APPENDICES

A. CYLINDRICAL AND SPHERICAL COORDINATE SYSTEMS

In Sec. 1.2 we learned that the Cartesian coordinate system is defined by a set of three mutually orthogonal surfaces, all of which are planes. The cylindrical and spherical coordinate systems also involve sets of three mutually orthogonal surfaces. For the cylindrical coordinate system, the three surfaces are a cylinder and two planes, as shown in Fig. A.1(a). One of these planes is the same as the $z =$ constant plane in the Cartesian coordinate system. The second plane contains the z axis and makes an angle ϕ with a reference plane, conveniently chosen to be the xz plane of the Cartesian coordinate system. This plane is therefore defined by $\phi =$ constant. The cylindrical surface has the z axis as its axis. Since the radial distance r from the z axis to points on the cylindrical surface is a constant, this surface is defined by $r =$ constant. Thus the three orthogonal surfaces defining the cylindrical coordinates of a point are $r =$ constant, $\phi =$ constant, and $z =$ constant. Only two of these coordinates (r and z) are distances; the third coordinate (ϕ) is an angle. We note that the entire space is spanned by varying r from 0 to ∞, ϕ from 0 to 2π, and z from $-\infty$ to ∞.

The origin is given by $r = 0, \phi = 0$, and $z = 0$. Any other point in space is given by the intersection of three mutually orthogonal surfaces obtained by incrementing the coordinates by appropriate amounts. For example, the intersection of the three surfaces $r = 2$, $\phi = \pi/4$, and $z = 3$ defines the point $A(2, \pi/4, 3)$, as shown in Fig. A.1(a). These three orthogonal surfaces define three curves that are mutually perpendicular. Two of these are straight lines and the third is a circle. We draw unit vectors, $\mathbf{i}_r$, $\mathbf{i}_\phi$, and $\mathbf{i}_z$ tangential to these

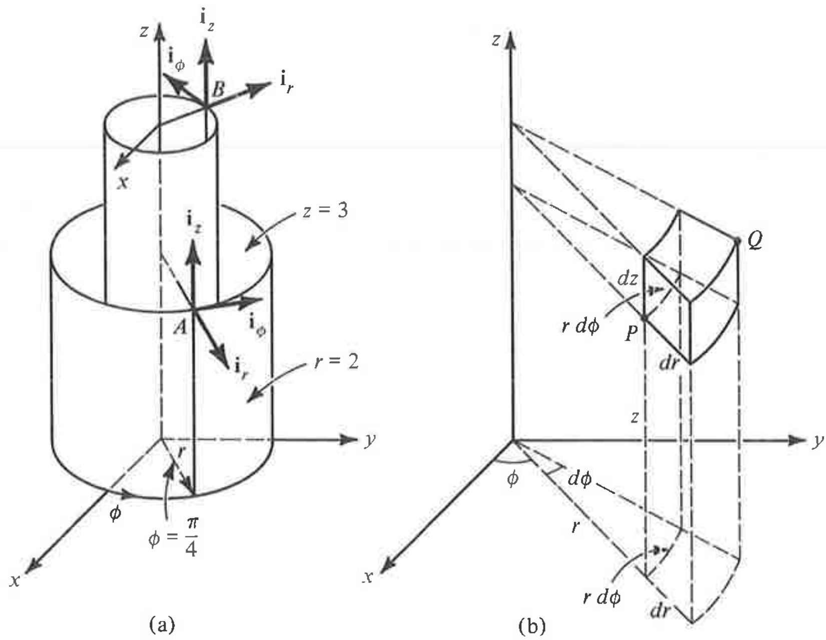

Figure A.1. Cylindrical coordinate system. (a) Orthogonal surfaces and unit vectors. (b) Differential volume formed by incrementing the coordinates.

curves at the point A and directed toward increasing values of r, ϕ, and z, respectively. These three unit vectors form a set of mutually orthogonal unit vectors in terms of which vectors drawn at A can be described. In a similar manner, we can draw unit vectors at any other point in the cylindrical coordinate system, as shown, for example, for point $B(1, 3\pi/4, 5)$ in Fig. A.1(a). It can now be seen that the unit vectors $\mathbf{i}_r$ and $\mathbf{i}_\phi$ at point B are not parallel to the corresponding unit vectors at point A. Thus unlike in the Cartesian coordinate system, the unit vectors $\mathbf{i}_r$ and $\mathbf{i}_\phi$ in the cylindrical coordinate system do not have the same directions everywhere, that is, they are not uniform. Only the unit vector $\mathbf{i}_z$, which is the same as in the Cartesian coordinate system, is uniform. Finally, we note that for the choice of ϕ as in Fig. A.1(a), that is, increasing from the positive x axis toward the positive y axis, the coordinate system is right-handed, that is, $\mathbf{i}_r \times \mathbf{i}_\phi = \mathbf{i}_z$.

To obtain expressions for the differential lengths, surfaces, and volume in the cylindrical coordinate system, we now consider two points $P(r, \phi, z)$ and $Q(r + dr, \phi + d\phi, z + dz)$ where Q is obtained by incrementing infinitesimally each coordinate from its value at P, as shown in Fig. A.1(b). The three orthogonal surfaces intersecting at P and the three orthogonal surfaces intersecting at Q define a box which can be considered to be rectangular since

dr, $d\phi$, and dz are infinitesimally small. The three differential length elements forming the contiguous sides of this box are $dr\,\mathbf{i}_r$, $r\,d\phi\,\mathbf{i}_\phi$, and $dz\,\mathbf{i}_z$. The differential length vector $d\mathbf{l}$ from P to Q is thus given by

$$d\mathbf{l} = dr\,\mathbf{i}_r + r\,d\phi\,\mathbf{i}_\phi + dz\,\mathbf{i}_z \tag{A.1}$$

The differential surfaces formed by pairs of the differential length elements are

$$\pm dS\,\mathbf{i}_z = \pm(dr)\,(r\,d\phi)\,\mathbf{i}_z = \pm dr\,\mathbf{i}_r \times r\,d\phi\,\mathbf{i}_\phi \tag{A.2a}$$

$$\pm dS\,\mathbf{i}_r = \pm(r\,d\phi)\,(dz)\,\mathbf{i}_r = \pm r\,d\phi\,\mathbf{i}_\phi \times dz\,\mathbf{i}_z \tag{A.2b}$$

$$\pm dS\,\mathbf{i}_\phi = \pm(dz)\,(dr)\,\mathbf{i}_\phi = \pm dz\,\mathbf{i}_z \times dr\,\mathbf{i}_r \tag{A.2c}$$

Finally, the differential volume dv formed by the three differential lengths is simply the volume of the box, that is,

$$dv = (dr)\,(r\,d\phi)\,(dz) = r\,dr\,d\phi\,dz \tag{A.3}$$

For the spherical coordinate system, the three mutually orthogonal surfaces are a sphere, a cone, and a plane, as shown in Fig. A.2(a). The plane is the same as the ϕ = constant plane in the cylindrical coordinate system.

