C H A P T E R 2

Maxwell's Equations in Integral Form

In Chapter 1, we learned the simple rules of vector algebra and familiarized ourselves with the basic concepts of fields in general, and then introduced electric and magnetic fields in terms of forces on charged particles. We now have the necessary background to introduce the additional tools required for the understanding of the various quantities associated with Maxwell's equations and then discuss Maxwell's equations. In particular, our goal in this chapter is to learn Maxwell's equations in integral form as a prerequisite to the derivation of their differential forms in the next chapter. Maxwell's equations in integral form govern the interdependence of certain field and source quantities associated with regions in space, that is, contours, surfaces, and volumes. The differential forms of Maxwell's equations, however, relate the characteristics of the field vectors at a given point to one another and to the source densities at that point.

Maxwell's equations in integral form are a set of four laws resulting from several experimental findings and a purely mathematical contribution. We shall, however, consider them as postulates and learn to understand their physical significance as well as their mathematical formulation. The source quantities involved in their formulation are charges and currents. The field quantities have to do with the line and surface integrals of the electric and magnetic field vectors. We shall therefore first introduce line and surface integrals and then consider successively the four Maxwell's equations in integral form.

2.1 THE LINE INTEGRAL

Line integral

To introduce the line integral, let us consider in a region of electric field **E** the movement of a test charge q from the point A to the point B along the path C as shown in Fig. 2.1(a). At each and every point along the path the electric field exerts a force on the test charge and, hence, does a certain amount of work in

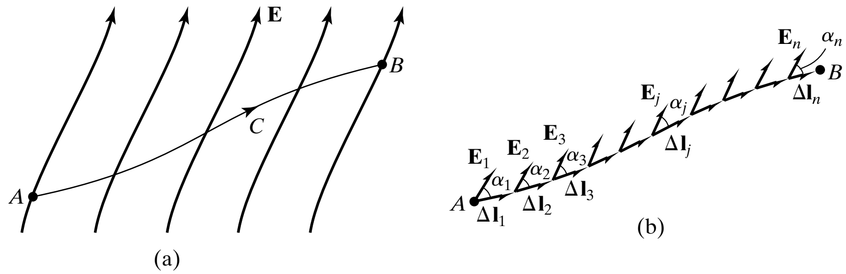

FIGURE 2.1

For evaluating the total amount of work done in moving a test charge along a path C from point A to point B in a region of electric field.

moving the charge to another point an infinitesimal distance away. To find the total amount of work done from A to B, we divide the path into a number of infinitesimal segments $\Delta\mathbf{l}_1, \Delta\mathbf{l}_2, \Delta\mathbf{l}_3, \ldots \Delta\mathbf{l}_n$, as shown in Fig. 2.1(b), find the infinitesimal amount of work done for each segment, and then add up the contributions from all the segments. Since the segments are infinitesimal in length, we can consider each of them to be straight and the electric field at all points within a segment to be the same and equal to its value at the start of the segment.

If we now consider one segment, say, the jth segment, and take the component of the electric field for that segment along the length of that segment, we obtain the result $E_j \cos \alpha_j$, where α_j is the angle between the direction of the electric field vector $\mathbf{E}_j$ at the start of that segment and the direction of that segment. Since the electric field intensity has the meaning of force per unit charge, the electric force along the direction of the jth segment is then equal to $qE_j \cos \alpha_j$. To obtain the work done in carrying the test charge along the length of the jth segment, we then multiply this electric force component by the length Δl_j of that segment. Thus, for the jth segment, we obtain the result for the work done by the electric field as

$$\Delta W_j = qE_j \cos \alpha_j \, \Delta l_j \tag{2.1}$$

If we do this for all the infinitesimal segments and add up all the contributions, we get the total work done by the electric field in moving the test charge from A to B along the path C to be

$$\begin{aligned} W_{AB} &= \Delta W_1 + \Delta W_2 + \Delta W_3 + \cdots + \Delta W_n \\ &= qE_1 \cos \alpha_1 \, \Delta l_1 + qE_2 \cos \alpha_2 \, \Delta l_2 + qE_3 \cos \alpha_3 \, \Delta l_3 + \cdots \\ &\quad + qE_n \cos \alpha_n \, \Delta l_n \\ &= q \sum_{j=1}^{n} E_j \cos \alpha_j \, \Delta l_j \\ &= q \sum_{j=1}^{n} (E_j)(\Delta l_j) \cos \alpha_j \end{aligned} \tag{2.2}$$

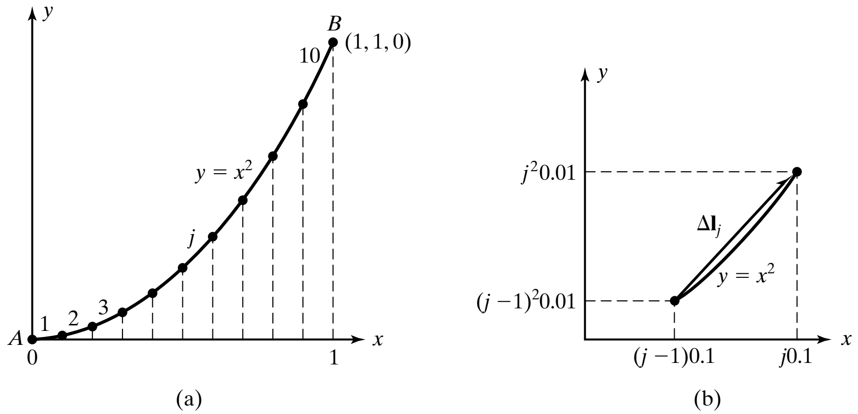

FIGURE 2.2

(a) Division of the path $y = x^2$ from $A(0, 0, 0)$ to $B(1, 1, 0)$ into 10 segments. (b) Length vector corresponding to the jth segment of part (a) approximated as a straight line.

Using the dot product operation between two vectors, we obtain

$$\boxed{W_{AB} = q\sum_{j=1}^{n} \mathbf{E}_j \cdot \Delta\mathbf{l}_j} \tag{2.3}$$

For a numerical example, let us consider the electric field given by

$$\mathbf{E} = y\mathbf{a}_y$$

and determine the work done by the field in the movement of 3 μC of charge from the point $A(0, 0, 0)$ to the point $B(1, 1, 0)$ along the parabolic path $y = x^2$, $z = 0$ shown in Fig. 2.2(a).

For convenience, we shall divide the path into 10 segments having equal projections along the x-axis, as shown in Fig. 2.2(a). We shall number the segments $1, 2, 3, \ldots, 10$. The coordinates of the starting and ending points of the jth segment are as shown in Fig. 2.2(b). The electric field at the start of the jth segment is given by

$$\mathbf{E}_j = (j - 1)^2 0.01\mathbf{a}_y$$

The length vector corresponding to the jth segment, approximated as a straight line connecting its starting and ending points, is

$$\Delta\mathbf{l}_j = 0.1\mathbf{a}_x + [j^2 - (j - 1)^2]0.01\mathbf{a}_y = 0.1\mathbf{a}_x + (2j - 1)0.01\mathbf{a}_y$$

The required work is then given by

$$\begin{aligned} W_{AB} &= 3 \times 10^{-6}\sum_{j=1}^{10} \mathbf{E}_j \cdot \Delta\mathbf{l}_j \\ &= 3 \times 10^{-6}\sum_{j=1}^{10} [(j - 1)^2 0.01\mathbf{a}_y] \cdot [0.1\mathbf{a}_x + (2j - 1)0.01\mathbf{a}_y] \end{aligned}$$

$$
\begin{aligned}
&= 3 \times 10^{-10} \sum_{j=1}^{10} (j-1)^2(2j-1) \\
&= 3 \times 10^{-10}[0 + 3 + 20 + 63 + 144 + 275 + 468 + 735 \\
&\quad + 1088 + 1539] \\
&= 3 \times 10^{-10} \times 4335 \text{ J} = 1.3005\ \mu\text{J}
\end{aligned}
$$

The result that we have just obtained for W_{AB} is approximate, since we divided the path from A to B into a finite number of segments. By dividing it into larger and larger numbers of segments, we can obtain more and more accurate results. In the limit that $n \to \infty$, the result converges to the exact value. The summation in (2.3) then becomes an integral, which represents exactly the work done by the field and is given by

$$
\boxed{W_{AB} = q \int_A^B \mathbf{E} \cdot d\mathbf{l}} \tag{2.4}
$$

The integral on the right side of (2.4) is known as the *line integral of* **E** *from A to B*, along the specified path.

Evaluation of line integral

We shall illustrate the evaluation of the line integral by computing the exact value of the work done by the electric field in the movement of the 3 μC charge for the path in Fig. 2.2(a). To do this, we note that at any arbitrary point on the curve $y = x^2$, $z = 0$,

$$
dy = 2x\, dx \qquad dz = 0
$$

so that the differential length vector tangential to the curve is given by

$$
\begin{aligned}
d\mathbf{l} &= dx\, \mathbf{a}_x + dy\, \mathbf{a}_y + dz\, \mathbf{a}_z \\
&= dx\, \mathbf{a}_x + 2x\, dx\, \mathbf{a}_y
\end{aligned}
$$

The value of $\mathbf{E} \cdot d\mathbf{l}$ at the point is

$$
\begin{aligned}
\mathbf{E} \cdot d\mathbf{l} &= y\mathbf{a}_y \cdot (dx\, \mathbf{a}_x + 2x\, dx\, \mathbf{a}_y) \\
&= x^2\mathbf{a}_y \cdot (dx\, \mathbf{a}_x + 2x\, dx\, \mathbf{a}_y) \\
&= 2x^3\, dx
\end{aligned}
$$

Thus, the required work is given by

$$
\begin{aligned}
W_{AB} &= q \int_{(0,0,0)}^{(1,1,0)} \mathbf{E} \cdot d\mathbf{l} = 3 \times 10^{-6} \int_0^1 2x^3\, dx \\
&= 3 \times 10^{-6} \left[\frac{2x^4}{4}\right]_0^1 = 1.5\ \mu\text{J}
\end{aligned}
$$

Note that we have evaluated the line integral by using x as the variable of integration. Alternatively, using y as the variable of integration, we obtain

$$\begin{aligned}
\mathbf{E} \cdot d\mathbf{l} &= y\mathbf{a}_y \cdot (dx\,\mathbf{a}_x + dy\,\mathbf{a}_y) \\
&= y\,dy \\
W_{AB} &= q\int_{(0,0,0)}^{(1,1,0)} \mathbf{E} \cdot d\mathbf{l} = 3 \times 10^{-6}\int_0^1 y\,dy \\
&= 3 \times 10^{-6}\left[\frac{y^2}{2}\right]_0^1 = 1.5\ \mu\text{J}
\end{aligned}$$

Thus, the integration can be performed with respect to x or y (or z in the three-dimensional case). What is important, however, is that the integrand must be expressed as a function of the variable of integration and the limits appropriate to that variable must be employed.

Voltage defined

Returning now to (2.4) and dividing both sides by q, we note that the line integral of $\mathbf{E}$ from A to B has the physical meaning of work per unit charge done by the field in moving the test charge from A to B. This quantity is known as the *voltage between A and B* along the specified path and is denoted by the symbol V_{AB}, having the units of volts. Thus,

$$\boxed{V_{AB} = \int_A^B \mathbf{E} \cdot d\mathbf{l}} \tag{2.5}$$

When the path under consideration is a closed path, that is, one that has no beginning or ending, such as a rubber band, as shown in Fig. 2.3, the line integral is written with a circle associated with the integral sign in the manner $\oint_C \mathbf{E} \cdot d\mathbf{l}$. The line integral of a vector around a closed path is known as the *circulation* of that vector. In particular, the line integral of $\mathbf{E}$ around a closed path is the work per unit charge done by the field in moving a test charge around the closed path. It is the voltage around the closed path and is also known as the *electromotive force*. We shall now consider an example of evaluating the line integral of a vector around a closed path.

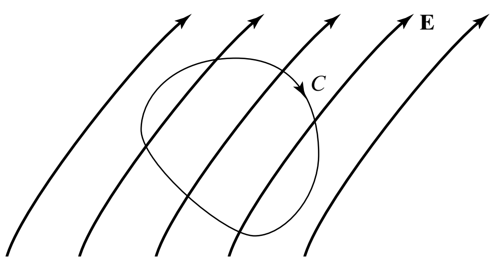

FIGURE 2.3

Closed path C in a region of electric field.

Example 2.1 Evaluation of line integral around a closed path

Let us consider the force field

$$\mathbf{F} = x\mathbf{a}_y$$

and evaluate $\oint_C \mathbf{F} \cdot d\mathbf{l}$, where C is the closed path $ABCDA$ shown in Fig. 2.4.

Noting that

$$\oint_{ABCDA} \mathbf{F} \cdot d\mathbf{l} = \int_A^B \mathbf{F} \cdot d\mathbf{l} + \int_B^C \mathbf{F} \cdot d\mathbf{l} + \int_C^D \mathbf{F} \cdot d\mathbf{l} + \int_D^A \mathbf{F} \cdot d\mathbf{l}$$

we simply evaluate each of the line integrals on the right side and add them up to obtain the required quantity.

First, we observe that since the entire closed path lies in the $z = 0$ plane, $dz = 0$ and $d\mathbf{l} = dx\,\mathbf{a}_x + dy\,\mathbf{a}_y$ for all four straight lines. Then for the side AB,

$$y = 1, \qquad dy = 0, \qquad d\mathbf{l} = dx\,\mathbf{a}_x + (0)\mathbf{a}_y = dx\,\mathbf{a}_x$$

$$\mathbf{F} \cdot d\mathbf{l} = (x\mathbf{a}_y) \cdot (dx\,\mathbf{a}_x) = 0$$

$$\int_A^B \mathbf{F} \cdot d\mathbf{l} = 0$$

For the side BC,

$$x = 3, \qquad dx = 0, \qquad d\mathbf{l} = (0)\mathbf{a}_x + dy\,\mathbf{a}_y = dy\,\mathbf{a}_y$$

$$\mathbf{F} \cdot d\mathbf{l} = (3\mathbf{a}_y) \cdot (dy\,\mathbf{a}_y) = 3\,dy$$

$$\int_B^C \mathbf{F} \cdot d\mathbf{l} = \int_1^5 3\,dy = 12$$

For the side CD,

$$y = 2 + x, \qquad dy = dx, \qquad d\mathbf{l} = dx\,\mathbf{a}_x + dx\,\mathbf{a}_y$$

$$\mathbf{F} \cdot d\mathbf{l} = (x\mathbf{a}_y) \cdot (dx\,\mathbf{a}_x + dx\,\mathbf{a}_y) = x\,dx$$

$$\int_C^D \mathbf{F} \cdot d\mathbf{l} = \int_3^1 x\,dx = -4$$

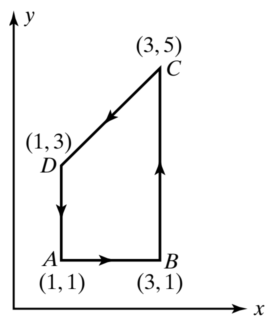

FIGURE 2.4

For evaluating the line integral of a vector field around a closed path.

For the side DA,

$$x = 1, \qquad dx = 0, \qquad d\mathbf{l} = (0)\mathbf{a}_x + dy\,\mathbf{a}_y$$
$$\mathbf{F}\cdot d\mathbf{l} = (\mathbf{a}_y)\cdot(dy\,\mathbf{a}_y) = dy$$
$$\int_D^A \mathbf{F}\cdot d\mathbf{l} = \int_3^1 dy = -2$$

Finally,

$$\oint_{ABCDA} \mathbf{F}\cdot d\mathbf{l} = 0 + 12 - 4 - 2 = 6$$

Conservative vs. nonconservative fields

In this example, we found that the line integral of $\mathbf{F}$ around the closed path C is nonzero. The field is then said to be a *nonconservative field*. For a nonconservative field, the line integral between two points, say, A and B, is dependent on the path followed from A to B. To show this, let us consider the two paths ACB and ADB, as shown in Fig. 2.5. Then we can write

$$\begin{aligned}\oint_{ACBDA} \mathbf{F}\cdot d\mathbf{l} &= \int_{ACB} \mathbf{F}\cdot d\mathbf{l} + \int_{BDA} \mathbf{F}\cdot d\mathbf{l} \\ &= \int_{ACB} \mathbf{F}\cdot d\mathbf{l} - \int_{ADB} \mathbf{F}\cdot d\mathbf{l}\end{aligned} \tag{2.6}$$

It can be easily seen that if $\oint_{ACBDA}\mathbf{F}\cdot d\mathbf{l}$ is not equal to zero, then $\int_{ACB}\mathbf{F}\cdot d\mathbf{l}$ is not equal to $\int_{ADB}\mathbf{F}\cdot d\mathbf{l}$. The two integrals are equal only if $\oint_{ACBDA}\mathbf{F}\cdot d\mathbf{l}$ is equal to zero, which is the case for *conservative fields*. Examples of conservative fields are Earth's gravitational field and the static electric field. An example of nonconservative fields is the time-varying electric field. Thus, in a time-varying electric field, the voltage between two points A and B is dependent on the path followed to evaluate the line integral of $\mathbf{E}$ from A to B, whereas in a static electric field, the voltage, more commonly known as the *potential difference,* between two points A and B is uniquely defined because the line integral of $\mathbf{E}$ from A to B is independent of the path followed from A to B.

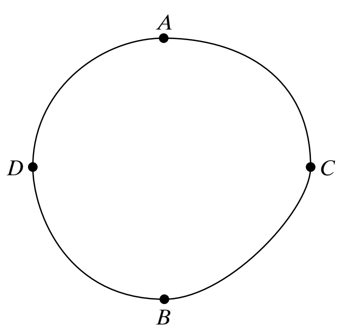

FIGURE 2.5

Two different paths from point A to point B.

K2.1. Line integral; Line integral of **E**; Voltage; Line integral around a closed path; Circulation; Line integral of **E** around a closed path; Electromotive force; Conservative vs. nonconservative fields.

