu_0\varepsilon_0}$ in the arguments of the functions f and g in both (3.78) and (3.79), we can now write (3.78) and (3.79) as

$$E_x(z, t) = Af\left(t - \frac{z}{v_p}\right) + Bg\left(t + \frac{z}{v_p}\right) \tag{3.82a}$$

$$H_y(z, t) = \frac{1}{\eta_0}\left[Af\left(t - \frac{z}{v_p}\right) - Bg\left(t + \frac{z}{v_p}\right)\right] \tag{3.82b}$$

Electromagnetic field due to the current sheet

4. Having learned that the solution to (3.72a) and (3.72b) consists of superposition of traveling waves propagating in the $+z$- and $-z$-directions, we now make use of this solution together with other considerations to find the electromagnetic field due to the infinite plane current sheet of Fig. 3.14, and with the current density given by (3.67). To do this, we observe the following:

(a) Since the current sheet, which is the source of waves, is in the $z = 0$ plane, there can be only a $(+)$ wave in the region $z > 0$ and only a $(-)$ wave in the region $z < 0$. Thus,

$$\mathbf{E}(z, t) = \begin{cases} Af\left(t - \dfrac{z}{v_p}\right)\mathbf{a}_x & \text{for} \quad z > 0 \\[2ex] Bg\left(t + \dfrac{z}{v_p}\right)\mathbf{a}_x & \text{for} \quad z < 0 \end{cases} \tag{3.83a}$$

$$\mathbf{H}(z, t) = \begin{cases} \dfrac{A}{\eta_0}f\left(t - \dfrac{z}{v_p}\right)\mathbf{a}_y & \text{for} \quad z > 0 \\[2ex] -\dfrac{B}{\eta_0}g\left(t + \dfrac{z}{v_p}\right)\mathbf{a}_y & \text{for} \quad z < 0 \end{cases} \tag{3.83b}$$

(b) Applying Faraday's law in integral form to the rectangular closed path *abcda* in Fig. 3.17 in the limit that the sides *bc* and *da* $\rightarrow 0$, with the sides *ab* and *dc* remaining on either side of the current sheet, we have

$$(ab)[E_x]_{z=0+} - (dc)[E_x]_{z=0-} = 0 \tag{3.84}$$

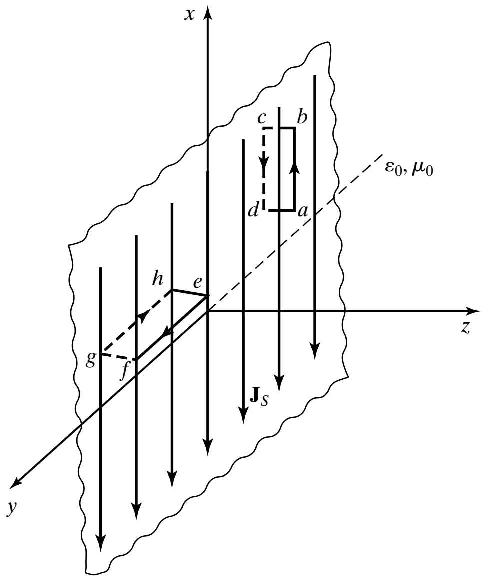

FIGURE 3.17

Rectangular closed paths with sides on either side of the infinite plane current sheet.

or $Af(t) = Bg(t)$. Thus, (3.83a) and (3.83b) reduce to

$$\mathbf{E}(z, t) = F\left(t \mp \frac{z}{v_p}\right)\mathbf{a}_x \qquad \text{for} \quad z \gtrless 0 \tag{3.85a}$$

$$\mathbf{H}(z, t) = \pm \frac{1}{\eta_0} F\left(t \mp \frac{z}{v_p}\right)\mathbf{a}_y \qquad \text{for} \quad z \gtrless 0 \tag{3.85b}$$

where we have used $Af(t) = Bg(t) = F(t)$.

(c) Applying Ampere's circuital law in integral form to the rectangular closed path *efghe* in Fig. 3.17 in the limit that the sides *fg* and *he* $\rightarrow 0$, with the sides *ef* and *hg* remaining on either side of the current sheet, we have

$$(ef)[H_y]_{z=0+} - (hg)[H_y]_{z=0-} = (ef)J_s(t) \tag{3.86}$$

or $(2/\eta_0)F(t) = J_S(t)$. Thus, $F(t) = (\eta_0/2)J_S(t)$, and (3.85a) and (3.85b) become

$$\boxed{\begin{aligned} \mathbf{E}(z, t) &= \frac{\eta_0}{2} J_S\left(t \mp \frac{z}{v_p}\right)\mathbf{a}_x \qquad \text{for } z \gtrless 0 \\ \mathbf{H}(z, t) &= \pm\frac{1}{2} J_S\left(t \mp \frac{z}{v_p}\right)\mathbf{a}_y \qquad \text{for } z \gtrless 0 \end{aligned}}$$
(3.87a), (3.87b)

Equations (3.87a) and (3.87b) represent the complete solution for the electromagnetic field due to the infinite plane current sheet of surface current density given by

$$\boxed{\mathbf{J}_S(t) = -J_S(t)\mathbf{a}_x \qquad \text{for} \quad z = 0} \tag{3.88}$$

The solution corresponds to uniform plane waves having their field components uniform in planes parallel to the current sheet and propagating to either side of the current sheet with the velocity $v_p(=c)$. The time variation of the electric field component E_x in a given $z =$ constant plane is the same as the current density variation delayed by the time $|z|/v_p$ and multiplied by $\eta_0/2$. The time variation of the magnetic field component in a given $z =$ constant plane is the same as the current density variation delayed by $|z|/v_p$ and multiplied by $\pm\frac{1}{2}$, depending on $z \gtrless 0$. Using these properties, one can construct plots of the field components versus time for fixed values of z and versus z for fixed values of t. We shall illustrate by means of an example.

Example 3.10 Plotting field variations for an infinite plane-sheet current source

Let us consider the function $J_S(t)$ in (3.88) to be that given in Fig. 3.18. We wish to find and sketch (a) E_x versus t for $z = 300$ m, (b) H_y versus t for $z = -450$ m, (c) E_x versus z for $t = 1$ μs, and (d) H_y versus z for $t = 2.5$ μs.

(a) Since $v_p = c = 3 \times 10^8$ m/s, the time delay corresponding to 300 m is 1 μs. Thus, the plot of E_x versus t for $z = 300$ m is the same as that of $J_S(t)$ multiplied by $\eta_0/2$, or 188.5, and delayed by 1 μs, as shown in Fig. 3.19(a).

(b) The time delay corresponding to 450 m is 1.5 μs. Thus, the plot of H_y versus t for $z = -450$ m is the same as that of $J_S(t)$ multiplied by $-1/2$ and delayed by 1.5 μs, as shown in Fig. 3.19(b).

(c) To sketch E_x versus z for a fixed value of t, say, t_1, we use the argument that a given value of E_x existing at the source at an earlier value of time, say, t_2, travels away from the source by the distance equal to $(t_1 - t_2)$ times v_p. Thus, at $t = 1$ μs, the values of E_x corresponding to points A and B in Fig. 3.18 move to the locations $z = \pm 300$ m and $z = \pm 150$ m, respectively, and the value of E_x corresponding to point C exists right at the source. Hence, the plot of E_x versus z for $t = 1$ μs is as shown in Fig. 3.19(c). Note that points beyond C in Fig. 3.18 correspond to $t > 1$ μs, and therefore they do not appear in the plot of Fig. 3.19(c).

(d) Using arguments as in part (c), we see that at $t = 2.5$ μs, the values of H_y corresponding to points A, B, C, D, and E in Fig. 3.18 move to the locations $z = \pm 750$ m, ± 600 m, ± 450 m, ± 300 m, and ± 150 m, respectively, as shown in Fig. 3.19(d). Note that the plot is an odd function of z, since the factor by which J_{S0} is multiplied to obtain H_y is $\pm\frac{1}{2}$, depending on $z \lessgtr 0$.

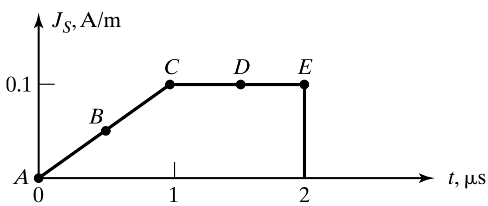

FIGURE 3.18
Plot of J_s versus t for Example 3.10.

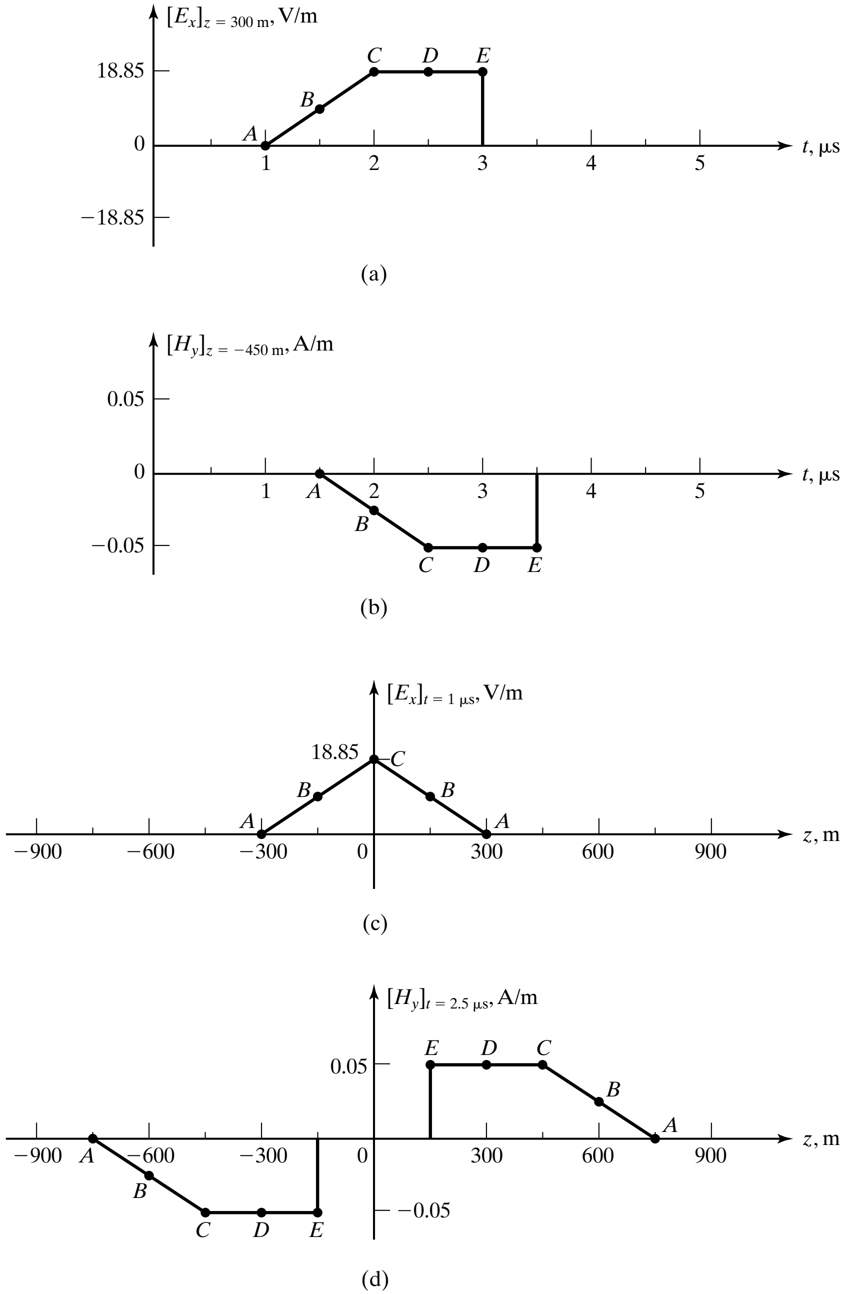

FIGURE 3.19

Plots of field components versus t for fixed values of z and versus z for fixed values of t for Example 3.10.

K3.4. Infinite plane current sheet; Uniform plane wave; Wave equation; Time domain; Traveling-wave functions; Velocity of propagation; Intrinsic impedance of free space; Time delay.

D3.11. For each of the following traveling-wave functions, find the velocity of propagation both in magnitude and direction: **(a)** $(0.05y - t)^2$; **(b)** $u(t + 0.02x)$; and **(c)** $\cos(2\pi \times 10^8 t - 2\pi z)$.

Ans. **(a)** $20\mathbf{a}_y$ m/s; **(b)** $-50\mathbf{a}_x$ m/s; **(c)** $10^8\mathbf{a}_z$ m/s.

D3.12. The time variation for $z = 0$ of a function $f(z, t)$ representing a traveling wave propagating in the $+z$-direction with velocity 200 m/s is shown in Fig. 3.20. Find the value of the function for each of the following cases: **(a)** $z = 300$ m, $t = 2.0$ s; **(b)** $z = -200$ m, $t = 0.4$ s; and **(c)** $z = 100$ m, $t = 0.5$ s.

Ans. **(a)** $0.25A$; **(b)** $0.6A$; **(c)** 0.

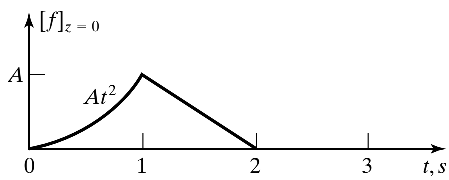

FIGURE 3.20
For Problem D3.12.

D3.13. The time variation for $z = 0$ of a function $g(z, t)$ representing a traveling wave propagating in the $-z$-direction with velocity 100 m/s is shown in Fig. 3.21. Find the value of the function for each of the following cases: **(a)** $z = 200$ m, $t = 0.2$ s; **(b)** $z = -300$ m, $t = 3.4$ s; and **(c)** $z = 100$ m, $t = 0.6$ s.

Ans. **(a)** $0.9A$; **(b)** $0.4A$; **(c)** A.

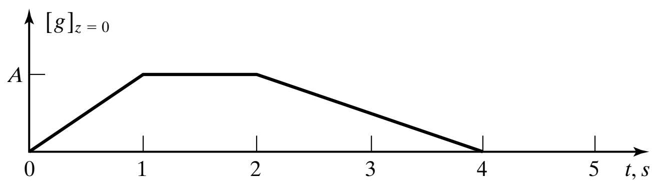

FIGURE 3.21
For Problem D3.13.