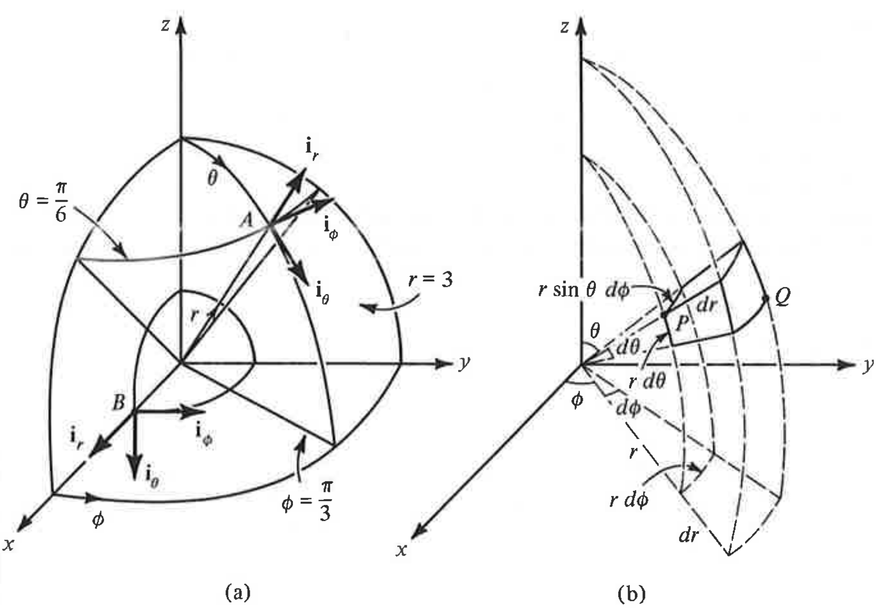

Figure A.2. Spherical coordinate system. (a) Orthogonal surfaces and unit vectors. (b) Differential volume formed by incrementing the coordinates.

The sphere has the origin as its center. Since the radial distance r from the origin to points on the spherical surface is a constant, this surface is defined by $r =$ constant. The spherical coordinate r should not be confused with the cylindrical coordinate r. When these two coordinates appear in the same expression, we shall use the subscripts c and s to distinguish between cylindrical and spherical. The cone has its vertex at the origin and its surface is symmetrical about the z axis. Since the angle θ is the angle that the conical surface makes with the z axis, this surface is defined by $\theta =$ constant. Thus the three orthogonal surfaces defining the spherical coordinates of a point are $r =$ constant, $\theta =$ constant, and $\phi =$ constant. Only one of these coordinates (r) is distance; the other two coordinates (θ and ϕ) are angles. We note that the entire space is spanned by varying r from 0 to ∞, θ from 0 to π, and ϕ from 0 to 2π.

The origin is given by $r = 0$, $\theta = 0$, and $\phi = 0$. Any other point in space is given by the intersection of three mutually orthogonal surfaces obtained by incrementing the coordinates by appropriate amounts. For example, the intersection of the three surfaces $r = 3$, $\theta = \pi/6$, and $\phi = \pi/3$ defines the point $A(3, \pi/6, \pi/3)$ as shown in Fig. A.2(a). These three orthogonal surfaces define three curves that are mutually perpendicular. One of these is a straight line and the other two are circles. We draw unit vectors $\mathbf{i}_r$, $\mathbf{i}_\theta$, and $\mathbf{i}_\phi$ tangential to these curves at point A and directed toward increasing values of r, θ, and ϕ, respectively. These three unit vectors form a set of mutually orthogonal unit vectors in terms of which vectors drawn at A can be described. In a similar manner, we can draw unit vectors at any other point in the spherical coordinate system, as shown, for example, for point $B(1, \pi/2, 0)$ in Fig. A.2(a). It can now be seen that these unit vectors at point B are not parallel to the corresponding unit vectors at point A. Thus in the spherical coordinate system all three unit vectors $\mathbf{i}_r$, $\mathbf{i}_\theta$, and $\mathbf{i}_\phi$ do not have the same directions everywhere, that is, they are not uniform. Finally, we note that for the choice of θ as in Fig. A.2(a), that is, increasing from the positive z axis toward the xy plane, the coordinate system is right-handed, that is, $\mathbf{i}_r \times \mathbf{i}_\theta = \mathbf{i}_\phi$.

To obtain expressions for the differential lengths, surfaces, and volume in the spherical coordinate system, we now consider two points $P(r, \theta, \phi)$ and $Q(r + dr, \theta + d\theta, \phi + d\phi)$ where Q is obtained by incrementing infinitesimally each coordinate from its value at P, as shown in Fig. A.2(b). The three orthogonal surfaces intersecting at P and the three orthogonal surfaces intersecting at Q define a box that can be considered to be rectangular since dr, $d\theta$, and $d\phi$ are infinitesimally small. The three differential length elements forming the contiguous sides of this box are $dr\,\mathbf{i}_r$, $r\,d\theta\,\mathbf{i}_\theta$, and $r \sin\theta\,d\phi\,\mathbf{i}_\phi$. The differential length vector $d\mathbf{l}$ from P to Q is thus given by

$$d\mathbf{l} = dr\,\mathbf{i}_r + r\,d\theta\,\mathbf{i}_\theta + r\sin\theta\,d\phi\,\mathbf{i}_\phi \tag{A.4}$$

The differential surfaces formed by pairs of the differential length elements are

$$\pm dS\, \mathbf{i}_\phi = \pm(dr)\,(r\, d\theta)\, \mathbf{i}_\phi = \pm dr\, \mathbf{i}_r \times r\, d\theta\, \mathbf{i}_\theta \tag{A.5a}$$

$$\pm dS\, \mathbf{i}_r = \pm(r\, d\theta)\,(r \sin\theta\, d\phi)\, \mathbf{i}_r = \pm r\, d\theta\, \mathbf{i}_\theta \times r \sin\theta\, d\phi\, \mathbf{i}_\phi \tag{A.5b}$$

$$\pm dS\, \mathbf{i}_\theta = \pm(r \sin\theta\, d\phi)\,(dr)\, \mathbf{i}_\theta = \pm r \sin\theta\, d\phi\, \mathbf{i}_\phi \times dr\, \mathbf{i}_r \tag{A.5c}$$

Finally, the differential volume dv formed by the three differential lengths is simply the volume of the box, that is,

$$dv = (dr)\,(r\, d\theta)\,(r \sin\theta\, d\phi) = r^2 \sin\theta\, dr\, d\theta\, d\phi \tag{A.6}$$

In the study of electromagnetics it is sometimes useful to be able to convert the coordinates of a point and vectors drawn at a point from one coordinate system to another, particularly from the Cartesian system to the cylindrical system and vice versa, and from the Cartesian system to the spherical system and vice versa. To derive first the relationships for the conversion of the coordinates, let us consider Fig. A.3(a) which illustrates the geometry pertinent to the coordinates of a point P in the three different coordinate systems. Thus from simple geometrical considerations, we have

$$x = r_c \cos\phi \qquad y = r_c \sin\phi \qquad z = z \tag{A.7}$$

$$x = r_s \sin\theta \cos\phi \qquad y = r_s \sin\theta \sin\phi \qquad z = r_s \cos\theta \tag{A.8}$$