D2.1. For each of the curves **(a)** $y = x^2, z = 0$, **(b)** $x^2 + y^2 = 2, z = 0$, and **(c)** $y = \sin 0.5\pi x, z = 0$ in a region of electric field $\mathbf{E} = y\mathbf{a}_x + x\mathbf{a}_y$, find the approximate value of the work done by the field in carrying a charge of 1 μC from the point (1, 1, 0) to the neighboring point on the curve, whose x coordinate is 1.1, by evaluating $\mathbf{E} \cdot \Delta\mathbf{l}$ along a straight line path.

Ans. **(a)** 0.31 μJ; **(b)** -0.0112 μJ; **(c)** 0.0877 μJ.

D2.2. For $\mathbf{F} = y(\mathbf{a}_x + \mathbf{a}_y)$, find $\int \mathbf{F} \cdot d\mathbf{l}$ for the straight-line paths between the following pairs of points from the first point to the second point: **(a)** (0, 0, 0) to (2, 0, 0); **(b)** (0, 2, 0) to (2, 2, 0); and **(c)** (2, 0, 0) to (2, 2, 0).

Ans. **(a)** 0; **(b)** 4; **(c)** 2.

2.2 THE SURFACE INTEGRAL

Surface integral

To introduce the surface integral, let us consider a region of magnetic field and an infinitesimal surface at a point in that region. Since the surface is infinitesimal, we can assume the magnetic flux density to be uniform on the surface, although it may be nonuniform over a wider region. If the surface is oriented normal to the magnetic field lines, as shown in Fig. 2.6(a), then the magnetic flux (webers) crossing the surface is simply given by the product of the surface area (meters squared) and the magnetic flux density (Wb/m^2) on the surface, that is, $B\,\Delta S$. If, however, the surface is oriented parallel to the magnetic field lines, as shown in Fig. 2.6(b), there is no magnetic flux crossing the surface. If the surface is oriented in such a manner that the normal to the surface makes an angle α with the magnetic field lines as shown in Fig. 2.6(c), then the amount of magnetic flux crossing the surface can be determined by considering that the

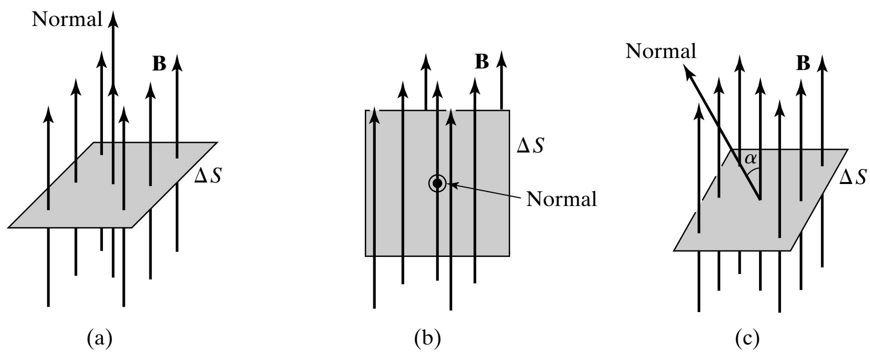

FIGURE 2.6

Infinitesimal surface ΔS in a magnetic field **B** oriented (a) normal to the field. (b) parallel to the field, and (c) with its normal making an angle a to the field.

component of **B** normal to the surface is $B \cos \alpha$ and the component tangential to the surface is $B \sin \alpha$. The component of **B** normal to the surface results in a flux of $(B \cos \alpha)\ \Delta S$ crossing the surface, whereas the component tangential to the surface does not contribute at all to the flux crossing the surface. Thus, the magnetic flux crossing the surface in this case is $(B \cos \alpha)\ \Delta S$. We can obtain this result alternatively by noting that the projection of the surface onto the plane normal to the magnetic field lines is $\Delta S \cos \alpha$. Hence, the magnetic flux crossing the surface ΔS is the same as that crossing normal to the area $\Delta S \cos \alpha$, that is, $B(\Delta S \cos \alpha)$ or $(B \cos \alpha)\ \Delta S$.

To aid further in the understanding of this concept, imagine raindrops falling vertically downward uniformly. If you hold a rectangular loop horizontally, the number of drops falling through the loop is simply equal to the area of the loop multipled by the density (number of drops per unit area) of the drops. If the loop is held vertically, no rain falls through the loop. If the loop is held at some angle to the horizontal, the number of drops falling through the loop is the same as that which would fall through another (smaller) loop, which is the projection of the slanted loop on to the horizontal plane.

Let us now consider a large surface S in the magnetic field region, as shown in Fig. 2.7. The magnetic flux crossing this surface can be found by dividing the surface into a number of infinitesimal surfaces $\Delta S_1, \Delta S_2, \Delta S_3, \ldots, \Delta S_n$, applying the result just obtained for each infinitesimal surface, and adding up the contributions from all the surfaces. To obtain the contribution from the jth surface, we draw the normal vector to that surface and find the angle α_j between the normal vector and the magnetic flux density vector $\mathbf{B}_j$ associated with that surface. Since the surface is infinitesimal, we can assume $\mathbf{B}_j$ to be the value of **B** at the centroid of the surface, and we can also erect the normal vector at that point. The contribution to the total magnetic flux from the jth infinitesimal surface is then given by

$$\Delta\psi_j = B_j \cos \alpha_j\ \Delta S_j \tag{2.7}$$

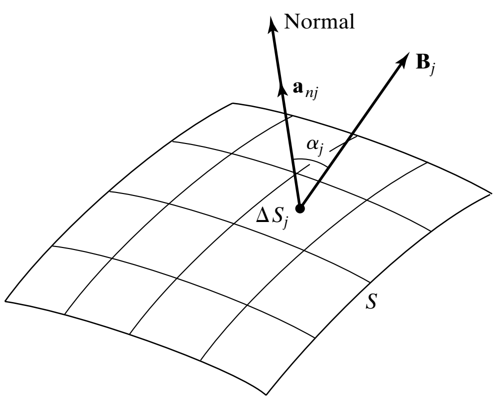

FIGURE 2.7

Division of a large surface S in a magnetic field region into a number of infinitesimal surfaces.

where the symbol ψ represents magnetic flux. The total magnetic flux crossing the surface S is then given by

$$\begin{aligned}[\psi]_S &= \Delta\psi_1 + \Delta\psi_2 + \Delta\psi_3 + \cdots + \Delta\psi_n \\ &= B_1 \cos\alpha_1\,\Delta S_1 + B_2\cos\alpha_2\,\Delta S_2 + B_3\cos\alpha_3\,\Delta S_3 + \cdots \\ &\quad + B_n\cos\alpha_n\,\Delta S_n \\ &= \sum_{j=1}^{n} B_j \cos\alpha_j\,\Delta S_j \\ &= \sum_{j=1}^{n} B_j(\Delta S_j)\cos\alpha_j\end{aligned} \tag{2.8}$$

Using the dot product operation between two vectors, we obtain

$$[\psi]_S = \sum_{j=1}^{n} \mathbf{B}_j \cdot \Delta S_j\,\mathbf{a}_{nj} \tag{2.9}$$

where $\mathbf{a}_{nj}$ is the unit vector normal to the surface ΔS_j. Furthermore, by using the concept of an infinitesimal surface vector as one having magnitude equal to the area of the surface and direction normal to the surface, that is,

$$\Delta\mathbf{S}_j = \Delta S_j\,\mathbf{a}_{nj} \tag{2.10}$$

we can write (2.9) as

$$\boxed{[\psi]_S = \sum_{j=1}^{n} \mathbf{B}_j \cdot \Delta\mathbf{S}_j} \tag{2.11}$$

For a numerical example, let us consider the magnetic field given by

$$\mathbf{B} = 3xy^2\mathbf{a}_z \text{ Wb/m}^2$$

and determine the magnetic flux crossing the portion of the xy-plane lying between $x = 0$, $x = 1$, $y = 0$, and $y = 1$. For convenience, we shall divide the surface into 25 equal areas, as shown in Fig. 2.8 (a). We shall designate the squares as 11, 12, . . . , 15, 21, 22, . . . , 55, where the first digit represents the number of the square in the x-direction and the second digit represents the number of the square in the y-direction. The x- and y-coordinates of the midpoint of the ijth square are $(2i - 1)0.1$ and $(2j - 1)0.1$, respectively, as shown in Fig. 2.8(b). The magnetic field at the center of the ijth square is then given by

$$\mathbf{B}_{ij} = 3(2i - 1)(2j - 1)^2 0.001\mathbf{a}_z$$

Since we have divided the surface into equal areas and since all areas are in the xy-plane,

$$\Delta\mathbf{S}_{ij} = 0.04\,\mathbf{a}_z \qquad \text{for all } i \text{ and } j$$

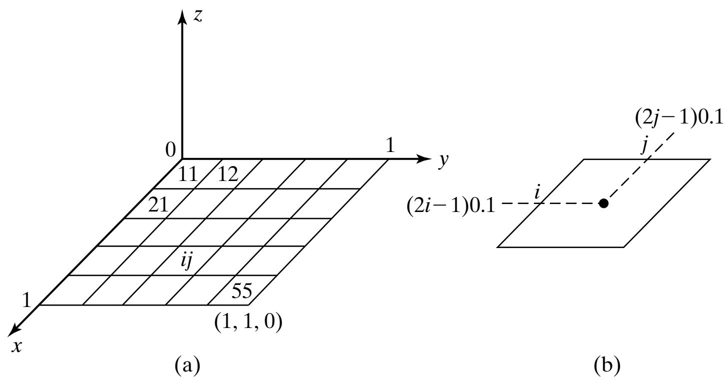

FIGURE 2.8

(a) Division of the portion of the xy-plane lying between $x = 0$, $x = 1$, $y = 0$, and $y = 1$ into 25 squares. (b) Area corresponding to the ijth square.

The required magnetic flux is then given by

$$\begin{aligned}
[\psi]_S &= \sum_{i=1}^{5}\sum_{j=1}^{5} \mathbf{B}_{ij} \cdot \Delta\mathbf{S}_{ij} \\
&= \sum_{i=1}^{5}\sum_{j=1}^{5} 3(2i - 1)(2j - 1)^2 0.001\mathbf{a}_z \cdot 0.04\mathbf{a}_z \\
&= 0.00012\sum_{i=1}^{5}\sum_{j=1}^{5}(2i - 1)(2j - 1)^2 \\
&= 0.00012(1 + 3 + 5 + 7 + 9)(1 + 9 + 25 + 49 + 81) \\
&= 0.495 \text{ Wb}
\end{aligned}$$

The result that we have just obtained for $[\psi]_S$ is approximate since we have divided the surface S into a finite number of areas. By dividing it into larger and larger numbers of squares, we can obtain more and more accurate results. In the limit that $n \to \infty$, the result converges to the exact value. The summation in (2.11) then becomes an integral that represents exactly the magnetic flux crossing the surface and is given by

$$\boxed{[\psi]_S = \int_S \mathbf{B} \cdot d\mathbf{S}} \tag{2.12}$$

where the symbol S associated with the integral sign denotes that the integration is performed over the surface S. The integral on the right side of (2.12) is known as the *surface integral of* **B** *over S*. The surface integral is a double integral since dS is equal to the product of two differential lengths.

Evaluation of surface integral

We shall illustrate the evaluation of the surface integral by computing the exact value of the magnetic flux crossing the surface in Fig. 2.8(a). To do this, we note that at any arbitrary point on the surface, the differential surface vector is given by

$$d\mathbf{S} = dx\,dy\,\mathbf{a}_z$$

The value of $\mathbf{B} \cdot d\mathbf{S}$ at the point is

$$\begin{aligned}\mathbf{B} \cdot d\mathbf{S} &= 3xy^2\,\mathbf{a}_z \cdot dx\,dy\,\mathbf{a}_z \\ &= 3xy^2\,dx\,dy\end{aligned}$$

Thus, the required magnetic flux is given by

$$\begin{aligned}[\psi]_S &= \int_S \mathbf{B} \cdot d\mathbf{S} \\ &= \int_{x=0}^{1}\int_{y=0}^{1} 3xy^2\,dx\,dy = 0.5\text{ Wb}\end{aligned}$$

When the surface under consideration is a closed surface, the surface integral is written with a circle associated with the integral sign in the manner $\oint_S \mathbf{B} \cdot d\mathbf{S}$. A closed surface is one that encloses a volume. Hence, if you are anywhere in that volume, you can get out of it only by making a hole in the surface, and vice versa. A simple example is the surface of a balloon inflated and tied up at the mouth. The surface integral of $\mathbf{B}$ over the closed surface S is simply the magnetic flux *emanating* from the volume bounded by the surface. Thus, whenever a closed surface integral is evaluated, the unit vectors normal to the differential surfaces are chosen to be pointing out of the volume, so as to give the outward flux of the vector field, unless specified otherwise. We shall now consider an example of evaluating $\oint_S \mathbf{B} \cdot d\mathbf{S}$.

Example 2.2 Evaluation of a closed surface integral

Let us consider the magnetic field

$$\mathbf{B} = (x + 2)\mathbf{a}_x + (1 - 3y)\mathbf{a}_y + 2z\mathbf{a}_z$$

and evaluate $\oint_S \mathbf{B} \cdot d\mathbf{S}$, where S is the surface of the cubical box bounded by the planes

$$\begin{matrix} x = 0 & x = 1 \\ y = 0 & y = 1 \\ z = 0 & z = 1 \end{matrix}$$

as shown in Fig. 2.9.

Noting that

$$\begin{aligned}\oint_S \mathbf{B} \cdot d\mathbf{S} = &\int_{abcd} \mathbf{B} \cdot d\mathbf{S} + \int_{efgh} \mathbf{B} \cdot d\mathbf{S} + \int_{adhe} \mathbf{B} \cdot d\mathbf{S} + \int_{bcgf} \mathbf{B} \cdot d\mathbf{S} \\ &+ \int_{aefb} \mathbf{B} \cdot d\mathbf{S} + \int_{dhgc} \mathbf{B} \cdot d\mathbf{S}\end{aligned}$$

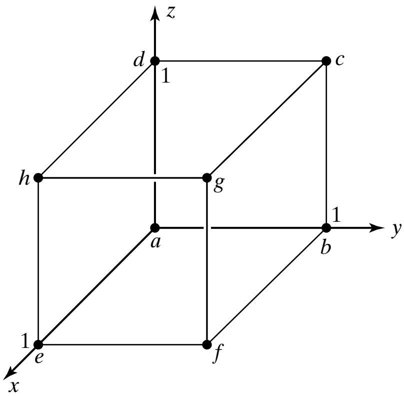

FIGURE 2.9

For evaluating the surface integral of a vector field over a closed surface.

we simply evaluate each of the surface integrals on the right side and add them up to obtain the required quantity. In doing so, we recognize that since the quantity we want is the magnetic flux out of the box, we should direct the unit normal vectors toward the outside of the box. Thus, for the surface *abcd*,

$$x = 0, \qquad \mathbf{B} = 2\mathbf{a}_x + (1 - 3y)\mathbf{a}_y + 2z\mathbf{a}_z, \qquad d\mathbf{S} = -dy\,dz\,\mathbf{a}_x$$

$$\mathbf{B} \cdot d\mathbf{S} = -2\,dy\,dz$$

$$\int_{abcd} \mathbf{B} \cdot d\mathbf{S} = \int_{z=0}^{1} \int_{y=0}^{1} (-2)\,dy\,dz = -2$$

For the surface *efgh*,

$$x = 1, \qquad \mathbf{B} = 3\mathbf{a}_x + (1 - 3y)\mathbf{a}_y + 2z\mathbf{a}_z, \qquad d\mathbf{S} = dy\,dz\,\mathbf{a}_x$$

$$\mathbf{B} \cdot d\mathbf{S} = 3\,dy\,dz$$

$$\int_{efgh} \mathbf{B} \cdot d\mathbf{S} = \int_{z=0}^{1} \int_{y=0}^{1} 3\,dy\,dz = 3$$

For the surface *adhe*,

$$y = 0, \qquad \mathbf{B} = (x + 2)\mathbf{a}_x + 1\mathbf{a}_y + 2z\mathbf{a}_z, \qquad d\mathbf{S} = -dz\,dx\,\mathbf{a}_y$$

$$\mathbf{B} \cdot d\mathbf{S} = -dz\,dx$$

$$\int_{aehd} \mathbf{B} \cdot d\mathbf{S} = \int_{x=0}^{1} \int_{z=0}^{1} (-1)\,dz\,dx = -1$$

For the surface *bcgf*,

$$y = 1, \qquad \mathbf{B} = (x + 2)\mathbf{a}_x - 2\mathbf{a}_y + 2z\mathbf{a}_z, \qquad d\mathbf{S} = dz\,dx\,\mathbf{a}_y$$

$$\mathbf{B} \cdot d\mathbf{S} = -2\,dz\,dx$$

$$\int_{bfgc} \mathbf{B} \cdot d\mathbf{S} = \int_{x=0}^{1} \int_{z=0}^{1} (-2)\,dz\,dx = -2$$

For the surface *aefb*,

$$z = 0, \qquad \mathbf{B} = (x + 2)\mathbf{a}_x + (1 - 3y)\mathbf{a}_y + 0\mathbf{a}_z, \qquad d\mathbf{S} = -dx\,dy\,\mathbf{a}_z$$

$$\mathbf{B}\cdot d\mathbf{S} = 0$$

$$\int_{aefb} \mathbf{B}\cdot d\mathbf{S} = 0$$

For the surface *dhgc*,

$$z = 1, \qquad \mathbf{B} = (x + 2)\mathbf{a}_x + (1 - 3y)\mathbf{a}_y + 2\mathbf{a}_z, \qquad d\mathbf{S} = dx\,dy\,\mathbf{a}_z$$

$$\mathbf{B}\cdot d\mathbf{S} = 2\,dx\,dy$$

$$\int_{dhgc} \mathbf{B}\cdot d\mathbf{S} = \int_{y=0}^{1}\int_{x=0}^{1} 2\,dx\,dy = 2$$

Finally,

$$\oint_S \mathbf{B}\cdot d\mathbf{S} = -2 + 3 - 1 - 2 + 0 + 2 = 0$$

K2.2. Surface integral; Surface integral of **B**; Magnetic flux; Surface integral over a closed surface.