3.5 SINUSOIDALLY TIME-VARYING UNIFORM PLANE WAVES IN FREE SPACE

Solution for the field for the sinusoidal case

In the previous section, we considered the current density on the infinite plane current sheet to be an arbitrary function of time and obtained the solution for the electromagnetic field. Of particular interest are fields varying sinusoidally with time. Sinusoidally time-varying fields are important because of their natural occurrence and ease of generation. For example, when we speak, we emit sine waves; when we tune our radio to a broadcast station, we receive sine waves; and so on. Also, any function for which the time variation is arbitrary can be expressed in terms of sinusoidally time-varying functions having a discrete or continuous spectrum of frequencies, depending on whether the function is periodic or aperiodic. Thus, if the response of a system to a sinusoidal excitation is known, its response for a nonsinusoidal excitation can be found. Sinusoidally time-varying fields are produced by a source whose current density varies sinusoidally with time. Thus, assuming the current density on the infinite plane sheet of Fig. 3.14 to be

$$\boxed{\mathbf{J}_S = -J_{S0} \cos \omega t \, \mathbf{a}_x \qquad \text{for} \quad z = 0} \tag{3.89}$$

where J_{S0} is the amplitude and ω is the radian frequency, we obtain the corresponding solution for the electromagnetic field by substituting $J_S(t) = J_{S0} \cos \omega t$ in (3.87a) and (3.87b):

$$\mathbf{E} = \frac{\eta_0 J_{S0}}{2} \cos(\omega t \mp \beta z) \, \mathbf{a}_x \qquad \text{for} \quad z \lessgtr 0 \tag{3.90a}$$

$$\mathbf{H} = \pm \frac{J_{S0}}{2} \cos(\omega t \mp \beta z) \, \mathbf{a}_y \qquad \text{for} \quad z \lessgtr 0 \tag{3.90b}$$

where

$$\boxed{\beta = \frac{\omega}{v_p}} \tag{3.91}$$

Properties and parameters of sinusoidal waves

Equations (3.90a) and (3.90b) represent sinusoidally time-varying uniform plane waves propagating away from the current sheet. The phenomenon is illustrated in Fig. 3.22, which shows sketches of the current density on the sheet and the distance variation of the electric and magnetic fields on either side of the current sheet for three values of t. It should be understood that in these sketches the field variations depicted along the z-axis hold also for any other line parallel to the z-axis. We shall now discuss in detail several important parameters and properties associated with the sinusoidal waves:

1. The argument $(\omega t \mp \beta z)$ of the cosine functions is the phase of the fields. We shall denote the phase by the symbol ϕ. Thus,

$$\boxed{\phi = \omega t \mp \beta z} \tag{3.92}$$

Note that ϕ is a function of t and z.

Frequency

2. Since

$$\boxed{\frac{\partial \phi}{\partial t} = \omega} \tag{3.93}$$

the rate of change of phase with time for a fixed value of z is equal to ω, the radian frequency of the wave. The linear frequency given by

$$\boxed{f = \frac{\omega}{2\pi}} \tag{3.94}$$

is the number of times the phase changes by 2π radians in one second for a fixed value of z. The situation is pertinent to an observer at a point in the field

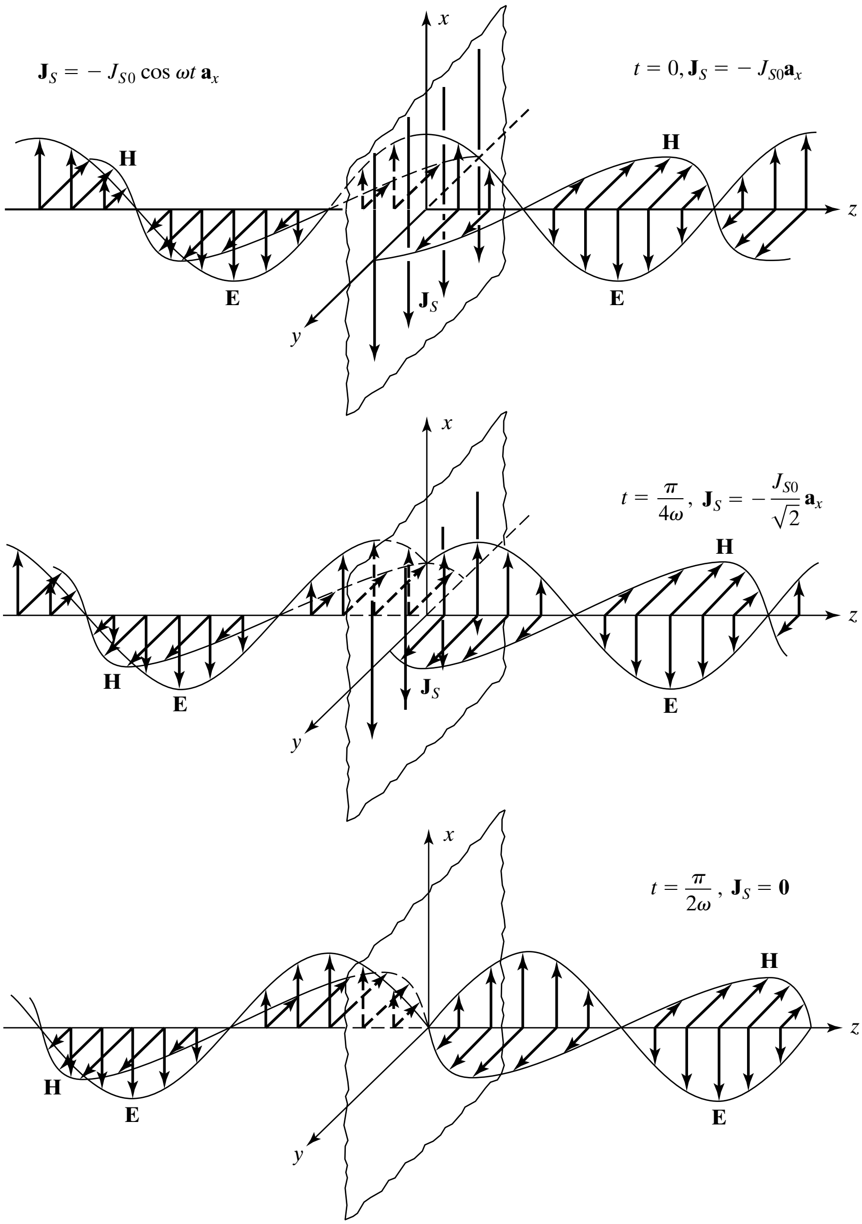

FIGURE 3.22

Time history of uniform plane electromagnetic wave radiating away from an infinite plane current sheet in free space.

region watching a *movie* of the field variations with time and counting the number of times in one second the field goes through a certain phase point, say, the positive maximum.

Phase constant

3. Since

$$\boxed{\frac{\partial \phi}{\partial z} = \mp\beta} \tag{3.95}$$

the magnitude of the rate of change of phase with distance z for a fixed value of time is equal to β, known as the *phase constant*. The situation is pertinent to taking a *still photograph* of the phenomenon at any given time along the z-axis, counting the number of radians of phase change in one meter.

Wavelength

4. It follows from property 3 that the distance, along the z-direction, in which the phase changes by 2π radians for a fixed value of time is equal to $2\pi/\beta$. This distance is known as the *wavelength*, denoted by the symbol λ. Thus,

$$\boxed{\lambda = \frac{2\pi}{\beta}} \tag{3.96}$$

It is the distance between two consecutive positive maximum points on the sinusoid, or between any other two points that are displaced from these two positive maximum points by the same distance and to the same side, as shown in Fig. 3.23.

Phase velocity

5. From (3.91), we note that the velocity of propagation of the wave is given by

$$\boxed{v_p = \frac{\omega}{\beta}} \tag{3.97}$$

Here, it is known as the *phase velocity*, since a constant value of phase progresses with that velocity along the z-direction. It is the velocity with which an observer has to move along the direction of propagation of the wave to be associated with a particular phase point on the moving sinusoid. Thus, it follows from (3.92) that

$$d(\omega t \mp \beta z) = 0$$

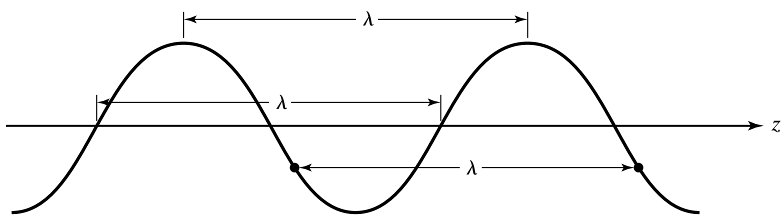

FIGURE 3.23

For explaining wavelength.

which gives

$$\omega\, dt \mp \beta\, dz = 0$$
$$\frac{dz}{dt} = \mp\frac{\omega}{\beta}$$

where the + and − signs correspond to (+) and (−) waves, respectively. We recall that for free space, $v_p = 1/\sqrt{\mu_0\varepsilon_0} = c = 3 \times 10^8$ m/s.

Classification of waves

6. From (3.96), (3.94), and (3.97), we note that

$$\lambda f = \left(\frac{2\pi}{\beta}\right)\left(\frac{\omega}{2\pi}\right) = \frac{\omega}{\beta}$$

or

$$\boxed{\lambda f = v_p} \tag{3.98}$$

Thus, the wavelength and frequency of a wave are not independent of each other, but are related through the phase velocity. This is not surprising because λ is a parameter governing the variation of the field with distance for a fixed time, f is a parameter governing the variation of the field with time for a fixed value of z, and we know from Maxwell's equations that the space and time variations of the fields are interdependent. For free space, (3.98) gives

$$\boxed{\lambda \text{ in meters} \times f \text{ in hertz} = 3 \times 10^8} \tag{3.99a}$$

or

$$\boxed{\lambda \text{ in meters} \times f \text{ in megahertz} = 300} \tag{3.99b}$$

It can be seen from these relationships that the higher the frequency, the shorter the wavelength. Waves are classified according to frequency or wavelength. Table 3.1 lists the commonly used designations for the various bands up to 300 GHz, where 1 GHz is 10^9 Hz. The corresponding frequency ranges

TABLE 3.1 Commonly Used Designations for the Various Frequency Ranges

Designation	Frequency Range	Wavelength Range
ELF (extremely low frequency)	30–3000 Hz	10,000–100 km
VLF (very low frequency)	3–30 kHz	100–10 km
LF (low frequency) or long waves	30–300 kHz	10–1 km
MF (medium frequency) or medium waves	300–3000 kHz	1000–100 m
HF (high frequency) or short waves	3–30 MHz	100–10 m
VHF (very high frequency)	30–300 MHz	10–1 m
UHF (ultrahigh frequency)	300–3000 MHz	100–10 cm
Microwaves	1–30 GHz	30–1 cm
Millimeter waves	30–300 GHz	10–1 mm

and wavelength ranges are also given. The frequencies above about 300 GHz fall into regions far infrared and beyond. The AM radio (550–1650 kHz) falls in the medium wave band, whereas the FM radio makes use of 88–108 MHz in the VHF band. The VHF TV channels 2–6 use 54–88 MHz, and 7–13 employ 174–216 MHz. The UHF TV channels are in the 470–890-MHz range. Microwave ovens operate at 2450 MHz. Police traffic radars operate at about 10.5 and 24.1 GHz. Various other ranges in Table 3.1 are used for various other applications too numerous to mention here.

Intrinsic impedance

7. The electric and magnetic fields are such that

$$\boxed{\frac{\text{amplitude of } \mathbf{E}}{\text{amplitude of } \mathbf{H}} = \eta_0} \tag{3.100}$$

We recall that η_0, the intrinsic impedance of free space, has a value approximately equal to 120π or 377 Ω.

8. The electric and magnetic fields have components lying in the planes of constant phase (z = constant planes) and perpendicular to each other and to the direction of propagation. In fact, the cross product of **E** and **H** results in a vector that is directed along the direction of propagation, as can be seen by noting that

$$\boxed{\mathbf{E} \times \mathbf{H} = \pm\frac{\eta_0 J_{s0}^2}{4}\cos^2(\omega t \mp \beta z)\,\mathbf{a}_z \qquad \text{for} \quad z \lessgtr 0} \tag{3.101}$$

We shall now consider two examples of the application of the properties we have learned thus far in this section.

Example 3.11 Finding parameters for a specified sinusoidal uniform plane-wave electric field

The electric field of a uniform plane wave is given by

$$\mathbf{E} = 10\sin(3\pi \times 10^8 t - \pi z)\,\mathbf{a}_x + 10\cos(3\pi \times 10^8 t - \pi z)\,\mathbf{a}_y \text{ V/m}$$

Let us find (a) the various parameters associated with the wave and (b) the corresponding magnetic field **H**.

(a) From the argument of the sine and cosine functions, we can identify the following:

$$\omega = 3\pi \times 10^8 \text{ rad/s}$$
$$\beta = \pi \text{ rad/m}$$

Then

$$f = \frac{\omega}{2\pi} = 1.5 \times 10^8 \text{ Hz}$$
$$\lambda = \frac{2\pi}{\beta} = 2 \text{ m}$$
$$v_p = \frac{\omega}{\beta} = 3 \times 10^8 \text{ m/s}$$

Note also that $\lambda f = 3 \times 10^8 = v_p$. In view of the minus sign associated with πz, the direction of propagation of the wave is the $+z$-direction.

(b) The unit vectors $\mathbf{a}_x$ and $\mathbf{a}_y$ associated with the first and second terms, respectively, tell us that the electric field contains components directed along the x- and y-directions. Using the properties 7 and 8 discussed earlier, we obtain the magnetic field of the wave to be

$$\begin{aligned}\mathbf{H} &= \frac{10}{377}\sin(3\pi \times 10^8 t - \pi z)\,\mathbf{a}_y + \frac{10}{377}\cos(3\pi \times 10^8 t - \pi z)(-\mathbf{a}_x) \\ &= -\frac{10}{377}\cos(3\pi \times 10^8 t - \pi z)\,\mathbf{a}_x + \frac{10}{377}\sin(3\pi \times 10^8 t - \pi z)\,\mathbf{a}_y \text{ A/m}\end{aligned}$$

Example 3.12 Electric field due to an array of two infinite plane current sheets

Principle of antenna array

An antenna array consists of two or more antenna elements spaced appropriately and excited with currents having the appropriate amplitudes and phases in order to obtain a desired radiation characteristic. To illustrate the principle of an antenna array, let us consider two infinite plane parallel current sheets, spaced $\lambda/4$ apart and carrying currents of equal amplitudes but out of phase by $\pi/2$ as given by the densities

$$\begin{aligned}\mathbf{J}_{S1} &= -J_{S0}\cos\omega t\,\mathbf{a}_x \qquad \text{for} \quad z = 0 \\ \mathbf{J}_{S2} &= -J_{S0}\sin\omega t\,\mathbf{a}_x \qquad \text{for} \quad z = \lambda/4\end{aligned}$$

and find the electric field due to the array of the two current sheets.