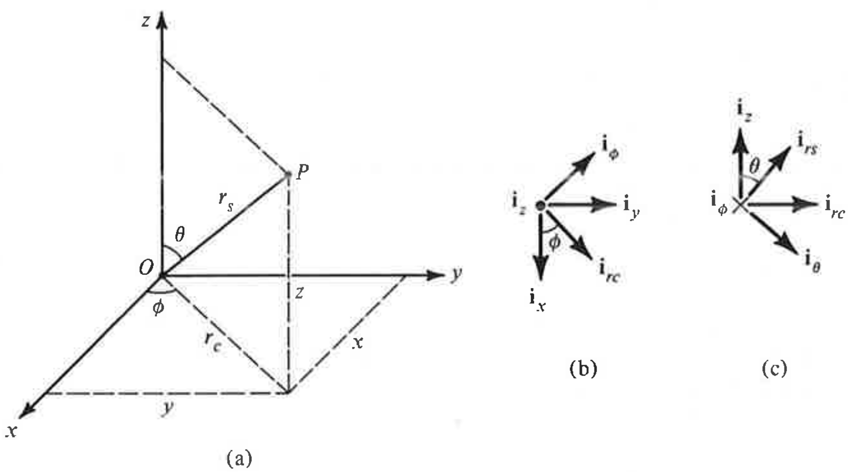

Figure A.3. (a) For conversion of coordinates of a point from one coordinate system to another. (b) and (c) For expressing unit vectors in cylindrical and spherical coordinate systems, respectively, in terms of unit vectors in the Cartesian coordinate system.

Conversely, we have

$$r_c = \sqrt{x^2 + y^2} \qquad \phi = \tan^{-1}\frac{y}{x} \qquad z = z \tag{A.9}$$

$$r_s = \sqrt{x^2 + y^2 + z^2} \qquad \theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z} \qquad \phi = \tan^{-1}\frac{y}{x} \tag{A.10}$$

Relationships (A.7) and (A.9) correspond to conversion from cylindrical coordinates to Cartesian coordinates and vice versa. Relationships (A.8) and (A.10) correspond to conversion from spherical coordinates to Cartesian coordinates and vice versa.

Considering next the conversion of vectors from one coordinate system to another, we note that in order to do this, we need to express each of the unit vectors of the first coordinate system in terms of its components along the unit vectors in the second coordinate system. From the definition of the dot product of two vectors, the component of a unit vector along another unit vector, that is, the cosine of the angle between the unit vectors, is simply the dot product of the two unit vectors. Thus considering the sets of unit vectors in the cylindrical and Cartesian coordinate systems, we have with the aid of Fig. A.3(b),

$$\mathbf{i}_{rc} \cdot \mathbf{i}_x = \cos\phi \qquad \mathbf{i}_{rc} \cdot \mathbf{i}_y = \sin\phi \qquad \mathbf{i}_{rc} \cdot \mathbf{i}_z = 0 \tag{A.11a}$$

$$\mathbf{i}_\phi \cdot \mathbf{i}_x = -\sin\phi \qquad \mathbf{i}_\phi \cdot \mathbf{i}_y = \cos\phi \qquad \mathbf{i}_\phi \cdot \mathbf{i}_z = 0 \tag{A.11b}$$

$$\mathbf{i}_z \cdot \mathbf{i}_x = 0 \qquad \mathbf{i}_z \cdot \mathbf{i}_y = 0 \qquad \mathbf{i}_z \cdot \mathbf{i}_z = 1 \tag{A.11c}$$

Similarly, for the sets of unit vectors in the spherical and Cartesian coordinate systems, we obtain with the aid of Fig. A.3(c) and Fig. A.3(b),

$$\mathbf{i}_{rs} \cdot \mathbf{i}_x = \sin\theta\cos\phi \qquad \mathbf{i}_{rs} \cdot \mathbf{i}_y = \sin\theta\sin\phi \qquad \mathbf{i}_{rs} \cdot \mathbf{i}_z = \cos\theta \tag{A.12a}$$

$$\mathbf{i}_\theta \cdot \mathbf{i}_x = \cos\theta\cos\phi \qquad \mathbf{i}_\theta \cdot \mathbf{i}_y = \cos\theta\sin\phi \qquad \mathbf{i}_\theta \cdot \mathbf{i}_z = -\sin\theta \tag{A.12b}$$

$$\mathbf{i}_\phi \cdot \mathbf{i}_x = -\sin\phi \qquad \mathbf{i}_\phi \cdot \mathbf{i}_y = \cos\phi \qquad \mathbf{i}_\phi \cdot \mathbf{i}_z = 0 \tag{A.12c}$$

We shall now illustrate the use of these relationships by means of an example.

Example A.1. Let us consider the vector $3\mathbf{i}_x + 4\mathbf{i}_y + 5\mathbf{i}_z$ at the point (3, 4, 5) and convert the vector to one in spherical coordinates.

First, from the relationships (A.10), we obtain the spherical coordinates of the point (3, 4, 5) to be

$$r_s = \sqrt{3^2 + 4^2 + 5^2} = 5\sqrt{2}$$

$$\theta = \tan^{-1}\frac{\sqrt{3^2+4^2}}{5} = \tan^{-1} 1 = 45^\circ$$

$$\phi = \tan^{-1}\frac{4}{3} = 53.13^\circ$$

Then noting from the relationships (A.12) that at the point under consideration,

$$\begin{aligned}
\mathbf{i}_x &= \sin\theta\cos\phi\,\mathbf{i}_{rs} + \cos\theta\cos\phi\,\mathbf{i}_\theta - \sin\phi\,\mathbf{i}_\phi \\
&= 0.3\sqrt{2}\,\mathbf{i}_{rs} + 0.3\sqrt{2}\,\mathbf{i}_\theta - 0.8\mathbf{i}_\phi \\
\mathbf{i}_y &= \sin\theta\sin\phi\,\mathbf{i}_{rs} + \cos\theta\sin\phi\,\mathbf{i}_\theta + \cos\phi\,\mathbf{i}_\phi \\
&= 0.4\sqrt{2}\,\mathbf{i}_{rs} + 0.4\sqrt{2}\,\mathbf{i}_\theta + 0.6\mathbf{i}_\phi \\
\mathbf{i}_z &= \cos\theta\,\mathbf{i}_{rs} - \sin\theta\,\mathbf{i}_\theta = 0.5\sqrt{2}\,\mathbf{i}_{rs} - 0.5\sqrt{2}\,\mathbf{i}_\theta
\end{aligned}$$

we obtain

$$\begin{aligned}
3\mathbf{i}_x + 4\mathbf{i}_y + 5\mathbf{i}_z &= (0.9\sqrt{2} + 1.6\sqrt{2} + 2.5\sqrt{2})\mathbf{i}_{rs} \\
&\quad + (0.9\sqrt{2} + 1.6\sqrt{2} - 2.5\sqrt{2})\mathbf{i}_\theta + (-2.4 + 2.4)\mathbf{i}_\phi = 5\sqrt{2}\,\mathbf{i}_{rs}
\end{aligned}$$

This result is to be expected since the given vector has components equal to the coordinates of the point at which it is specified. Hence its magnitude is equal to the distance of the point from the origin, that is, the spherical coordinate r of the point and its direction is along the line drawn from the origin to the point, that is, along the unit vector $\mathbf{i}_{rs}$ at that point. In fact, the given vector is a particular case of the vector $x\mathbf{i}_x + y\mathbf{i}_y + z\mathbf{i}_z = r_s\mathbf{i}_{rs}$, known as the "position vector," since it is the same as the vector drawn from the origin to the point (x, y, z). ■

REVIEW QUESTIONS

A.1. What are the three orthogonal surfaces defining the cylindrical coordinate system?