D2.3. Given $\mathbf{B} = (y\mathbf{a}_x - x\mathbf{a}_y)$ Wb/m^2, find by evaluating $\mathbf{B}\cdot\Delta\mathbf{S}$ the approximate absolute value of the magnetic flux crossing from one side to the other side of an infinitesimal surface of area 0.001 m^2 at the point (1, 2, 1) for each of the following orientations of the surface: **(a)** in the $x = 1$ plane; **(b)** on the surface $2x^2 + y^2 = 6$; and **(c)** normal to the unit vector $\frac{1}{3}(2\mathbf{a}_x + \mathbf{a}_y + 2\mathbf{a}_z)$.

Ans. **(a)** 2×10^{-3} Wb; **(b)** $(1/\sqrt{2}) \times 10^{-3}$ Wb; **(c)** 10^{-3} Wb.

D2.4. For the vector field $\mathbf{A} = x(\mathbf{a}_x + \mathbf{a}_y)$, find the absolute value of $\int \mathbf{A}\cdot d\mathbf{S}$ over the following plane surfaces: **(a)** square having the vertices at (0, 0, 0), (0, 2, 0), (0, 2, 2), and (0, 0, 2); **(b)** square having the vertices at (2, 0, 0), (2, 2, 0), (2, 2, 2), and (2, 0, 2); **(c)** square having the vertices at (0, 0, 0), (2, 0, 0), (2, 0, 2), and (0, 0, 2); and **(d)** triangle having the vertices at (0, 0, 0), (2, 0, 0), and (0, 0, 2).

Ans. **(a)** 0; **(b)** 8; **(c)** 4; **(d)** $\frac{4}{3}$.

2.3 FARADAY'S LAW

In the preceding two sections, we introduced the line and surface integrals. We are now ready to consider Maxwell's equations in integral form. The first equation, which we shall discuss in this section, is a consequence of an experimental finding by Michael Faraday in 1831 that time-varying magnetic fields give rise to electric fields and, hence, it is known as *Faraday's law*. Faraday discovered that when the magnetic flux enclosed by a loop of wire changes with time, a current is produced in the loop, indicating that a voltage or an *electromotive force*, abbreviated as emf, is induced around the loop. The variation of the magnetic flux can result from the time variation of the magnetic flux enclosed by a fixed loop

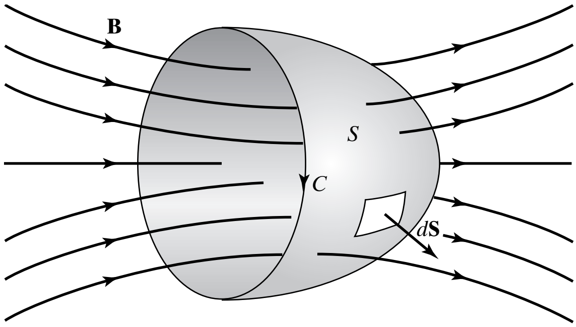

FIGURE 2.10

For illustrating Faraday's law.

or from a moving loop in a static magnetic field or from a combination of the two, that is, a moving loop in a time-varying magnetic field.

Statement of Faraday's law

In mathematical form, Faraday's law is given by

$$\boxed{\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S}} \tag{2.13}$$

where S is a surface bounded by the closed path C, as shown in Fig. 2.10. In words, Faraday's law states that *the electromotive force around a closed path is equal to the negative of the time rate of change of the magnetic flux enclosed by that path*. There are certain procedures and observations of interest pertinent to the application of (2.13). We shall discuss these next.

Right-hand screw rule

1. The magnetic flux on the right side is to be evaluated in accordance with the *right-hand screw rule* (R.H.S. rule), a convention that is applied consistently for all electromagnetic field laws involving integration over surfaces bounded by closed paths. The right-hand screw rule consists of imagining a right-hand screw being turned around the closed path, as illustrated in Fig. 2.11 for two opposing senses of paths, and using the resulting direction of advance of the screw to evaluate the surface integral. The application of this rule to the geometry of

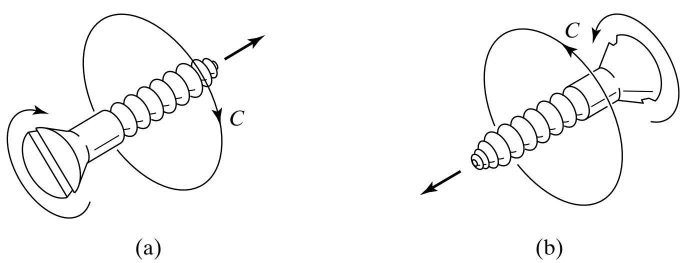

FIGURE 2.11

Right-hand screw rule convention employed in the formulation of electomagnetic field laws.

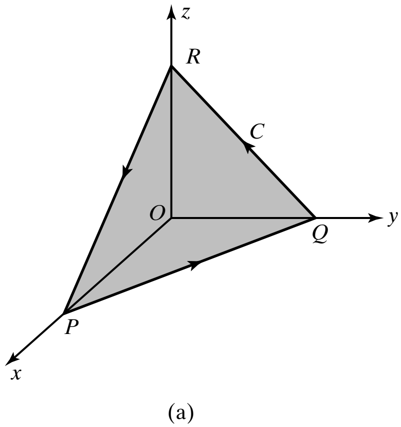

FIGURE 2.12

(a) A plane surface and (b) a combination of three plane surfaces, bounded by the closed path C.

Fig. 2.10 means that in evaluating the surface integral of **B** over S, the normal vector to the differential surface dS should be directed as shown in that figure.

2. In evaluating the surface integral of **B**, any surface *S bounded by C* can be employed. For example, if the loop C is a planar loop, it is not necessary to consider the plane surface having the loop as its perimeter. One can consider a curved surface bounded by C or any combination of plane (or plane and curved) surfaces which together are bounded by C, and which is sometimes a more desirable choice. To illustrate this point, consider the planar loop $PQRP$ in Fig. 2.12 (a). The most obvious surface bounded by this loop is the plane surface PQR inclined to the coordinate planes. Now imagine this plane surface to be an elastic sheet glued to the perimeter and pushed in toward the origin so as to conform to the coordinate planes. Then we obtain the combination of the plane surfaces OPQ, OQR, and ORP, as shown in Fig. 2.12(b), which together constitute a surface also bounded by the loop. To evaluate the surface integral of **B** for the surface in Fig. 2.12(a), we need to make use of the $d\mathbf{S}$ vector on that slant surface. On the other hand, for the geometry in Fig. 2.12(b), we can use the (simpler) $d\mathbf{S}$ vectors associated with the coordinate planes. The fact that any surface S bounded by a closed path C can be employed to evaluate the magnetic flux enclosed by C implies that the magnetic flux through all such surfaces is the same in order for the emf around C to be unique. As we shall learn in Section 2.4, it is a fundamental property of the magnetic field that the magnetic flux is the same through all surfaces bounded by a given closed path.

3. The closed path C on the left side need not represent a loop of wire, but can be an imaginary contour. It means that the time-varying magnetic flux induces an electric field in the region and this results in an emf around the closed path. If a wire is placed in the position occupied by the closed path, the emf will produce a current in the loop simply because the charges in the wire are constrained to move along the wire.

Lenz's law

4. The minus sign on the right side together with the right-hand screw rule ensures that *Lenz's law* is always satisfied. Lenz's law states that *the sense of the induced emf is such that any current it produces tends to* oppose the change *in the magnetic flux producing it*. It is important to note that the induced emf acts to oppose the change in the flux and not the flux itself. To clarify this, let us consider that the flux is into the paper and increasing with time. Then the induced emf acts to produce flux out of the paper. On the other hand, if the same flux is decreasing with time, then the induced emf acts to produce flux into the paper.

Faraday's law for N-turn coil

5. If the loop C contains more than one turn, such as in an N-turn coil, then the surface S bounded by the periphery of the loop takes the shape of a spiral ramp, as shown in Fig. 2.13 (a) for N equal to 2. This surface can be visualized by taking two paper plates, cutting each of them along a radius, as shown in Figs. 2.13(b) and (c), and joining the edge BO of the plate in (c) to the edge $A'O$ of the plate in (b). For a tightly wound coil, this is equivalent to the situation in which N separate, identical, single-turn loops are stacked so that the emf induced in the N-turn coil is N times that induced in one turn. Thus, for an N-turn coil,

$$\boxed{\text{emf} = -N\frac{d\psi}{dt}} \tag{2.14}$$

where ψ is the magnetic flux computed as though the coil is a one-turn coil.

We shall now consider two examples to illustrate the determination of induced emf using Faraday's law, the first involving a stationary loop in a time-varying magnetic field and the second involving a moving conductor in a static magnetic field.

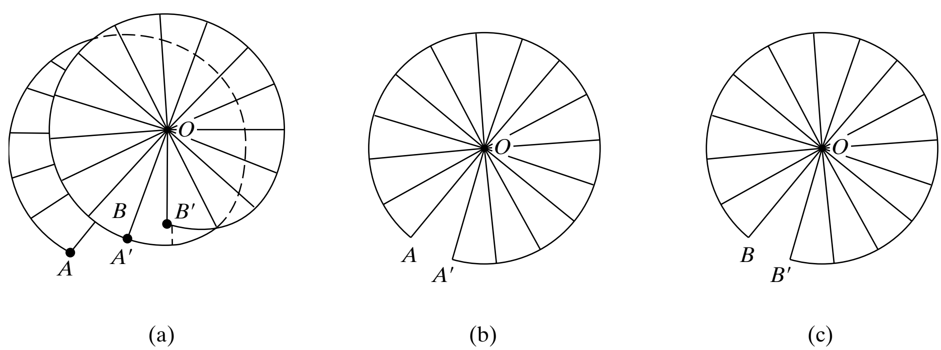

FIGURE 2.13

For illustrating the surface bounded by a loop containing two turns.

Example 2.3 Induced emf around a rectangular loop in a time-varying magnetic field

Stationary loop in a time-varying magnetic field

A time-varying magnetic field is given by

$$\mathbf{B} = B_0 \cos \omega t \, \mathbf{a}_y$$

where B_0 is a constant. It is desired to find the induced emf around the rectangular loop C in the xz-plane bounded by the lines $x = 0$, $x = a$, $z = 0$, and $z = b$, as shown in Fig. 2.14.

Choosing $d\mathbf{S} = dx\,dz\,\mathbf{a}_y$ in accordance with the right-hand screw rule and using the plane surface S bounded by the loop, we obtain the magnetic flux enclosed by the loop to be

$$\begin{aligned}\psi &= \int_S \mathbf{B} \cdot d\mathbf{S} = \int_{z=0}^{b} \int_{x=0}^{a} B_0 \cos \omega t \, \mathbf{a}_y \cdot dx\,dz\,\mathbf{a}_y \\ &= B_0 \cos \omega t \int_{z=0}^{b} \int_{x=0}^{a} dx\,dz = abB_0 \cos \omega t\end{aligned}$$

Note that since the magnetic flux density is uniform and normal to the plane of the loop, this result could have been obtained by simply multiplying the area ab of the loop by the component $B_0 \cos \omega t$ of the flux density vector. The induced emf around the loop is then given by

$$\begin{aligned}\oint_C \mathbf{E} \cdot d\mathbf{l} &= -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{S} \\ &= -\frac{d}{dt}[abB_0 \cos \omega t] = abB_0 \omega \sin \omega t\end{aligned}$$

The time variations of the magnetic flux enclosed by the loop and the induced emf around the loop are shown in Fig. 2.15. It can be seen that when the magnetic flux enclosed by the loop into the paper is *decreasing* with time, the induced emf is positive, thereby producing a clockwise current if the loop were a wire. This polarity of the current gives rise to a magnetic field directed into the paper inside the loop and, hence, acts to *oppose the decrease* of the magnetic flux enclosed by the loop. When the magnetic flux enclosed by the loop into the paper is *increasing* with time, the induced emf is negative, thereby producing a counterclockwise current around the loop. This polarity of the current gives rise to a magnetic field directed out of the paper inside the loop and hence acts to *oppose the increase* of the magnetic flux enclosed by the loop. These observations are consistent with Lenz's law.

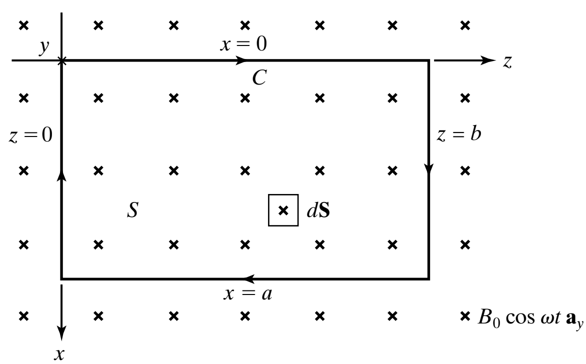

FIGURE 2.14

Rectangular loop in the xz-plane situated in a time-varying magnetic field.

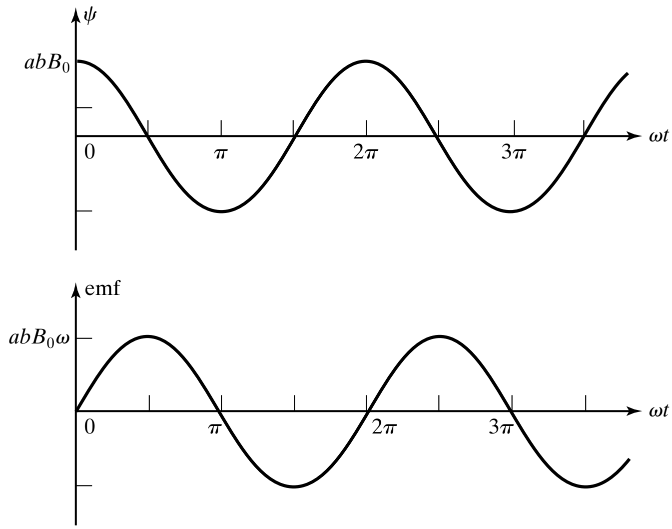

FIGURE 2.15

Time variations of magnetic flux ψ enclosed by the loop of Fig. 2.14 and the resulting induced emf around the loop.

Example 2.4 Induced emf around an expanding loop in a uniform static magnetic field

Moving conductor in a static magnetic field

A rectangular loop of wire with three sides fixed and the fourth side movable is situated in a plane perpendicular to a uniform magnetic field $\mathbf{B} = B_0\mathbf{a}_z$, as illustrated in Fig. 2.16. The movable side consists of a conducting bar moving with a velocity v_0 in the y-direction. It is desired to find the emf induced around the closed path C of the loop.

Letting the position of the movable side at any time t be $y_0 + v_0t$, considering $d\mathbf{S} = dx\,dy\,\mathbf{a}_z$ in accordance with the right-hand screw rule, and using the plane surface S bounded by the loop, we obtain the magnetic flux enclosed by the loop to be

$$\begin{aligned}\int_S \mathbf{B}\cdot d\mathbf{S} &= \int_S B_0\mathbf{a}_z\cdot dx\,dy\,\mathbf{a}_z\\ &= \int_{x=0}^{l}\int_{y=0}^{y_0+v_0t} B_0\,dx\,dy\\ &= B_0 l(y_0 + v_0t)\end{aligned}$$

Note that this result could also have been obtained as the product of the area of the loop $l(y_0 + v_0t)$ and the flux density B_0, because of the uniformity of the flux density within

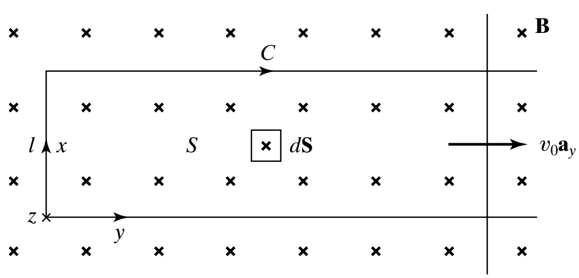

FIGURE 2.16

Rectangular loop of wire with a movable side situated in a uniform magnetic field.

the area of the loop and its perpendicularity to the plane of the loop. The emf induced around C is given by

$$\begin{aligned}\oint_C \mathbf{E}\cdot d\mathbf{l} &= -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S}\\ &= -\frac{d}{dt}[B_0 l(y_0 + v_0 t)]\\ &= -B_0 l v_0\end{aligned}$$

Note that if the bar is moving to the right, the induced emf is negative and produces a current in the sense opposite to that of C. This polarity of the current is such that it gives rise to a magnetic field directed out of the paper inside the loop. The flux of this magnetic field is in opposition to the flux of the original magnetic field and hence tends to *oppose the increase* in the magnetic flux enclosed by the loop. On the other hand, if the bar is moving to the left, v_0 is negative, the induced emf is positive, and produces current in the same sense as that of C. This polarity of current is such that it gives rise to a magnetic field directed into the paper inside the loop. The flux of this magnetic field is in augmentation to the flux of the original magnetic field and hence tends to *oppose the decrease* in the magnetic flux enclosed by the loop. These observations are once again consistent with Lenz's law.

It is also of interest to note that the induced emf can also be interpreted as being due to the electric field induced in the moving bar by virtue of its motion perpendicular to the magnetic field. Thus, a charge Q in the bar experiences a force $\mathbf{F} = Q\mathbf{v} \times \mathbf{B}$ or $Qv_0\mathbf{a}_y \times B_0\mathbf{a}_z = Qv_0B_0\mathbf{a}_x$. To an observer moving with the bar, this force appears as an electric force due to an electric field $\mathbf{F}/Q = v_0B_0\mathbf{a}_x$. Viewed from inside the loop, this electric field is in the counterclockwise sense. Hence, the induced emf, which is the line integral of $\mathbf{E}$ along the bar, is given by

$$\int_{x=0}^{l} v_0B_0\mathbf{a}_x \cdot dx\,\mathbf{a}_x = \int_{x=0}^{l} v_0B_0\,dx = v_0B_0l$$

in the counterclockwise sense (i.e., opposite to C), consistent with the result deduced from Faraday's law. This concept of induced emf is known as the *motional emf concept,* which is employed widely in the study of electromechanics.