We apply the result given by (3.90a) to each current sheet separately and then use superposition to find the required total electric field due to the array of the two current sheets. Thus, for the current sheet in the $z = 0$ plane, we have

$$\mathbf{E}_1 = \begin{cases} \dfrac{\eta_0 J_{S0}}{2}\cos(\omega t - \beta z)\,\mathbf{a}_x & \text{for} \quad z > 0 \\ \dfrac{\eta_0 J_{S0}}{2}\cos(\omega t + \beta z)\,\mathbf{a}_x & \text{for} \quad z < 0 \end{cases}$$

For the current sheet in the $z = \lambda/4$ plane, we have

$$\begin{aligned}\mathbf{E}_2 &= \begin{cases} \dfrac{\eta_0 J_{S0}}{2}\sin\left[\omega t - \beta\left(z - \dfrac{\lambda}{4}\right)\right]\mathbf{a}_x & \text{for} \quad z > \dfrac{\lambda}{4} \\ \dfrac{\eta_0 J_{S0}}{2}\sin\left[\omega t + \beta\left(z - \dfrac{\lambda}{4}\right)\right]\mathbf{a}_x & \text{for} \quad z < \dfrac{\lambda}{4} \end{cases} \\ &= \begin{cases} \dfrac{\eta_0 J_{S0}}{2}\sin\left(\omega t - \beta z + \dfrac{\pi}{2}\right)\mathbf{a}_x & \text{for} \quad z > \dfrac{\lambda}{4} \\ \dfrac{\eta_0 J_{S0}}{2}\sin\left(\omega t + \beta z - \dfrac{\pi}{2}\right)\mathbf{a}_x & \text{for} \quad z < \dfrac{\lambda}{4} \end{cases} \\ &= \begin{cases} \dfrac{\eta_0 J_{S0}}{2}\cos(\omega t - \beta z)\,\mathbf{a}_x & \text{for} \quad z > \dfrac{\lambda}{4} \\ -\dfrac{\eta_0 J_{S0}}{2}\cos(\omega t + \beta z)\,\mathbf{a}_x & \text{for} \quad z > \dfrac{\lambda}{4} \end{cases}\end{aligned}$$

Now, using superposition, we find the total electric field due to the two current sheets to be

$$\begin{aligned}\mathbf{E} &= \mathbf{E}_1 + \mathbf{E}_2 \\ &= \begin{cases} \eta_0 J_{S0} \cos(\omega t - \beta z)\,\mathbf{a}_x & \text{for} \quad z > \lambda/4 \\ \eta_0 J_{S0} \sin \omega t \sin \beta z\,\mathbf{a}_x & \text{for} \quad 0 < z < \lambda/4 \\ \mathbf{0} & \text{for} \quad z < 0 \end{cases}\end{aligned}$$

Thus, the total field is zero in the region $z < 0$ due to the phase opposition of the individual fields, and, hence, there is no radiation toward that side of the array. In the region $z > \lambda/4$, the total field is twice that of the field of a single sheet due to the individual fields being in phase. The phenomenon is illustrated in Fig. 3.24, which shows the individual fields E_{x1} and E_{x2} and the total field $E_x = E_{x1} + E_{x2}$ for a few values of t. The result that we have obtained here for the total field due to the array of two current sheets, spaced $\lambda/4$ apart and fed with currents of equal amplitudes but out of phase by $\pi/2$, is said to correspond to an "endfire" radiation pattern.

K3.5. Sinusoidal waves; Phase; Frequency; Wavelength; Phase velocity; Frequency times wavelength; Intrinsic impedance; Antenna array.

D3.14. For a sinusoidally time-varying uniform plane wave propagating in free space, find the following: **(a)** the frequency f, if the phase of the field at a point is observed to change by 3π rad in 0.1 μs; **(b)** the wavelength λ, if the phase of the field at a particular value of time is observed to change by 0.04π in a distance of 1 m along the direction of propagation of the wave; **(c)** the frequency f, if the wavelength is 25 m; and **(d)** the wavelength λ, if the frequency is 5 MHz.

Ans **(a)** 15 MHz; **(b)** 50 m; **(c)** 12 MHz; **(d)** 60 m.

D3.15. The magnetic field of a uniform plane wave in free space is given by

$$\mathbf{H} = H_0 \cos(6\pi \times 10^8 t + 2\pi y)\,\mathbf{a}_x \text{ A/m}$$

Find unit vectors along the following: **(a)** the direction of propagation of the wave; **(b)** the direction of the magnetic field at $t = 0$, $y = 0$; and **(c)** the direction of the electric field at $t = 0$, $y = 0$.

Ans. **(a)** $-\mathbf{a}_y$; **(b)** $\mathbf{a}_x$; **(c)** $-\mathbf{a}_z$.

D3.16. For the array of two infinite plane current sheets of Example 3.12, assume that

$$\mathbf{J}_{S2} = -kJ_{S0} \sin \omega t\,\mathbf{a}_x \qquad \text{for} \quad z = \lambda/4$$

where $|k| \leq 1$. Find the value of k for each of the following values of the ratio of the amplitude of the electric-field intensity for $z > \lambda/4$ to the amplitude of the electric-field intensity for $z < 0$: **(a)** 1/3; **(b)** 3; and **(c)** 7.

Ans. **(a)** $-1/2$; **(b)** 1/2; **(c)** 3/4.

3.6 POLARIZATION OF SINUSOIDALLY TIME-VARYING VECTOR FIELDS

Returning now to the solution for the uniform plane wave fields given by (3.90a) and (3.90b), we can talk about *wave polarization*. Polarization is the characteristic that describes how the tip of a sinusoidally time-varying field vector at a point in space changes position with time. In the case of waves, when we

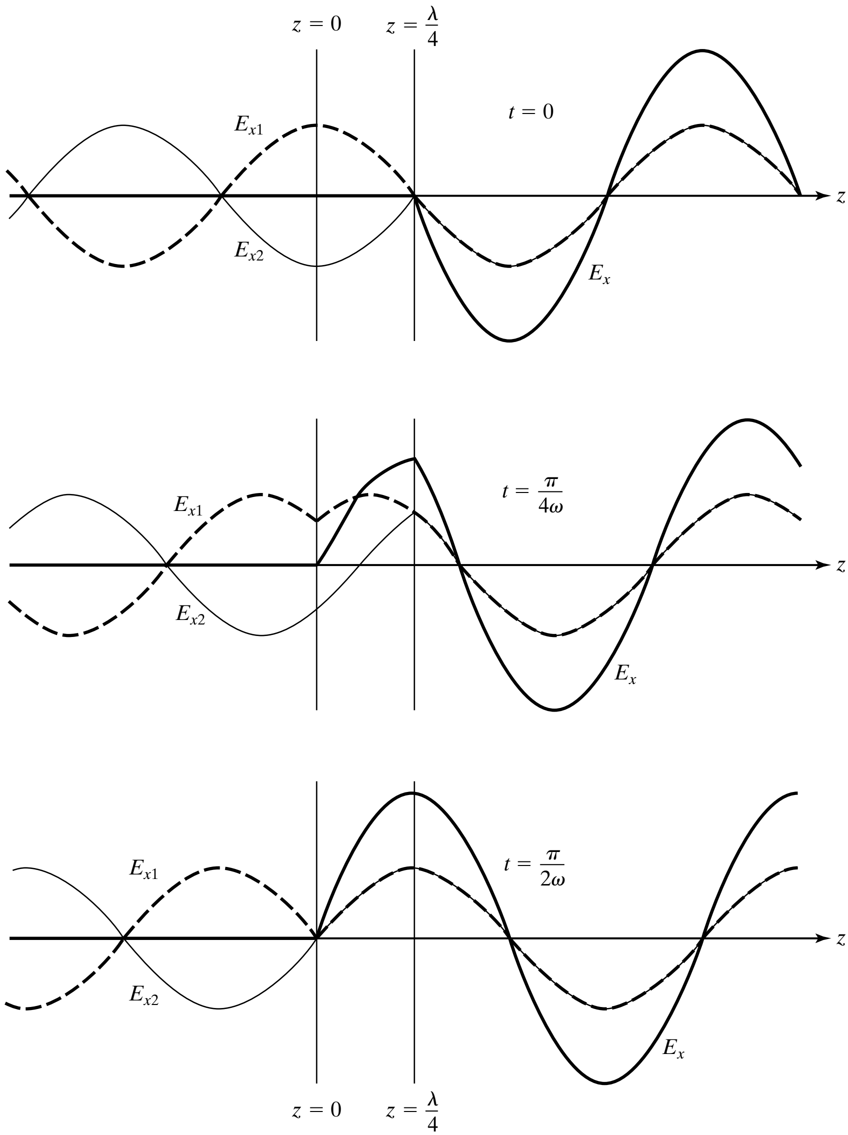

FIGURE 3.24

Time history of individual fields and the total field due to an array of two infinite plane parallel current sheets.

talk about polarization, we refer to the electric field associated with the wave. The electric field given by (3.90a) has only an x-component. We can visualize the sinusoidal variation with time of this field at a particular point in the field region by a vector changing in magnitude and direction, as shown in Fig. 3.25(a). Since the tip of the vector moves back and forth along a line, which in this case

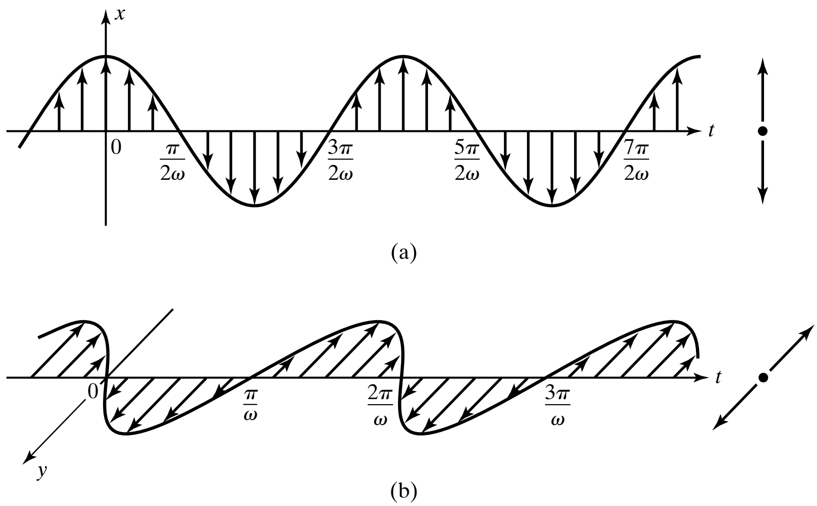

FIGURE 3.25

(a) Time variation of a linearly polarized vector in the x-direction. (b) Time variation of a linearly polarized vector in the y-direction.

is parallel to the x-axis, the field is said to be *linearly polarized* in the x-direction. Similarly, the sinusoidal variation with time of a field having a y-component only can be visualized by a vector changing its magnitude and direction, as shown in Fig. 3.25(b). Since the tip of the vector moves back and forth parallel to the y-axis, the field is said to be linearly polarized in the y-direction.

For fields having more than one component, the polarization can be linear, circular, or elliptical, that is, the tip of the field vector can describe a straight line, a circle, or an ellipse with time, as shown in Fig. 3.26, depending on the relative amplitudes and phase angles of the component vectors. Note that in the case of linear polarization, the direction of the vector remains along a straight line, but its magnitude changes with time. For circular polarization, the magnitude remains constant, but its direction changes with time. Elliptical polarization is characterized by both magnitude and direction of the vector changing with time. Let us consider two components and discuss the different cases.

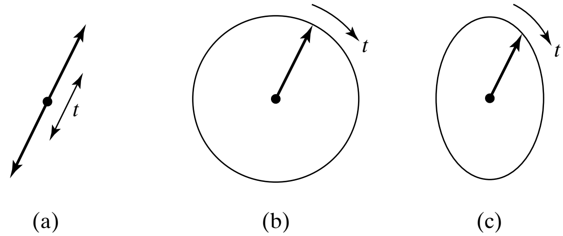

FIGURE 3.26

(a) Linear, (b) circular, and (c) elliptical polarizations.

Linear polarization

If the two component sinusoidally time-varying vectors have arbitrary amplitudes but are in phase or in phase opposition as, for example,

$$\mathbf{F}_1 = F_1 \cos(\omega t + \phi)\, \mathbf{a}_x \tag{3.102a}$$
$$\mathbf{F}_2 = \pm F_2 \cos(\omega t + \phi)\, \mathbf{a}_y \tag{3.102b}$$

then the sum vector $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$ is linearly polarized in a direction making an angle

$$\alpha = \tan^{-1}\frac{F_y}{F_x} = \pm\tan^{-1}\frac{F_2}{F_1} \tag{3.103}$$

with the x-direction, as shown in the series of sketches in Fig. 3.27 for the in-phase case illustrating the time history of the magnitude and direction of $\mathbf{F}$ over an interval of one period. The reasoning can be extended to two (or more) linearly polarized vectors that are not necessarily along the coordinate axes, but are all *in phase*. Thus, the sum vector of any number of linearly polarized vectors having different directions and amplitudes but in phase is also a linearly polarized vector.

Circular polarization

If the two component sinusoidally time-varying vectors have equal amplitudes, differ in direction by 90°, and differ in phase by $\pi/2$, as, for example,

$$\mathbf{F}_1 = F_0 \cos(\omega t + \phi)\, \mathbf{a}_x \tag{3.104a}$$
$$\mathbf{F}_2 = F_0 \sin(\omega t + \phi)\, \mathbf{a}_y \tag{3.104b}$$

then, to determine the *polarization* of the sum vector $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$, we note that the magnitude of $\mathbf{F}$ is given by

$$|\mathbf{F}| = |F_0 \cos(\omega t + \phi)\, \mathbf{a}_x + F_0 \sin(\omega t + \phi)\, \mathbf{a}_y| = F_0 \tag{3.105}$$

and that the angle α which $\mathbf{F}$ makes with $\mathbf{a}_x$ is given by

$$\alpha = \tan^{-1}\frac{F_y}{F_x} = \tan^{-1}\frac{F_0 \sin(\omega t + \phi)}{F_0 \cos(\omega t + \phi)} = \omega t + \phi \tag{3.106}$$

Thus, the sum vector rotates with constant magnitude F_0 and at a rate of ω rad/s, so that its tip describes a circle. The sum vector is then said to be *circularly polarized*.

