9. STATIC AND QUASISTATIC FIELDS

In the preceding five chapters we studied the principles of propagation, transmission, and radiation of electromagnetic waves. These phenomena are based upon the interaction between the time-varying electric and magnetic fields as indicated by Maxwell's equations that we introduced in Chaps. 2 and 3. To conclude our study of the elements of engineering electromagnetics, we shall devote this chapter to static fields, that is, fields independent of time, and quasistatic fields, which are low-frequency extensions of static fields. Since we have already built up many of the concepts and tools of engineering electromagnetics in the previous chapters, our goal in this chapter will be to start with Maxwell's equations, set the time variations equal to zero, and proceed with a logical development of the topics.

Perhaps the most important quantity in the study of static fields is the electric potential, a scalar that is related to the static electric field intensity through a vector operation known as the "gradient." We shall introduce the gradient and the electric potential at the outset and illustrate the computation of the static electric field through the use of the potential concept. We shall then consider the solution of two important differential equations involving the potential, known as "Poisson's equation" and "Laplace's equation," which have applications in electronic devices, among others. We shall then extend our study to the quasistatic case, illustrating the determination of low-frequency behavior of physical structures via the quasistatic field approach, and we shall finally conclude the chapter with a discussion of magnetic circuits.

9.1 GRADIENT AND ELECTRIC POTENTIAL

For static fields, $\partial/\partial t = 0$, and Maxwell's curl equations given for time-varying fields by

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{9.1}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{9.2}$$

reduce to

$$\nabla \times \mathbf{E} = 0 \tag{9.3}$$

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{9.4}$$

respectively. Equation (9.3) states that the curl of the static electric field is equal to zero. If the curl of a vector is zero, then that vector can be expressed as the "gradient" of a scalar, since the curl of the gradient of a scalar is identically equal to zero. The gradient of a scalar, say Φ, denoted $\nabla\Phi$ (del Φ) is given in Cartesian coordinates by

$$\begin{aligned}\nabla\Phi &= \left(\mathbf{i}_x \frac{\partial}{\partial x} + \mathbf{i}_y \frac{\partial}{\partial y} + \mathbf{i}_z \frac{\partial}{\partial z}\right)\Phi \\ &= \frac{\partial \Phi}{\partial x}\mathbf{i}_x + \frac{\partial \Phi}{\partial y}\mathbf{i}_y + \frac{\partial \Phi}{\partial z}\mathbf{i}_z\end{aligned} \tag{9.5}$$

The curl of $\nabla\Phi$ is then given by

$$\begin{aligned}\nabla \times \nabla\Phi &= \begin{vmatrix} \mathbf{i}_x & \mathbf{i}_y & \mathbf{i}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (\nabla\Phi)_x & (\nabla\Phi)_y & (\nabla\Phi)_z \end{vmatrix} \\ &= \begin{vmatrix} \mathbf{i}_x & \mathbf{i}_y & \mathbf{i}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial \Phi}{\partial x} & \dfrac{\partial \Phi}{\partial y} & \dfrac{\partial \Phi}{\partial z} \end{vmatrix} \\ &= 0\end{aligned} \tag{9.6}$$

To discuss the physical interpretation of the gradient, we note that

$$\begin{aligned}\nabla\Phi \cdot d\mathbf{l} &= \left(\frac{\partial \Phi}{\partial x}\mathbf{i}_x + \frac{\partial \Phi}{\partial y}\mathbf{i}_y + \frac{\partial \Phi}{\partial z}\mathbf{i}_z\right) \cdot (dx\,\mathbf{i}_x + dy\,\mathbf{i}_y + dz\,\mathbf{i}_z) \\ &= \frac{\partial \Phi}{\partial x}dx + \frac{\partial \Phi}{\partial y}dy + \frac{\partial \Phi}{\partial z}dz \\ &= d\Phi\end{aligned} \tag{9.7}$$

Let us consider a surface on which Φ is equal to a constant, say Φ_0, and a point P on that surface as shown in Fig. 9.1(a). If we now consider another point Q_1 on the same surface and an infinitesimal distance away from P, $d\Phi$ between these two points is zero since Φ is constant on the surface. Thus for the vector $d\mathbf{l}_1$ drawn from P to Q_1, $[\nabla\Phi]_P \cdot d\mathbf{l}_1 = 0$ and hence $[\nabla\Phi]_P$ is perpendicular to $d\mathbf{l}_1$. Since this is true for all points $Q_1, Q_2, Q_3, \ldots$ on the

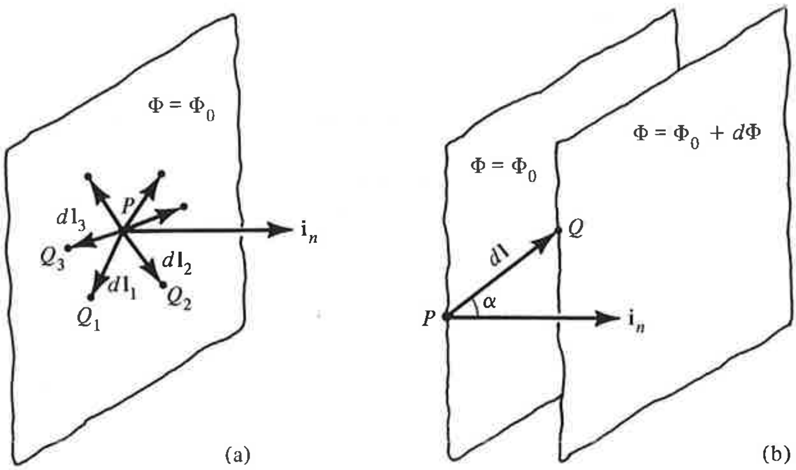


Figure 9.1. For discussing the physical interpretation of the gradient of a scalar function.

constant Φ surface, it follows that $[\nabla\Phi]_P$ must be normal to all possible infinitesimal displacement vectors $d\mathbf{l}_1, d\mathbf{l}_2, d\mathbf{l}_3, \ldots$ drawn at P and hence is normal to the surface. Denoting $\mathbf{i}_n$ to be the unit normal vector to the surface at P, we then have

$$[\nabla\Phi]_P = |\nabla\Phi|_P\,\mathbf{i}_n \tag{9.8}$$

Let us now consider two surfaces on which Φ is constant, having values Φ_0 and $\Phi_0 + d\Phi$, as shown in Fig. 9.1(b). Let P and Q be points on the $\Phi = \Phi_0$ and $\Phi = \Phi_0 + d\Phi$ surfaces, respectively, and $d\mathbf{l}$ be the vector drawn from P to Q. Then from (9.7) and (9.8),

$$\begin{aligned} d\Phi &= [\nabla\Phi]_P \cdot d\mathbf{l} \\ &= |\nabla\Phi|_P\,\mathbf{i}_n \cdot d\mathbf{l} \\ &= |\nabla\Phi|_P\, dl \cos\alpha \end{aligned} \tag{9.9}$$

where α is the angle between $\mathbf{i}_n$ at P and $d\mathbf{l}$. Thus

$$|\nabla\Phi|_P = \frac{d\Phi}{dl\cos\alpha} \tag{9.10}$$

Since $dl \cos \alpha$ is the distance between the two surfaces along $\mathbf{i}_n$ and hence is the shortest distance between them, it follows that $|\nabla\Phi|_P$ is the maximum rate of increase of Φ at the point P. Thus the gradient of a scalar function Φ at a point is a vector having magnitude equal to the maximum rate of increase of Φ at that point and is directed along the direction of the maximum rate of increase, which is normal to the constant Φ surface passing through that point. This concept of the gradient of a scalar function is often utilized to find a unit vector normal to a given surface. We shall illustrate this by means of an example.