A.2. What are the limits of variation of the cylindrical coordinates?

A.3. Which of the unit vectors in the cylindrical coordinate system are not uniform?

A.4. State whether the vector $3\mathbf{i}_r + 4\mathbf{i}_\phi + 5\mathbf{i}_z$ at the point $(1, 0, 2)$ and the vector $3\mathbf{i}_r + 4\mathbf{i}_\phi + 5\mathbf{i}_z$ at the point $(2, \pi/2, 3)$ are equal or not.

A.5. What are the differential length vectors in cylindrical coordinates?

A.6. What are the three orthogonal surfaces defining the spherical coordinate system?

A.7. What are the limits of variation of the spherical coordinates?

A.8. Which of the unit vectors in the spherical coordinate system are not uniform?

A.9. State if the vector $3\mathbf{i}_r + 4\mathbf{i}_\theta$ at the point $(1, \pi/2, 0)$ and the vector $3\mathbf{i}_r + 4\mathbf{i}_\theta$ at the point $(2, 0, \pi/2)$ are equal or not.

A.10. What are the differential length vectors in spherical coordinates?

A.11. Outline the procedure for converting a vector at a point from one coordinate system to another.

A.12. What is the expression for the position vector in the cylindrical coordinate system?

PROBLEMS

A.1. Express in terms of Cartesian coordinates the vector drawn from the point $P(2, \pi/3, 1)$ to the point $Q(4, 2\pi/3, 2)$ in cylindrical coordinates.

A.2. Express in terms of Cartesian coordinates the vector drawn from the point $P(1, \pi/3, \pi/4)$ to the point $Q(2, 2\pi/3, 3\pi/4)$ in spherical coordinates.

A.3. Determine if the vector $\mathbf{i}_r + \mathbf{i}_\phi + 2\mathbf{i}_z$ at the point $(1, \pi/4, 2)$ and the vector $\sqrt{2}\,\mathbf{i}_r + 2\mathbf{i}_z$ at the point $(2, \pi/2, 3)$ are equal or not.

A.4. Determine if the vector $3\mathbf{i}_r + \sqrt{3}\,\mathbf{i}_\theta - 2\mathbf{i}_\phi$ at the point $(2, \pi/3, \pi/6)$ and the vector $\mathbf{i}_r + \sqrt{3}\,\mathbf{i}_\theta - 2\sqrt{3}\,\mathbf{i}_\phi$ at the point $(1, \pi/6, \pi/3)$ are equal or not.

A.5. Find the dot and cross products of the unit vector $\mathbf{i}_r$ at the point $(1, 0, 0)$ and the unit vector $\mathbf{i}_\phi$ at the point $(2, \pi/4, 1)$ in cylindrical coordinates.

A.6. Find the dot and cross products of the unit vector $\mathbf{i}_r$ at the point $(1, \pi/4, 0)$ and the unit vector $\mathbf{i}_\theta$ at the point $(2, \pi/2, \pi/2)$ in spherical coordinates.

A.7. Convert the vector $5\mathbf{i}_x + 12\mathbf{i}_y + 6\mathbf{i}_z$ at the point $(5, 12, 4)$ to one in cylindrical coordinates.

A.8. Convert the vector $3\mathbf{i}_x + 4\mathbf{i}_y - 5\mathbf{i}_z$ at the point $(3, 4, 5)$ to one in spherical coordinates.

B. CURL, DIVERGENCE, AND GRADIENT IN CYLINDRICAL AND SPHERICAL COORDINATE SYSTEMS

In Secs. 3.1, 3.4, and 9.1 we introduced the curl, divergence, and gradient, respectively, and derived the expressions for them in the Cartesian coordinate system. In this appendix we shall derive the corresponding expressions in the cylindrical and spherical coordinate systems. Considering first the cylindrical coordinate system, we recall from Appendix A that the infinitesimal box defined by the three orthogonal surfaces intersecting at point $P(r, \theta, \phi)$ and the three orthogonal surfaces intersecting at point $Q(r + dr, \phi + d\phi, z + dz)$ is as shown in Fig. B.1.

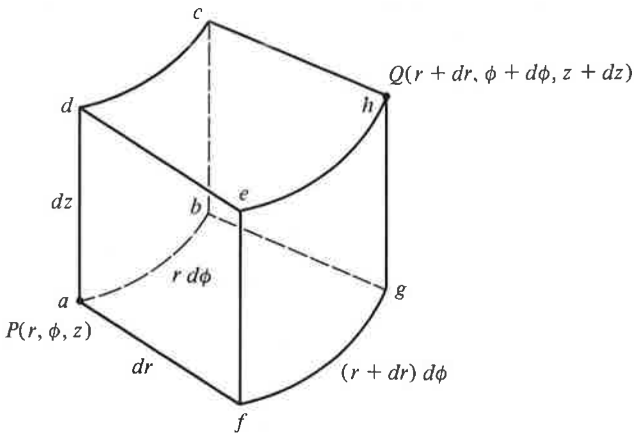

Figure B.1. Infinitesimal box formed by incrementing the coordinates in the cylindrical coordinate system.

From the basic definition of the curl of a vector introduced in Sec. 3.3 and given by

$$\nabla \times \mathbf{A} = \lim_{\Delta S \to 0} \left[\frac{\oint_C \mathbf{A} \cdot d\mathbf{l}}{\Delta S} \right]_{\max} \mathbf{i}_n \tag{B.1}$$

we find the components of $\nabla \times \mathbf{A}$ as follows with the aid of Fig. B.1:

$$\begin{aligned}
(\nabla \times \mathbf{A})_r &= \lim_{\substack{d\phi \to 0 \\ dz \to 0}} \frac{\oint_{abcda} \mathbf{A} \cdot d\mathbf{l}}{\text{area } abcd} \\
&= \lim_{\substack{d\phi \to 0 \\ dz \to 0}} \frac{\left\{ \begin{array}{l} [A_\phi]_{(r,z)} r\, d\phi + [A_z]_{(r,\phi+d\phi)}\, dz \\ - [A_\phi]_{(r,z+dz)} r\, d\phi - [A_z]_{(r,\phi)}\, dz \end{array} \right\}}{r\, d\phi\, dz} \\
&= \lim_{d\phi \to 0} \frac{[A_z]_{(r,\phi+d\phi)} - [A_z]_{(r,\phi)}}{r\, d\phi} + \lim_{dz \to 0} \frac{[A_\phi]_{(r,z)} - [A_\phi]_{(r,z+dz)}}{dz} \\
&= \frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}
\end{aligned} \tag{B.2a}$$