In the two examples we just discussed, we have implicitly illustrated the principles behind two of the practical applications of Faraday's law. These are pertinent to the reception of radio and TV signals using a loop antenna and electromechanical energy conversion.

Principle of loop antenna

That the arrangement considered in Example 2.3 illustrates the principle of a loop antenna can be seen by noting that if the loop C were in the xy-plane or in the yz-plane, no emf would be induced in it since the magnetic flux density is then parallel to the plane of the loop and no flux is enclosed by the loop. In fact, for any arbitrary orientation of the loop, only that component of $\mathbf{B}$ normal to the plane of the loop contributes to the magnetic flux enclosed by the loop and, hence, to the emf induced in the loop. Thus, for a given magnetic field, the voltage induced in the loop varies as the orientation of the loop is changed, with the maximum occurring when the loop is in the plane perpendicular to the magnetic field. Pocket AM radios generally contain a type of loop antenna consisting of many turns of wire wound around a bar of magnetic material, and TV receivers generally employ a single-turn circular loop for UHF channels. Thus, for maximum signals to be received, the AM radios and the TV loop antennas need to be oriented appropriately. Another point of interest evident from Example 2.3 is that the induced emf is proportional to ω, the radian frequency of the source of the magnetic field. Hence, for the same voltage to be induced for a given amplitude B_0 of the magnetic flux density, the area of the loop times the number of turns is inversely proportional to the frequency.

Locating a radio transmitter

What is undesirable for one purpose can sometimes be used to advantage for another purpose. The fact that no voltage is induced in the loop antenna when the magnetic field is parallel to the plane of the loop is useful for locating the transmitter of a radio wave. Since the magnetic field of an incoming radio wave is perpendicular to its direction of propagation, no voltage is induced in the loop when its axis is along the direction of the transmitter. For a transmitter on Earth's surface, it is then sufficient to use two spaced vertical loop antennas and find their orientations for which no signals are received. By then producing backward along the axes of the two loop antennas, as shown by the top view in Fig. 2.17, the location of the transmitter can be determined.

Electromechanical energy conversion

That the arrangement considered in Example 2.4 is a simple example of an electromechanical energy converter can be seen by recognizing that in view of the current flow in the moving bar, the bar is acted on by a magnetic force. Since for positive v_0, the current flows in the loop in the sense opposite to that of C and hence in the positive x-direction in the moving bar, and since the magnetic field is in the z-direction, the magnetic force is exerted in the $\mathbf{a}_x \times \mathbf{a}_z$ or $-\mathbf{a}_y$-direction. Thus, to keep the bar moving, an external force must be exerted in the $+\mathbf{a}_y$-direction, thereby requiring mechanical work to be done by an external agent. It is this mechanical work that is converted into electrical energy in the loop.

Principle of rotating generator

What we have just discussed is the principle of generation of electric power by linear motion of a conductor in a magnetic field. Practical electric generators are of the rotating type. The principle of a rotating generator can be illustrated by considering a rectangular loop of wire situated symmetrically about the z-axis and rotating with angular velocity ω around the z-axis in a constant magnetic field $\mathbf{B} = B_0\mathbf{a}_x$, as shown in Fig. 2.18(a). Then noting from the view in Fig. 2.18(b) that the magnetic flux ψ enclosed by the loop at any arbitrary value of time is the same as that enclosed by its projection onto the yz-plane at that time, we obtain $\psi = B_0 A \cos \omega t$, where A is the area of the loop and the situation

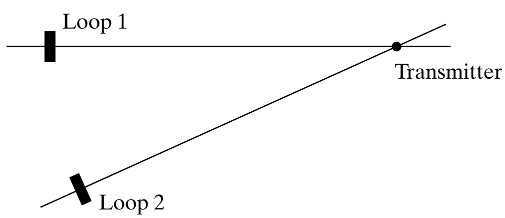

FIGURE 2.17

Top view of an arrangement consisting of two loop antennas for locating a transmitter of radio waves.

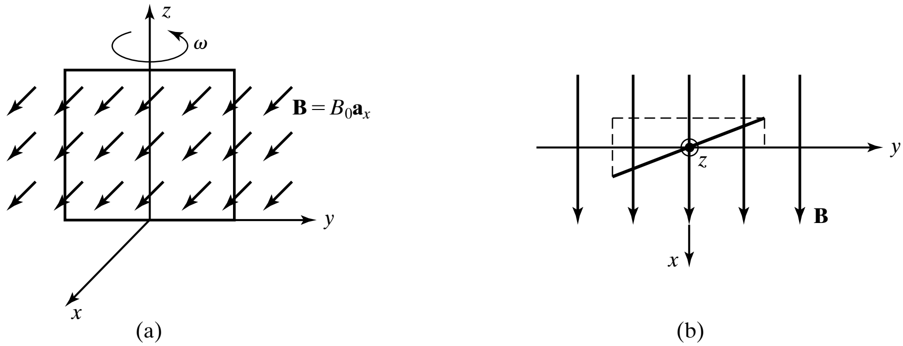

FIGURE 2.18

For illustrating the principle of a rotating generator.

shown in Fig. 2.18(a) is assumed for $t = 0$. The emf induced in the loop is $-d\psi/dt$, or $\omega B_0 A \sin \omega t$. Thus, the rotating loop in the constant magnetic field produces an alternating voltage. The same result can be achieved by a stationary loop in a rotating magnetic field. In fact, in most generators, a stationary member, or stator, carries the coils in which the voltage is induced, and a rotating member, or rotor, provides the magnetic field. As in the case of the arrangement of Example 2.4, a certain amount of mechanical work must be done to keep the loop rotating. It is this mechanical work, which is supplied by the prime mover (such as a turbine in the case of a hydroelectric generator or the engine of an automobile in the case of its alternator) turning the rotor, that is converted into electrical energy.

Magnetic levitation

There are numerous other applications of Faraday's law, but we shall discuss only one more before we conclude this section. This is the phenomenon of *magnetic levitation*, a basis for rapid transit systems employing trains that hover over their guideways and do not touch the rail, among other applications. Magnetic levitation arises from a combination of Faraday's law and Ampère's force law. It can be explained and demonstrated through a series of simple experiments, culminating in a current-carrying coil lifting up above a metallic plate, as described in the following:

1. Consider a pair of coils (30 to 50 turns of No. 26 wire of about 4-in. diameter) attached to nails on a piece of wood, as shown in Fig. 2.19. Set to zero the output of a variable power supply obtained by connecting a variac to the 110-V ac mains. Connect one output terminal (A) of the variac to the beginnings (C_1 and C_2) of both coils and the second output terminal (B) to the ends (D_1 and D_2) of both coils so that currents flow in the two coils in the same sense. Apply some voltage to the coils by turning up the variac and note the attraction between the coils. Repeat the experiment by connecting A to C_1 and D_2 and B to C_2 and D_1, so that currents in the two coils flow in opposite senses, and note repulsion this time. What we have just described is Ampère's force law at work. If the currents flow in the same sense, the magnetic force is one of attraction,

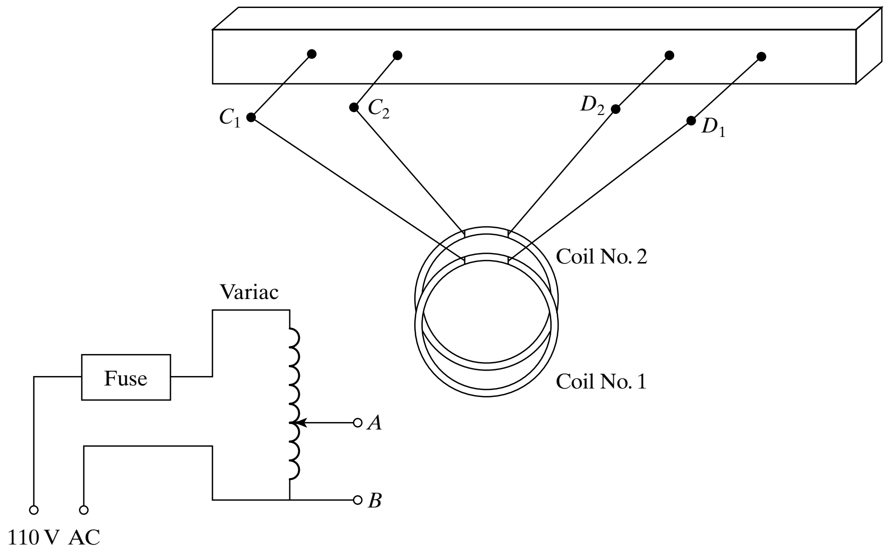

FIGURE 2.19

Experimental setup for demonstration of Ampère's force law, Faraday's law, and the principle of magnetic levitation.

and if the currents flow in opposite senses, it is one of repulsion, as shown in Figs. 2.20(a) and (b), respectively, for straight wires, for the sake of simplicity.[1]

2. Connect coil No. 2 to the variac and coil No. 1 to an oscilloscope to observe the induced voltage in coil No. 1, thereby demonstrating Faraday's law. Note the change in the induced voltage as the variac voltage is changed. Note also the change in the induced voltage by keeping the variac voltage constant and moving coil No. 1 away from coil No. 2 and/or turning it about the vertical.

3. Connect coil No. 2 to the variac and leave coil No. 1 open-circuited. Observe that no action takes place as the variac voltage is applied to coil No. 2. This is because although a voltage is induced in coil No. 1, no current flows in it.

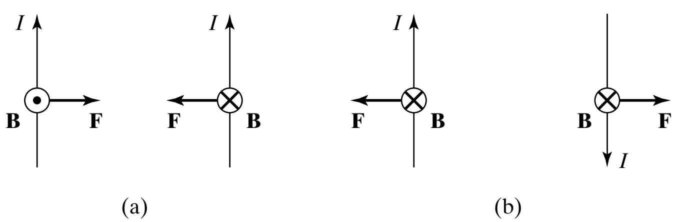

FIGURE 2.20

For explaining (a) force of attraction for currents flowing in the same sense and (b) force of repulsion for currents flowing in opposite senses.

[1]See L. Pearce Williams, "André-Marie Ampère," *Scientific American*, January 1989, pp. 90–97, for an interesting account of Ampère's experiments involving helical and spiral coils.

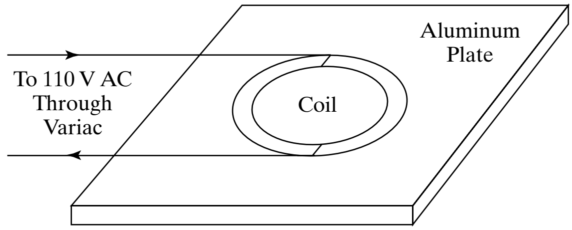

FIGURE 2.21

Setup for demonstrating magnetic levitation.

Now short circuit coil No. 1 and repeat the experiment to note repulsion. This is due to the induced voltage in coil No. 1 causing a current flow in it in the sense opposite to that in coil No. 2, and, hence, is a result of the combination of Faraday's law and Ampère's force law. That the force is one of repulsion can be deduced by writing circuit equations and showing that the current in the short-circuited coil does indeed flow in the sense opposite to that in the excited coil. However, it can be explained with the aid of physical reasoning as follows. When both coils are excited in the same sense in part (1) of the demonstration, the magnetic flux linking each coil is the sum of two fluxes in the same sense, due to the two currents. When the two coils are excited in opposite senses, the magnetic flux linking each coil is the algebraic sum of two fluxes in opposing senses, due to the two currents. Therefore, for the same source voltage and for the same pair of coils, the currents that flow in the coils in the second case have to be greater than those in the first case, for the induced voltage in each coil to equal the applied voltage. Thus, the force of repulsion in the second case is greater than the force of attraction in the first case. Consider now the case of one of the coils excited by source voltage, say, V_g, and the other short-circuited. Then the situation can be thought of as the first coil excited by $V_g/2$ and $V_g/2$ in series, and the second coil excited by $V_g/2$ and $-V_g/2$ in series, thereby resulting in a force of attraction and a force of repulsion. Since the force of repulsion is greater than the force of attraction, the net force, according to superposition, is one of repulsion.

4. Now to demonstrate actual levitation, place a smaller coil (about 30 turns of No. 28 wire of about 2-in. diameter) on a heavy aluminum plate (5 in. $\times$ 5 in. $\times$ $\frac{1}{2}$ in.), as shown in Fig. 2.21. Applying only the minimum necessary voltage and turning the variac only momentarily to avoid overheating, pass current through the coil from the variac to see the coil levitate. This levitation is due to the repulsive action between the current in the coil and the induced currents in the metallic plate. Since the plate is heavy and cannot move, the alternative is for the coil to lift up.

K2.3. Faraday's law; Right-hand screw rule; Lenz's law; Faraday's law for N-turn coil; Motional emf concept; Principle of loop antenna; Electromechanical energy conversion; Rotating generator; Magnetic levitation.

D2.5. Given $\mathbf{B} = B_0(\sin \omega t\ \mathbf{a}_x - \cos \omega t\ \mathbf{a}_y)$ Wb/m^2, find the induced emf around each of the following closed paths: **(a)** the rectangular path from (0, 0, 0) to (0, 1, 0) to (0, 1, 1) to (0, 0, 1) to (0, 0, 0); **(b)** the triangular path from (1, 0, 0) to (0, 1, 0) to

$(0, 0, 1)$ to $(1, 0, 0)$; and **(c)** the rectangular path from $(0, 0, 0)$ to $(1, 1, 0)$ to $(1, 1, 1)$ to $(0, 0, 1)$ to $(0, 0, 0)$.

Ans. **(a)** $-\omega B_0 \cos \omega t$ V; **(b)** $-\dfrac{\omega B_0}{\sqrt{2}} \cos(\omega t - \pi/4)$ V; **(c)** $-\sqrt{2}\, \omega B_0 \cos(\omega t + \pi/4)$ V.

D2.6. A square loop lies in the xy-plane forming the closed path C connecting the points $(0, 0, 0)$, $(1, 0, 0)$, $(1, 1, 0)$, $(0, 1, 0)$, and $(0, 0, 0)$, in that order. A magnetic field $\mathbf{B}$ exists in the region. From considerations of Lenz's law, determine whether the induced emf around the closed path C at $t = 0$ is positive, negative, or zero for each of the following magnetic fields, where B_0 is a positive constant: **(a)** $\mathbf{B} = B_0 t \mathbf{a}_z$; **(b)** $\mathbf{B} = B_0 \cos(2\pi t + 60°)\, \mathbf{a}_z$; and **(c)** $\mathbf{B} = B_0 e^{-t^2} \mathbf{a}_z$.

Ans. **(a)** negative; **(b)** positive; **(c)** zero.

D2.7. For $\mathbf{B} = B_0 \cos \omega t\, \mathbf{a}_z$ Wb/m^2, find the induced emf around the following closed paths: **(a)** the closed path comprising the straight lines successively connecting the points $(0, 0, 0)$, $(1, 0, 0)$, $(1, 1, 0)$, $(0, 1, 0)$, $(0, 0, 0.001)$, and $(0, 0, 0)$; **(b)** the closed path comprising the straight lines successively connecting the points $(0, 0, 0)$, $(1, 0, 0)$, $(1, 1, 0)$, $(0, 1, 0)$, $(0, 0, 0.001)$, $(1, 0, 0.001)$, $(1, 1, 0.001)$, $(0, 1, 0.001)$, $(0, 0, 0.002)$, and $(0, 0, 0)$ with a slight kink in the straight line at the point $(0, 0, 0.001)$ to avoid touching the point; and **(c)** the closed path comprising the helical path $r = 1/\sqrt{\pi}$, $\phi = 1000\pi z$ from $(1/\sqrt{\pi}, 0, 0)$ to $(1/\sqrt{\pi}, 0, 0.01)$ and the straight-line path from $(1/\sqrt{\pi}, 0, 0.01)$ to $(1/\sqrt{\pi}, 0, 0)$ with slight kinks to avoid touching the helical path.

Ans. **(a)** $\omega B_0 \sin \omega t$ V; **(b)** $2\omega B_0 \sin \omega t$ V; **(c)** $5\omega B_0 \sin \omega t$ V.

2.4 AMPÈRE'S CIRCUITAL LAW

In the preceding section, we introduced Faraday's law, one of Maxwell's equations, in integral form. In this section, we introduce another of Maxwell's equations in integral form. This equation, known as *Ampère's circuital law,* is a combination of an experimental finding of Oersted that electric currents generate magnetic fields and a mathematical contribution of Maxwell that time-varying electric fields give rise to magnetic fields. It is this contribution of Maxwell that led to the prediction of electromagnetic wave propagation even before the phenomenon was discovered experimentally. In mathematical form, Ampère's circuital law is analogous to Faraday's law and is given by

$$\boxed{\oint_C \frac{\mathbf{B}}{\mu_0} \cdot d\mathbf{l} = [I_c]_S + \frac{d}{dt} \int_S \varepsilon_0 \mathbf{E} \cdot d\mathbf{S}} \tag{2.15}$$

where S is a surface bounded by C.

The quantity $\oint_C \frac{\mathbf{B}}{\mu_0} \cdot d\mathbf{l}$ on the left side of (2.15) is the line integral of the vector field $\mathbf{B}/\mu_0$ around the closed path C. We learned in Section 2.1 that the quantity $\oint_C \mathbf{E} \cdot d\mathbf{l}$ has the physical meaning of work per unit charge associated with the movement of a test charge around the closed path C. The quantity $\oint_C \frac{\mathbf{B}}{\mu_0} \cdot d\mathbf{l}$ does not have a similar physical meaning. This is because magnetic force on a moving charge is directed perpendicular to the direction of motion of the charge, as well as to the direction of the magnetic field, and hence does not do work in the movement of the charge. The vector $\mathbf{B}/\mu_0$ is known as the "magnetic field intensity vector" and is denoted by the symbol $\mathbf{H}$. By recalling from (1.78) that $\mathbf{B}$ has the units of [(permeability)(current)(length)] per [(distance)^2], we note that the quantity $\mathbf{H}$ has the units of current per unit distance or amp/m. This gives the units of current or amp to $\oint_C \mathbf{H} \cdot d\mathbf{l}$. In analogy with the name "electromotive force" for $\oint_C \mathbf{E} \cdot d\mathbf{l}$, the quantity $\oint_C \mathbf{H} \cdot d\mathbf{l}$ is known as the "magnetomotive force," abbreviated as mmf.