Example 9.1. Let us find the unit vector normal to the surface $y = x^2$ at the point (2, 4, 1) by using the concept of the gradient of a scalar.

Writing the equation for the surface as

$$x^2 - y = 0$$

we note that the scalar function that is constant on the surface is given by

$$\Phi(x, y, z) = x^2 - y$$

The gradient of the scalar function is then given by

$$\begin{aligned}\nabla\Phi &= \nabla(x^2 - y)\\ &= \frac{\partial(x^2 - y)}{\partial x}\mathbf{i}_x + \frac{\partial(x^2 - y)}{\partial y}\mathbf{i}_y + \frac{\partial(x^2 - y)}{\partial z}\mathbf{i}_z\\ &= 2x\mathbf{i}_x - \mathbf{i}_y\end{aligned}$$

The value of the gradient at the point (2, 4, 1) is $2(2)\mathbf{i}_x - \mathbf{i}_y = 4\mathbf{i}_x - \mathbf{i}_y$. Thus the required unit vector is

$$\mathbf{i}_n = \pm\frac{4\mathbf{i}_x - \mathbf{i}_y}{|4\mathbf{i}_x - \mathbf{i}_y|} = \pm\left(\frac{4}{\sqrt{17}}\mathbf{i}_x - \frac{1}{\sqrt{17}}\mathbf{i}_y\right) \qquad \blacksquare$$

Returning to Maxwell's curl equation for the static electric field given by (9.3), we can now express $\mathbf{E}$ as the gradient of a scalar function, say, Φ. The question then arises as to what this scalar function is. To obtain the answer, let us consider a region of static electric field. Then we can draw a set of surfaces orthogonal everywhere to the field lines, as shown in Fig. 9.2. These surfaces correspond to the constant Φ surfaces. Since on any such surface $\mathbf{E} \cdot d\mathbf{l} = 0$, no work is involved in the movement of a test charge from one point to another on the surface. Such surfaces are known as the "equipotential surfaces." Since they are orthogonal to the field lines, they may physically be occupied by conductors without affecting the field distribution.

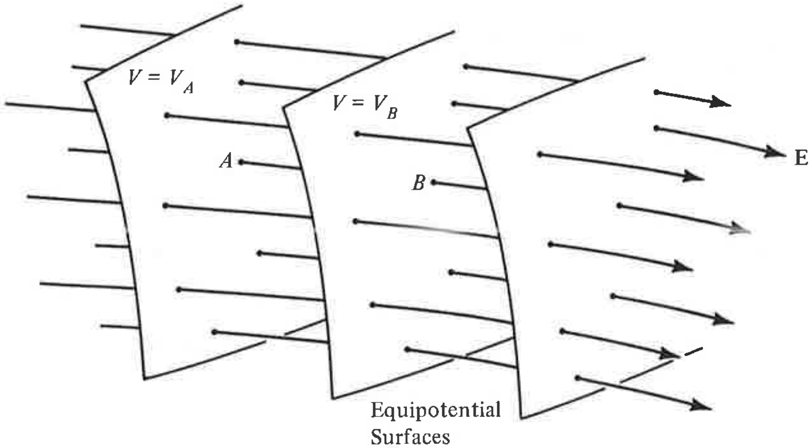


Figure 9.2. A set of equipotential surfaces in a region of static electric field.

Movement of a test charge from a point, say A, on one equipotential surface to a point, say B, on another equipotential surface involves an amount of work per unit charge equal to $\int_A^B \mathbf{E} \cdot d\mathbf{l}$ to be done by the field. This quantity is known as the "electric potential difference" between the points A and B and is denoted by the symbol $[V]_A^B$. It has the units of volts. There is a potential drop from A to B if work is done by the field and a potential rise if work is done against the field by an external agent. The situation is similar to that in the earth's gravitational field for which there is a potential drop associated with the movement of a mass from a point of higher elevation to a point of lower elevation and a potential rise for just the opposite case.

It is convenient to define an "electric potential" associated with each point. The potential at point A, denoted V_A, is simply the potential difference between point A and a reference point, say O. It is the amount of work per unit charge done by the field in connection with the movement of a test charge from A to O, or the amount of work per unit charge done against the field by an external agent in moving the test charge from O to A. Thus

$$V_A = \int_A^O \mathbf{E} \cdot d\mathbf{l} = -\int_O^A \mathbf{E} \cdot d\mathbf{l} \tag{9.11}$$

and

$$\begin{aligned} [V]_A^B = \int_A^B \mathbf{E} \cdot d\mathbf{l} &= \int_A^O \mathbf{E} \cdot d\mathbf{l} + \int_O^B \mathbf{E} \cdot d\mathbf{l} \\ &= \int_A^O \mathbf{E} \cdot d\mathbf{l} - \int_B^O \mathbf{E} \cdot d\mathbf{l} \\ &= V_A - V_B \end{aligned} \tag{9.12}$$

If we now consider points A and B to be separated by infinitesimal length $d\mathbf{l}$ from A to B, then the incremental potential drop from A to B is $\mathbf{E}_A \cdot d\mathbf{l}$, or the incremental potential rise dV along the length $d\mathbf{l}$ is given by

$$dV = -\mathbf{E}_A \cdot d\mathbf{l} \tag{9.13}$$

Writing

$$dV = [\nabla V]_A \cdot d\mathbf{l} \tag{9.14}$$

in accordance with (9.7), we then have

$$[\nabla V]_A \cdot d\mathbf{l} = -\mathbf{E}_A \cdot d\mathbf{l} \tag{9.15}$$

Since (9.15) is true at any point A in the static electric field, it follows that

$$\mathbf{E} = -\nabla V \tag{9.16}$$

Thus we have obtained the result that the static electric field is the negative of the gradient of the electric potential.

Before proceeding further we note that the potential difference we have defined here has the same meaning as the voltage between two points, defined in Sec. 2.1. We, however, recall that the voltage between two points A and B in a time-varying field is in general dependent on the path followed from A to B to evaluate $\int_A^B \mathbf{E} \cdot d\mathbf{l}$ since according to Faraday's law

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S} \tag{9.17}$$

is not in general equal to zero. On the other hand, the potential difference (or voltage) between two points A and B in a static electric field is independent of the path followed from A to B to evaluate $\int_A^B \mathbf{E} \cdot d\mathbf{l}$ since for static fields, $\partial/\partial t = 0$, and (9.17) reduces to

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0 \tag{9.18}$$

Thus the potential difference between two points in a static electric field has a unique value. Fields for which the line integral around a closed path is zero are known as "conservative" fields. The static electric field is a conservative field. The earth's gravitational field is another example of a conservative field since the work done in moving a mass around a closed path is equal to zero.