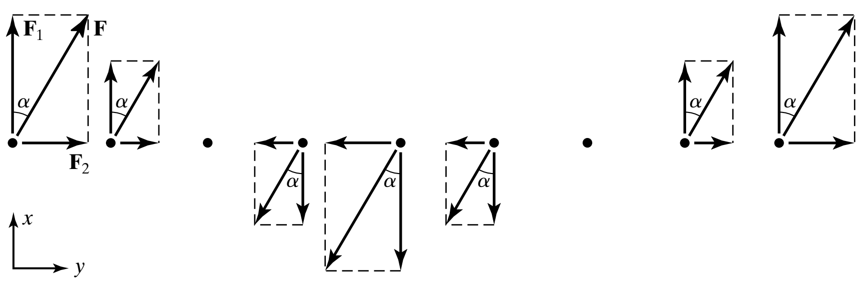

FIGURE 3.27

The sum vector of two linearly polarized vectors in phase is a linearly polarized vector.

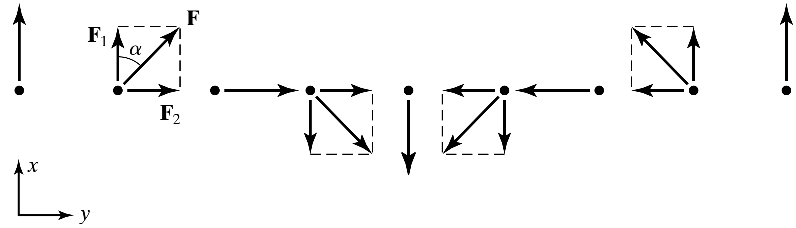

FIGURE 3.28
Circular polarization.

The series of sketches in Fig. 3.28 illustrates the time history of the magnitude and direction of **F** over an interval of one period.

The reasoning can be generalized to two linearly polarized vectors not necessarily along the coordinate axes. Thus, if two linearly polarized vectors satisfy the three conditions of (1) equal amplitudes, (2) perpendicularity in direction, and (3) phase difference of 90°, then their sum vector is circularly polarized. We shall illustrate this by means of an example.

Example 3.13 Determination of the polarization of the sum of two linearly polarized vectors

Suppose we are given two vectors

$$\mathbf{F}_1 = (3\mathbf{a}_x - 4\mathbf{a}_z)\cos 2\pi \times 10^6 t$$
$$\mathbf{F}_2 = 5\mathbf{a}_y \sin 2\pi \times 10^6 t$$

at a point. Note that the vector $\mathbf{F}_1$ consists of two components (x and z) that are in phase opposition. Hence, it is linearly polarized, but along the direction of $(3\mathbf{a}_x - 4\mathbf{a}_z)$. The vector $\mathbf{F}_2$ is linearly polarized along the y-direction. We wish to determine the polarization of the vector $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$.

Since the two linearly polarized vectors $\mathbf{F}_1$ and $\mathbf{F}_2$ are not in phase, we rule out the possibility of **F** being linearly polarized. In fact, since $\mathbf{F}_1$ varies with time in a cosine manner, whereas $\mathbf{F}_2$ varies in a sine manner, we note that $\mathbf{F}_1$ and $\mathbf{F}_2$ differ in phase by 90°. The amplitude of $\mathbf{F}_1$ is $\sqrt{3^2 + (-4)^2}$, or 5, which is equal to that of $\mathbf{F}_2$. Also,

$$\mathbf{F}_1 \cdot \mathbf{F}_2 = (3\mathbf{a}_x - 4\mathbf{a}_z) \cdot 5\mathbf{a}_y = 0$$

so that $\mathbf{F}_1$ and $\mathbf{F}_2$ are perpendicular. Thus, $\mathbf{F}_1$ and $\mathbf{F}_2$ satisfy all three conditions for the sum of two linearly polarized vectors to be circularly polarized. Therefore, **F** is circularly polarized.

Alternatively, we observe that

$$\begin{aligned} |\mathbf{F}| &= |\mathbf{F}_1 + \mathbf{F}_2| \\ &= |3\cos 2\pi \times 10^6 t\, \mathbf{a}_x + 5\sin 2\pi \times 10^6 t\, \mathbf{a}_y - 4\cos 2\pi \times 10^6 t\, \mathbf{a}_z| \\ &= (25\cos^2 2\pi \times 10^6 t + 25\sin^2 2\pi \times 10^6 t)^{1/2} \\ &= \sqrt{25} = 5, \text{ a constant with time} \end{aligned}$$

Hence, **F** is circularly polarized.

Elliptical polarization

For the general case in which the conditions for the sum vector to be linearly polarized or circularly polarized are not satisfied, the sum vector is *elliptically polarized*; that is, its tip describes an ellipse. Thus, linear and circular polarizations are special cases of elliptical polarization. For example, the ellipse described by the tip of the vector resulting from the superposition of two sinusoidally time-varying, orthogonal component vectors $\mathbf{F}_1 = A \cos (\omega t + \phi)\, \mathbf{a}_x$ and $\mathbf{F}_2 = B \cos (\omega t + \theta)\, \mathbf{a}_y$ of the same frequency, for values of $A = 40$, $B = 60$, $\phi = 60°$, and $\theta = 105°$, is shown in Fig. 3.29, where the component vectors and the resultant vector are also shown for one value of time, and the interval between the dots is one-hundredth of the period $2\pi/\omega$.

Relevance of polarization in reception of radio waves

An example in which polarization is relevant is in the reception of radio waves. If the incoming signal is linearly polarized, then for *maximum* voltage to be induced in a *linear* receiving antenna, the antenna must be oriented parallel to the direction of polarization of the signal. Any other orientation of the antenna will result in a smaller induced voltage, since the antenna "sees" only that component of the electric field parallel to itself. In particular, if the antenna is in the plane perpendicular to the direction of polarization of the incoming signal, *no* voltage is induced. On the other hand, if the incoming signal is circularly or elliptically polarized, a voltage is induced in the antenna, except for one orientation that is along the line perpendicular to the plane of the circle or the ellipse.

Finally, in the case of circular and elliptical polarizations, since the circle or the ellipse can be traversed in one of two opposite senses, we talk of right-handed or clockwise polarization and left-handed or counterclockwise polarization. The convention is that if in a given constant phase plane, the tip of the field vector of a circularly polarized wave rotates with time in the clockwise sense as seen looking along the direction of propagation of the wave, the wave is said to be right circularly polarized. If the tip of the field vector rotates in the counterclockwise sense, the wave is said to be left circularly polarized. Similar considerations hold for elliptically polarized waves, which arise due to the superposition of two linearly polarized waves in the general case.

For example, for the uniform plane wave of Example 3.11, The two components of $\mathbf{E}$ are equal in amplitude, perpendicular, and out of phase by 90°. Therefore, the wave is circularly polarized. To determine if the polarization is right-handed or left-handed, we look at the electric field vectors in the $z = 0$

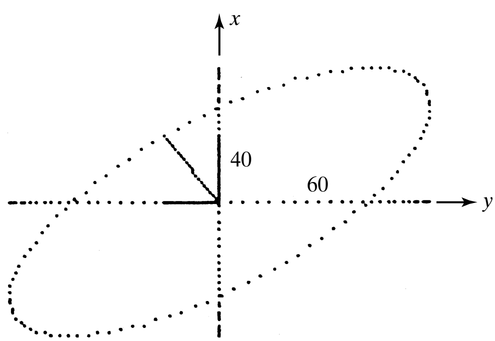

FIGURE 3.29

Ellipse traced by the tip of the vector $40 \cos (\omega t + 60°)\, \mathbf{a}_x + 60 \cos (\omega t + 105°)\, \mathbf{a}_y$.

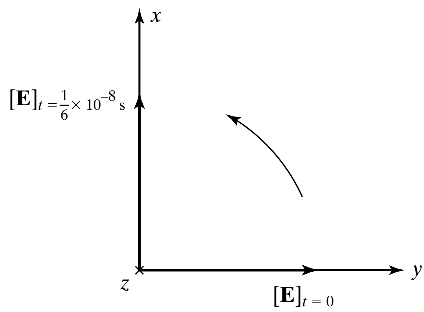

FIGURE 3.30

For the determination of the sense of circular polarization for the field of Ex. 3.11.

plane for two values of time, $t = 0$ and $t = \frac{1}{6} \times 10^{-8}$ s $(3\pi \times 10^8 t = \pi/2)$. These are shown in Fig. 3.30. As time progresses, the tip of the vector rotates in the counterclockwise sense, as seen looking in the $+z$-direction. Hence, the wave is left circularly polarized.

K3.6. Polarization; Linear polarization; Circular polarization; Elliptical polarization.

D3.17. Two sinusoidally time-varying vector fields are given by

$$\mathbf{F}_1 = F_0 \cos (2\pi \times 10^8 t - 2\pi z)\, \mathbf{a}_x$$
$$\mathbf{F}_2 = F_0 \cos (2\pi \times 10^8 t - 3\pi z)\, \mathbf{a}_y$$

Find the polarization of $\mathbf{F}_1 + \mathbf{F}_2$ at each of the following points: **(a)** (3, 4, 0); **(b)** (3, −2, 0.5); **(c)** (−2, 1, 1); and **(d)** (−1, −3, 0.2).

Ans. **(a)** Linear; **(b)**circular; **(c)** linear; **(d)** elliptical.

D3.18. A sinusoidally time-varying vector field is given at a point by $\mathbf{F} = 1 \cos (\omega t + 60°)\, \mathbf{a}_x + 1 \cos (\omega t + \alpha)\, \mathbf{a}_y$. Find the value(s) of α between 0° and 360° for each of the following cases: **(a) F** is linearly polarized along a line lying in the second and fourth quadrants; **(b) F** is circularly polarized with the sense of rotation from the $+x$-direction toward the $+y$-direction with time; and **(c) F** is circularly polarized with the sense of rotation from the $+y$-direction toward the $+x$-direction with time.

Ans. **(a)** 240°; **(b)** 330°; **(c)** 150°.

3.7 POWER FLOW AND ENERGY STORAGE

In Sec. 3.4, we obtained the solution for the electromagnetic field due to an infinite plane current sheet situated in the $z = 0$ plane, for arbitrary time variation, and then in Sec. 3.5 we considered the solution for the sinusoidal case. For a surface current flowing in the negative x-direction, we found the electric field on the sheet to be directed in the positive x-direction. Since the current is flowing against the force due to the electric field, a certain amount of work must be done by the source of the current to maintain the current flow on the sheet. Let us consider a rectangular area of length Δx and width Δy on the current sheet as shown in Fig. 3.31. Since the current density is $J_{S0} \cos \omega t$, the charge crossing

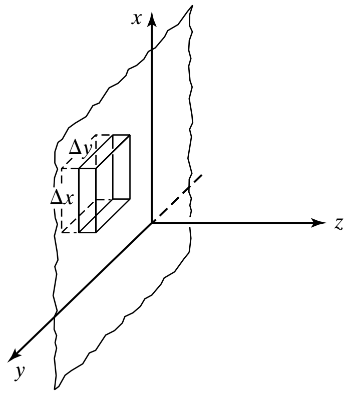

FIGURE 3.31

For the determination of power flow density associated with the electromagnetic field.

the width Δy in time dt is $dq = J_{S0}\,\Delta y \cos \omega t\, dt$. The force exerted on this charge by the electric field is given by

$$\mathbf{F} = dq\,\mathbf{E} = J_{S0}\,\Delta y \cos \omega t\, dt\, E_x \mathbf{a}_x \tag{3.107}$$

The amount of work required to be done against the electric field in displacing this charge by the distance Δx is

$$dw = F_x\,\Delta x = J_{S0} E_x \cos \omega t\, dt\, \Delta x\, \Delta y \tag{3.108}$$

Thus the power supplied by the source of the current in maintaining the surface current over the area $\Delta x\, \Delta y$ is

$$\frac{dw}{dt} = J_{S0} E_x \cos \omega t\, \Delta x\, \Delta y \tag{3.109}$$

Recalling that E_x on the sheet is $\eta_0 \dfrac{J_{S0}}{2} \cos \omega t$, we obtain

$$\frac{dw}{dt} = \eta_0 \frac{J_{S0}^2}{2} \cos^2 \omega t\, \Delta x\, \Delta y \tag{3.110}$$

We would expect the power given by (3.110) to be carried by the electromagnetic wave, half of it to either side of the current sheet. To investigate this, we note that the quantity $\mathbf{E} \times \mathbf{H}$ has the units of

$$\begin{aligned} \frac{\text{newtons}}{\text{coulomb}} \times \frac{\text{amperes}}{\text{meter}} &= \frac{\text{newtons}}{\text{coulomb}} \times \frac{\text{coulomb}}{\text{second-meter}} \times \frac{\text{meter}}{\text{meter}} \\ &= \frac{\text{newton-meters}}{\text{second}} \times \frac{1}{(\text{meter})^2} = \frac{\text{watts}}{(\text{mater})^2} \end{aligned}$$

which represents power density. Let us then consider the rectangular box enclosing the area $\Delta x\, \Delta y$ on the current sheet and with its sides almost touching the current sheet on either side of it, as shown in Fig. 3.31. Evaluating the surface integral of $\mathbf{E} \times \mathbf{H}$ over the surface of the rectangular box, we obtain the power flow out of the box as

$$\begin{aligned}\oint \mathbf{E} \times \mathbf{H} \cdot d\mathbf{S} &= \eta_0 \frac{J_{S0}^2}{4} \cos^2 \omega t \, \mathbf{a}_z \cdot \Delta x \, \Delta y \mathbf{a}_z \\ &\quad + \left(-\eta_0 \frac{J_{S0}^2}{4} \cos^2 \omega t \, \mathbf{a}_z\right) \cdot (-\Delta x \, \Delta y \, \mathbf{a}_z) \\ &= \eta_0 \frac{J_{S0}^2}{2} \cos^2 \omega t \, \Delta x \, \Delta y \end{aligned} \tag{3.111}$$

This result is exactly equal to the power supplied by the current source as given by (3.110).