The quantity $[I_c]_S$ on the right side of (2.15) is the current due to flow of free charges crossing the surface S. It can be a convection current such as due to motion of a charged cloud in space, or a conduction current due to motion of charges in a conductor. Although $[I_c]_S$ can be filamentary current, surface current, or volume current, or a combination of these, it is formulated in terms of the volume current density vector, $\mathbf{J}$, in the manner

$$[I_c]_S = \int_S \mathbf{J} \cdot d\mathbf{S} \tag{2.16}$$

Just as the surface integral of the magnetic flux density vector $\mathbf{B}$ (Wb/m^2) over a surface S gives the magnetic flux (Wb) crossing that surface, the surface integral of $\mathbf{J}$ (A/m^2) over a surface S gives the current (A) crossing that surface.

The quantity $\int_S \varepsilon_0 \mathbf{E} \cdot d\mathbf{S}$ on the right side of (2.15) is the flux of the vector field $\varepsilon_0\mathbf{E}$ crossing the surface S. The vector $\varepsilon_0\mathbf{E}$ is known as the "displacement vector" or the "displacement flux density vector" and is denoted by the symbol $\mathbf{D}$. By recalling from (1.62) that $\mathbf{E}$ has the units of (charge) per [(permittivity)(distance)^2], we note that the quantity $\mathbf{D}$ has the units of charge per unit area or C/m^2. Hence the quantity $\int_S \varepsilon_0 \mathbf{E} \cdot d\mathbf{S}$, that is, the displacement flux, has the units of charge, and the quantity $\frac{d}{dt}\int_S \varepsilon_0 \mathbf{E} \cdot d\mathbf{S}$ has the units of $\frac{d}{dt}$ (charge) or current and is known as the "displacement current." Physically, it is not a current in the sense that it does not represent the flow of charges, but mathematically it is equivalent to a current crossing the surface S.

Statement of Ampère's circuital law

Replacing $\mathbf{B}/\mu_0$ and $\varepsilon_0\mathbf{E}$ in (2.15) by $\mathbf{H}$ and $\mathbf{D}$, respectively, and using (2.16), we rewrite Ampère's circuital law as

$$\boxed{\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt}\int_S \mathbf{D} \cdot d\mathbf{S}} \tag{2.17}$$

In words, (2.17) states that *the magnetomotive force around a closed path C is equal to the algebraic sum of the current due to flow of charges and the displacement current bounded by C.* The situation is illustrated in Fig. 2.22.

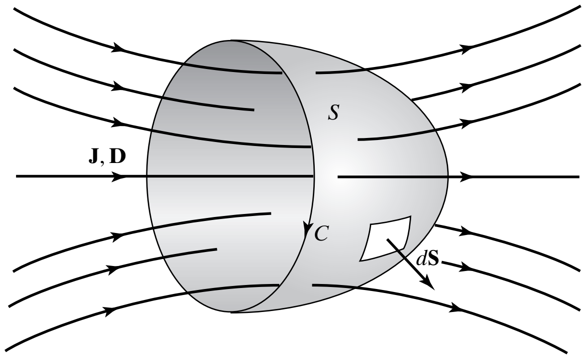

FIGURE 2.22

For illustrating Ampère's circuital law.

As in the case of Faraday's law, there are certain procedures and observations pertinent to the application of (2.17). These are as follows.

1. The surface integrals on the right side of (2.17) are to be evaluated in accordance with the R.H.S. rule, which means that for the geometry of Fig. 2.22, the normal vector to the differential surface dS should be directed as shown in the figure.
2. In evaluating the surface integrals, *any surface S bounded by C* can be employed. However, *the same surface* must be employed for the two surface integrals. It is not correct to consider two different surfaces to evaluate the two surface integrals, although both surfaces may be bounded by C.

Observation 2 implies that for the mmf around C to be unique, the sum of the two currents (current due to flow of charges and displacement current) through all possible surfaces bounded by C is the same. Let us now consider two surfaces S_1 and S_2 bounded by the closed paths C_1 and C_2, respectively, as shown in Fig. 2.23, where C_1 and C_2 are traversed in opposite senses and touch each other so that S_1 and S_2 together form a closed surface. The situation may be imagined by considering the closed surface to be that of a potato and C_1 and C_2 to be two rubber bands around the potato.

Applying Ampère's circuital law to C_1 and S_1 and noting that $d\mathbf{S}_1$ is chosen in accordance with the R.H.S. rule, we have

$$\oint_{C_1} \mathbf{H} \cdot d\mathbf{l} = \int_{S_1} \mathbf{J} \cdot d\mathbf{S}_1 + \frac{d}{dt}\int_{S_1} \mathbf{D} \cdot d\mathbf{S}_1 \tag{2.18a}$$

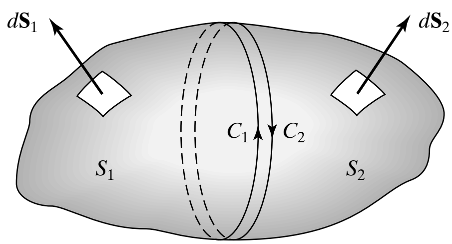

FIGURE 2.23

Two closed paths C_1 and C_2 touching each other and bounding the surfaces S_1 and S_2, respectively, which together form a closed surface.

Similarly, applying Ampère's circuital law to C_2 and S_2 and noting again that $d\mathbf{S}_2$ is chosen in accordance with the R.H.S. rule, we have

$$\oint_{C_2} \mathbf{H} \cdot d\mathbf{l} = \int_{S_1} \mathbf{J} \cdot d\mathbf{S}_2 + \frac{d}{dt}\int_{S_2} \mathbf{D} \cdot d\mathbf{S}_2 \tag{2.18b}$$

Now adding (2.18a) and (2.18b), we obtain

$$0 = \oint_{S_1+S_2} \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt}\oint_{S_1+S_2} \mathbf{D} \cdot d\mathbf{S} \tag{2.19}$$

where the left side results from the fact that C_1 and C_2 are actually the same path but traversed in opposite senses, so that the two line integrals are the negatives of each other.

Since the closed surface $S_1 + S_2$ can be of any size and shape, we can generalize (2.19) to write

$$\oint_S \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt}\oint_S \mathbf{D} \cdot d\mathbf{S} = 0$$

or

$$\boxed{\frac{d}{dt}\oint_S \mathbf{D} \cdot d\mathbf{S} = -\oint_S \mathbf{J} \cdot d\mathbf{S}} \tag{2.20}$$

Thus, the displacement current emanating from a closed surface is equal to the current due to charges flowing into the volume bounded by that closed surface.

Capacitor circuit

An important example of the property given by (2.20) at work is in a capacitor circuit, as shown in Fig. 2.24. In this circuit, the time-varying voltage source sets up a time-varying electric field between the plates of the capacitor

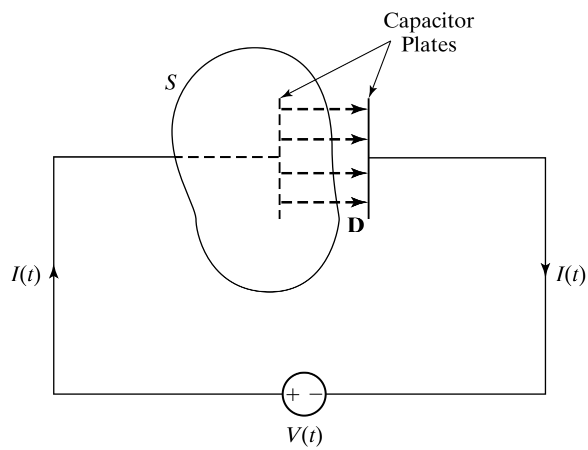

FIGURE 2.24

Capacitor circuit for illustrating that the displacement current from one plate to the other is equal to the wire current.

and directed from one plate to the other. Therefore, one can talk about displacement current crossing a surface between the plates. According to (2.20) applied to a closed surface S enclosing one of the plates, as shown in the figure,

$$\frac{d}{dt}\oint_S \mathbf{D}\cdot d\mathbf{S} = I(t) \tag{2.21}$$

where $I(t)$ is the current (due to flow of charges in the wire) drawn from the voltage source. Neglecting fringing effects and assuming that the electric field is normal to the plates and uniform, we have, from (2.21),

$$\frac{d}{dt}\oint_S \mathbf{D}\cdot d\mathbf{S} = \frac{d}{dt}(DA) = I(t) \tag{2.22}$$

where A is the area of each plate. Thus, where the wire current ends on one of the plates, the displacement current takes over and completes the circuit to the second plate.

Radiation from antenna

Let us now return to Ampère's circuital law (2.17) and examine it together with Faraday's law (2.13). To do this, we repeat the two laws

$$\boxed{\begin{aligned}\oint_C \mathbf{E}\cdot d\mathbf{l} &= -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S} \qquad (2.23)\\ \oint_C \mathbf{H}\cdot d\mathbf{l} &= \int_S \mathbf{J}\cdot d\mathbf{S} + \frac{d}{dt}\int_S \mathbf{D}\cdot d\mathbf{S} \qquad (2.24)\end{aligned}}$$

and observe that time-varying electric and magnetic fields are interdependent, since according to Faraday's law (2.23), a time-varying magnetic field produces an electric field, whereas according to Ampère's circuital law (2.24), a time-varying electric field gives rise to a magnetic field. In addition, Ampère's circuital law tells us that an electric current generates a magnetic field. These properties from the basis for the phenomena of radiation and propagation of electromagnetic waves. To provide a simplified, qualitative explanation of radiation from an antenna, we begin with a piece of wire carrying a time-varying current, $I(t)$, as shown in Fig. 2.25. Then, the time-varying current generates a time-varying magnetic field $\mathbf{H}(t)$, which surrounds the wire. Time-varying electric and magnetic fields, $\mathbf{E}(t)$ and $\mathbf{H}(t)$, are then produced in succession, as shown by two views in Fig. 2.25, thereby giving rise to electromagnetic waves. Thus, just as water waves are produced when a rock is thrown in a pool of water, electromagnetic waves are radiated when a piece of wire in space is excited by a time-varying current.

K2.4. Ampère's circuital law; Magnetic field intensity; Magnetomotive force; Displacement flux density; Displacement current; Capacitor circuit; Radiation from an antenna.

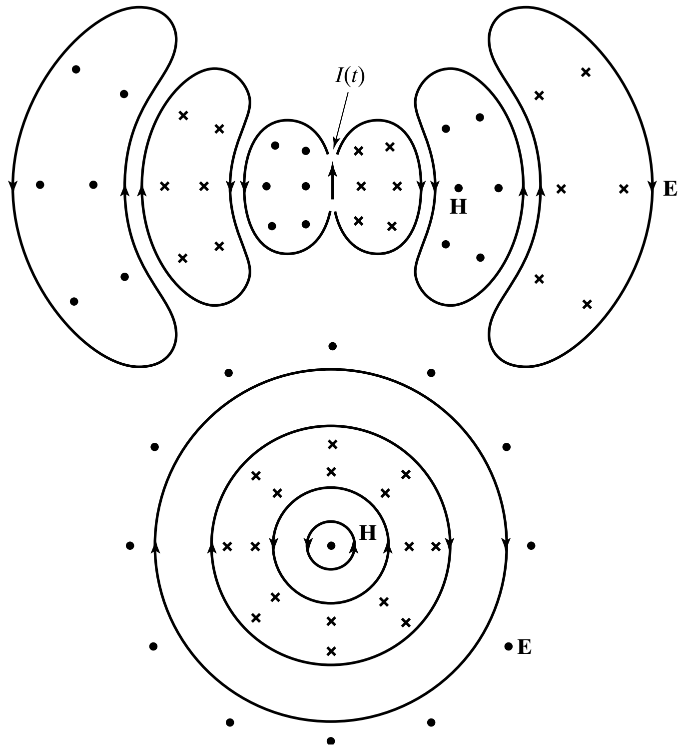

FIGURE 2.25

Two views of a simplified depiction of electromagnetic wave radiation from a piece of wire carrying a time-varying current.

D2.8. For $\mathbf{E} = E_0 t e^{-t^2} \mathbf{a}_z$ in free space, find the displacement current crossing an area of 0.1 m^2 in the xy-plane from the $-z$-side to the $+z$-side for each of the following values of t: **(a)** $t = 0$; **(b)** $t = 1/\sqrt{2}$ s; and **(c)** $t = 1$ s.

Ans. **(a)** $0.1\,\varepsilon_0 E_0$ A; **(b)** 0; **(c)** $-0.1e^{-1}\varepsilon_0 E_0$ A.

D2.9. Three point charges $Q_1(t)$, $Q_2(t)$, and $Q_3(t)$ situated at the corners of an equilateral triangle of sides 1 m are connected to each other by wires along the sides of the triangle. Currents of I A and $3I$ A flow from Q_1 to Q_2 and Q_1 to Q_3, respectively. The displacement current emanating from a spherical surface of radius 0.1 m and centered at Q_2 is $-2I$ A. Find the following: **(a)** the current flowing from Q_2 to Q_3; **(b)** the displacement current emanating from the spherical surface of radius 0.1 m and centered at Q_1; and **(c)** the displacement current emanating from the spherical surface of radius 0.1 m and centered at Q_3.

Ans. **(a)** $3I$ A; **(b)** $-4I$ A; **(c)** $6I$ A.

2.5 GAUSS' LAWS

In the previous two sections, we learned two of the four Maxwell's equations. These two equations have to do with the line integrals of the electric and magnetic fields around closed paths. The remaining two Maxwell's equations are pertinent to the surface integrals of the electric and magnetic fields over closed surfaces. These are known as *Gauss' laws*.

Gauss' law for the electric field states that *the displacement flux emanating from a closed surface S is equal to the charge contained within the volume V bounded by that surface.* This statement, although familiarly known as Gauss' law, has its origin in experiments conducted by Faraday. In mathematical form, it is given by

Gauss' law for the electric field

$$\boxed{\oint_S \mathbf{D} \cdot d\mathbf{S} = [Q]_V} \tag{2.25}$$

The quantity $[Q]_V$ is the charge contained within the volume V bounded by S. Although $[Q]_V$ can be a point charge, surface charge, or volume charge, or a combination of these, it is formulated as the volume integral of the volume charge density ρ, that is, in the manner

$$[Q]_V = \int_V \rho \, dv \tag{2.26}$$

The volume integral is a triple integral since dv is the product of three differential lengths. For an illustration of the evaluation of a volume integral, let us consider

Evaluation of volume integral

$$\rho = (x + y + z) \text{ C/m}^3$$

and the cubical volume V bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, and $z = 1$. Then the charge Q contained within the cubical volume is given by

$$\begin{aligned}
Q = \int_V \rho \, dv &= \int_{x=0}^{1} \int_{y=0}^{1} \int_{z=0}^{1} (x + y + z) \, dx \, dy \, dz \\
&= \int_{x=0}^{1} \int_{y=0}^{1} \left[xz + yz + \frac{z^2}{2} \right]_{z=0}^{1} dx \, dy \\
&= \int_{x=0}^{1} \int_{y=0}^{1} \left(x + y + \frac{1}{2} \right) dx \, dy \\
&= \int_{x=0}^{1} \left[xy + \frac{y^2}{2} + \frac{y}{2} \right]_{y=0}^{1} dx \\
&= \int_{x=0}^{1} (x + 1) \, dx \\
&= \left[\frac{x^2}{2} + x \right]_{x=0}^{1} \\
&= \tfrac{3}{2} \text{ C}
\end{aligned}$$

We may now write Gauss' law for the electric field (2.25) in the manner

$$\boxed{\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho\, dv} \tag{2.27}$$

where we recall that

$$\mathbf{D} = \varepsilon_0 \mathbf{E}$$

and it is understood that $\int_V \rho\, dv$, although formulated in terms of the volume charge density ρ, represents the algebraic sum of all free charges contained within V. The situation is illustrated in Fig. 2.26.

Gauss' law for the magnetic field

Gauss' law for the magnetic field is analogous to Gauss' law for the electric field and is given by

$$\boxed{\oint_S \mathbf{B} \cdot d\mathbf{S} = 0} \tag{2.28}$$

In words, (2.28) states that *the magnetic flux emanating from a closed surface is equal to zero.* In physical terms, (2.28) signifies that magnetic charges do not exist and magnetic flux lines are closed. Whatever magnetic flux enters (or leaves) a certain part of a closed surface must leave (or enter) through the remainder of the closed surface, as illustrated in Fig. 2.27.

This property of the magnetic field is sometimes useful in the computation of magnetic flux crossing a given surface (which is not closed). For example, to find the magnetic flux crossing the slanted plane surface S_1 in Fig. 2.28, it is not necessary to evaluate formally the surface integral of $\mathbf{B}$ over that surface. Since the slant surface S_1 and the three surfaces S_2, S_3, and S_4 in the coordinate planes together form a closed surface, the required flux is the same as the net flux crossing the surfaces S_2, S_3, and S_4. In fact, the net flux crossing the surfaces S_2, S_3, and S_4 is the same as that crossing any nonplanar surface having the same periphery as that of S_1. Thus, as already pointed out in Section 2.3, it is a fundamental property of the magnetic field that the magnetic flux is the same through

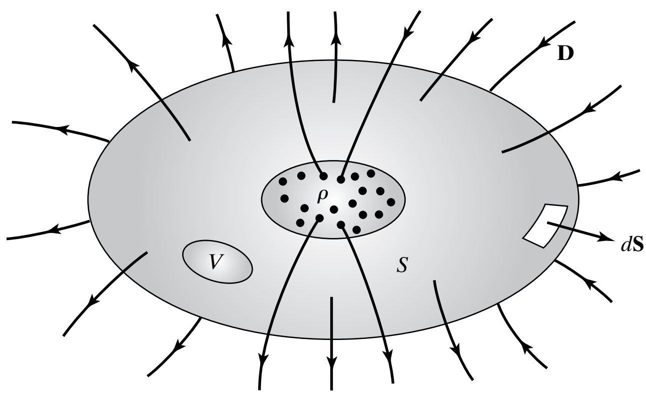

FIGURE 2.26

For illustrating Gauss' law for the electric field.