Returning now to the discussion of electric potential, let us consider the electric field of a point charge and investigate the electric potential due to the point charge. To do this, we recall from Sec. 1.5 that the electric field intensity due to a point charge Q is directed radially away from the point charge and its magnitude is $Q/4\pi\epsilon_0 R^2$ where R is the radial distance from the point charge. Since the equipotential surfaces are everywhere orthogonal to the field lines, it then follows that they are spherical surfaces centered at the point charge, as shown by the cross-sectional view in Fig. 9.3. If we now consider two equipotential surfaces of radii R and $R + dR$, the potential drop from the surface of radius R to the surface of radius $R + dR$ is $\frac{Q}{4\pi\epsilon_0 R^2}\,dR$ or, the incremental potential rise dV is given by

$$\begin{aligned} dV &= -\frac{Q}{4\pi\epsilon_0 R^2}\,dR \\ &= d\left(\frac{Q}{4\pi\epsilon_0 R} + C\right) \end{aligned} \tag{9.19}$$

where C is a constant. Thus

$$V(R) = \frac{Q}{4\pi\epsilon_0 R} + C \tag{9.20}$$

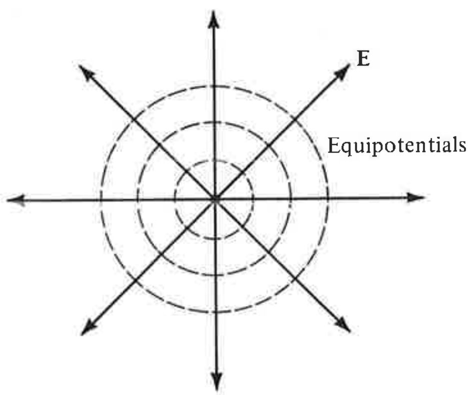


Figure 9.3. Cross-sectional view of equipotential surfaces and electric field lines for a point charge.

We can conveniently set C equal to zero by noting that it is equal to $V(\infty)$ and by choosing $R = \infty$ for the reference point. Thus we obtain the electric potential due to a point charge Q to be

$$V = \frac{Q}{4\pi\epsilon_0 R} \tag{9.21}$$

We note that the potential drops off inversely with the radial distance away from the point charge.

Equation (9.21) is often the starting point for the computation of the potential field due to static charge distributions and the subsequent determination of the electric field by using (9.16). We shall illustrate this by considering the case of the electric dipole in the following example and we shall include a few other cases in the problems.

Example 9.2. As we have learned in Sec. 5.2, the electric dipole consists of two equal and opposite point charges. Let us consider a static electric dipole consisting of point charges Q and $-Q$ situated on the z axis at $z = d/2$ and $z = -d/2$, respectively, as shown in Fig. 9.4(a) and find the potential and hence the electric field at distances far from the dipole.

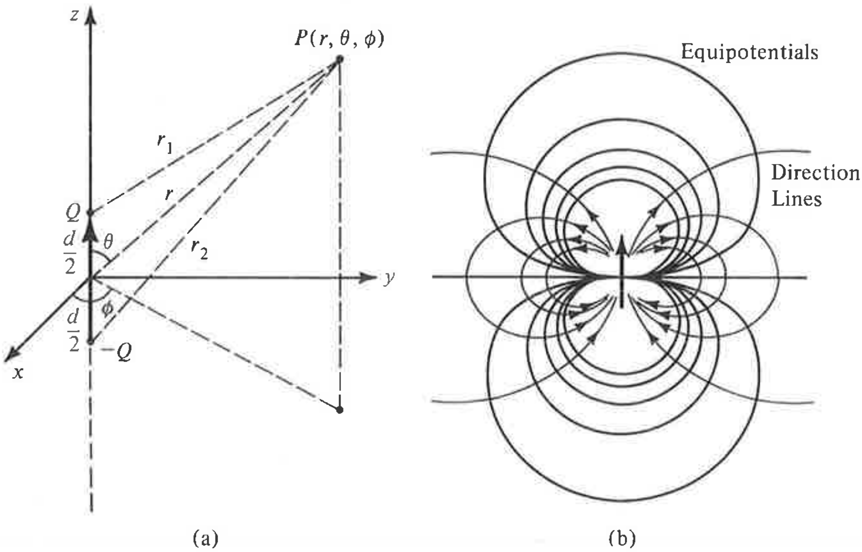


Figure 9.4. (a) Geometry pertinent to the determination of the electric field due to an electric dipole. (b) Cross sections of equipotential surfaces and direction lines of the electric field for the electric dipole.

First we note that in view of the symmetry associated with the dipole around the z axis, it is convenient to use the spherical coordinate system. As discussed in Appendix A, the spherical coordinates of a point P are the distance r from the origin O to the point P, the angle θ which the line OP makes with the z axis, and the angle ϕ which the line from the origin to the projection of P onto the xy plane makes with the x axis as shown in Fig. 9.4(a). Denoting the distance from the point charge Q to P to be r_1 and the distance from the point charge $-Q$ to P to be r_2, we write the expression for the electric potential at P due to the electric dipole as

$$\begin{aligned} V &= \frac{Q}{4\pi\epsilon_0 r_1} + \frac{-Q}{4\pi\epsilon_0 r_2} \\ &= \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \end{aligned}$$

For a point P far from the dipole, that is, for $r \gg d$, the lines drawn from the two charges to the point are almost parallel. Hence

$$r_1 \approx r - \frac{d}{2}\cos\theta$$

$$r_2 \approx r + \frac{d}{2}\cos\theta$$

and

$$\frac{1}{r_1} - \frac{1}{r_2} = \frac{r_2 - r_1}{r_1 r_2} \approx \frac{d\cos\theta}{r^2}$$

so that

$$V \approx \frac{Qd\cos\theta}{4\pi\epsilon_0 r^2} = \frac{\mathbf{p}\cdot\mathbf{i}_r}{4\pi\epsilon_0 r^2}$$

where $\mathbf{p} = Qd\mathbf{i}_z$ is the dipole moment of the electric dipole. Thus the potential field of the electric dipole drops off inversely with the square of the distance from the dipole.

Now, from (9.16) and noting from Appendix B that the gradient of a scalar in spherical coordinates is given by

$$\nabla\Phi = \frac{\partial\Phi}{\partial r}\mathbf{i}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\mathbf{i}_\theta + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\mathbf{i}_\phi$$

we obtain the electric field intensity due to the dipole to be

$$\mathbf{E} = -\nabla V = -\frac{\partial}{\partial r}\left(\frac{Qd\cos\theta}{4\pi\epsilon_0 r^2}\right)\mathbf{i}_r - \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{Qd\cos\theta}{4\pi\epsilon_0 r^2}\right)\mathbf{i}_\theta$$

$$= \frac{Qd}{4\pi\epsilon_0 r^3}(2\cos\theta\,\mathbf{i}_r + \sin\theta\,\mathbf{i}_\theta)$$

We note that this result agrees with the one obtained directly in (8.8) in Sec. 8.1.