$$\begin{aligned}
(\nabla \times \mathbf{A})_\phi &= \lim_{\substack{dz \to 0 \\ dr \to 0}} \frac{\oint_{adefa} \mathbf{A} \cdot d\mathbf{l}}{\text{area } adef} \\
&= \lim_{\substack{dz \to 0 \\ dr \to 0}} \frac{\left\{ \begin{array}{l} [A_z]_{(r,\phi)}\, dz + [A_r]_{(\phi,z+dz)}\, dr \\ - [A_z]_{(r+dr,\phi)}\, dz - [A_r]_{(\phi,z)}\, dr \end{array} \right\}}{dr\, dz} \\
&= \lim_{dz \to 0} \frac{[A_r]_{(\phi,z+dz)} - [A_r]_{(\phi,z)}}{dz} + \lim_{dr \to 0} \frac{[A_z]_{(r,\phi)} - [A_z]_{(r+dr,\phi)}}{dr} \\
&= \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}
\end{aligned} \tag{B.2b}$$

$$\begin{aligned}
(\nabla \times \mathbf{A})_z &= \lim_{\substack{dr \to 0 \\ d\phi \to 0}} \frac{\oint_{afgba} \mathbf{A} \cdot d\mathbf{l}}{\text{area } afgb} \\
&= \lim_{\substack{dr \to 0 \\ d\phi \to 0}} \frac{\left\{ \begin{array}{l} [A_r]_{(\phi,z)}\, dr + [A_\phi]_{(r+dr,z)}(r + dr)\, d\phi \\ - [A_r]_{(\phi+d\phi,z)}\, dr - [A_\phi]_{(r,z)} r\, d\phi \end{array} \right\}}{r\, dr\, d\phi} \\
&= \lim_{dr \to 0} \frac{[rA_\phi]_{(r+dr,z)} - [rA_\phi]_{(r,z)}}{r\, dr} + \lim_{d\phi \to 0} \frac{[A_r]_{(\phi,z)} - [A_r]_{(\phi+d\phi,z)}}{r\, d\phi} \\
&= \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi) - \frac{1}{r}\frac{\partial A_r}{\partial \phi}
\end{aligned} \tag{B.2c}$$

Combining (B.2a), (B.2b) and (B.2c), we obtain the expression for the curl of a vector in cylindrical coordinates as

$$\nabla \times \mathbf{A} = \left[\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right]\mathbf{i}_r + \left[\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right]\mathbf{i}_\phi + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial \phi}\right]\mathbf{i}_z = \begin{vmatrix} \frac{\mathbf{i}_r}{r} & \mathbf{i}_\phi & \frac{\mathbf{i}_z}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix} \tag{B.3}$$

To find the expression for the divergence, we make use of the basic definition of the divergence of a vector, introduced in Sec. 3.6 and given by

$$\nabla \cdot \mathbf{A} = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta v} \tag{B.4}$$

Evaluating the right side of (B.4) for the box of Fig. B.1, we obtain

$$\begin{aligned}\nabla \cdot \mathbf{A} &= \lim_{\substack{dr \to 0 \\ d\phi \to 0 \\ dz \to 0}} \frac{\left\{\begin{matrix}[A_r]_{r+dr}(r+dr)\,d\phi\,dz - [A_r]_r r\,d\phi\,dz + [A_\phi]_{\phi+d\phi}\,dr\,dz \\ - [A_\phi]_\phi\,dr\,dz + [A_z]_{z+dz} r\,dr\,d\phi - [A_z]_z r\,dr\,d\phi\end{matrix}\right\}}{r\,dr\,d\phi\,dz} \\ &= \lim_{dr \to 0} \frac{[rA_r]_{r+dr} - [rA_r]_r}{r\,dr} + \lim_{d\phi \to 0} \frac{[A_\phi]_{\phi+d\phi} - [A_\phi]_\phi}{r\,d\phi} \\ &\quad + \lim_{dz \to 0} \frac{[A_z]_{z+dz} - [A_z]_z}{dz} \\ &= \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}\end{aligned} \tag{B.5}$$

To obtain the expression for the gradient of a scalar, we recall from Appendix A that in cylindrical coordinates,

$$d\mathbf{l} = dr\,\mathbf{i}_r + r\,d\phi\,\mathbf{i}_\phi + dz\,\mathbf{i}_z \tag{B.6}$$

and hence

$$\begin{aligned}d\Phi &= \frac{\partial \Phi}{\partial r}\,dr + \frac{\partial \Phi}{\partial \phi}\,d\phi + \frac{\partial \Phi}{\partial z}\,dz \\ &= \left(\frac{\partial \Phi}{\partial r}\mathbf{i}_r + \frac{1}{r}\frac{\partial \Phi}{\partial \phi}\mathbf{i}_\phi + \frac{\partial \Phi}{\partial z}\mathbf{i}_z\right) \cdot (dr\,\mathbf{i}_r + r\,d\phi\,\mathbf{i}_\phi + dz\,\mathbf{i}_z) \\ &= \nabla\Phi \cdot d\mathbf{l}\end{aligned} \tag{B.7}$$

Thus

$$\nabla\Phi = \frac{\partial\Phi}{\partial r}\mathbf{i}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\phi}\mathbf{i}_\phi + \frac{\partial\Phi}{\partial z}\mathbf{i}_z \tag{B.8}$$

Turning now to the spherical coordinate system, we recall from Appendix A that the infinitesimal box defined by the three orthogonal surfaces intersecting at $P(r, \theta, \phi)$ and the three orthogonal surfaces intersecting at $Q(r + dr, \theta + d\theta, \phi + d\phi)$ is as shown in Fig. B.2. From the basic definition of the curl of a vector given by (B.1), we then find the components of $\nabla \times \mathbf{A}$ as follows with the aid of Fig. B.2:

$$(\nabla \times \mathbf{A})_r = \lim_{\substack{d\theta\to 0\\ d\phi\to 0}} \frac{\oint_{abcda} \mathbf{A}\cdot d\mathbf{l}}{\text{area } abcd}$$

$$= \lim_{\substack{d\theta\to 0\\ d\phi\to 0}} \frac{\left\{\begin{matrix}[A_\theta]_{(r,\phi)}r\,d\theta + [A_\phi]_{(r,\theta+d\theta)}r\sin(\theta+d\theta)\,d\phi \\ -[A_\theta]_{(r,\phi+d\phi)}r\,d\theta - [A_\phi]_{(r,\theta)}r\sin\theta\,d\phi\end{matrix}\right\}}{r^2\sin\theta\,d\theta\,d\phi}$$

$$= \lim_{d\theta\to 0} \frac{[A_\phi\sin\theta]_{(r,\theta+d\theta)} - [A_\phi\sin\theta]_{(r,\theta)}}{r\sin\theta\,d\theta}$$

$$+ \lim_{d\phi\to 0} \frac{[A_\theta]_{(r,\phi)} - [A_\theta]_{(r,\phi+d\phi)}}{r\sin\theta\,d\phi}$$

$$= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\phi\sin\theta) - \frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial\phi} \tag{B.9a}$$