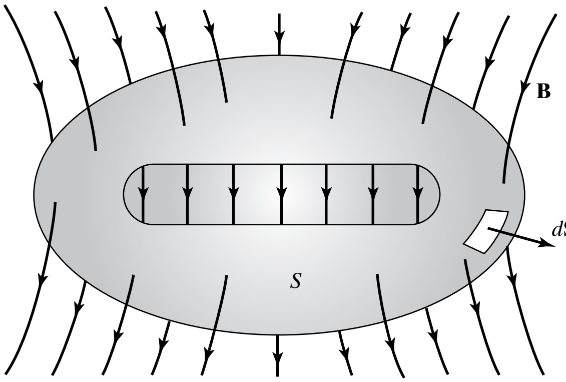

FIGURE 2.27

For illustrating Gauss' law for the magnetic field.

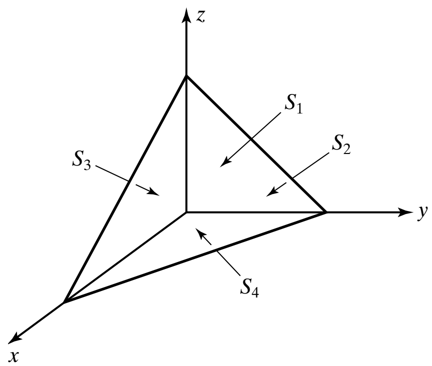

FIGURE 2.28

Slanted plane surface S_1 and surface S_2, S_3, and S_4 in the coordinate planes.

all surfaces bounded by a closed path, and hence *any surface S bounded by closed path C* can be used in Faraday's law.

In view of the foregoing discussion, it can be seen that Gauss' law for the magnetic field is not independent of Faraday's law. To show this mathematically, we consider the geometry shown in Fig. 2.23 and apply Faraday's law to the two closed paths to write

$$\oint_{C_1} \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_{S_1} \mathbf{B} \cdot d\mathbf{S}_1$$
$$\oint_{C_2} \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_{S_2} \mathbf{B} \cdot d\mathbf{S}_2$$

Adding the two equations, we obtain

$$0 = -\frac{d}{dt}\oint_{S_1+S_2} \mathbf{B} \cdot d\mathbf{S}$$

or

$$\oint_{S_1+S_2} \mathbf{B} \cdot d\mathbf{S} = \text{constant with time} \tag{2.29}$$

Since there is no experimental evidence that the right side of (2.29) is nonzero, it follows that

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$$

where we have replaced $S_1 + S_2$ by S.

K2.5. Volume integral; Gauss' law for the electric field; Gauss' law for the magnetic field.

D2.10. Several types of charge are located, in Cartesian coordinates, as follows: a point charge of 1 μC at $(1, 1, -1.5)$, a line charge of uniform density 2 μC/m along the straight line from $(-1, -1, -1)$ to $(3, 3, 3)$, and a surface charge of uniform density -1 $\mu\text{C/m}^2$ on that part of the plane $x = 0$ between $z = -1$ and $z = 1$. Find the displacement flux emanating from each of the following closed surfaces: **(a)** surface of the cubical box bounded by the planes $x = \pm 2$, $y = \pm 2$, and $z = \pm 2$; **(b)** surface of the cylindrical box of radius 2 m, having the z-axis as its axis and lying between $z = -2$ and $z = 2$; and **(c)** surface of the octahedron having its vertices at $(3, 0, 0)$, $(-3, 0, 0)$, $(0, 3, 0)$, $(0, -3, 0)$, $(0, 0, 3)$, and $(0, 0, -3)$.
Ans. **(a)** 3.3923 μC; **(b)** 1.3631 μC; **(c)** -3.0718 μC.

D2.11. Magnetic fluxes of absolute values ψ_1, ψ_2, and ψ_3 cross three surfaces S_1, S_2, and S_3, respectively, constituting a closed surface S. If $\psi_1 + \psi_2 + \psi_3 = \psi_0$, find the smallest of ψ_1, ψ_2, and ψ_3 for each of the following cases: **(a)** ψ_1, ψ_2, and ψ_3 are in arithmetic progression; **(b)** $1/\psi_1$, $1/\psi_2$, and $1/\psi_3$ are in arithmetic progression; and **(c)** $\ln \psi_1$, $\ln \psi_2$, and $\ln \psi_3$ are in arithmetic progression.
Ans. **(a)** $\frac{1}{6}\psi_0$; **(b)** $\frac{1}{2 + 2\sqrt{2}}\psi_0$; **(c)** $\frac{1}{3 + \sqrt{5}}\psi_0$.

2.6 THE LAW OF CONSERVATION OF CHARGE

Law of Conservation of Charge

Just as Gauss's law for magnetic field is not independent of Faraday's law, Gauss' law for the electric field is not independent of Ampère's circuital law in view of the *law of conservation of charge*. The law of conservation of charge states that *the net current due to flow of charges emanating from a closed surface S is equal to the time rate of decrease of the charge within the volume V bounded by S*. It is given in mathematical form by

$$\boxed{\oint_S \mathbf{J} \cdot d\mathbf{S} = -\frac{d}{dt}\int_V \rho \, dv} \tag{2.30}$$

As illustrated in Fig. 2.29, this law follows from the property that electric charge is conserved. If the charge in a given volume is decreasing with time at a certain rate, there must be a net outflow of the charge at the same rate. Since current is defined to be the rate of flow of charge, (2.30) then follows. As in the case of (2.17), it is understood that $\oint_S \mathbf{J} \cdot d\mathbf{S}$ in (2.30), although formulated in terms of $\mathbf{J}$, represents the algebraic sum of all currents due to flow of charges crossing S.

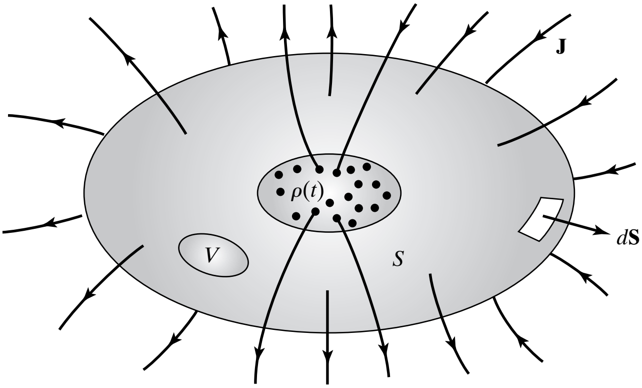

FIGURE 2.29
For illustrating the law of conservation of charge.

Comparing (2.20) and (2.30), we obtain

$$\frac{d}{dt}\oint_S \mathbf{D}\cdot d\mathbf{S} = \frac{d}{dt}\int_V \rho\, dv$$

$$\frac{d}{dt}\left(\oint_S \mathbf{D}\cdot d\mathbf{S} - \int_V \rho\, dv\right) = 0 \tag{2.31}$$

$$\oint_S \mathbf{D}\cdot d\mathbf{S} - \int_V \rho\, dv = \text{constant with time}$$

Since there is no experimental evidence that the right side of (2.31) is nonzero, it follows that

$$\oint_S \mathbf{D}\cdot d\mathbf{S} = \int_V \rho\, dv$$

Thus, since (2.20) follows from Ampère's circuital law, Gauss' law for the electric field follows from Ampère's circuital law with the aid of the law of conservation of charge.

We shall now illustrate the combined application of Gauss' law for the electric field, the law of conservation of charge, and Ampère's circuital law by means of an example.

Example 2.5 Combined application of several of Maxwell's equations in integral form

Let us consider current I A flowing from a point charge $Q(t)$ at the origin to infinity along a semi-infinitely long straight wire occupying the positive z-axis, and find $\oint_C \mathbf{H}\cdot d\mathbf{l}$, where C is a circular path of radius a lying in the xy-plane and centered at the point charge, as shown in Fig. 2.30.

Considering the hemispherical surface S bounded by C, and above the xy-plane, as shown in Fig. 2.30, and applying Ampère's circuital law, we obtain

$$\oint_C \mathbf{H}\cdot d\mathbf{l} = I + \frac{d}{dt}\int_S \mathbf{D}\cdot d\mathbf{S} \tag{2.32}$$

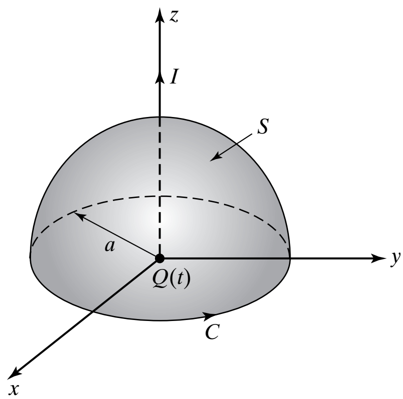

FIGURE 2.30

Semi-infinitely long wire of current I, with a point charge $Q(t)$ at the origin.

From Gauss' law for the electric field, the displacement flux emanating from a spherical surface centered at the point charge is equal to Q. In view of the spherical symmetry of the electric field about the point charge, half of the flux goes through the hemispherical surface. Thus,

$$\int_S \mathbf{D} \cdot d\mathbf{S} = \frac{Q}{2} \tag{2.33}$$

From the law of conservation of charge applied to a spherical surface centered at the point charge,

$$I = -\frac{dQ}{dt} \tag{2.34}$$

Substituting (2.33) into (2.32) and then using (2.34), we obtain

$$\begin{aligned}\oint_C \mathbf{H} \cdot d\mathbf{l} &= I + \frac{d}{dt}\left(\frac{Q}{2}\right) \\ &= I + \frac{1}{2}\frac{dQ}{dt} \\ &= I + \frac{1}{2}(-I) \\ &= \frac{I}{2}\end{aligned}$$

It should be noted that the same result holds for any contour C lying in any plane passing through the origin and surrounding the point charge $Q(t)$ and the wire in the right-hand sense as seen looking along the positive z-axis.

K2.6. Law of conservation of charge.

D2.12. Three point charges $Q_1(t)$, $Q_2(t)$, and $Q_3(t)$ are situated at the vertices of a triangle and are connected by means of wires carrying currents. A current I A flows from Q_1 to Q_2 and $3I$ A flows from Q_2 to Q_3. The charge Q_3 is increasing with time at the rate of $5I$ C/s. Find the following: **(a)** dQ_1/dt; **(b)** dQ_2/dt; and **(c)** the current flowing from Q_1 to Q_3.

Ans. **(a)** $-3I$ C/s; **(b)** $-2I$ C/s; **(c)** $2I$ A.

2.7 APPLICATION TO STATIC FIELDS

Collecting together Faraday's law (2.13), Ampere's circuital law (2.17), Gauss' law for the electric field (2.27), and Gauss' law for the magnetic field (2.28), we have the four Maxwell's equations in integral form given by

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S} \tag{2.35a}$$

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt}\int_S \mathbf{D} \cdot d\mathbf{S} \tag{2.35b}$$

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho\, dv \tag{2.35c}$$

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0 \tag{2.35d}$$

whereas the law of conservation of charge is given by

$$\oint_S \mathbf{J} \cdot d\mathbf{S} = -\frac{d}{dt}\int_V \rho\, dv \tag{2.36}$$

Maxwell's equations in integral form for static fields

For static fields, that is, for $d/dt = 0$, Maxwell's equations in integral form become

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0 \tag{2.37a}$$

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S} \tag{2.37b}$$

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho\, dv \tag{2.37c}$$

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0 \tag{2.37d}$$

whereas the law of conservation of charge becomes

$$\oint_S \mathbf{J} \cdot d\mathbf{S} = 0 \tag{2.38}$$

It can be immediately seen from (2.37a)–(2.37d) that the interdependence between the electric and magnetic fields no longer exists. Equation (2.37a) tells us simply that the static electric field is a conservative field. Similarly, (2.37d) tells us that the magnetic flux is the same through all surfaces bounded by a closed path. On the other hand, (2.37c) and (2.37b) enable us to find the static electric and magnetic fields for certain time-invariant charge and current distributions, respectively. These distributions must be such that the resulting electric and magnetic fields possess symmetry to be able to replace the integrals on the left sides of (2.37c) and (2.37b) by algebraic expressions involving the components of electric and magnetic fields, respectively.

In addition, in the case of (2.37b), the current on the right side must be uniquely given for a given closed path C, which property is ensured by (2.38). An example in which this current is uniquely given is that of the infinitely long wire in Fig. 2.31(a). This is because the current crossing all possible surfaces bounded by the closed path C is equal to I since the wire, being infinitely long, pierces through all such surfaces. This can also be seen in a different manner by imagining the closed path to be a rigid loop and visualizing that the loop cannot be moved to one side of the wire without cutting the wire. On the other hand, if the wire is finitely long, as shown in Fig. 2.31(b), it can be seen that for some surfaces bounded by C, the wire pierces through the surface, whereas for some other surfaces, it does not. Alternatively, a rigid loop occupying the closed path can be moved to one side of the wire without cutting the wire. Thus, for this case, there is no unique value of the wire current enclosed by C and hence (2.37b) cannot be used to determine **H**. The problem here is that (2.38) is not satisfied, since for current to flow in the finitely long wire, there must be time-varying charges at the two ends, thereby giving rise to time-varying electric field. Hence, a displacement current exists in addition to the wire current

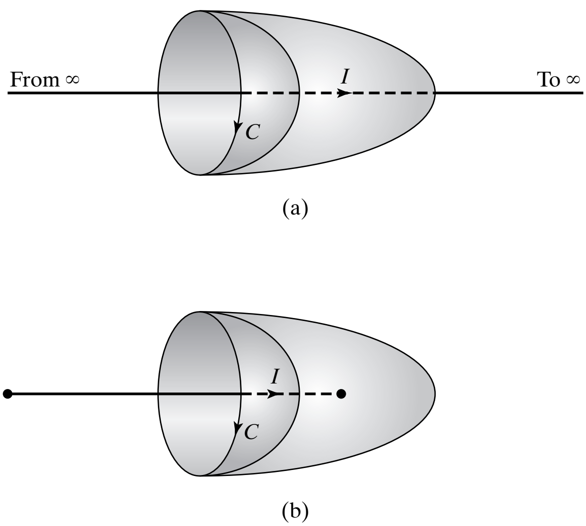

FIGURE 2.31

For illustrating that the current enclosed by a closed path C is uniquely given in (a) but not in (b).

such that the algebraic sum of the two currents crossing all surfaces bounded by C is the same and requires the use of (2.17).

We shall now illustrate the application of (2.37c) and (2.37b) by means of some examples.

Example 2.6 Electric field due to an infinitely long line charge using Gauss' law

D *due to a line charge*

Let us consider charge distributed uniformly with density ρ_{L0} C/m along the z-axis and find the electric field due to the infinitely long line charge using (2.37c).

Let us consider the closed surface S of a cylinder of radius r, with the line charge as its axis and extending from $z = 0$ to $z = l$, as shown in Fig. 2.32. Then according to (2.37c),

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \rho_{L0} l \qquad (2.39)$$

Although this result is valid for any closed surface enclosing the portion of the line charge from $z = 0$ to $z = l$, we have chosen the particular surface in Fig. 2.32 to be able to reduce the surface integral of **D** in (2.37c), and hence in (2.39), to an algebraic quantity. To do this, we note the following:

(a) In view of the uniform charge density, the entire line charge can be thought of as the superposition of pairs of equal point charges located at equal distances above and below any given point on the z-axis. Hence the field due to the entire line charge has only a radial component independent of ϕ and z.

(b) In view of (a), the contribution to the closed surface integral from the top and bottom surfaces of the cylindrical box is zero.

Thus, we have

$$\mathbf{D} = D_r(r)\mathbf{a}_r$$

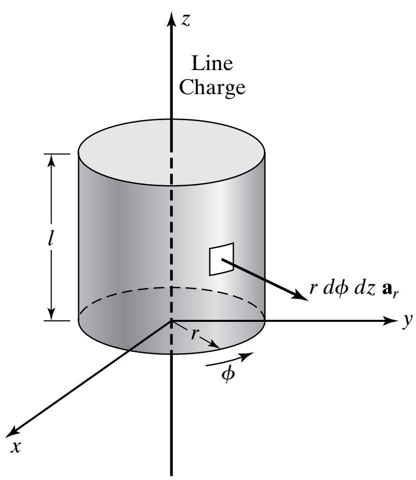

FIGURE 2.32

For the determination of electric field due to an infinitely long line charge of uniform density ρ_{L0} C/m.

and

$$\begin{aligned}\oint_S \mathbf{D}\cdot d\mathbf{S} &= \int_{\phi=0}^{2\pi}\int_{z=0}^{l} D_r(r)\mathbf{a}_r \cdot r\, d\phi\, dz\, \mathbf{a}_r \\ &= 2\pi r l D_r(r)\end{aligned} \tag{2.40}$$

Comparing (2.39) and (2.40), we obtain

$$\begin{aligned}2\pi r l D_r(r) &= \rho_{L0} l \\ D_r(r) &= \frac{\rho_{L0}}{2\pi r}\end{aligned}$$

$$\boxed{\mathbf{D} = \frac{\rho_{L0}}{2\pi r}\mathbf{a}_r} \tag{2.41}$$

The field varies inversely with the radial distance away from the line charge.

Example 2.7 Electric field due to a spherical volume charge using Gauss' law

D *due to a spherical volume charge*

Let us consider charge distributed uniformly with density ρ_0 C/m^3 in the spherical region $r \leq a$, as shown by the cross-sectional view in Fig. 2.33, and find the electric field due to the spherical charge by using (2.37c).