Finally, a sketch of the direction lines of the electric field and of the cross sections of the equipotential surfaces ($\cos\theta/r^2 =$ constant) is shown in Fig. 9.4(b). Although it is possible to derive the equation for the direction lines, it is not essential to do so since they can be sketched by recognizing that (a) they must originate from the positive charge and end on the negative charge and (b) they must be everywhere perpendicular to the equipotential surfaces. ■

9.2 POISSON'S EQUATION

In the previous section we learned that for the static electric field, $\nabla \times \mathbf{E}$ is equal to zero, and hence

$$\mathbf{E} = -\nabla V$$

Substituting this result into Maxwell's divergence equation for **D**, and assuming ϵ to be uniform, we obtain

$$\begin{aligned}\nabla \cdot \mathbf{D} &= \nabla \cdot \epsilon\mathbf{E} = \epsilon\nabla \cdot \mathbf{E} \\ &= \epsilon\nabla \cdot (-\nabla V) = \rho\end{aligned}$$

or

$$\nabla \cdot \nabla V = -\frac{\rho}{\epsilon}$$

The quantity $\nabla \cdot \nabla V$ is known as the "Laplacian" of V, denoted $\nabla^2 V$ (del squared V). Thus we have

$$\nabla^2 V = -\frac{\rho}{\epsilon} \tag{9.22}$$

This equation is known as "Poisson's equation." It governs the relationship between the volume charge density ρ in a region and the potential in that region. In Cartesian coordinates,

$$\begin{aligned}\nabla^2 V &= \nabla \cdot \nabla V \\ &= \left(\mathbf{i}_x \frac{\partial}{\partial x} + \mathbf{i}_y \frac{\partial}{\partial y} + \mathbf{i}_z \frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial V}{\partial x}\mathbf{i}_x + \frac{\partial V}{\partial y}\mathbf{i}_y + \frac{\partial V}{\partial z}\mathbf{i}_z\right) \\ &= \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}\end{aligned} \tag{9.23}$$

and Poisson's equation becomes

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho}{\epsilon} \tag{9.24}$$

For the one-dimensional case in which V varies with x only, $\partial^2 V/\partial y^2$ and $\partial^2 V/\partial z^2$ are both equal to zero, and (9.24) reduces to

$$\frac{\partial^2 V}{\partial x^2} = \frac{d^2 V}{dx^2} = -\frac{\rho}{\epsilon} \tag{9.25}$$

To illustrate an example of the application of (9.25), let us consider the space charge layer in a p–n junction semiconductor with zero bias, as shown in Fig. 9.5(a), in which the region $x < 0$ is doped p-type and the region $x > 0$ is doped n-type. To review briefly the formation of the space charge layer, we note that since the density of the holes on the p side is larger than that on the n side, there is a tendency for the holes to diffuse to the n side and recombine with the electrons. Similarly, there is a tendency for the electrons on the n side to diffuse to the p side and recombine with the holes. The diffusion of holes leaves behind negatively charged acceptor atoms and the

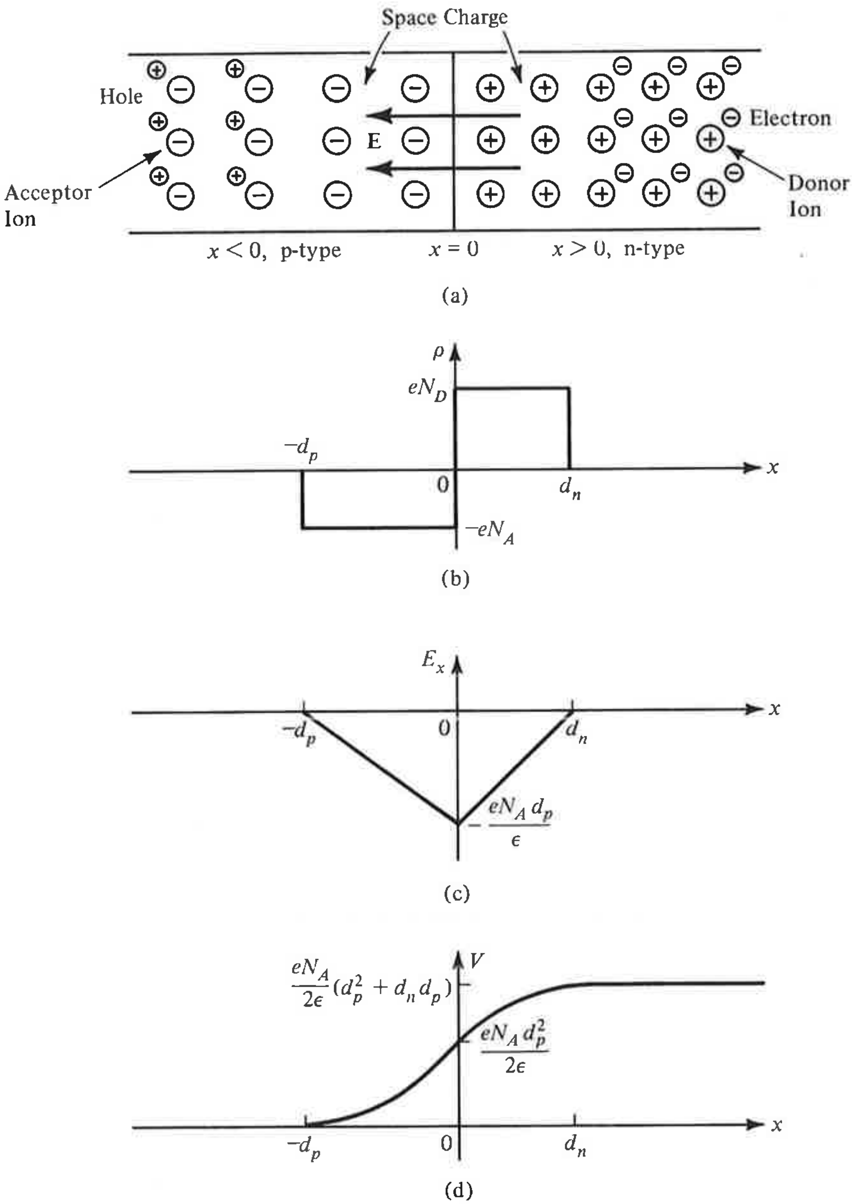


Figure 9.5. For illustrating the application of Poisson's equation for the determination of the potential distribution for a *p–n* junction semiconductor.

diffusion of electrons leaves behind positively charged donor atoms. Since these acceptor and donor atoms are immobile, a space charge layer, also known as the "depletion layer," is formed in the region of the junction with negative charges on the p side and positive charges on the n side. This space charge gives rise to an electric field directed from the n side of the junction to the p side so that it opposes diffusion of the mobile carriers across the junction thereby resulting in an equilibrium.

For simplicity, let us consider an abrupt junction, that is, a junction in which the impurity concentration is constant on either side of the junction. Let N_A and N_D be the acceptor and donor ion concentrations, respectively, and d_p and d_n be the widths in the p and n regions, respectively, of the depletion layer. The space charge density ρ is then given by

$$\rho = \begin{cases} -eN_A & \text{for } -d_p < x < 0 \\ eN_D & \text{for } \quad 0 < x < d_n \end{cases} \tag{9.26}$$

as shown in Fig. 9.5(b), where e is the electronic charge. Since the semiconductor is electrically neutral, the total acceptor charge must be equal to the total donor charge, that is,

$$eN_A d_p = eN_D d_n \tag{9.27}$$

Substituting (9.26) into (9.25), we obtain

$$\frac{d^2V}{dx^2} = \begin{cases} \dfrac{eN_A}{\epsilon} & \text{for } -d_p < x < 0 \\ -\dfrac{eN_D}{\epsilon} & \text{for } \quad 0 < x < d_n \end{cases} \tag{9.28}$$

This equation governs the potential distribution in the depletion layer.

To solve (9.28) for V, we integrate it once and obtain

$$\frac{dV}{dx} = \begin{cases} \dfrac{eN_A}{\epsilon}x + C_1 & \text{for } -d_p < x < 0 \\ -\dfrac{eN_D}{\epsilon}x + C_2 & \text{for } \quad 0 < x < d_n \end{cases}$$

where C_1 and C_2 are constants of integration. To evaluate C_1 and C_2, we note that since $\mathbf{E} = -\nabla V = -(\partial V/\partial x)\mathbf{i}_x$, $\partial V/\partial x$ is simply equal to $-E_x$. Since the electric field lines begin on the positive charges and end on the negative charges, the field and hence $\partial V/\partial x$ must vanish at $x = -d_p$ and $x = d_n$, giving us

$$\frac{dV}{dx} = \begin{cases} \dfrac{eN_A}{\epsilon}(x + d_p) & \text{for } -d_p < x < 0 \\ -\dfrac{eN_D}{\epsilon}(x - d_n) & \text{for } \quad 0 < x < d_n \end{cases} \tag{9.29}$$

The field intensity, that is, $-(dV/dx)$, may now be sketched as a function of x as shown in Fig. 9.5(c).