Instantaneous Poynting vector

We now interpret the quantity $\mathbf{E} \times \mathbf{H}$ as the power flow density vector associated with the electromagnetic field. It is known as the *Poynting vector* after J.H. Poynting and is denoted by the symbol $\mathbf{P}$. Thus,

$$\boxed{\mathbf{P} = \mathbf{E} \times \mathbf{H}} \tag{3.112}$$

In particular, it is the instantaneous Poynting vector, since $\mathbf{E}$ and $\mathbf{H}$ are instantaneous field vectors. Although we have here introduced the Poynting vector by considering the specific case of the electromagnetic field due to the infinite plane current sheet, the interpretation that $\oint_S \mathbf{E} \times \mathbf{H} \cdot d\mathbf{S}$ is equal to the power flow out of the closed surface S is applicable in the general case.

Example 3.14 Distance variations of fields far from a physical antenna

Far from a physical antenna, that is, at a distance of several wavelengths from the antenna, the radiated electromagnetic waves are approximately uniform plane waves with their constant phase surfaces lying normal to the radial directions away from the antenna, as shown for two directions in Fig. 3.32. We wish to show from the Poynting vector and physical considerations that the electric and magnetic fields due to the antenna vary inversely proportional to the radial distance away from the antenna.

From considerations of electric and magnetic fields of a uniform plane wave, the Poynting vector is directed everywhere in the radial direction, indicating power flow radially away from the antenna, and is proportional to the square of the magnitude of the electric field intensity. Let us now consider two spherical surfaces of radii r_a and r_b and centered at the antenna, and insert a cone through these two surfaces such that the vertex is at the antenna, as shown in Fig. 3.32. Then the power crossing the portion of the spherical surface of radius r_b inside the cone must be the same as the power crossing the portion of the spherical surface of radius r_a inside the cone. Since these surface areas are proportional to the square of the radius and since the surface integral of the Poynting vector gives the power, the Poynting vector must be inversely proportional to the square of the radius. This in turn means that the electric field intensity and hence the magnetic field intensity must be inversely proportional to the radius.

Thus, from these simple considerations, we have established that far from a radiating antenna the electromagnetic field is inversely proportional to the radial distance

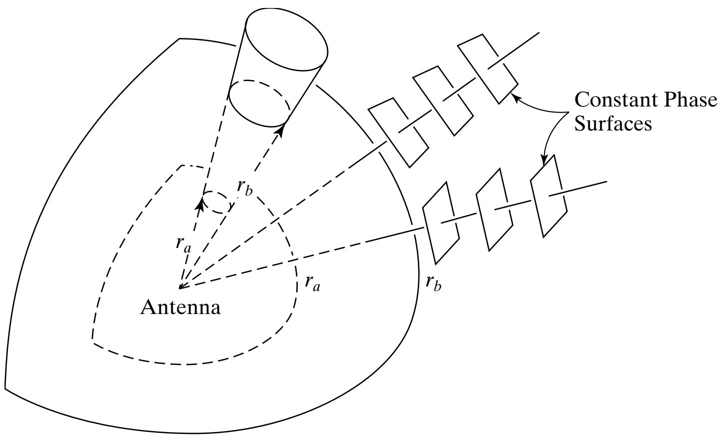

FIGURE 3.32

Radiation of electromagnetic waves far from a physical antenna.

away from the antenna. This reduction of the field intensity inversely proportional to the distance is known as the "free space reduction." For example, let us consider communication from earth to the moon. The distance from the earth to the moon is approximately 38×10^4 km or 38×10^7 m. Hence, the free space reduction factor for the field intensity is $10^{-7}/38$ or, in terms of decibels, the reduction is $20 \log_{10} 38 \times 10^7$, or 171.6 db.

Returning to the electromagnetic field due to the infinite plane current sheet, let us consider the region $z > 0$. The magnitude of the Poynting vector in this region is given by

$$P_z = E_x H_y = \eta_0 \frac{J_{S0}^2}{4} \cos^2 (\omega t - \beta z) \tag{3.113}$$

The variation of P_z with z for $t = 0$ is shown in Fig. 3.33. If we now consider a rectangluar box lying between $z = z$ and $z = z + \Delta z$ planes and having dimensions

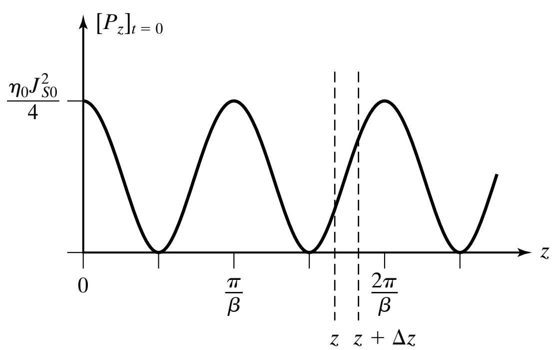

FIGURE 3.33

For the discussion of energy storage in electric and magnetic fields.

Δx and Δy in the x and y directions, respectively, we would in general obtain a nonzero result for the power flowing out of the box, since $\partial P_z/\partial z$ is not everywhere zero. Thus there is some energy stored in the volume of the box. We then ask ourselves the question, "Where does this energy reside?" A convenient way of interpretation is to attribute the energy storage to the electric and magnetic fields.

To discuss the energy storage in the electric and magnetic fields further, we evaluate the power flow out of the rectangular box. Thus

$$\begin{aligned}
\oint_S \mathbf{P} \cdot d\mathbf{S} &= [P_z]_{z+\Delta z}\,\Delta x\,\Delta y - [P_z]_z\,\Delta x\,\Delta y \\
&= \frac{[P_z]_{z+\Delta z} - [P_z]_z}{\Delta z}\,\Delta x\,\Delta y\,\Delta z \\
&= \frac{\partial P_z}{\partial z}\,\Delta v
\end{aligned}$$

where Δv is the volume of the box. Letting P_z equal to E_xH_y and using (3.72a) and (3.72b), we obtain

$$\begin{aligned}
\oint_S \mathbf{P} \cdot d\mathbf{S} &= \frac{\partial}{\partial z}[E_xH_y]\,\Delta v \\
&= \left(H_y\frac{\partial E_x}{\partial z} + E_x\frac{\partial H_y}{\partial z}\right)\Delta v \\
&= \left(-H_y\frac{\partial B_y}{\partial t} - E_x\frac{\partial D_x}{\partial t}\right)\Delta v \\
&= -\mu_0 H_y\frac{\partial H_y}{\partial t}\,\Delta v - \varepsilon_0 E_x\frac{\partial E_x}{\partial t}\,\Delta v \\
&= -\frac{\partial}{\partial t}\left(\frac{1}{2}\mu_0 H_y^2\,\Delta v\right) - \frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon_0 E_x^2\,\Delta v\right)
\end{aligned} \tag{3.114}$$

Equation (3.114) tells us that the power flow out of the box is equal to the sum of the time rates of decrease of the quantities $\frac{1}{2}\varepsilon_0 E_x^2\,\Delta v$ and $\frac{1}{2}\mu_0 H_y^2\,\Delta v$. These quantities are obviously the energies stored in the electric and magnetic fields, respectively, in the volume of the box. It then follows that the energy densities associated with the electric and magnetic fields in free space are $\frac{1}{2}\varepsilon_0 E_x^2$ and $\frac{1}{2}\mu_0 H_y^2$, respectively, having the units J/m^3. Although we have obtained these results by considering the particular case of the uniform plane wave, they hold in general.

Poynting's theorem

Equation (3.114) is a special case of a theorem known as the *Poynting's theorem*. To derive Poynting's theorem for the general case, we make use of the vector identity.

$$\boldsymbol{\nabla} \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot \boldsymbol{\nabla} \times \mathbf{E} - \mathbf{E} \cdot \boldsymbol{\nabla} \times \mathbf{H} \tag{3.115}$$

and Maxwell's curl equations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

to obtain

$$\begin{aligned}
\nabla \cdot (\mathbf{E} \times \mathbf{H}) &= \mathbf{H} \cdot \left(-\mu_0 \frac{\partial \mathbf{H}}{\partial t}\right) - \mathbf{E} \cdot \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right) \\
&= -\mu_0 \mathbf{H} \cdot \frac{\partial \mathbf{H}}{\partial t} - \mathbf{E} \cdot \mathbf{J} - \varepsilon_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} \\
&= -\mathbf{E} \cdot \mathbf{J} - \mu_0 \frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{H} \cdot \mathbf{H}\right) - \varepsilon_0 \frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{E} \cdot \mathbf{E}\right) \\
&= -\mathbf{E} \cdot \mathbf{J} - \frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon_0 E^2\right) - \frac{\partial}{\partial t}\left(\frac{1}{2}\mu_0 H^2\right)
\end{aligned} \tag{3.116}$$

Substituting $\mathbf{P}$ for $\mathbf{E} \times \mathbf{H}$ and taking the volume integral of both sides of (3.116) over the volume V, we obtain

$$\int_V (\nabla \cdot \mathbf{P})\, dv = -\int_V (\mathbf{E} \cdot \mathbf{J})\, dv - \int_V \frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon_0 E^2\right) dv - \int_V \frac{\partial}{\partial t}\left(\frac{1}{2}\mu_0 H^2\right) dv \tag{3.117}$$

Interchanging the differentiation operation with time and integration operation over volume in the second and third terms on the right side and replacing the volume integral on the left side by a closed surface integral in accordance with the divergence theorem, we get

$$\boxed{\oint_S \mathbf{P} \cdot d\mathbf{S} = -\int_V (\mathbf{E} \cdot \mathbf{J})\, dv - \frac{\partial}{\partial t}\int_V \frac{1}{2}\varepsilon_0 E^2\, dv - \frac{\partial}{\partial t}\int_V \frac{1}{2}\mu_0 H^2\, dv} \tag{3.118}$$

where S is the surface bounding the volume V. Equation (3.118) is the Poynting's theorem for the general case. Since it should hold for any size V, it follows that the electric stored energy density and the magnetic stored energy density in free space are given by

$$w_e = \frac{1}{2}\varepsilon_0 E^2 \tag{3.119a}$$

$$w_m = \frac{1}{2}\mu_0 H^2 \tag{3.119b}$$

respectively. The quantity $\mathbf{E} \cdot \mathbf{J}$, having the units (V/m) (A/m^2) or W/m^3, is the power density associated with the work done by the field, having to do with the current flow.

Time-average Poynting vector

Returning now to (3.113), we can talk about the time-average value of P_z, denoted $\langle P_z \rangle$. It is the value of P_z averaged over one period of the sinusoidal time variation of the source; that is,

$$\boxed{\langle P_z \rangle = \frac{1}{T}\int_0^T P_z(t)\,dt} \tag{3.120}$$

where $T(=1/f)$ is the period. From (3.113), we have

$$\begin{aligned}\langle P_z \rangle &= \left\langle \eta_0 \frac{J_{S0}^2}{4}\cos^2(\omega t - \beta z)\right\rangle \\ &= \eta_0 \frac{J_{S0}^2}{4}\left\langle \frac{1 + \cos 2(\omega t - \beta z)}{2}\right\rangle \\ &= \eta_0 \frac{J_{S0}^2}{8}\end{aligned} \tag{3.121}$$

This can be expressed in the manner

$$\begin{aligned}\langle P_z \rangle &= \eta_0 \frac{J_{S0}^2}{8} \\ &= \frac{1}{2}\mathrm{Re}\left[\left(\frac{\eta_0 J_{S0}}{2}e^{-j\beta z}\right)\left(\frac{J_{S0}}{2}e^{j\beta z}\right)\right] \\ &= \mathrm{Re}\left[\frac{1}{2}\bar{E}_x \bar{H}_y^*\right]\end{aligned} \tag{3.122}$$

where $\bar{E}_x$ and $\bar{H}_y$ are the phasor electric and magnetic field components, respectively. See Appendix A for phasors. In terms of vector quantities,

$$\begin{aligned}\langle \mathbf{P} \rangle &= \mathrm{Re}\left[\frac{1}{2}\bar{E}_x \mathbf{a}_x \times (\bar{H}_y \mathbf{a}_y)^*\right] \\ &= \mathrm{Re}\left[\frac{1}{2}\bar{\mathbf{E}} \times \bar{\mathbf{H}}^*\right] \\ &= \mathrm{Re}[\bar{\mathbf{P}}]\end{aligned} \tag{3.123}$$

which is the time-average Poynting vector, where

$$\bar{\mathbf{P}} = \frac{1}{2}\bar{\mathbf{E}} \times \bar{\mathbf{H}}^* \tag{3.124}$$

is the complex Poynting vector.

K3.7. Power flow; Poynting vector; Poynting's theorem; Electric stored energy density; Magnetic stored energy density; Time-average Poynting vector.

D3.19. The magnetic field associated with a uniform plane wave propagating in the $+z$-direction in free space is given by

$$\mathbf{H} = H_0 \cos(6\pi \times 10^7 t - 0.2\pi z)\, \mathbf{a}_y \text{ A/m}$$

Find the following: **(a)** the instantaneous power flow across a surface of area 1 m^2 in the $z = 0$ plane at $t = 0$; **(b)** the instantaneous power flow across a surface of area 1 m^2 in the $z = 0$ plane at $t = (1/8)\ \mu\text{s}$; and **(c)** the time-average power flow across a surface of area 1 m^2 in the $z = 0$ plane.

Ans. **(a)** $120\pi H_0^2$ W; **(b)** 0 W; **(c)** $60\pi H_0^2$ W.

D3.20. Find the time-average values of the following: **(a)** $A \sin \omega t \sin 3\omega t$; **(b)** $A(\cos^2 \omega t - 0.5 \sin^2 2\omega t)$; and **(c)** $A \sin^6 \omega t$.

Ans. **(a)** 0; **(b)** $0.25A$; **(c)** $0.3125A$.

SUMMARY

We have in this chapter derived the differential forms of Maxwell's equations from their integral forms, which we introduced in Chapter 2. For the general case of electric and magnetic fields having all three components, each of them dependent on all coordinates and time, Maxwell's equations in differential form are given as follows in words and in mathematical form.