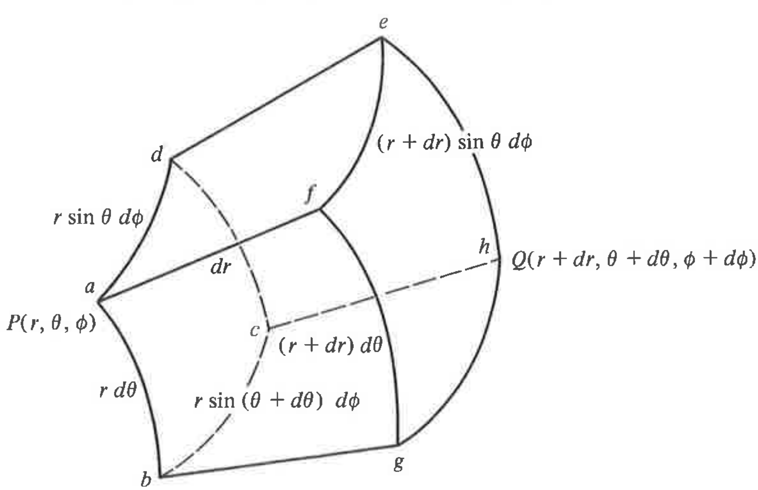

Figure B.2. Infinitesimal box formed by incrementing the coordinates in the spherical coordinate system.

$$(\nabla \times \mathbf{A})_\theta = \lim_{\substack{d\phi \to 0 \\ dr \to 0}} \frac{\oint_{adefa} \mathbf{A} \cdot d\mathbf{l}}{\text{area } adef}$$

$$= \lim_{\substack{d\phi \to 0 \\ dr \to 0}} \frac{\left\{ \begin{array}{l} [A_\phi]_{(r,\theta)} r \sin\theta \, d\phi + [A_r]_{(\theta,\phi+d\phi)} \, dr \\ - [A_\phi]_{(r+dr,\theta)}(r+dr)\sin\theta \, d\phi - [A_r]_{(\theta,\phi)} \, dr \end{array} \right\}}{r \sin\theta \, dr \, d\phi}$$

$$= \lim_{d\phi \to 0} \frac{[A_r]_{(\theta,\phi+d\phi)} - [A_r]_{(\theta,\phi)}}{r \sin\theta \, d\phi} + \lim_{dr \to 0} \frac{[rA_\phi]_{(r,\theta)} - [rA_\phi]_{(r+dr,\theta)}}{r \, dr}$$

$$= \frac{1}{r\sin\theta} \frac{\partial A_r}{\partial \phi} - \frac{1}{r} \frac{\partial}{\partial r}(rA_\phi) \tag{B.9b}$$

$$(\nabla \times \mathbf{A})_\phi = \lim_{\substack{dr \to 0 \\ d\theta \to 0}} \frac{\oint_{afgba} \mathbf{A} \cdot d\mathbf{l}}{\text{area } afgb}$$

$$= \lim_{\substack{dr \to 0 \\ d\theta \to 0}} \frac{\left\{ \begin{array}{l} [A_r]_{(\theta,\phi)} \, dr + [A_\theta]_{(r+dr,\phi)}(r+dr) \, d\theta \\ - [A_r]_{(\theta+d\theta,\phi)} \, dr - [A_\theta]_{(r,\phi)} r \, d\theta \end{array} \right\}}{r \, dr \, d\theta}$$

$$= \lim_{dr \to 0} \frac{[rA_\theta]_{(r+dr,\phi)} - [rA_\theta]_{(r,\phi)}}{r \, dr} + \lim_{d\theta \to 0} \frac{[A_r]_{(\theta,\phi)} \, dr - [A_r]_{(\theta+d\theta,\phi)} \, dr}{r \, d\theta}$$

$$= \frac{1}{r} \frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r} \frac{\partial A_r}{\partial \theta} \tag{B.9c}$$

Combining (B.9a), (B.9b), and (B.9c), we obtain the expression for the curl of a vector in spherical coordinates as

$$\nabla \times \mathbf{A} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(A_\phi \sin\theta) - \frac{\partial A_\theta}{\partial\phi}\right]\mathbf{i}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)\right]\mathbf{i}_\theta + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]\mathbf{i}_\phi$$

$$= \begin{vmatrix} \dfrac{\mathbf{i}_r}{r^2\sin\theta} & \dfrac{\mathbf{i}_\theta}{r\sin\theta} & \dfrac{\mathbf{i}_\phi}{r} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix} \tag{B.10}$$

To find the expression for the divergence, we make use of the basic definition of the divergence of a vector given by (B.4) and by evaluating its right side for the box of Fig. B.2, we obtain

$$\nabla \cdot \mathbf{A} = \lim_{\substack{dr\to 0\\ d\theta\to 0\\ d\phi\to 0}} \frac{\begin{Bmatrix}[A_r]_{r+dr}(r+dr)^2 \sin\theta\, d\theta\, d\phi - [A_r]_r r^2 \sin\theta\, d\theta\, d\phi \\ + [A_\theta]_{\theta+d\theta} r \sin(\theta+d\theta)\, dr\, d\phi - [A_\theta]_\theta r \sin\theta\, dr\, d\phi \\ + [A_\phi]_{\phi+d\phi} r\, dr\, d\theta - [A_\phi]_\phi r\, dr\, d\theta\end{Bmatrix}}{r^2 \sin\theta\, dr\, d\theta\, d\phi}$$

$$= \lim_{dr\to 0} \frac{[r^2 A_r]_{r+dr} - [r^2 A_r]_r}{r^2\, dr} + \lim_{d\theta\to 0} \frac{[A_\theta \sin\theta]_{\theta+d\theta} - [A_\theta \sin\theta]_\theta}{r \sin\theta\, d\theta}$$

$$+ \lim_{d\phi\to 0} \frac{[A_\phi]_{\phi+d\phi} - [A_\phi]_\phi}{r\sin\theta\, d\phi}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi} \tag{B.11}$$

To obtain the expression for the gradient of a scalar, we recall from Appendix A that in spherical coordinates,

$$d\mathbf{l} = dr\,\mathbf{i}_r + r\,d\theta\,\mathbf{i}_\theta + r\sin\theta\,d\phi\,\mathbf{i}_\phi \tag{B.12}$$

and hence

$$d\Phi = \frac{\partial\Phi}{\partial r}dr + \frac{\partial\Phi}{\partial\theta}d\theta + \frac{\partial\Phi}{\partial\phi}d\phi$$

$$= \left(\frac{\partial\Phi}{\partial r}\mathbf{i}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\mathbf{i}_\theta + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\mathbf{i}_\phi\right)\cdot(dr\,\mathbf{i}_r + r\,d\theta\,\mathbf{i}_\theta + r\sin\theta\,d\phi\,\mathbf{i}_\phi)$$

$$= \nabla\Phi \cdot d\mathbf{l} \tag{B.13}$$

Thus

$$\nabla\Phi = \frac{\partial\Phi}{\partial r}\mathbf{i}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\mathbf{i}_\theta + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\mathbf{i}_\phi \tag{B.14}$$

REVIEW QUESTIONS

B.1. Briefly discuss the basic definition of the curl of a vector.

B.2. Justify the application of the basic definition of the curl of a vector to determine separately the individual components of the curl.

B.3. How would you generalize the interpretations for the components of the curl of a vector in terms of the lateral derivatives involving the components of the vector to hold in cylindrical and spherical coordinate systems?

B.4. Briefly discuss the basic definition of the divergence of a vector.