Proceeding further, we integrate (9.29) and obtain

$$V = \begin{cases} \frac{eN_A}{2\epsilon}(x + d_p)^2 + C_3 & \text{for } -d_p < x < 0 \\ -\frac{eN_D}{2\epsilon}(x - d_n)^2 + C_4 & \text{for } \quad 0 < x < d_n \end{cases}$$

where C_3 and C_4 are constants of integration. To evaluate C_3 and C_4, we first set the potential at $x = -d_p$ arbitrarily equal to zero to obtain C_3 equal to zero. Then we make use of the condition that the potential be continuous at $x = 0$, since the discontinuity in dV/dx at $x = 0$ is finite, to obtain

$$\frac{eN_A}{2\epsilon} d_p^2 = -\frac{eN_D}{2\epsilon} d_n^2 + C_4$$

or

$$C_4 = \frac{e}{2\epsilon}(N_A d_p^2 + N_D d_n^2)$$

Substituting this value for C_4 and setting C_3 equal to zero in the expression for V, we get the required solution

$$V = \begin{cases} \frac{eN_A}{2\epsilon}(x + d_p)^2 & \text{for } -d_p < x < 0 \\ -\frac{eN_D}{2\epsilon}(x^2 - 2xd_n) + \frac{eN_A}{2\epsilon} d_p^2 & \text{for } \quad 0 < x < d_n \end{cases} \tag{9.30}$$

The variation of potential with x as given by (9.30) is shown in Fig. 9.5(d).

We can proceed further and find the width $d = d_p + d_n$ of the depletion layer by setting $V(d_n)$ equal to the contact potential, V_0, that is, the potential difference across the depletion layer resulting from the electric field in the layer. Thus

$$\begin{aligned} V_0 = V(d_n) &= \frac{eN_D}{2\epsilon} d_n^2 + \frac{eN_A}{2\epsilon} d_p^2 \\ &= \frac{e}{2\epsilon} \frac{N_D(N_A + N_D)}{N_A + N_D} d_n^2 + \frac{e}{2\epsilon} \frac{N_A(N_A + N_D)}{N_A + N_D} d_p^2 \\ &= \frac{e}{2\epsilon} \frac{N_A N_D}{N_A + N_D}(d_n^2 + d_p^2 + 2d_n d_p) \\ &= \frac{e}{2\epsilon} \frac{N_A N_D}{N_A + N_D} d^2 \end{aligned}$$

where we have used (9.27). Finally, we obtain the result that

$$d = \sqrt{\frac{2\epsilon V_0}{e}\left(\frac{1}{N_A} + \frac{1}{N_D}\right)}$$

which tells us that the depletion layer width is smaller the heavier the doping is. This property is used in tunnel diodes to achieve layer widths on the order of 10^{-6} cm by heavy doping as compared to widths on the order of 10^{-4} cm in ordinary p–n junctions.

We have just illustrated an example of the application of Poisson's equation involving the solution for the potential distribution for a given charge distribution. Poisson's equation is even more useful for the solution of problems in which the charge distribution is the quantity to be determined given the functional dependence of the charge density on the potential. We shall, however, not pursue this topic any further.

9.3 LAPLACE'S EQUATION

In the previous section we derived Poisson's equation

$$\nabla^2 V = -\frac{\rho}{\epsilon}$$

If the charge density in a region is zero, then Poisson's equation reduces to

$$\nabla^2 V = 0 \tag{9.31}$$

This equation is known as "Laplace's equation." It governs the behavior of the potential in a charge-free region. In Cartesian coordinates, it is given by

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \tag{9.32}$$

Laplace's equation is also satisfied by the potential in conductors under steady current condition. For the steady current condition, $\partial\rho/\partial t = 0$ and the continuity equation given for the time-varying case by

$$\nabla \cdot \mathbf{J}_c + \frac{\partial \rho}{\partial t} = 0$$

reduces to

$$\nabla \cdot \mathbf{J}_c = 0 \tag{9.33}$$

Replacing $\mathbf{J}_c$ by $\sigma\mathbf{E} = -\sigma\nabla V$ where σ is the conductivity of the conductor and assuming σ to be constant, we obtain

$$\nabla \cdot \sigma\mathbf{E} = \sigma\nabla \cdot \mathbf{E} = -\sigma\nabla \cdot \nabla V = -\sigma\nabla^2 V = 0$$

or

$$\nabla^2 V = 0$$

The problems for which Laplace's equation is applicable consist of finding the potential distribution in the region between two conductors given the charge distribution on the surfaces of the conductors or the potentials of the conductors or a combination of the two. The procedure involves the solving of Laplace's equation subject to the boundary conditions on the surfaces of the conductors. The electric field intensity between the conductors is then found by using $\mathbf{E} = -\nabla V$, from which the conduction current density is obtained by using $\mathbf{J}_c = \sigma \mathbf{E}$, if the medium is a conductor. We shall illustrate this by means of an example involving variation of V in one dimension.

Example 9.3. Let us consider two infinite, plane, parallel, perfectly conducting plates occupying the planes $x = 0$ and $x = d$ and kept at potentials $V = 0$ and $V = V_0$, respectively, as shown by the cross-sectional view in Fig. 9.6, and find the solution for Laplace's equation in the region between the plates. The arrangement may be considered an idealization of two parallel plates having dimensions very large compared to the spacing between them.

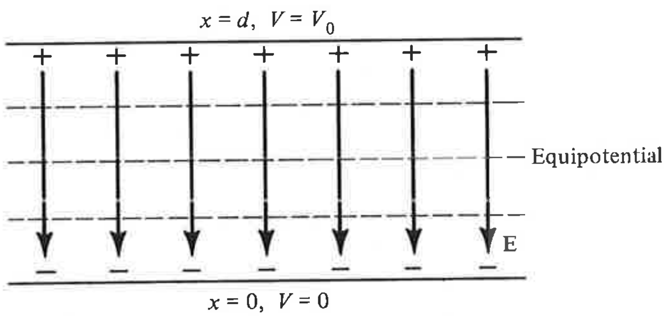


Figure 9.6. For illustrating the solution of Laplace's equation in one dimension.

The potential is obviously a function of x only and hence (9.32) reduces to

$$\frac{\partial^2 V}{\partial x^2} = \frac{d^2 V}{dx^2} = 0$$

Integrating this equation twice, we obtain

$$V(x) = Ax + B$$

where A and B are constants of integration. To determine the values of A and B, we make use of the boundary conditions for V, that is,

$$V = 0 \qquad \text{for } x = 0$$
$$V = V_0 \qquad \text{for } x = d$$

giving us

$$0 = A(0) + B \qquad \text{or} \qquad B = 0$$

$$V_0 = A(d) + B = Ad \qquad \text{or} \qquad A = \frac{V_0}{d}$$

Thus the required solution for the potential is given by

$$V = \frac{V_0}{d}x \qquad \text{for } 0 < x < d$$

which tells us that the equipotentials are planes parallel to the conductors, as shown in Fig. 9.6.