As in Example 2.6, we shall once again choose a surface S that enables the replacement of the surface integral in (2.37c) by an algebraic quantity. To do this, we note from considerations of symmetry, and of the spherical charge as a superposition of point charges, that $\mathbf{D}$ possesses only an r-component dependent on r only. Thus,

$$\mathbf{D} = D_r(r)\mathbf{a}_r$$

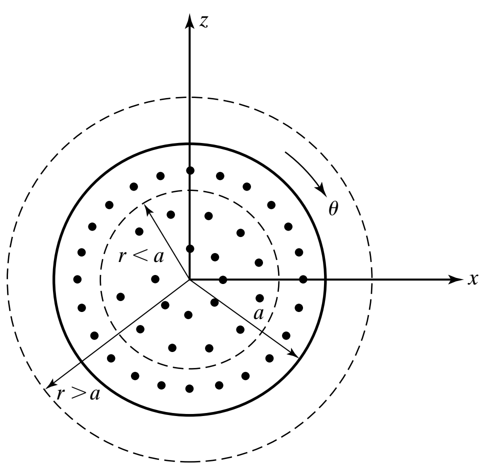

FIGURE 2.33

For the determination of electric field due to a spherical charge of uniform density ρ_0 C/m^3.

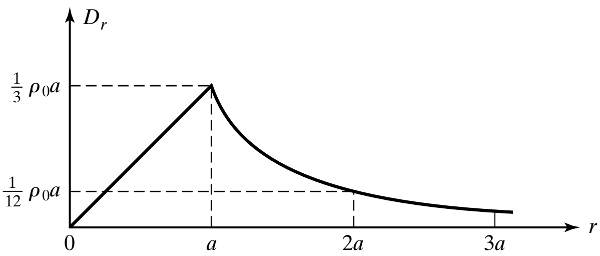

FIGURE 2.34
Variation of D_r with r for the spherical charge of Fig. 2.33.

Choosing, then, a spherical surface of radius r centered at the origin, we obtain

$$\begin{aligned}\oint_S \mathbf{D} \cdot d\mathbf{S} &= \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} D_r(r)\mathbf{a}_r \cdot r^2 \sin\theta \, d\theta \, d\phi \, \mathbf{a}_r \\ &= 4\pi r^2 D_r(r)\end{aligned} \tag{2.42}$$

Noting that the charge exists only for $r < a$, and with uniform density, we obtain the charge enclosed by the spherical surface to be

$$\int_V \rho \, dv = \begin{cases} \frac{4}{3}\pi r^3 \rho_0 & \text{for} \quad r \le a \\ \frac{4}{3}\pi a^3 \rho_0 & \text{for} \quad r \ge a \end{cases} \tag{2.43}$$

Substituting (2.42) and (2.43) into (2.37c), we get

$$4\pi r^2 D_r(r) = \begin{cases} \frac{4}{3}\pi r^3 \rho_0 & \text{for} \quad r \le a \\ \frac{4}{3}\pi a^3 \rho_0 & \text{for} \quad r \ge a \end{cases}$$

$$D_r(r) = \begin{cases} \dfrac{\rho_0 r}{3} & \text{for} \quad r \le a \\ \dfrac{\rho_0 a^3}{3r^2} & \text{for} \quad r \ge a \end{cases}$$

$$\boxed{\mathbf{D} = \begin{cases} \dfrac{\rho_0 r}{3}\mathbf{a}_r & \text{for} \quad r \le a \\ \dfrac{\rho_0 a^3}{3r^2}\mathbf{a}_r & \text{for} \quad r \ge a \end{cases}} \tag{2.44}$$

The variation of D_r with r is shown plotted in Fig. 2.34.

Example 2.8 Magnetic field due to cylindrical wire of current using Ampere's circuital law

$\mathbf{H}$ due to a cylindrical wire of current

Let us consider current flowing with uniform density $\mathbf{J} = J_0\mathbf{a}_z$ A/m^2 in an infinitely long solid cylindrical wire of radius a with its axis along the z-axis, as shown by the cross-sectional view in Fig. 2.35. We wish to find the magnetic field everywhere using (2.37b).

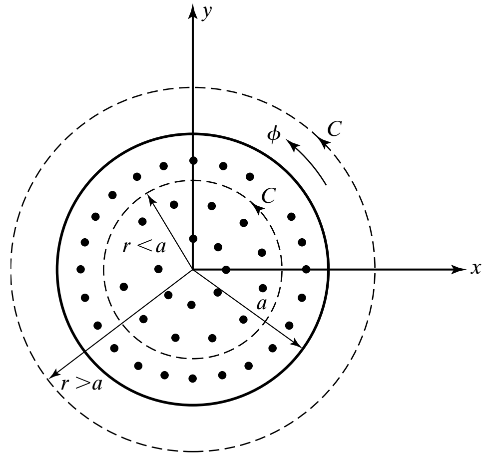

FIGURE 2.35

For the determination of magnetic field due to an infinitely long solid cylindrical wire of uniform current density $J_0\mathbf{a}_z$ A/m^2.

The current distribution can be thought of as the superposition of infinitely long filamentary wires parallel to the z-axis. Then in view of the symmetry about the z-axis and from the nature of the magnetic field due to an infinitely long wire given by (1.79), we can say that the required $\mathbf{H}$ has only a ϕ component dependent on r only. Thus,

$$\mathbf{H} = H_\phi(r)\mathbf{a}_\phi$$

Choosing, then, a circular closed path C of radius r lying in the xy-plane and centered at the origin, we obtain

$$\begin{aligned} \oint_C \mathbf{H} \cdot d\mathbf{l} &= \int_{\phi=0}^{2\pi} H_\phi(r)\mathbf{a}_\phi \cdot r\, d\phi\, \mathbf{a}_\phi \\ &= 2\pi r H_\phi(r) \end{aligned} \tag{2.45}$$

Considering the plane surface bounded by C, and noting that the current exists only for $r < a$, we obtain the current enclosed by the closed path to be

$$\begin{aligned} \int_S \mathbf{J} \cdot d\mathbf{S} &= \begin{cases} \displaystyle\int_{r=0}^{r}\int_{\phi=0}^{2\pi} J_0\mathbf{a}_z \cdot r\, dr\, d\phi\, \mathbf{a}_z & \text{for} \quad r \le a \\ \displaystyle\int_{r=0}^{a}\int_{\phi=0}^{2\pi} J_0\mathbf{a}_z \cdot r\, dr\, d\phi\, \mathbf{a}_z & \text{for} \quad r \ge a \end{cases} \\ &= \begin{cases} J_0\pi r^2 & \text{for } r \le a \\ J_0\pi a^2 & \text{for } r \ge a \end{cases} \end{aligned} \tag{2.46}$$

Substituting (2.45) and (2.46) into (2.37b), we get

$$2\pi r H_\phi = \begin{cases} J_0\pi r^2 & \text{for} \quad r \le a \\ J_0\pi a^2 & \text{for} \quad r \ge a \end{cases}$$

$$H_\phi = \begin{cases} \dfrac{J_0 r}{2} & \text{for} \quad r \le a \\ \dfrac{J_0 a^2}{2r} & \text{for} \quad r \ge a \end{cases}$$

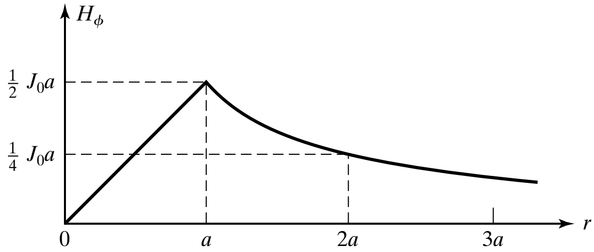

FIGURE 2.36
Variation of H_ϕ with r for the cylindrical wire of current of Fig. 2.35.

$$\boxed{\mathbf{H} = \begin{cases} \dfrac{J_0 r}{2}\mathbf{a}_\phi & \text{for} \quad r \le a \\ \dfrac{J_0 a^2}{2r}\mathbf{a}_\phi & \text{for} \quad r \ge a \end{cases}} \tag{2.47}$$

The variation of H_ϕ with r is shown plotted in Fig. 2.36.

K2.7. Maxwell's equations in integral form for static fields; Uniqueness of current enclosed by a closed path; **D** due to symmetrical charge distributions; **H** due to symmetrical current distributions.

D2.13. Charge is distributed with uniform density ρ_0 C/m^3 inside a regular solid of edges a. Find the displacement flux emanating from one side of the solid for each of the following shapes of the solid: **(a)** tetrahedron; **(b)** cube; and **(c)** octahedron.

Ans. **(a)** $0.0295\rho_0 a^3$ C; **(b)** $0.1667\rho_0 a^3$ C; **(c)** $0.0589\rho_0 a^3$ C.

D2.14. The cross section of an infinitely long solid wire having the z-axis as its axis is a regular polygon of sides a. Current flows in the wire with uniform density $J_0\mathbf{a}_z$ A/m^2. Find the line integral of **H** along one side of the polygon and traversed in the sense of increasing ϕ for each of the following shapes of the polygon: **(a)** equilateral triangle; **(b)** square; and **(c)** octagon.

Ans. **(a)** $0.1443a^2J_0$ A; **(b)** $0.25a^2J_0$ A; **(c)** $0.6036a^2J_0$ A.

SUMMARY

We first learned in this chapter how to evaluate line and surface integrals of vector quantities, and then we introduced Maxwell's equations in integral form. These equations, which form the basis of electromagnetic field theory, are given as follows in words and in mathematical form:

Faraday's law. The electromotive force around a closed path C is equal to the negative of the time rate of change of the magnetic flux enclosed by that path; that is,

$$\oint_C \mathbf{E}\cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S} \tag{2.48}$$

Ampère's circuital law. The magnetomotive force around a closed path C is equal to the sum of the current enclosed by that path due to the actual flow of charges and the displacement current due to the time rate of change of the displacement flux enclosed by that path; that is,

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt}\int_S \mathbf{D} \cdot d\mathbf{S} \tag{2.49}$$

Gauss' law for the electric field. The displacement flux emanating from a closed surface S is equal to the charge enclosed by that surface; that is,

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho\, dv \tag{2.50}$$

Gauss' law for the magnetic field. The magnetic flux emanating from a closed surface S is equal to zero; that is,

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0 \tag{2.51}$$

An auxiliary equation, the law of conservation of charge, is given by

$$\oint_S \mathbf{J} \cdot d\mathbf{S} = -\frac{d}{dt}\int_V \rho\, dv \tag{2.52}$$

In words, (2.52) states that the current due to flow of charges emanating from a closed surface is equal to the time rate of decrease of the charge enclosed by that surface.

In using (2.48)–(2.52), we recall that

$$\mathbf{D} = \varepsilon_0 \mathbf{E} \tag{2.53}$$

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} \tag{2.54}$$

In evaluating the right sides of (2.48) and (2.49), the normal vectors to the surfaces must be chosen such that they are directed in the right-hand sense, that is, toward the side of advance of a right-hand screw as it is turned around C. In (2.50), (2.51), and (2.52), it is understood that the surface integrals are evaluated so as to find the flux outward from the volume bounded by the surface. We also learned that (2.51) is not independent of (2.48) and that (2.50) follows from (2.49) with the aid of (2.52).

Finally, we discussed several applications of Maxwell's equations, including the computation of static electric and magnetic fields due to symmetrical charge and current distributions, respectively.

REVIEW QUESTIONS

Q2.1. How do you find the work done in moving a test charge by an infinitesimal distance in an electric field? What is the amount of work involved in moving the test charge normal to the electric field?

Q2.2. What is the physical interpretation of the line integral of $\mathbf{E}$ between two points A and B?

Q2.3. How do you find the approximate value of the line integral of a vector field along a given path? How do you find the exact value of the line integral?

Q2.4. Discuss conservative versus nonconservative fields, giving examples.

Q2.5. How do you find the magnetic flux crossing an infinitesimal surface?

Q2.6. What is the magnetic flux crossing an infinitesimal surface oriented parallel to the magnetic flux density vector? For what orientation of the infinitesimal surface relative to the magnetic flux density vector is the magnetic flux crossing the surface a maximum?

Q2.7. How do you find the approximate value of the surface integral of a vector field over a given surface? How do you find the exact value of the surface integral?

Q2.8. Provide physical interpretations for the closed surface integrals of any two vectors of your choice.

Q2.9. State Faraday's law.

Q2.10. What are the different ways in which an emf is induced around a loop?

Q2.11. Discuss the right-hand screw rule convention associated with the application of Faraday's law.

Q2.12. To find the induced emf around a planar loop, is it necessary to consider the magnetic flux crossing the plane surface bounded by the loop? Explain.

Q2.13. What is Lenz's law?

Q2.14. Discuss briefly the motional emf concept.

Q2.15. How would you orient a loop antenna to obtain maximum signal from an incident electromagnetic wave that has its magnetic field directed along the north–south line?

Q2.16. State three applications of Faraday's law.

Q2.17. State Ampère's circuital law.

Q2.18. What is displacement current? Compare and contrast displacement current with current due to flow of charges.

Q2.19. Is it meaningful to consider two different surfaces bounded by a closed path to compute the two different currents on the right side of Ampère's circuital law to find $\oint \mathbf{H} \cdot d\mathbf{l}$ around the closed path?

Q2.20. Discuss the relationship between the displacement current emanating from a closed surface and the current due to flow of charges emanating from the same closed surface.

Q2.21. Give an example involving displacement current.

Q2.22. Discuss briefly the principle of radiation from a wire carrying a time-varying current.

Q2.23. State Gauss' law for the electric field.

Q2.24. How do you evaluate a volume integral?

Q2.25. State Gauss' law for the magnetic field.

Q2.26. What is the physical interpretation of Gauss' law for the magnetic field?

Q2.27. Discuss the dependence of Gauss' law for the magnetic field on Faraday's law.

Q2.28. State the law of conservation of charge.

Q2.29. How is Gauss' law for the electric field dependent on Ampère's circuital law?

Q2.30. Summarize Maxwell's equations in integral form for time-varying fields.

Q2.31. Summarize Maxwell's equations in integral form for static fields.

Q2.32. Are static electric and magnetic fields interdependent? Explain.

Q2.33. Discuss briefly the application of Gauss' law for the electric field to determine the electric field due to charge distributions.

Q2.34. When can you say that the current in a wire enclosed by a closed path is uniquely defined? Give two examples.

Q2.35. Give an example in which the current in a wire enclosed by a closed path is not uniquely defined. Is it correct to apply Ampère's circuital law for the static case in such a situation? Explain.

Q2.36. Discuss briefly the application of Ampère's circuital law to determine the magnetic field due to current distributions.

PROBLEMS

Section 2.1

P2.1. Evaluation of line integral in Cartesian coordinates. For the vector field $\mathbf{F} = y\mathbf{a}_x - z\mathbf{a}_y + x\mathbf{a}_z$, find $\int_{(0,0,0)}^{(1,1,1)} \mathbf{F} \cdot d\mathbf{l}$ for each of the following paths from $(0, 0, 0)$ to $(1, 1, 1)$: **(a)** $x = y = z$ and **(b)** $x = y = z^3$.

P2.2. Evaluation of line integral around a closed path in Cartesian coordinates. Given $\mathbf{F} = xy\mathbf{a}_x + yz\mathbf{a}_y + zx\mathbf{a}_z$, find $\oint_C \mathbf{F} \cdot d\mathbf{l}$, where C is the closed path comprising the straight lines from $(0, 0, 0)$ to $(1, 1, 1)$, from $(1, 1, 1)$ to $(1, 1, 0)$, and from $(1, 1, 0)$ to $(0, 0, 0)$.

P2.3. Evaluation of line integral in Cartesian coordinates. For the vector field $\mathbf{F} = \cos y\ \mathbf{a}_x - x \sin y\ \mathbf{a}_y$, find $\int_{(0,0,0)}^{(1,2\pi,1)} \mathbf{F} \cdot d\mathbf{l}$ in each of the following ways: **(a)** along the straight-line path between the two points; **(b)** along the curved path $x = z = \sin(y/4)$ between the two points; and **(c)** without choosing any particular path. Is the vector field conservative or nonconservative? Explain.

P2.4. Evaluation of line integral around closed path in cylindrical coordinates. Given $\mathbf{A} = 2r \sin\phi\ \mathbf{a}_r + r^2\mathbf{a}_\phi + z\mathbf{a}_z$ in cylindrical coordinates, find $\oint_C \mathbf{A} \cdot d\mathbf{l}$, where C is the closed path comprising the straight line from $(0, 0, 0)$ to $(1, 0, 0)$, the circular arc from $(1, 0, 0)$ to $(1, \pi/2, 0)$ through $(1, \pi/4, 0)$, the straight line from $(1, \pi/2, 0)$ to $(1, \pi/2, 1)$, and the straight line from $(1, \pi/2, 1)$ to $(0, 0, 0)$.

P2.5. Evaluation of line integral in spherical coordinates. Given $\mathbf{A} = e^{-r}(\cos\theta\ \mathbf{a}_r + \sin\theta\ \mathbf{a}_\theta) + r \sin\theta\ \mathbf{a}_\phi$ in spherical coordinates, find $\int \mathbf{A} \cdot d\mathbf{l}$ for each of the following paths: **(a)** straight-line path from $(0, 0, 0)$ to $(2, 0, 0)$; **(b)** circular arc from $(2, 0, \pi/4)$ to $(2, \pi/2, \pi/4)$ through $(2, \pi/4, \pi/4)$; and **(c)** circular arc from $(2, \pi/6, 0)$ to $(2, \pi/6, \pi/2)$ through $(2, \pi/6, \pi/4)$.

Section 2.2

P2.6. Evaluation of a closed surface integral in Cartesian coordinates. Given $\mathbf{A} = x^2yz\mathbf{a}_x + y^2zx\mathbf{a}_y + z^2xy\mathbf{a}_z$, evaluate $\oint_S \mathbf{A}\cdot d\mathbf{S}$, where S is the surface of the cubical box bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, and $z = 1$.

P2.7. Evaluation of a closed surface integral in Cartesian coordinates. Given $\mathbf{A} = (x^2y + 2)\mathbf{a}_x + 3\mathbf{a}_y - 2xyz\mathbf{a}_z$, evaluate $\oint_S \mathbf{A}\cdot d\mathbf{S}$, where S is the surface of the rectangular box bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 2$, $z = 0$, and $z = 3$.

P2.8. Evaluation of a closed surface integral in cylindrical coordinates. Given $\mathbf{A} = r\cos\phi\,\mathbf{a}_r - r\sin\phi\,\mathbf{a}_\phi$ in cylindrical coordinates, evaluate $\oint_S \mathbf{A}\cdot d\mathbf{S}$, where S is the surface of the box bounded by the plane surfaces $\phi = 0$, $\phi = \pi/2$, $z = 0$, $z = 1$, and the cylindrical surface $r = 2$, $0 < \phi < \pi/2$.