B.5. How would you generalize the interpretation for the divergence of a vector in terms of the longitudinal derivatives involving the components of the vector to hold in cylindrical and spherical coordinate systems?

B.6. Provide general interpretation for the components of the gradient of a scalar.

PROBLEMS

B.1. Find the curl and the divergence for each of the following vectors in cylindrical coordinates: (a) $r \cos \phi \, \mathbf{i}_r - r \sin \phi \, \mathbf{i}_\phi$; (b) $\frac{1}{r}\mathbf{i}_r$; (c) $\frac{1}{r}\mathbf{i}_\phi$.

B.2. Find the gradient for each of the following scalar functions in cylindrical coordinates: (a) $\frac{1}{r} \cos \phi$; (b) $r \sin \phi$.

B.3. Find the expansion for the Laplacian, that is, the divergence of the gradient, of a scalar in cylindrical coordinates.

B.4. Find the curl and the divergence for each of the following vectors in spherical coordinates: (a) $r^2\mathbf{i}_r + r \sin \theta \, \mathbf{i}_\theta$; (b) $\frac{e^{-r}}{r}\mathbf{i}_\theta$; (c) $\frac{1}{r^2}\mathbf{i}_r$.

B.5. Find the gradient for each of the following scalar functions in spherical coordinates: (a) $\frac{\sin \theta}{r}$; (b) $r \cos \theta$.

B.6. Find the expansion for the Laplacian, that is, the divergence of the gradient, of a scalar in spherical coordinates.

C. UNITS AND DIMENSIONS

In 1960 the International System of Units was given official status at the Eleventh General Conference on weights and measures held in Paris, France. This system of units is an expanded version of the rationalized meter-kilogram-second-ampere (MKSA) system of units and is based on six fundamental or basic units. The six basic units are the units of length, mass, time, current, temperature, and luminous intensity.

The international unit of length is the meter. It is exactly 1,650,763.73 times the wavelength in vacuum of the radiation corresponding to the unperturbed transition between the levels $2p_{10}$ and $5d_5$ of the atom of krypton-86, the orange-red line. The international unit of mass is the kilogram. It is the mass of the International Prototype Kilogram which is a particular cylinder of platinum-iridium alloy preserved in a vault at Sevres, France, by the International Bureau of Weights and Measures. The international unit of time is the second. It is equal to 9,192,631,770 times the period corresponding to the frequency of the transition between the hyperfine levels $F = 4$, $M = 0$ and $F = 3$, $M = 0$ of the fundamental state $^2S_{1/2}$ of the cesium–133 atom unperturbed by external fields.

To present the definition for the international unit of current, we first define the newton, which is the unit of force, derived from the fundamental units meter, kilogram, and second in the following manner. Since velocity is rate of change of distance with time, its unit is meter per second. Since acceleration is rate of change of velocity with time, its unit is meter per second per second or meter per second squared. Since force is mass times acceleration, its unit is kilogram-meter per second squared, also known as the newton. Thus, the newton is that force which imparts an acceleration of 1 meter per second squared to a mass of 1 kilogram. The international unit of current, which is the ampere, can now be defined. It is the constant current which when maintained in two straight, infinitely long, parallel conductors of negligible cross section and placed one meter apart in vacuum produces a force of 2×10^{-7} newtons per meter length of the conductors.

The international unit of temperature is the Kelvin degree. It is based on the definition of the thermodynamic scale of temperature by means of the triple-point of water as a fixed fundamental point to which a temperature of exactly 273.16 degrees Kelvin is attributed. The international unit of luminous intensity is the candela. It is defined such that the luminance of a blackbody radiator at the freezing temperature of platinum is 60 candelas per square centimeter.

We have just defined the six basic units of the International System of Units. Two supplementary units are the radian and the steradian for plane angle and solid angle, respectively. All other units are derived units. For example, the unit of charge which is the coulomb is the amount of charge transported in 1 second by a current of 1 ampere; the unit of energy which is the joule is the work done when the point of application of a force of 1 newton is displaced a distance of 1 meter in the direction of the force; the unit of power which is the watt is the power which gives rise to the production of energy at the rate of 1 joule per second; the unit of electric potential difference which is the volt is the difference of electric potential between two points of a conducting wire carrying constant current of 1 ampere when the power dissipated between these points is equal to 1 watt; and so on. The units for the various quantities used in this book are listed in Table C.1, together with the symbols of the quantities and their dimensions.

Dimensions are a convenient means of checking the possible validity of a derived equation. The dimension of a given quantity can be expressed as some combination of a set of fundamental dimensions. These fundamental dimensions are mass (M), length (L) and time (T). In electromagnetics, it is the usual practice to consider the charge (Q), instead of the current, as the additional fundamental dimension. For the quantities listed in Table C.1, these four dimensions are sufficient. Thus, for example, the dimension of velocity is length (L) divided by time (T), that is LT^{-1}; the dimension of acceleration is length (L) divided by time squared (T^2), that is, LT^{-2}; the dimension of force is mass (M) times acceleration (LT^{-2}), that is, MLT^{-2}; the dimension of ampere is charge (Q) divided by time (T), that is, QT^{-1}; and so on.

To illustrate the application of dimensions for checking the possible validity of a derived equation, let us consider the equation for the phase velocity of an electromagnetic wave in free space, given by

$$v_p = \frac{1}{\sqrt{\mu_0\epsilon_0}}$$

We know that the dimension of v_p is LT^{-1}. Hence we have to show that the dimension of $1/\sqrt{\mu_0\epsilon_0}$ is also LT^{-1}. To do this, we note from Coulomb's law that

$$\epsilon_0 = \frac{Q_1Q_2}{4\pi FR^2}$$

Hence, the dimension of ϵ_0 is $Q^2/[(MLT^{-2})(L^2)]$ or $M^{-1}L^{-3}T^2Q^2$. We note from Ampere's law of force applied to two infinitesimal current elements parallel to each other and normal to the line joining them that

$$\mu_0 = \frac{4\pi FR^2}{(I_1\,dl_1)(I_2\,dl_2)}$$

Hence the dimension of μ_0 is $[(MLT^{-2})(L^2)]/(QT^{-1}L)^2$ or MLQ^{-2}. Now we obtain the dimension of $1/\sqrt{\mu_0\epsilon_0}$ as $1/\sqrt{(M^{-1}L^{-3}T^2Q^2)(MLQ^{-2})}$ or LT^{-1}, which is the same as the dimension of v_p. It should, however, be noted that the test for the equality of the dimensions of the two sides of a derived equation is not a sufficient test to establish the equality of the two sides since any dimensionless constants associated with the equation may be in error.

It is not always necessary to refer to the table of dimensions for checking the possible validity of a derived equation. For example, let us assume that we have derived the expression for the characteristic impedance of a transmission line, i.e., $\sqrt{\mathcal{L}/\mathcal{C}}$ and we wish to verify that $\sqrt{\mathcal{L}/\mathcal{C}}$ does indeed have the dimension of impedance. To do this, we write

$$\sqrt{\frac{\mathcal{L}}{\mathcal{C}}} = \sqrt{\frac{\omega\mathcal{L}1}{\omega\mathcal{C}1}} = \sqrt{\frac{\omega L}{\omega C}} = \sqrt{(\omega L)\left(\frac{1}{\omega C}\right)}$$

We now recognize from our knowledge of circuit theory that both ωL and $1/\omega C$, being the reactances of L and C, respectively, have the dimension of impedance. Hence we conclude that $\sqrt{\mathcal{L}/\mathcal{C}}$ has the dimension of $\sqrt{(\text{impedance})^2}$ or impedance.