Proceeding further, we obtain

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial x}\mathbf{i}_x = -\frac{V_0}{d}\mathbf{i}_x \qquad \text{for } 0 < x < d$$

This field is uniform and directed from the higher potential plate to the lower potential plate, as shown in Fig. 9.6. The surface charge densities on the two plates are given by

$$[\rho_S]_{x=0} = [\mathbf{D}]_{x=0} \cdot \mathbf{i}_x = -\frac{\epsilon V_0}{d}\mathbf{i}_x \cdot \mathbf{i}_x = -\frac{\epsilon V_0}{d}$$

$$[\rho_S]_{x=d} = [\mathbf{D}]_{x=d} \cdot (-\mathbf{i}_x) = -\frac{\epsilon V_0}{d}\mathbf{i}_x \cdot (-\mathbf{i}_x) = \frac{\epsilon V_0}{d}$$

The magnitude of the surface charge per unit area on either plate is $Q = |\rho_S|(1) = \epsilon V_0/d$, and the capacitance per unit area of the plates, that is, the ratio of Q to V_0, is equal to ϵ/d.

If the medium between the plates is a conductor, then the conduction current density is given by

$$\mathbf{J}_c = \sigma\mathbf{E} = -\frac{\sigma V_0}{d}\mathbf{i}_x$$

The conduction current from the higher potential plate to the lower potential plate per unit area of the plates is $I_c = |\mathbf{J}_c|(1) = \sigma V_0/d$, and the conductance per unit area of the plates, that is, the ratio of I_c to V_0, is equal to σ/d. ■

We have just illustrated the solution of Laplace's equation by considering an example involving the variation of V in one dimension only. Before going on to the solution of Laplace's equation in two dimensions, a brief discussion of the applicability of Laplace's equation in the determination of transmission-line parameters and field maps is in order. To do this, we recall that a transmission line is characterized by fields that are entirely transverse to its axis. Hence in any given transverse plane, that is, cross-sectional plane, $\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$ and $\mathbf{E}$ possesses the same spatial characteristics in the transverse dimensions as those of a static field although it is time-varying. Thus by solving Laplace's equation in the cross-sectional plane, subject to the boundary conditions at the conductors of the line, we can obtain the field map consisting of equipotential "lines" and electric field lines. The equipotential lines, being everywhere orthogonal to the electric field lines, are identical to the magnetic field lines. Conversely, the graphical field mapping technique discussed in Sec. 6.3 is equally applicable to the solution of Laplace's equation if we recognize that the magnetic field lines are equivalent to equipotential lines. A comparison of the results of Example 9.3 with the parallel-plate transmission line case in Sec. 6.2 serves as an example for this discussion.

Returning to the solution of Laplace's equation, we now consider its solution in two dimensions, say x and y. The potential, being independent of z, then satisfies the equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \tag{9.34}$$

Equation (9.34) is a partial differential equation in two dimensions. As we have already discussed in Sec. 4.4, the technique by means of which it is solved is the "separation of variables" technique. It consists of assuming that the solution for the potential is the product of two functions, one of which is a function of x only and the second is a function of y only. Denoting these functions to be X and Y, respectively, we have

$$V(x, y) = X(x)Y(y) \tag{9.35}$$

Substituting this assumed solution into the differential equation, we obtain

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0$$

Dividing both sides by XY and rearranging, we get

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} \tag{9.36}$$

The left side of (9.36) is a function of x only; the right side is a function of y only. Thus (9.36) states that a function of x only is equal to a function of y only. A function of x only other than a constant cannot be equal to a function of y only other than the same constant for all values of x and y. For example,

$2x$ is equal to $4y$ for only those pairs of values of x and y for which $x = 2y$. Since we are seeking a solution that is good for all pairs of x and y, the only solution that satisfies (9.36) is that for which each side of (9.36) is equal to a constant. Denoting this constant to be α^2, we have

$$\frac{d^2X}{dx^2} = \alpha^2 X \tag{9.37a}$$

and

$$\frac{d^2Y}{dy^2} = -\alpha^2 Y \tag{9.37b}$$

Thus we have obtained two ordinary differential equations involving separately the variables x and y, starting with the partial differential equation involving both of the variables x and y. It is for this reason that the method is known as the separation of variables technique.

The solutions for (9.37a) and (9.37b) are given by

$$X(x) = \begin{cases} Ae^{\alpha x} + Be^{-\alpha x} & \text{for } \alpha \neq 0 \\ A_0 x + B_0 & \text{for } \alpha = 0 \end{cases} \tag{9.38a}$$

where A, B, A_0, and B_0 are arbitrary constants, and

$$Y(y) = \begin{cases} C \cos \alpha y + D \sin \alpha y & \text{for } \alpha \neq 0 \\ C_0 y + D_0 & \text{for } \alpha = 0 \end{cases} \tag{9.38b}$$

where C, D, C_0, and D_0 are arbitrary constants. Substituting (9.38a) and (9.38b) into (9.35), we obtain

$$V(x, y) = \begin{cases} (Ae^{\alpha x} + Be^{-\alpha x})(C \cos \alpha y + D \sin \alpha y) & \text{for } \alpha \neq 0 \\ (A_0 x + B_0)(C_0 y + D_0) & \text{for } \alpha = 0 \end{cases} \tag{9.39}$$

Equation (9.39) is the general solution for Laplace's equation in the two dimensions x and y. The arbitrary constants are evaluated from the boundary conditions specified for a given problem. We shall now consider two examples.

Example 9.4. Let us consider an infinitely long rectangular slot cut in a semi-infinite plane conducting slab held at zero potential, as shown by the cross-sectional view, transverse to the slot, in Fig. 9.7. With reference to the coordinate system shown in the figure, assume that a potential distribution $V = V_0 \sin(\pi y/b)$, where V_0 is a constant, is created at the mouth $x = a$ of the slot by the application of a potential to an appropriately shaped conductor away from the mouth of the slot not shown in the figure. We wish to find the potential distribution in the slot.

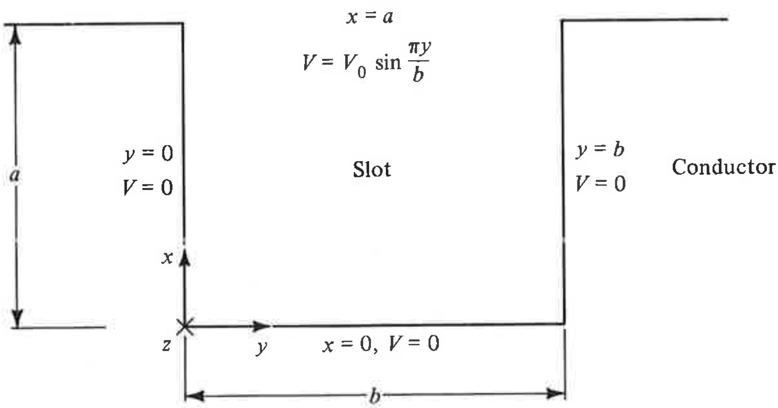


Figure 9.7. Cross-sectional view of a rectangular slot cut in a semi-infinite plane conducting slab at zero potential. The potential at the mouth of the slot is $V_0 \sin (\pi y/b)$ volts.