P2.9. Evaluation of a closed surface integral in spherical coordinates. Given $\mathbf{A} = r^2\mathbf{a}_r + r\sin\theta\,\mathbf{a}_\theta$ in spherical coordinates, find $\oint_S \mathbf{A}\cdot d\mathbf{S}$, where S is the surface of that part of the spherical volume of radius unity and lying in the first octant.

Section 2.3

P2.10. Induced emf around a closed path in a time-varying magnetic field. Find the induced emf around the rectangular closed path C connecting the points $(0, 0, 0)$, $(a, 0, 0)$, $(a, b, 0)$, $(0, b, 0)$, and $(0, 0, 0)$, in that order, for each of the following magnetic fields:

(a) $\mathbf{B} = \dfrac{B_0a^2}{(x + a)^2}e^{-t}\mathbf{a}_z$

(b) $\mathbf{B} = B_0 \sin\dfrac{\pi x}{a}\cos\omega t\,\mathbf{a}_z$

P2.11. Induced emf around a moving loop in a static magnetic field. A magnetic field is given in the xz-plane by $\mathbf{B} = (B_0/x)\mathbf{a}_y$ Wb/m^2, where B_0 is a constant. A rigid rectangular loop is situated in the xz-plane and with its corners at the points (x_0, z_0), $(x_0, z_0 + b)$, $(x_0 + a, z_0 + b)$, and $(x_0 + a, z_0)$. If the loop is moving in that plane with a velocity $\mathbf{v} = v_0\mathbf{a}_x$ m/s, where v_0 is a constant, find by using Faraday's law the induced emf around the loop in the sense defined by connecting the above points in succession. Discuss your result by using the motional emf concept.

P2.12. Induced emf around a closed path in a time-varying magnetic field. A magnetic field is given in the xz-plane by $\mathbf{B} = B_0\cos\pi(x - v_0t)\,\mathbf{a}_y$ Wb/m^2. Consider a rigid square loop situated in the xz-plane with its vertices at $(x, 0, 1)$, $(x, 0, 2)$, $(x + 1, 0, 2)$, and $(x + 1, 0, 1)$. **(a)** Find the expression for the emf induced around the loop in the sense defined by connecting the above points in succession. **(b)** What would be the induced emf if the loop is moving with the velocity $\mathbf{v} = v_0\mathbf{a}_x$ m/s instead of being stationary?

P2.13. Induced emf around a swinging loop in a static magnetic field. A rigid rectangular loop of metallic wire is hung by pivoting one side along the x-axis, as shown in Fig. 2.37. The loop is free to swing about the pivoted side without friction under the influence of gravity and in the presence of a uniform magnetic field $\mathbf{B} = B_0\mathbf{a}_z$ Wb/m^2. If the loop is given a slight angular displacement and released, show that the emf induced around the closed path C of the loop is approximately equal to $B_0ab\omega$, where ω is the angular velocity of swing of the loop

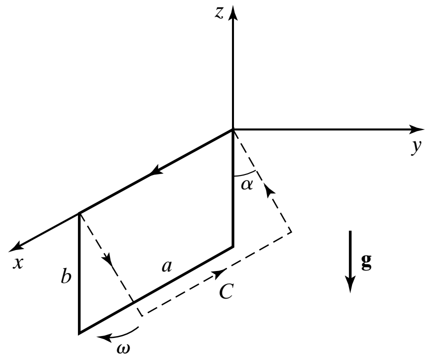

FIGURE 2.37
For Problem P2.13.

toward the vertical. Does the loop swing faster or slower than in the absence of the magnetic field? Explain.

P2.14. A conducting bar rolling down inclined rails in a uniform static magnetic field. A rigid conducting bar of length L, mass M, and electrical resistance R rolls without friction down two parallel conducting rails that are inclined at an angle α with the horizontal, as shown in Fig. 2.38. The rails are of negligible resistance and are joined at the bottom by another conductor, also of negligible resistance, so that the total resistance of the loop formed by the rolling bar and the three other sides is R. The entire arrangement is situated in a region of uniform static magnetic field $\mathbf{B} = B_0\mathbf{a}_z$ Wb/m^2, directed vertically downward. Assume the bar to be rolling down with uniform velocity $\mathbf{v}$ parallel to the rails under the influence of Earth's gravity (acting in the positive z-direction) and the magnetic force due to the current in the loop produced by the induced emf. Show that v is equal to $(MgR/B_0^2L^2)\tan\alpha\sec\alpha$.

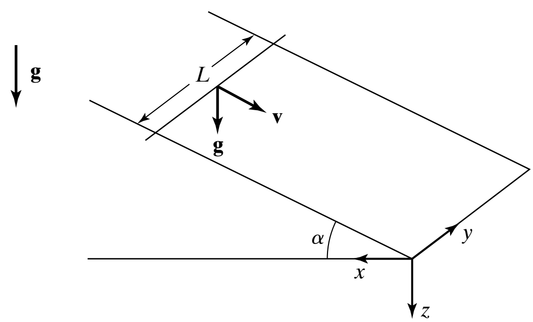

FIGURE 2.38
For Problem P2.14.

P2.15. Induced emf around a revolving loop in a static magnetic field. A rigid rectangular loop of base b and height h situated normal to the xy-plane and with its sides pivoted to the z-axis revolves about the z-axis with angular velocity ω rad/s in the sense of increasing ϕ, as shown in Fig. 2.39. Find the induced emf around the closed path C of the loop for each of the following magnetic fields: **(a)** $\mathbf{B} = B_0\mathbf{a}_y$ Wb/m^2 and **(b)** $\mathbf{B} = B_0(y\mathbf{a}_x - x\mathbf{a}_y)$ Wb/m^2. Assume the loop to be in the xz-plane at $t = 0$.

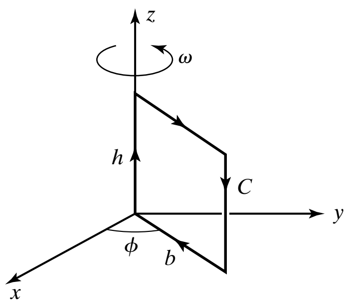

FIGURE 2.39
For Problem P2.15.

P2.16. Induced emf around a loop in a time-varying magnetic field for several cases. A rigid rectangular loop of area A is situated in the xz-plane and symmetrically about the z-axis, as shown in Fig. 2.40, in a region of magnetic field $\mathbf{B} = B_0(\sin \omega t\, \mathbf{a}_x + \cos \omega t\, \mathbf{a}_y)$ Wb/m^2. Find the induced emf around the closed path C of the loop for each of the following cases: **(a)** the loop is stationary; **(b)** the loop revolves around the z-axis in the sense of increasing ϕ with uniform angular velocity of ω rad/s; and **(c)** the loop revolves around the z-axis in the sense of decreasing ϕ with uniform angular velocity of ω rad/s. For parts (b) and (c), assume that the loop is in the xz-plane at $t = 0$.

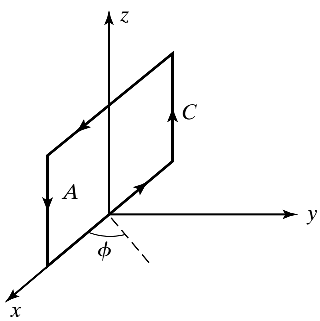

FIGURE 2.40
For Problem P2.16.

Section 2.4

P2.17. Application of Ampere's circuital law in integral form. Given that $\mathbf{H} = \pm H_0(t \mp \sqrt{\mu_0\varepsilon_0}\, z)^2\mathbf{a}_y$ and $\mathbf{D} = \sqrt{\mu_0\varepsilon_0}\, H_0(t \mp \sqrt{\mu_0\varepsilon_0}\, z)^2\mathbf{a}_x$ for $z \gtrless 0$, find the current due to flow of charges enclosed by the rectangular closed path from $(0, 0, 1)$ to $(0, 1, 1)$ to $(0, 1, -1)$ to $(0, 0, -1)$ to $(0, 0, 1)$.

P2.18. Application of Ampere's circuital law in integral form. A current density due to flow of charges is given by $\mathbf{J} = -(x\mathbf{a}_x + y\mathbf{a}_y + z^2\mathbf{a}_z)$ A/m^2. Find the displacement current emanating from each of the following closed surfaces: **(a)** the surface of the cubical box bounded by the planes $x = \pm 2$, $y = \pm 2$, and $z = \pm 2$, and **(b)** the surface of the cylindrical box bounded by the surfaces $r = 1$, $z = 0$, and $z = 2$.

P2.19. Finding rms value of current drawn from voltage source connected to a capacitor. A voltage source connected to a parallel-plate capacitor by means of wires sets up a uniform electric field of $E = 180 \sin 2\pi \times 10^6 t \sin 4\pi \times 10^6 t$ V/m between the plates of the capacitor and normal to the plates. Assume that no field exists outside the region between the plates. If the area of each plate is $0.1\ \text{m}^2$ and the medium between the plates is free space, find the root-mean-square value of the current drawn from the voltage source.

P2.20. Finding rms value of current drawn from voltage source connected to a capacitor. Assume that the time variation of the electric field in Problem P3.19 is as shown in Fig. 2.41. Find and plot versus time the current drawn from the voltage source. What is the root-mean-square value of the current?

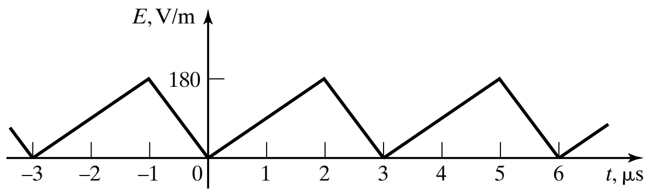

FIGURE 2.41
For Problem P2.20.

Section 2.5

P2.21. Finding displacement flux emanating from a surface enclosing charge. For each of the following charge distributions, find the displacement flux emanating from the surface enclosing the charge: **(a)** $\rho(x, y, z) = \rho_0(3 - x^2 - y^2 - z^2)$ for the cubical box bounded by $x = \pm 1$, $y = \pm 1$, and $z = \pm 1$; and **(b)** $\rho(x, y, z) = \rho_0(xyz)$ for $x > 0$, $y > 0$, $z > 0$, and $x^2 + y^2 + z^2 < 1$.

P2.22. Finding displacement flux emanating from a surface enclosing charge. For each of the following charge distributions, find the displacement flux emanating from the surface enclosing the charge: **(a)** $\rho(r, \phi, z) = \rho_0 e^{-r^2}$ for $r < 1, 0 < z < 1$ in cylindrical coordinates; and **(b)** $\rho(r, \theta, \phi) = (\rho_0/r)\sin^2\theta$ for $r < 1, 0 < \theta < \pi/2$ in spherical coordinates.

P2.23. Application of Gauss' law for the magnetic field in integral form. Using the property that $\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$, find the absolute value of the magnetic flux crossing that portion of the surface $y = \sin x$ bounded by $x = 0$, $x = \pi$, $z = 0$, and $z = 1$ for $\mathbf{B} = B_0(y\mathbf{a}_x - x\mathbf{a}_y)\ \text{Wb/m}^2$.

Section 2.6

P2.24. Application of the law of conservation of charge. Given $\mathbf{J} = (x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z)\ \text{A/m}^2$, find the time rate of decrease of the charge contained within each of the following volumes: **(a)** volume bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, and $z = 1$; **(b)** volume bounded by the cylinders $r = 1$ and $r = 2$ and the planes $z = 0$ and $z = 1$; and **(c)** volume bounded by the spherical surfaces $r = 1$ and $r = 2$ and the conical surface $\theta = \pi/3$.

P2.25. Combined application of several of Maxwell's equations in integral form. Current I flows along a straight wire from a point charge $Q_1(t)$ located at the origin to a point charge $Q_2(t)$ located at $(0, 0, 1)$. Find the line integral of $\mathbf{H}$ along the square closed path having the vertices at $(1, 1, 0)$, $(-1, 1, 0)$, $(-1, -1, 0)$, and $(1, -1, 0)$ and traversed in that order.

P2.26. Combined application of several of Maxwell's equations in integral form. Current I flows along a straight wire from a point charge $Q_1(t)$ at the origin to a point charge $Q_2(t)$ at the point $(2, 2, 2)$. Find the line integral of $\mathbf{H}$ around the triangular closed path having the vertices at (3, 0, 0), (0, 3, 0), and (0, 0, 3) and traversed in that order.

Section 2.7

P2.27. Application of Gauss' law for the electric field in integral form and symmetry. Charge is distributed with density $\rho(x, y, z)$ in a cubical box bounded by the planes $x = \pm 1$ m, $y = \pm 1$ m, and $z = \pm 1$ m. Find the displacement flux emanating from one side of the box for each of the following cases: **(a)** $\rho(x, y, z) = (3 - x^2 - y^2 - z^2)$ C/m^3 and **(b)** $\rho(x, y, z) = \sqrt{|xyz|}$ C/m^3.

P2.28. Electric field due to a cylindrical charge distribution using Gauss' law. Charge is distributed with density $\rho_0 e^{-r^2}$ C/m^3 in the cylindrical region $r < 1$. Find **D** everywhere.

P2.29. Electric field due to a spherical charge distribution using Gauss' law. Charge is distributed with uniform density ρ_0C/m^3 in the region $a < r < 2a$ in spherical coordinates. Find **D** everywhere and plot D_r versus r.

P2.30. Application of Ampere's circuital law in integral form and symmetry. Current flows with density $\mathbf{J}(x, y)$ in an infinitely long thick wire having the z-axis as its axis. The cross section of the wire in the xy-plane is the square bounded by $x = \pm 1$ m and $y = \pm 1$ m. Find the line integral of **H** along one side of the square and traversed in the sense of increasing ϕ for each of the following cases: **(a)** $\mathbf{J}(x, y) = (|x| + |y|)\mathbf{a}_z$ A/m^2 and **(b)** $\mathbf{J}(x, y) = x^2y^2\mathbf{a}_z$ A/m^2.

P2.31. Magnetic field due to a solid wire of current using Ampere's circuital law. Current flows with density $\mathbf{J} = J_0(r/a)\mathbf{a}_z$ A/m^2 along an infinitely long solid cylindrical wire of radius a having the z-axis as its axis. Find **H** everywhere and plot H_ϕ versus r.

P2.32. Magnetic field for a coaxial cable using Ampere's circuital law. A coaxial cable consists of an inner conductor of radius $3a$ and an outer conductor of inner radius $4a$ and outer radius $5a$. Assume the cable to be infinitely long and its axis to be along the z-axis. Current I flows with uniform density in the $+z$-direction in the inner conductor and returns with uniform density in the $-z$-direction in the outer conductor. Find **H** everywhere and plot H_ϕ versus r.

REVIEW PROBLEMS

R2.1. Determination of a specified static vector field to be a conservative field. Show that the vector field given by

$$\mathbf{F} = \cos\theta \sin\phi \, \mathbf{a}_r - \sin\theta \sin\phi \, \mathbf{a}_\theta + \cot\theta \cos\theta \, \mathbf{a}_\phi$$

is a conservative field. Then find the value of $\int \mathbf{F} \cdot d\mathbf{l}$ from the point $(1, \pi/6, \pi/3)$ to the point $(4, \pi/3, \pi/6)$.

R2.2. Induced emf around an expanding loop in a nonuniform static magnetic field. In Fig. 2.42, a rectangular loop of wire with three sides fixed and the fourth side movable is situated in a plane perpendicular to a nonuniform magnetic field $\mathbf{B} = B_0 y \mathbf{a}_z$ Wb/m^2, where B_0 is a constant. The position of the movable side is varied with time in the manner $y = y_0 + a \cos \omega t$, where $a < y_0$. Find the induced emf around the closed path C of the loop. Verify that Lenz's law is satisfied. Show also that the induced emf consists of two frequency components, ω and 2ω.

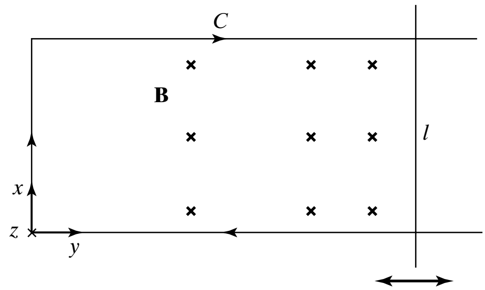

FIGURE 2.42
For Problem R2.2.

R2.3. Finding amplitude of current from sinusoidal voltage source connected to a capacitor. A voltage source is connected by means of wires to a parallel-plate capacitor made up of circular plates of radii a in the $z = 0$ and $z = d$ planes and having their centers on the z-axis. The electric field between the plates is given by

$$\mathbf{E} = E_0 \sin\frac{\pi r}{2a} \cos \omega t \, \mathbf{a}_z \qquad \text{for} \quad r < a$$

Find the amplitude of the current drawn from the voltage source, assuming the region between the plates to be free space and that no field exists outside this region.

R2.4. Combined application of several of Maxwell's equations in integral form. Current I flows along a straight wire from a point charge $Q_1(t)$ located at one of the vertices of a cube to a point charge $Q_2(t)$ at the center of the cube. Find the absolute value of the line integral of $\mathbf{H}$ around the periphery of one of the three sides of the cube not containing the vertex at which Q_1 is located.

R2.5. Electric field due to a spherical charge distribution using Gauss' law. Charge is distributed with density $\rho = \rho_0(r/a)^2$, where ρ_0 is a constant, in the spherical region $r < a$. Find $\mathbf{D}$ everywhere and plot D_r versus r.

R2.6. Magnetic field in the hollow region of wire bounded by two parallel cylindrical surfaces. Current flows axially with uniform density $\mathbf{J}_0$ A/m^2 in the region between two infinitely long parallel, cylindrical surfaces of radii a and b $(<a)$, and with their axes separated by the vector distance $\mathbf{c}$, where $|\mathbf{c}| < (a - b)$. Find the magnetic field intensity in the current-free region inside the cylindrical surface of radius b.