TABLE C.1. Symbols, Units, and Dimensions of Various Quantities

Quantity	*Symbol*	*Unit*	*Dimensions*
Admittance	$\bar{Y}$	mho	$M^{-1}L^{-2}TQ^2$
Area	A	square meter	L^2
Attenuation constant	α	neper/meter	L^{-1}
Capacitance	C	farad	$M^{-1}L^{-2}T^2Q^2$
Capacitance per unit length	$\mathcal{C}$	farad/meter	$M^{-1}L^{-3}T^2Q^2$

TABLE C.1. Continued

Quantity	*Symbol*	*Unit*	*Dimensions*
Cartesian coordinates	x	meter	L
	y	meter	L
	z	meter	L
Characteristic admittance	Y_0	mho	$M^{-1}L^{-2}TQ^2$
Characteristic impedance	Z_0	ohm	$ML^2T^{-1}Q^{-2}$
Charge	Q, q	coulomb	Q
Conductance	G	mho	$M^{-1}L^{-2}TQ^2$
Conductance per unit length	$\mathcal{G}$	mho/meter	$M^{-1}L^{-3}TQ^2$
Conduction current density	$\mathbf{J}_c$	ampere/square meter	$L^{-2}T^{-1}Q$
Conductivity	σ	mho/meter	$M^{-1}L^{-3}TQ^2$
Current	I	ampere	$T^{-1}Q$
Cutoff frequency	f_c	hertz	T^{-1}
Cutoff wavelength	λ_c	meter	L
Cylindrical coordinates	r, r_c	meter	L
	ϕ	radian	—
	z	meter	L
Differential length element	$d\mathbf{l}$	meter	L
Differential surface element	$d\mathbf{S}$	square meter	L^2
Differential volume element	dv	cubic meter	L^3
Directivity	D	—	—
Displacement flux density	$\mathbf{D}$	coulomb/square meter	$L^{-2}Q$
Electric dipole moment	$\mathbf{p}$	coulomb-meter	LQ
Electric field intensity	$\mathbf{E}$	volt/meter	$MLT^{-2}Q^{-1}$
Electric potential	V	volt	$ML^2T^{-2}Q^{-1}$
Electric susceptibility	χ_e	—	—
Electron density	N	(meter)$^{-3}$	L^{-3}
Electronic charge	e	coulomb	Q
Energy	W	joule	ML^2T^{-2}
Energy density	w	joule/cubic meter	$ML^{-1}T^{-2}$
Force	$\mathbf{F}$	newton	MLT^{-2}
Frequency	f	hertz	T^{-1}
Group velocity	v_g	meter/second	LT^{-1}
Guide impedance	η_g	ohm	$ML^2T^{-1}Q^{-2}$
Guide wavelength	λ_g	meter	L
Impedance	$\bar{Z}$	ohm	$ML^2T^{-1}Q^{-2}$
Inductance	L	henry	ML^2Q^{-2}
Inductance per unit length	$\mathcal{L}$	henry/meter	MLQ^{-2}
Intrinsic impedance	η	ohm	$ML^2T^{-1}Q^{-2}$
Length	l	meter	L
Line charge density	ρ_L	coulomb/meter	$L^{-1}Q$
Magnetic dipole moment	$\mathbf{m}$	ampere-square meter	$L^2T^{-1}Q$
Magnetic field intensity	$\mathbf{H}$	ampere/meter	$L^{-1}T^{-1}Q$
Magnetic flux	ψ	weber	$ML^2T^{-1}Q^{-1}$
Magnetic flux density	$\mathbf{B}$	tesla or weber/square meter	$MT^{-1}Q^{-1}$
Magnetic susceptibility	χ_m	—	—
Magnetic vector potential	$\mathbf{A}$	weber/meter	$MLT^{-1}Q^{-1}$

TABLE C.1. Continued

Quantity	*Symbol*	*Unit*	*Dimensions*
Magnetization current density	$\mathbf{J}_m$	ampere/square meter	$L^{-2}T^{-1}Q$
Magnetization vector	$\mathbf{M}$	ampere/meter	$L^{-1}T^{-1}Q$
Mass	m	kilogram	M
Mobility	μ	square meter/volt-second	$M^{-1}TQ$
Permeability	μ	henry/meter	MLQ^{-2}
Permeability of free space	μ_0	henry/meter	MLQ^{-2}
Permittivity	ϵ	farad/meter	$M^{-1}L^{-3}T^2Q^2$
Permittivity of free space	ϵ_0	farad/meter	$M^{-1}L^{-3}T^2Q^2$
Phase constant	β	radian/meter	L^{-1}
Phase velocity	v_p	meter/second	LT^{-1}
Plasma frequency	f_N	hertz	T^{-1}
Polarization current density	$\mathbf{J}_p$	ampere/square meter	$L^{-2}T^{-1}Q$
Polarization vector	$\mathbf{P}$	coulomb/square meter	$L^{-2}Q$
Power	P	watt	ML^2T^{-3}
Power density	p	watt/square meter	MT^{-3}
Poynting vector	$\mathbf{P}$	watt/square meter	MT^{-3}
Propagation constant	$\bar{\gamma}$	complex neper/meter	L^{-1}
Propagation vector	$\boldsymbol{\beta}$	radian/meter	L^{-1}
Radian frequency	ω	radian/second	T^{-1}
Radiation resistance	R_{rad}	ohm	$ML^2T^{-1}Q^{-2}$
Reactance	X	ohm	$ML^2T^{-1}Q^{-2}$
Reflection coefficient	Γ	—	—
Refractive index	n	—	—
Relative permeability	μ_r	—	—
Relative permittivity	ϵ_r	—	—
Reluctance	$\mathcal{R}$	ampere (turn)/weber	$M^{-1}L^{-2}Q^2$
Resistance	R	ohm	$ML^2T^{-1}Q^{-2}$
Skin depth	δ	meter	L
Spherical coordinates	r, r_s	meter	L
	θ	radian	—
	ϕ	radian	—
Standing wave ratio	SWR	—	—
Surface charge density	ρ_S	coulomb/square meter	$L^{-2}Q$
Surface current density	$\mathbf{J}_S$	ampere/meter	$L^{-1}T^{-1}Q$
Susceptance	B	mho	$M^{-1}L^{-2}TQ^2$
Time	t	second	T
Transmission coefficient	τ	—	—
Unit normal vector	$\mathbf{i}_n$	—	—
Velocity	v	meter/second	LT^{-1}
Velocity of light in free space	c	meter/second	LT^{-1}
Voltage	V	volt	$ML^2T^{-2}Q^{-1}$
Volume	V	cubic meter	L^3
Volume charge density	ρ	coulomb/cubic meter	$L^{-3}Q$
Volume current density	$\mathbf{J}$	ampere/square meter	$L^{-2}T^{-1}Q$
Wavelength	λ	meter	L
Work	W	joule	ML^2T^{-2}