Since the slot is infinitely long in the z direction with uniform cross section, the problem is two dimensional in x and y and the general solution for V given by (9.39) is applicable. The boundary conditions are

$$V = 0 \qquad \text{for } y = 0,\ 0 < x < a \tag{9.40a}$$

$$V = 0 \qquad \text{for } y = b,\ 0 < x < a \tag{9.40b}$$

$$V = 0 \qquad \text{for } x = 0,\ 0 < y < b \tag{9.40c}$$

$$V = V_0 \sin \frac{\pi y}{b} \qquad \text{for } x = a,\ 0 < y < b \tag{9.40d}$$

The solution corresponding to $\alpha = 0$ does not fit the boundary conditions since V is required to be zero for two values of y and in the range $0 < x < a$. Hence we can ignore that solution and consider only the solution for $\alpha \neq 0$.

Applying the boundary condition (9.40a), we have

$$0 = (Ae^{\alpha x} + Be^{-\alpha x})(C) \qquad \text{for } 0 < x < a$$

The only way of satisfying this equation for a range of values of x is by setting $C = 0$. Next, applying the boundary condition (9.40c), we have

$$0 = (A + B)D \sin \alpha y \qquad \text{for } 0 < y < b$$

This requires that $(A + B)D = 0$, which can be satisfied by either $A + B = 0$ or $D = 0$. We, however, rule out $D = 0$ since it results in a trivial solution of zero for the potential. Hence we set

$$A + B = 0 \qquad \text{or} \qquad B = -A$$

Thus the solution for V reduces to

$$V(x, y) = (Ae^{\alpha x} - Ae^{-\alpha x})D \sin \alpha y$$
$$= A' \sinh \alpha x \sin \alpha y \tag{9.41}$$

where $A' = 2AD$.

Next, applying boundary condition (9.40b) to (9.41), we obtain

$$0 = A' \sinh \alpha x \sin \alpha b \qquad \text{for } 0 < x < a$$

To satisfy this equation without obtaining a trivial solution of zero for the potential, we set

$$\sin \alpha b = 0$$

or

$$\alpha b = n\pi \qquad n = 1, 2, 3, \ldots$$

$$\alpha = \frac{n\pi}{b} \qquad n = 1, 2, 3, \ldots$$

Since several values of α satisfy the boundary condition, several solutions are possible for the potential. To take this into account, we write the solution as the superposition of all these solutions multiplied by different arbitrary constants. In this manner, we obtain

$$V(x, y) = \sum_{n=1,2,3,\ldots}^{\infty} A'_n \sinh \frac{n\pi x}{b} \sin \frac{n\pi y}{b} \qquad \text{for } 0 < y < b \tag{9.42}$$

Finally, applying the boundary condition (9.40d) to (9.42), we get

$$V_0 \sin \frac{\pi y}{b} = \sum_{n=1,2,3,\ldots}^{\infty} A'_n \sinh \frac{n\pi a}{b} \sin \frac{n\pi y}{b} \qquad \text{for } 0 < y < b \tag{9.43}$$

On the right side of (9.43) we have an infinite series of sine terms in y, but on the left side we have only one sine term in y. Equating the coefficients of the sine terms having the same arguments, we obtain

$$A'_n \sinh \frac{n\pi a}{b} = \begin{cases} V_0 & \text{for } n = 1 \\ 0 & \text{for } n \neq 1 \end{cases}$$

or

$$A'_1 = \frac{V_0}{\sinh (\pi a/b)}$$

$$A'_n = 0 \qquad \text{for } n \neq 1$$

Substituting this result in (9.42), we obtain the required solution for V as

$$V(x, y) = V_0 \frac{\sinh (\pi x/b)}{\sinh (\pi a/b)} \sin \frac{\pi y}{b} \tag{9.44}$$

We may now compute the potential at any point inside the slot given the values of a, b, and V_0. For example, for $a = b$, that is, for a square slot, (9.44) gives the potential at the center of the slot to be $0.1993V_0$. ■

Example 9.5. Let us assume that the rectangular slot of Fig. 9.7 is covered at the mouth $x = a$ by a conducting plate that is kept at a potential $V = V_0$, making sure that the edges touching the corners of the slot are insulated, as shown in Fig. 9.8(a), and find the solution for the potential in the slot for this new boundary condition.

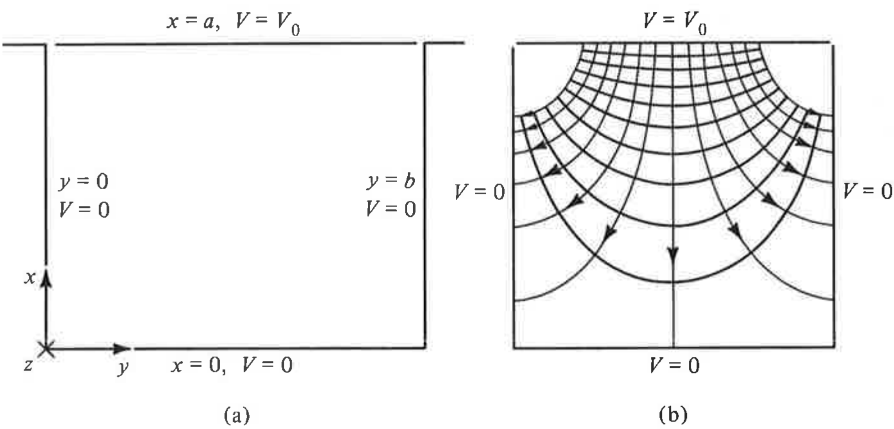


Figure 9.8. (a) Cross-sectional view of a rectangular slot in a semi-infinite plane conducting slab at zero potential and covered at the mouth by a conducting plate kept at a potential V_0. (b) Equipotentials and direction lines of electric field in the slot for the case $b/a = 1$.

Since the boundary conditions (9.40a)–(9.40c) remain the same, all we need to do to find the required solution for the potential is to substitute the new boundary condition

$$V = V_0 \qquad \text{for } x = a,\ 0 < y < b$$

in (9.42) and evaluate the coefficients A'_n. Thus we have

$$V_0 = \sum_{n=1,2,3,\ldots}^{\infty} A'_n \sinh \frac{n\pi a}{b} \sin \frac{n\pi y}{b} \qquad \text{for } 0 < y < b \tag{9.45}$$

In this equation we have an infinite series on the right side, but the left side is a constant. Thus we cannot hope to obtain A'_n by simply comparing the coefficients of the sine terms having like arguments as in Example 9.4. If we do so, we get the ridiculous answer of $V_0 = 0$ and all $A'_n = 0$ since there is no constant term on the right side and there are no sine terms on the left side.