Faraday's law. The curl of the electric field intensity is equal to the negative of the time derivative of the magnetic flux density; that is,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{3.125}$$

Ampère's circuital law. The curl of the magnetic field intensity is equal to the sum of the current density due to flow of charges and the displacement current density, which is the time derivative of the displacement flux density; that is,

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{3.126}$$

Gauss' law for the electric field. The divergence of the displacement flux density is equal to the charge density; that is,

$$\nabla \cdot \mathbf{D} = \rho \tag{3.127}$$

Gauss' law for the magnetic field. The divergence of the magnetic flux density is equal to zero; that is,

$$\nabla \cdot \mathbf{B} = 0 \tag{3.128}$$

Auxiliary to (3.125)–(3.128), the continuity equation is given by

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \tag{3.129}$$

This equation, which is the differential form of the law of conservation of charge, states that the sum of the divergence of the current density due to flow of charges and the time derivative of the charge density is equal to zero. Also, we recall that

$$\mathbf{D} = \varepsilon_0 \mathbf{E}$$
$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0}$$

which relate **D** and **H** to **E** and **B**, respectively, for free space.

We have learned that the basic definitions of curl and divergence, which have enabled us to discuss their physical interpretations with the aid of the curl and divergence meters, are

$$\nabla \times \mathbf{A} = \lim_{\Delta S \to 0} \left[\frac{\oint_C \mathbf{A} \cdot d\mathbf{l}}{\Delta S} \right]_{\max} \mathbf{a}_n$$
$$\nabla \cdot \mathbf{A} = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta v}$$

Thus, the curl of a vector field at a point is a vector whose magnitude is the circulation of that vector field per unit area, with the area oriented so as to maximize this quantity and in the limit that the area shrinks to the point. The direction of the vector is normal to the area in the aforementioned limit and in the right-hand sense. The divergence of a vector field at a point is a scalar quantity equal to the net outward flux of that vector field per unit volume in the limit that the volume shrinks to the point. In Cartesian coordinates, the expansions for curl and divergence are

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$
$$= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \mathbf{a}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \mathbf{a}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \mathbf{a}_z$$
$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Thus, Maxwell's equations in differential form relate the spatial variations of the field vectors at a point to their temporal variations and to the charge and current densities at that point.

We have also learned two theorems associated with curl and divergence. These are the Stokes' theorem and the divergence theorem given, respectively, by

$$\oint_C \mathbf{A} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S}$$
$$\oint_S \mathbf{A} \cdot d\mathbf{S} = \int_V (\nabla \cdot \mathbf{A})\, dv$$

Stokes' theorem enables us to replace the line integral of a vector around a closed path by the surface integral of the curl of that vector over any surface bounded by that closed path, and vice versa. The divergence theorem enables us to replace the surface integral of a vector over a closed surface by the volume integral of the divergence of that vector over the volume bounded by the closed surface and vice versa.

Next, we studied the principles of uniform plane waves. Uniform plane waves are a building block in the study of electromagnetic wave propagation. They are the simplest type of solutions resulting from the coupling of the electric and magnetic fields in Maxwell's curl equations. Their electric and magnetic fields are perpendicular to each other and to the direction of propagation. The fields are *uniform* in the *planes* perpendicular to the direction of propagation.

We first obtained the uniform plane wave solution to Maxwell's equations in time domain in free space by considering an infinite plane current sheet in the xy-plane with uniform surface current density given by

$$\mathbf{J}_S = -J_S(t)\mathbf{a}_x \text{ A/m}$$

and deriving the electromagnetic field due to the current sheet to be

$$\mathbf{E} = \frac{\eta_0}{2} J_S\left(t \mp \frac{z}{v_p}\right)\mathbf{a}_x \quad \text{for} \quad z \gtrless 0 \tag{3.130a}$$

$$\mathbf{H} = \pm \frac{1}{2} J_S\left(t \mp \frac{z}{v_p}\right)\mathbf{a}_y \quad \text{for} \quad z \gtrless 0 \tag{3.130b}$$

where

$$v_p = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

and

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

are the velocity of propagation and intrinsic impedance, respectively. In (3.130a) and (3.130b), the arguments $(t - z/v_p)$ and $(t + z/v_p)$ represent wave motion in the positive z-direction and the negative z-direction, respectively, with the velocity v_p. Thus, (3.130a) and (3.130b) correspond to waves propagating away from the current sheet to either side of it. Since the fields are uniform in constant-z planes, they represent uniform plane waves. We discussed how to plot the variations of the field components versus t for fixed values of z and versus z for fixed values of t, for a given function $J_S(t)$.

We then extended the solution to sinusoidally time-varying uniform plane waves by considering the current density on the infinite plane sheet to be

$$\mathbf{J}_S = -J_{S0} \cos \omega t \, \mathbf{a}_x \text{ A/m}$$

and obtaining the corresponding field to be

$$\mathbf{E} = \frac{\eta_0 J_{S0}}{2} \cos (\omega t \mp \beta z) \, \mathbf{a}_x \qquad \text{for} \quad z \lessgtr 0 \tag{3.131a}$$

$$\mathbf{H} = \pm \frac{J_{S0}}{2} \cos (\omega t \mp \beta z) \, \mathbf{a}_y \qquad \text{for} \quad z \lessgtr 0 \tag{3.131b}$$

where

$$\beta = \frac{\omega}{v_p} = \omega \sqrt{\mu_0 \varepsilon_0} \tag{3.132}$$

We discussed several important parameters and properties associated with these waves, including polarization. The quantity β is the phase constant, that is, the magnitude of the rate of change of phase with distance along the direction of propagation, for a fixed time. The velocity v_p, which from (3.132) is given by

$$v_p = \frac{\omega}{\beta} \tag{3.133}$$

is known as the phase velocity, because it is the velocity with which a particular constant phase progresses along the direction of propagation. The wavelength λ, that is, the distance along the direction of propagation in which the phase changes by 2π radians, for a fixed time, is given by

$$\lambda = \frac{2\pi}{\beta} \tag{3.134}$$

The wavelength is related to the frequency f in a simple manner as given by

$$v_p = \lambda f \tag{3.135}$$

which follows from (3.133) and (3.135) and is a result of the fact that the time and space variations of the fields are interdependent. We also discussed the principle of antenna array.

Polarization of sinusoidally time-varying vector fields was then considered. In the general case, the polarization is elliptical, that is, the tip of the field vector describes an ellipse with time. Linear and circular polarizations are special cases.

Finally, we learned that there is power flow and energy storage associated with the wave propagation that accounts for the work done in maintaining the current flow on the sheet. The power flow density is given by the Poynting vector

$$\mathbf{P} = \mathbf{E} \times \mathbf{H}$$

and the energy densities associated with the electric and magnetic fields are given, respectively, by

$$w_e = \frac{1}{2}\varepsilon_0 E^2$$

$$w_m = \frac{1}{2}\mu_0 H^2$$

The surface integral of the Poynting vector over a given closed surface gives the total power flow out of the volume bounded by that surface.

REVIEW QUESTIONS

Q3.1. State Faraday's law in differential form for the simple case of $\mathbf{E} = E_x(z, t)\mathbf{a}_x$. How is it derived from Faraday's law in integral form?

Q3.2. State Faraday's law in differential form for the general case of an arbitrary electric field. How is it derived from its integral form?

Q3.3. What is meant by the net right-lateral differential of the x- and y-components of a vector normal to the z-direction? Give an example in which the net right-lateral differential of E_x and E_y normal to the z-direction is zero, although the individual derivatives are nonzero.

Q3.4. What is the determinant expansion for the curl of a vector in Cartesian coordinates?

Q3.5. State Ampère's circuital law in differential form for the general case of an arbitrary magnetic field. How is it derived from its integral form?

Q3.6. State Ampère's circuital law in differential form for the simple case of $\mathbf{H} = H_y(z, t)\mathbf{a}_y$. How is it derived from Ampère's circuital law in differential form for the general case?

Q3.7. If a pair of **E** and **B** at a point satisfies Faraday's law in differential form, does it necessarily follow that it also satisfies Ampère's circuital law in differential form, and vice versa?

Q3.8. State Gauss' law for the electric field in differential form. How is it derived from its integral form?

Q3.9. What is meant by the net longitudinal differential of the components of a vector field? Give an example in which the net longitudinal differential of the components of a vector field is zero, although the individual derivatives are nonzero.

Q3.10. What is the expansion for the divergence of a vector in Cartesian coordinates?

Q3.11. State Gauss' law for the magnetic field in differential form. How is it derived from its integral form?

Q3.12. How can you determine if a given vector can represent a magnetic field?

Q3.13. State the continuity equation and discuss its physical interpretation.

Q3.14. Summarize Maxwell's equations in differential form.

Q3.15. State and briefly discuss the basic definition of the curl of a vector.

Q3.16. What is a curl meter? How does it help visualize the behavior of the curl of a vector field?

Q3.17. Provide two examples of physical phenomena in which the curl of a vector field is nonzero.

Q3.18. State and briefly discuss the basic definition of the divergence of a vector.

Q3.19. What is a divergence meter? How does it help visualize the behavior of the divergence of a vector field?

Q3.20. Provide two examples of physical phenomena in which the divergence of a vector field is nonzero.

Q3.21. State Stokes' theorem and discuss its application.

Q3.22. State the divergence theorem and discuss its application.

Q3.23. What is the divergence of the curl of a vector?

Q3.24. What is a uniform plane wave? Why is the study of uniform plane waves important?

Q3.25. Outline the procedure for obtaining from the two Maxwell's curl equations the particular differential equation for the special case of $\mathbf{J} = J_x(z)\mathbf{a}_x$.

Q3.26. State the wave equation for the case of $\mathbf{E} = E_x(z, t)\mathbf{a}_x$. Describe the procedure for its solution.

Q3.27. Discuss by means of an example how a function $f(t - z\sqrt{\mu_0\varepsilon_0})$ represents a traveling wave propagating in the positive z-direction.

Q3.28. Discuss by means of an example how a function $g(t + z\sqrt{\mu_0\varepsilon_0})$ represents a traveling wave propagating in the negative z-direction.

Q3.29. What is the significance of the intrinsic impedance of free space? What is its value?

Q3.30. Summarize the procedure for obtaining the solution for the electromagnetic field due to the infinite plane sheet of uniform time-varying current density.

Q3.31. State and discuss the solution for the electromagnetic field due to the infinite plane sheet of current density $\mathbf{J}_S(t) = -J_S(t)\mathbf{a}_x$ for $z = 0$.

Q3.32. Discuss the parameters ω, β, and v_p associated with sinusoidally time-varying uniform plane waves.

Q3.33. Define wavelength. What is the relationship among wavelength, frequency, and phase velocity?

Q3.34. Discuss the classification of waves according to frequency, giving examples of their application in the different frequency ranges.

Q3.35. How is the direction of propagation of a uniform plane wave related to the directions of its fields?

Q3.36. Discuss the principle of an antenna array with the aid of an example.

Q3.37. A sinusoidally time-varying vector is expressed in terms of its components along the x-, y-, and z-axes. What is the polarization of each of the components?

Q3.38. What are the conditions for the sum of two linearly polarized sinusoidally time-varying vectors to be circularly polarized?

Q3.39. What is the polarization for the general case of the sum of two sinusoidally time-varying linearly polarized vectors having arbitrary amplitudes, phase angles, and directions?

Q3.40. Discuss the relevance of polarization in the reception of radio waves.

Q3.41. Discuss right-handed and left-handed circular polarizations associated with sinusoidally time-varying uniform plane waves.

Q3.42. What is the Poynting vector? What is the physical interpretation of the Poynting vector over a closed surface?

Q3.43. Discuss how the fields far from a physical antenna vary inversely with the distance from the antenna.

Q3.44. Discuss the interpretation of energy storage in the electric and magnetic fields of a uniform plane wave. What are the energy densities associated with the electric and magnetic fields?

Q3.45. State Poynting's theorem. How is it derived from Maxwell's curl equations?

Q3.46. What is the time-average Poynting vector? How is it expressed in terms of the complex electric and magnetic fields?

PROBLEMS

Section 3.1

P3.1. **Evaluating curls of vector fields.** Find the curls of the following vector fields:

(a) $zx\mathbf{a}_x + xy\mathbf{a}_y + yz\,\mathbf{a}_z$
(b) $\cos y\mathbf{a}_x - x\sin y\,\mathbf{a}_y$
(c) $(e^{-r^2}/r)\mathbf{a}_\phi$ in cylindrical coordinates
(d) $2r\cos\theta\,\mathbf{a}_r + r\mathbf{a}_\theta$ in spherical coordinates

P3.2. **Finding B for a given E from Faraday's law in differential form.** For each of the following electric fields, find **B** that satisfies Faraday's law in differential form:

(a) $\mathbf{E} = E_0 \cos 3\pi z \cos 9\pi \times 10^8 t\,\mathbf{a}_x$
(b) $\mathbf{E} = E_0\mathbf{a}_y \cos[3\pi \times 10^8 t + 0.2\pi(4x + 3z)]$

P3.3. **Simplified forms of Maxwell's curl equations for special cases.** Obtain the simplified differential equations for the following cases: (**a**) Ampère's circuital law for $\mathbf{H} = H_x(z, t)\mathbf{a}_x$ and (**b**) Faraday's law for $\mathbf{E} = E_\phi(r, t)\mathbf{a}_\phi$ in cylindrical coordinates.

P3.4. **Simultaneous satisfaction of Faraday's and Ampere's circuital laws by E and B.** For the electric field $\mathbf{E} = E_0 e^{-\alpha z}\cos\omega t\,\mathbf{a}_x$ in free space ($\mathbf{J} = \mathbf{0}$), find **B** that satisfies Faraday's law in differential form and then determine if the pair of **E** and **B** satisfy Ampère's circuital law in differential form.

P3.5. **Satisfaction of Maxwell's curl equations for a specified electric field.** For the electric field $\mathbf{E} = E_0\cos(\omega t - \alpha y - \beta z)\,\mathbf{a}_x$ in free space ($\mathbf{J} = \mathbf{0}$), find the necessary condition relating $\alpha, \beta, \omega, \mu_0$, and ε_0 for the field to satisfy both of Maxwell's curl equations.

P3.6. **Satisfaction of Maxwell's curl equations for a specified electric field.** For the electric field $\mathbf{E} = E_0 e^{-kx}\cos(2 \times 10^8 t - y)\,\mathbf{a}_z$ in free space ($\mathbf{J} = \mathbf{0}$), find the value(s) of k for which the field satisfies both of Maxwell's curl equations.

P3.7. **Magnetic fields of current distributions from Ampere's circuital law in differential form.** For each of the following current distributions, find the corresponding magnetic field intensity using Ampère's circuital law in differential form without the displacement current density term, and plot both the current density and the magnetic field intensity components versus the appropriate coordinate:

(a) $\mathbf{J} = \begin{cases} J_0 \dfrac{z}{a} \mathbf{a}_x & \text{for} \quad -a < z < a \\ \mathbf{0} & \text{otherwise} \end{cases}$

(b) $\mathbf{J} = \begin{cases} J_0 \mathbf{a}_\phi & \text{for} \quad a < r < 2a \\ \mathbf{0} & \text{otherwise} \end{cases}$

in cylindrical coordinates

Section 3.2

P3.8. Evaluating divergences of vector fields. Find the divergences of the following vector fields:

(a) $zx\mathbf{a}_x + xy\mathbf{a}_y + yz\mathbf{a}_z$

(b) $3\mathbf{a}_x + (y - 3)\mathbf{a}_y + (2 + z)\mathbf{a}_z$

(c) $r \sin \phi \, \mathbf{a}_\phi$ in cylindrical coordinates

(d) $r \cos \theta \, (\cos \theta \, \mathbf{a}_r - \sin \theta \, \mathbf{a}_\theta)$ in spherical coordinates

P3.9. Electric fields of charge distributions from Gauss' law in differential form. For each of the following charge distributions, find the corresponding displacement flux density using Gauss' law for the electric field in differential form, and plot both the charge density and the displacement flux density component versus the appropriate coordinate:

(a) $\rho = \begin{cases} \rho_0 \dfrac{x}{a} & \text{for} \quad -a < x < a \\ 0 & \text{otherwise} \end{cases}$

(b) $\rho = \begin{cases} \rho_0 \dfrac{r}{a} & \text{for} \quad 0 < r < a \\ 0 & \text{otherwise} \end{cases}$

in cylindrical coordinates

P3.10. Realizability of vector fields as certain types of fields. For each of the following vector fields, find the value of the constant k for which the vector field can be realized as a magnetic field or as a current density in the absence of charge accumulation (or depletion):

(a) $(1/y^k)(2x\mathbf{a}_x + y\mathbf{a}_y)$

(b) $r(\sin k\phi \, \mathbf{a}_r + \cos k\phi \, \mathbf{a}_\phi)$ in cylindrical coordinates

(c) $[1 + (2/r^3)]\cos \theta \, \mathbf{a}_r + k[1 - (1/r^3)] \sin \theta \, \mathbf{a}_\theta$ in spherical coordinates

P3.11. Realizability of vector fields as certain types of fields. Determine which of the following static fields can be realized both as an electric field in a charge-free region and a magnetic field in a current-free region:

(a) $y\mathbf{a}_x + x\mathbf{a}_y$

(b) $[1 + (1/r^2)] \cos \phi \, \mathbf{a}_r - [1 - (1/r^2)] \sin \phi \, \mathbf{a}_\phi$ in cylindrical coordinates

(c) $r \sin \theta \, \mathbf{a}_\theta$ in spherical coordinates

Section 3.3

P3.12. Interpretation of curl with the aid of curl meter and by expansion. With the aid of the curl meter and also by expansion in the Cartesian coordinate system, discuss the curl of the velocity vector field associated with the flow of water in the stream of Fig. 3.8(a), except that the velocity v_z varies in a nonlinear manner from zero at the banks to a maximum of v_0 at the center given by

$$\mathbf{v} = \frac{4v_0}{a^2}(ax - x^2)\mathbf{a}_z$$

P3.13. Interpretation of curl with the aid of curl meter and by expansion. With the aid of the curl meter and also by expansion in the cylindrical coordinate system, discuss the curl of the linear velocity vector field associated with points inside Earth due to its spin motion.

P3.14. Interpretation of divergence with the aid of divergence meter and by expansion. Discuss the divergences of the following vector fields with the aid of the divergence meter and also by expansion in the appropriate coordinate system: **(a)** the position vector field associated with points in three-dimensional space and **(b)** the linear velocity vector field associated with points inside Earth due to its spin motion.

P3.15. Verification of Stokes' theorem. Verify Stokes' theorem for the following cases: **(a)** the vector field $\mathbf{A} = zx\mathbf{a}_x + xy\mathbf{a}_y + yz\mathbf{a}_z$ and the closed path comprising the straight lines from (0, 0, 0) to (0, 1, 0), from (0, 1, 0) to (0, 1, 1), and from (0, 1, 1) to (0, 0, 0) and **(b)** the vector field $\mathbf{A} = \cos y\,\mathbf{a}_x - x \sin y\,\mathbf{a}_y$ independent of a closed path.

P3.16. Verification of the divergence theorem. Verify the divergence theorem for the following cases: **(a)** the vector field $xy\mathbf{a}_x + yz\mathbf{a}_y + zx\mathbf{a}_z$ and the cubical box bounded by the planes $x = 0, x = 1, y = 0, y = 1, z = 0$, and $z = 1$ and **(b)** the vector field $y^2\mathbf{a}_y - 2yz\mathbf{a}_z$ and the closed surface bounded by the planes $x = 0, y = 0, z = 0, z = 1$ and $x + y = 1$.

Section 3.4

P3.17. Simplified forms of Maxwell's curl equations for special case of J. From Maxwell's curl equations, obtain the particular differential equations for the case of $\mathbf{J} = J_z(y, t)\mathbf{a}_z$.

P3.18. Plotting of functions of time and distance. For each of the following functions, plot the value of the function versus z for the two specified values of time and discuss the traveling-wave nature of the function.

(a) $e^{-|t-z|}$; $t = 0, t = 1$ s

(b) $(2 \times 10^8 t + z)[u(2 \times 10^8 t + z) - u(2 \times 10^8 t + z - 3)]$;
$t = 0, t = 10^{-8}$ s

P3.19. Writing traveling wave functions for specified time and distance variations. Write expressions for traveling-wave functions corresponding to the following cases: **(a)** time variation at $x = 0$ in the manner $10u(t)$ and propagating in the $-x$-direction with velocity 0.5 m/s; **(b)** time variation at $y = 0$ in the manner $t \sin 20t$ and propagating in the $+y$-direction with velocity 4 m/s; and **(c)** distance variation at $t = 0$ in the manner $5z^3e^{-z^2}$ and propagating in the $-z$-direction with velocity 2 m/s.

P3.20. Plotting time and distance variations of a traveling wave. The variation with z for $t = 0$ of a function $f(z, t)$ representing a traveling wave propagating in the $+z$-direction with velocity 100 m/s is shown in Fig. 3.34. Find and sketch: **(a)** f versus z for $t = 1$ s; **(b)** f versus t for $z = 0$; and **(c)** f versus t for $z = 200$ m.

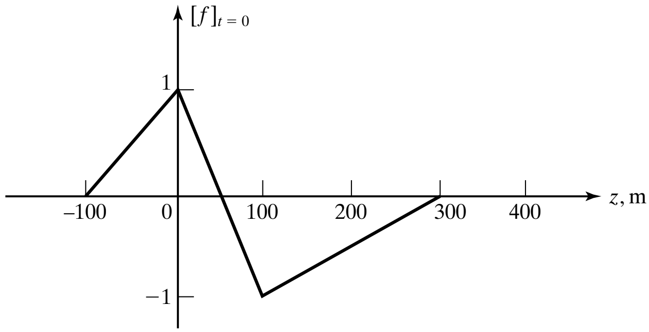

FIGURE 3.34
For Problems 3.20 and 3.21

P3.21. Plotting time and distance variations of a traveling wave. Repeat Problem P3.20 if the function f represents a traveling wave propagating in the $-z$-direction with velocity 100 m/s.

P3.22. Plotting field variations for a specified infinite plane-sheet current source. An infinite plane sheet lying in the $z = 0$ plane in free space carries a surface current of density $\mathbf{J}_S = -J_S(t)\mathbf{a}_x$, where $J_S(t)$ is as shown in Fig. 3.35. Find and sketch **(a)** E_x versus t in the $z = 300$ m plane; **(b)** E_x versus z for $t = 2\ \mu$s; and **(c)** H_y versus z for $t = 4\ \mu$s.

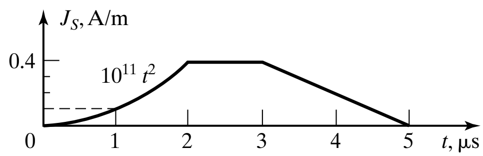

FIGURE 3.35
For Problem 3.22

P3.23. Plotting field variations for a specified infinite plane-sheet current source. An infinite plane sheet of current density $\mathbf{J}_S = -J_S(t)\mathbf{a}_x$ A/m where $J_S(t)$ is as shown in Fig. 3.36, lies in the $z = 0$ plane in free space. Find and sketch: **(a)** E_x versus t in the $z = 300$ m plane; **(b)** H_y versus t in the $z = -600$ m plane; **(c)** E_x versus z for $t = 3\ \mu$s; and **(d)** H_y versus z for $t = 4\ \mu$s.

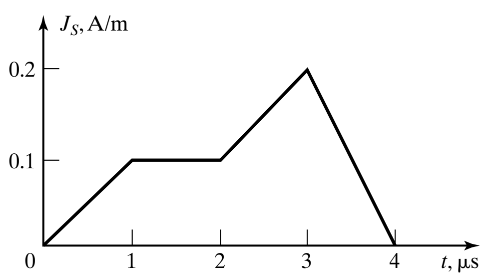

FIGURE 3.36
For Problem P3.23

P3.24. Source and more field variations from a given field variation of a uniform plane wave. The time variation of the electric-field intensity E_x in the $z = 300$ m plane of a uniform plane wave propagating away from an infinite plane current sheet of current density $\mathbf{J}_S(t) = -J_S(t)\mathbf{a}_x$ lying in the $z = 0$ plane in free space is given by the periodic function shown in Fig. 3.37. Find and sketch **(a)** J_S versus t; **(b)** E_x versus t in $z = -600$ m plane; **(c)** E_x versus z for $t = 2$ μs; and **(d)** H_y versus z for $t = 3$ μs.

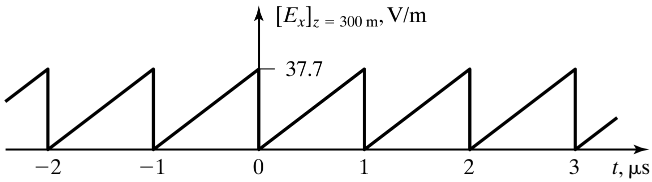

FIGURE 3.37
For Problem P3.24

Section 3.5

P3.25. Finding parameters for a specified sinusoidal uniform plane-wave electric field. The electric-field intensity of a uniform plane wave propagating in free space is given by

$$\mathbf{E} = 37.7 \cos (9\pi \times 10^7 t + 0.3\pi y)\, \mathbf{a}_x \text{ V/m}$$

Find: **(a)** the frequency; **(b)** the wavelength; **(c)** the direction of propagation of the wave; and **(d)** the associated magnetic-field intensity vector **H**.

P3.26. Writing field expressions for an infinite plane current sheet source. Given $\mathbf{J}_S = 0.2(\sqrt{3}\mathbf{a}_x + \mathbf{a}_y) \cos 6\pi \times 10^9 t$ A/m in the $z = 0$ plane in free space, find **E** and **H** for $z \lessgtr 0$. Use the following three steps, which are generalizations of the solution to the electromagnetic field due to the infinite plane current sheet in the $z = 0$ plane:

1. Write the expression for **H** on the sheet and on either side of it, by noting that $[\mathbf{H}]_{z=0\pm} = \frac{1}{2}\mathbf{J}_S \times (\pm\mathbf{a}_z) = \frac{1}{2}\mathbf{J}_s \times \mathbf{a}_n$, where $\mathbf{a}_n$ is the unit vector normal to the sheet and directed toward the side of interest.
2. Extend the result of step 1 to write the expression for **H** everywhere, that is, for $z \lessgtr 0$, considering the traveling wave character of the fields.
3. Write the solution for **E** everywhere by noting that **(a)** the amplitude of $\mathbf{E} = \eta_0 \times$ the amplitude of **H**, and **(b)** the direction of **E**, the direction of **H**, and the direction of propagation constitute a right-handed orthogonal set, so that $\mathbf{E} = \eta_0 \mathbf{H} \times \mathbf{a}_n$.

P3.27. Writing field expressions for an infinite plane current sheet source. Given $\mathbf{J}_S = 0.2 \sin 15\pi \times 10^7 t\, \mathbf{a}_y$ A/m in the $x = 0$ plane in free space, find **E** and **H** for $x \lessgtr 0$. Use the three steps outlined in Problem P3.26, except that the current sheet is in the $x = 0$ plane.

P3.28. Electric field due to an array of two infinite plane current sheets. The current densities of two infinite, plane, parallel current sheets are given by

$$\mathbf{J}_{S1} = -J_{S0} \cos \omega t\, \mathbf{a}_x \quad \text{in the } z = 0 \text{ plane}$$
$$\mathbf{J}_{S2} = -kJ_{S0} \cos \omega t\, \mathbf{a}_x \quad \text{in the } z = \lambda/2 \text{ plane}$$

Find the electric-field intensities in the three regions: **(a)** $z < 0$; **(b)** $0 < z < \lambda/2$; and **(c)** $z > \lambda/2$.

P3.29. **An array of three infinite plane current sheets.** The current densities of three infinite plane, parallel, current sheets are given by

$$\begin{aligned} \mathbf{J}_{S1} &= -J_{S0} \cos \omega t \, \mathbf{a}_x && \text{in the } z = 0 \text{ plane} \\ \mathbf{J}_{S2} &= -kJ_{S0} \sin \omega t \, \mathbf{a}_x && \text{in the } z = \lambda/4 \text{ plane} \\ \mathbf{J}_{S3} &= -2kJ_{s0} \cos \omega t \, \mathbf{a}_x && \text{in the } z = \lambda/2 \text{ plane} \end{aligned}$$

Obtain the expression for the ratio of the amplitude of the electric field in the region $z > \lambda/2$ to the amplitude of the electric field in the region $z < 0$. Then find the ratio for each of the following values of k: **(a)** $k = -1$, **(b)** $k = 1/2$, and **(c)** $k = 1$. Find the value(s) of k for each of the following values of the ratio: **(a)** 1/3 and **(b)** 3.

Section 3.6

P3.30. **Determination of polarization for combinations of linearly polarized vectors.** Three sinusoidally time-varying linearly polarized vector fields are given at a point by

$$\begin{aligned} \mathbf{F}_1 &= \sqrt{3}\,\mathbf{a}_x \cos (2\pi \times 10^6 t + 30^\circ) \\ \mathbf{F}_2 &= \mathbf{a}_z \cos (2\pi \times 10^6 t + 30^\circ) \\ \mathbf{F}_3 &= \left(\frac{1}{2}\mathbf{a}_x + \sqrt{3}\,\mathbf{a}_y + \frac{\sqrt{3}}{2}\mathbf{a}_z\right) \cos (2\pi \times 10^6 t - 60^\circ) \end{aligned}$$

Determine the polarizations of the following: **(a)** $\mathbf{F}_1 + \mathbf{F}_2$; **(b)** $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$; and **(c)** $\mathbf{F}_1 - \mathbf{F}_2 + \mathbf{F}_3$.

P3.31. **Polarization of sum of two linearly polarized vector fields.** Two sinusoidally time-varying, linearly polarized vector fields are given at a point by

$$\begin{aligned} \mathbf{F}_1 &= (C\mathbf{a}_x + C\mathbf{a}_y + \mathbf{a}_z) \cos 2\pi \times 10^6 t \\ \mathbf{F}_2 &= (C\mathbf{a}_x + \mathbf{a}_y - 2\mathbf{a}_z) \sin 2\pi \times 10^6 t \end{aligned}$$

where C is a constant. **(a)** Determine the polarization of the vector $\mathbf{F}_1 + \mathbf{F}_2$ for $C = 2$. **(b)** Find the value(s) of C for which the tip of the vector $\mathbf{F}_1 + \mathbf{F}_2$ traces a circle with time.

P3.32. **Unit vectors along the hour and minute hands of an analog watch.** Consider an analog watch that keeps accurate time and assume the origin to be at the center of the dial, the x-axis passing through the 12 mark, and the y-axis passing through the 3 mark. **(a)** Write the expression for the time-varying unit vector directed along the hour hand of the watch. **(b)** Write the expression for the time-varying unit vector directed along the minute hand of the watch. **(c)** Obtain the specific expression for these unit vectors when the hour hand and the minute hand are aligned exactly and between the 5 and 6 marks.

P3.33. **Field expressions for sinusoidal uniform plane wave for specified characteristics.** Write the expressions for the electric- and magnetic-field intensities of a sinusoidally time-varying uniform plane wave propagating in free space and having the following characteristies: **(a)** $f = 100$ MHz; **(b)** direction of propagation is the $+z$-direction; and **(c)** polarization is right circular with the electric field in the $z = 0$ plane at $t = 0$ having an x-component equal to E_0 and a y-component equal to $0.75E_0$.

P3.34. Determination of sense of polarization for several cases of sinusoidal traveling waves. For each of the following fields, determine if the polarization is right- or left-circular or elliptical.

(a) $E_0 \cos(\omega t - \beta y)\,\mathbf{a}_z + E_0 \sin(\omega t - \beta y)\,\mathbf{a}_x$

(b) $E_0 \cos(\omega t + \beta x)\,\mathbf{a}_y + E_0 \sin(\omega t + \beta x)\,\mathbf{a}_z$

(c) $E_0 \cos(\omega t + \beta y)\,\mathbf{a}_x - 2E_0 \sin(\omega t + \beta y)\,\mathbf{a}_z$

(d) $E_0 \cos(\omega t - \beta x)\,\mathbf{a}_z - E_0 \sin(\omega t - \beta x + \pi/4)\,\mathbf{a}_y$

P3.35. Uniform plane-wave field in terms of right and left circularly polarized components. Express each of the following uniform plane wave electric fields as a superposition of right- and left-circularly polarized fields:

(a) $E_0\mathbf{a}_x \cos(\omega t + \beta z)$

(b) $E_0\mathbf{a}_x \cos(\omega t - \beta z + \pi/3) - E_0\mathbf{a}_y \cos(\omega t - \beta z + \pi/6)$

Section 3.7.

P3.36. Instantaneous and time-average Poynting vectors for specified electric fields. For each of the following electric-field intensities for a uniform plane wave in free space, find the instantaneous and time-average Poynting vectors:

(a) $\mathbf{E} = E_0 \cos(\omega t - \beta z)\,\mathbf{a}_x + 2E_0 \cos(\omega t - \beta z)\,\mathbf{a}_y$

(b) $\mathbf{E} = E_0 \cos(\omega t - \beta z)\,\mathbf{a}_x - E_0 \sin(\omega t - \beta z)\,\mathbf{a}_y$

(c) $\mathbf{E} = E_0 \cos(\omega t - \beta z)\,\mathbf{a}_x + 2E_0 \sin(\omega t - \beta z)\,\mathbf{a}_y$

P3.37. Poynting vector and power flow for a coaxial cable. The electric and magnetic fields in a coaxial cable, an arrangement of two coaxial perfectly conducting cylinders of radii a and b $(>a)$, are given by

$$\mathbf{E} = \frac{V_0}{r \ln(b/a)} \cos \omega(t - z\sqrt{\mu_0\varepsilon_0})\,\mathbf{a}_r \qquad \text{for} \quad a < r < b$$

$$\mathbf{H} = \frac{I_0}{2\pi r} \cos \omega(t - z\sqrt{\mu_0\varepsilon_0})\,\mathbf{a}_\theta \qquad \text{for} \quad a < r < b$$

where V_0 and I_0 are constants and the axis of the cylinders is the z-axis. **(a)** Find the instantaneous and time-average Poynting vectors associated with the fields. **(b)** Find the time-average power flow along the coaxial cable.

P3.38. Power radiated for specified radiation fields of an antenna. The electric- and magnetic-field intensities in the radiation field of an antenna located at the origin are given in spherical coordinates by

$$\mathbf{E} = E_0 \frac{\sin\theta}{r} \cos \omega(t - r\sqrt{\mu_0\varepsilon_0})\,\mathbf{a}_\theta \text{ V/m}$$

$$\mathbf{H} = \frac{E_0}{\sqrt{\mu_0/\varepsilon_0}} \frac{\sin\theta}{r} \cos \omega(t - r\sqrt{\mu_0\varepsilon_0})\,\mathbf{a}_\theta \text{ A/m}$$

Find: **(a)** the instantaneous Poynting vector; **(b)** the instantaneous power radiated by the antenna by evaluating the surface integral of the instantaneous Poynting vector over a spherical surface of radius r centered at the antenna and enclosing the antenna; and **(c)** the time-average power radiated by the antenna.

P3.39. Energy storage associated with a charge distribution. A volume charge distribution is given in spherical coordinates by

$$\rho = \begin{cases} \rho_0(r/a)^2 & \text{for} \quad r < a \\ 0 & \text{for} \quad r > a \end{cases}$$

(a) Find the energy stored in the electric field of the charge distribution. **(b)** Find the work required to rearrange the charge distribution with uniform density in the region $r < a$.

P3.40. Energy storage associated with a current distribution. A current distribution is given in cylindrical coordinates by

$$\mathbf{J} = \begin{cases} J_0\mathbf{a}_z & \text{for} \quad r < 30a \\ -J_0\mathbf{a}_z & \text{for} \quad 4a < r < 5a \end{cases}$$

Find the energy stored in the magnetic field of the current distribution per unit length in the z-direction.

REVIEW PROBLEMS

R3.1. Satisfaction of Maxwell's curl equations for a specified electric field. Find the numerical value(s) of k, if any, such that the electric field in free space ($\mathbf{J} = \mathbf{0}$) given by

$$\mathbf{E} = E_0 \sin 6x \sin (3 \times 10^9 t - kz)\, \mathbf{a}_y$$

satisfies both of Maxwell's curl equations.

R3.2. Satisfaction of Maxwell's curl equations for fields in a rectangular cavity resonator. The rectangular cavity resonator is a box comprising the region $0 < x < a$, $0 < y < b$, and $0 < z < d$, and bounded by metallic walls on all of its six sides. The time-varying electric and magnetic fields inside the resonator are given by

$$\mathbf{E} = E_0 \sin \frac{\pi x}{a} \sin \frac{\pi z}{d} \cos \omega t\, \mathbf{a}_y$$
$$\mathbf{H} = H_{01} \sin \frac{\pi x}{a} \cos \frac{\pi z}{d} \sin \omega t\, \mathbf{a}_x - H_{02} \cos \frac{\pi x}{a} \sin \frac{\pi z}{d} \sin \omega t\, \mathbf{a}_z$$

where E_0, H_{01}, and H_{02} are constants and ω is the radian frequency of oscillation. Find the value of ω that satisfies both of Maxwell's curl equations. The medium inside the resonator is free space.

R3.3. Electric field of a charge distribution from Gauss' law in differential form. The x-variation of charge density independent of y and z is shown in Fig. 3.38. Find and sketch the resulting D_x versus x.

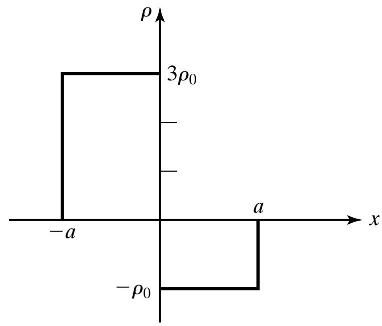

FIGURE 3.38
For Problem R3.3

R3.4. Classification of static vector fields. With respect to the properties of physical realizability as electric and magnetic fields, static vector fields can be classified into four groups: **(i)** electric field only; **(ii)** magnetic field only; **(iii)** electric field in a charge-free region or a magnetic field in a current-free region; and **(iv)** none of the preceding three. For each of the following fields, determine the group to which it belongs:

(a) $x\mathbf{a}_x + y\mathbf{a}_y$

(b) $(x^2 - y^2)\mathbf{a}_x - 2xy\mathbf{a}_y + 4\mathbf{a}_z$

(c) $\dfrac{e^{-r}}{r}\mathbf{a}_\phi$ in cylindrical coordinates

(d) $\dfrac{1}{r}(\cos\phi\,\mathbf{a}_r + \sin\phi\,\mathbf{a}_\phi)$ in cylindrical coordinates

R3.5. Finding traveling-wave functions from specified sums of the functions. Figures 3.39(a) and (b) show the distance variations at $t = 0$ and $t = 1$ s, respectively, of the sum of two functions $f(z, t)$ and $g(z, t)$, each of duration not exceeding 3 s, and representing traveling waves propagating in the $+z$- and $-z$-directions, respectively, with velocity 100 m/s. Find and sketch f and g versus t for $z = 0$.

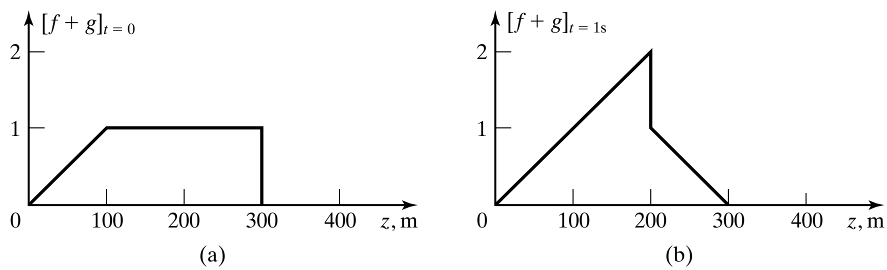

FIGURE 3.39
For Problem R3.5

R3.6. Plotting more field variations from a given field variation of a uniform plane wave. The time-variation of the electric field E_x in the $z = 600$ m plane of a uniform plane wave propagating away from an infinite plane current sheet lying in the $z = 0$ plane is given by the periodic function shown in Fig. 3.40. Find and sketch the following: (a) E_x versus t in the $z = 200$ m plane; (b) E_x versus z for $t = 0$; and (c) H_y versus z for $t = 1/3\ \mu$s.

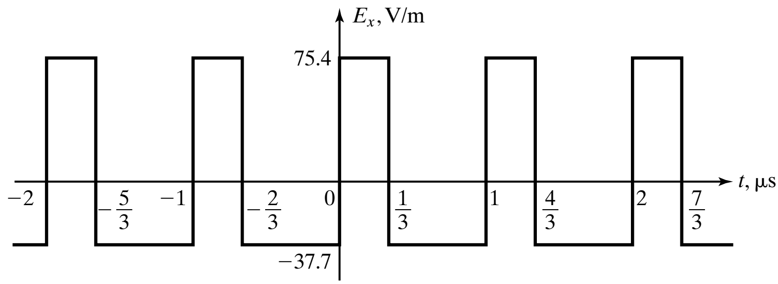

FIGURE 3.40

For Problem R3.6.

R3.7. An array of two infinite plane current sheets. For the array of two infinite plane current sheets of Example 3.12, assume that

$$\mathbf{J}_{S2} = -J_{S0} \sin(\omega t + \alpha)\, \mathbf{a}_x \text{ for } z = \lambda/4$$

Obtain the expression for the ratio of the amplitude of the electric field in the region $z > \lambda/4$ to the amplitude of the electric field in the region $z < 0$. Then find the following: **(a)** the value of the ratio for $\alpha = \pi/4$; and **(b)** the value of α for $0 < \alpha < \pi/2$, for the ratio to be equal to 2.

R3.8. A superposition of two infinite plane current sheets. Given $\mathbf{J}_{S1} = 0.2 \cos 6\pi \times 10^8 t\, \mathbf{a}_x$ A/m in the $y = 0$ plane and $\mathbf{J}_{S2} = 0.2 \cos 6\pi \times 10^8 t\, \mathbf{a}_z$ A/m in the $y = 0.25$ m plane, find **E** and **H** in the two regions $y < 0$ and $y > 0.25$ m. Discuss the polarizations of the fields and the time-average power flow in both regions. Note that the two current densities are directed perpendicular to each other.

R3.9. Elliptical polarization. The components of a sinusoidally time-varying vector field are given at a point by

$$F_x = 1 \cos \omega t$$
$$F_y = 1 \cos(\omega t + 60°)$$

Show that the field is elliptically polarized in the xy-plane with the equation of the ellipse given by $x^2 - xy + y^2 = 3/4$. Further show that the axial ratio (ratio of the major axis to the minor axis) of the ellipse is $\sqrt{3}$ and the tilt angle (angle made by the major axis with the x-axis) is 45°.

R3.10. Work associated with rearranging a charge distribution. Find the amount of work required for rearranging a uniformly distributed surface charge Q of radius a into a uniformly distributed volume charge of radius a.