The way out of the dilemma is to make use of the so-called orthogonality property of sine functions, given by

$$\int_{y=0}^{p} \sin\frac{n\pi y}{p}\sin\frac{m\pi y}{p}\,dy = \begin{cases} 0 & n \neq m \\ \dfrac{p}{2} & n = m \end{cases}$$

where m and n are integers. Multiplying both sides of (9.45) by $\sin\dfrac{m\pi y}{b}\,dy$ and integrating between the limits 0 and b, we have

$$\int_{y=0}^{b} V_0 \sin\frac{m\pi y}{b}\,dy = \int_{y=0}^{b} \sum_{n=1,2,3,\ldots}^{\infty} A'_n \sinh\frac{n\pi a}{b}\sin\frac{n\pi y}{b}\sin\frac{m\pi y}{b}\,dy$$

The integration and summation on the right side can be interchanged, giving us

$$\int_{y=0}^{b} V_0 \sin\frac{m\pi y}{b}\,dy = \sum_{n=1,2,3,\ldots}^{\infty} A'_n \sinh\frac{n\pi a}{b}\int_{y=0}^{b} \sin\frac{n\pi y}{b}\sin\frac{m\pi y}{b}\,dy$$

or

$$\frac{V_0 b}{m\pi}(1 - \cos m\pi) = \left(A'_m \sinh\frac{m\pi a}{b}\right)\frac{b}{2}$$

$$A'_m = \begin{cases} \dfrac{4V_0}{m\pi}\,\dfrac{1}{\sinh(m\pi a/b)} & \text{for } m \text{ odd} \\ 0 & \text{for } m \text{ even} \end{cases}$$

Substituting this result in (9.42), we obtain the required solution for the potential inside the slot as

$$V = \sum_{n=1,3,5,\ldots}^{\infty} \frac{4V_0}{n\pi}\,\frac{\sinh(n\pi x/b)}{\sinh(n\pi a/b)}\sin\frac{n\pi y}{b} \tag{9.46}$$

The numerical values of potentials may now be computed for points inside the slot for given values of a, b, and V_0, and equipotentials may be sketched by joining points having approximately the same potential values. The electric field lines can then be drawn orthogonal to the equipotentials. The resulting sketches for a square slot are shown in Fig. 9.8(b). ■

9.4 COMPUTER SOLUTION OF LAPLACE'S EQUATION*

In the previous section we illustrated the solution of the two-dimensional Laplace's equation in Cartesian coordinates x and y. In this section we shall discuss the approximate solution of the two-dimensional Laplace's equation

*This section may be omitted without loss of continuity.

which forms the basis for adaptation to digital computers. To illustrate the principle behind the approximate solution, let us suppose that we know the potentials V_1, V_2, V_3, and V_4 at four points equidistant from a point $P(0, 0, 0)$ and lying on mutually perpendicular axes, which we call x and y, passing through P as shown in Fig. 9.9. We wish to find the potential V_0 at P in terms of V_1, V_2, V_3, and V_4.

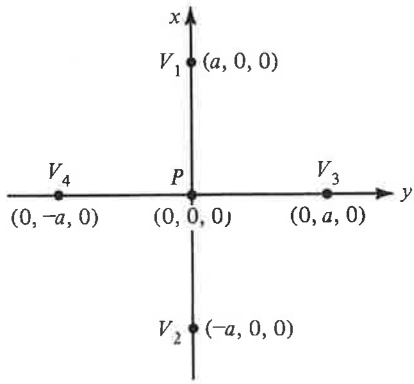


Figure 9.9. For illustrating the principle behind the approximate solution of Laplace's equation in two dimensions.

Assuming no variation of V in the z direction, we require that

$$[\nabla^2 V]_P = \left[\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right]_{(0,0,0)} = 0 \tag{9.47}$$

To solve this equation approximately for V_0, we note that

$$\begin{aligned}\left[\frac{\partial^2 V}{\partial x^2}\right]_{(0,0,0)} &= \left[\frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right)\right]_{(0,0,0)} \\ &\approx \frac{1}{a}\left\{\left[\frac{\partial V}{\partial x}\right]_{(a/2,0,0)} - \left[\frac{\partial V}{\partial x}\right]_{(-a/2,0,0)}\right\} \\ &\approx \frac{1}{a}\left\{\frac{[V]_{(a,0,0)} - [V]_{(0,0,0)}}{a} - \frac{[V]_{(0,0,0)} - [V]_{(-a,0,0)}}{a}\right\} \\ &= \frac{1}{a^2}[(V_1 - V_0) - (V_0 - V_2)] \\ &= \frac{1}{a^2}(V_1 + V_2 - 2V_0) \end{aligned} \tag{9.48a}$$

Similarly,

$$\left[\frac{\partial^2 V}{\partial y^2}\right]_{(0,0,0)} \approx \frac{1}{a^2}(V_3 + V_4 - 2V_0) \tag{9.48b}$$

Substituting (9.48a) and (9.48b) into (9.47) and rearranging, we obtain

$$V_0 \approx \frac{1}{4}(V_1 + V_2 + V_3 + V_4) \tag{9.49}$$

Thus the potential at P is approximately equal to the average of the potentials at the four equidistant points lying along mutually perpendicular axes through P. The result becomes more and more accurate as the spacing a becomes less and less.

Equation (9.49) forms the basis for the computer solution of Laplace's equation. To illustrate the technique, let us consider the problem of Example 9.4 and assume $a = b$, that is, a square slot. We can then divide the area of the slot into a 4×4 grid of squares, as shown in Fig. 9.10. If we assume V_0 to be 100 V, then the potentials at the five grid points along the mouth of the slot are $100 \sin 0$, $100 \sin \frac{\pi}{4}$, $100 \sin \frac{\pi}{2}$, $100 \sin \frac{3\pi}{4}$, and $100 \sin \pi$ or 0, 70.71, 100, 70.71, and 0 V, respectively, as shown in the figure. The potentials at the grid points along the remaining three sides of the slot are all zero. The exact values of potentials at the grid points inside the slot computed from the

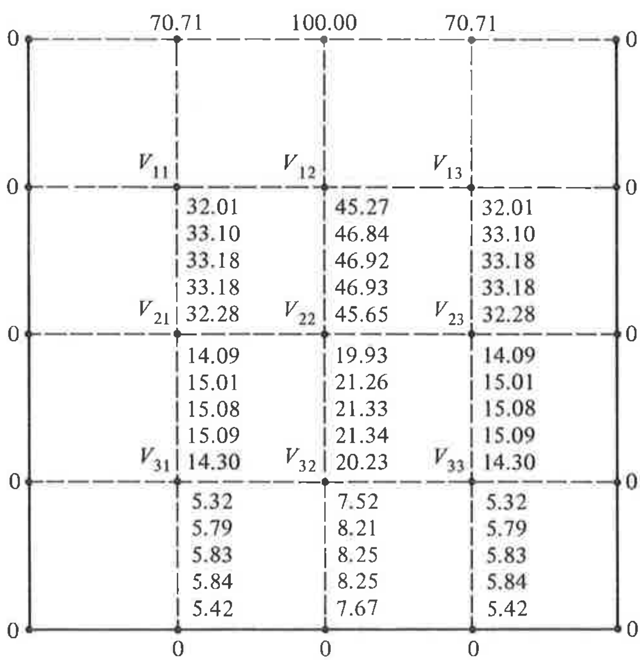


Figure 9.10. For illustrating the computer solution of Laplace's equation in two dimensions.

analytical solution given by (9.44) are shown by the upper rows of numbers beside the grid points for later comparison with those obtained by the computer solution technique.

The computer solution consists of finding the potentials at the nine grid points inside the slot from the given values at the grid points on the boundaries of the slot. Irrespective of how this is achieved, we must obtain a final set of values such that the potential at each grid point inside the slot is the average of the potentials at the neighboring four grid points, or the sum of the four neighboring potentials is equal to four times the potential at the grid point, in accordance with (9.49). The simplest technique adaptable to computer solution is to start with values of zero for all unknown potentials. Each unknown potential is then replaced by the average of the four neighboring potentials by traversing the grid in a systematic manner and by replacing in this process old values with new values as they are computed, until a set of values satisfying (9.49) at each grid point, to within a specified error, is obtained. Any symmetry associated with the problem, as in the present case, can be utilized to advantage for achieving a reduction in the number of computations.

The method we just discussed is known as the "iteration" technique since it involves the iterative process of converging an initially assumed solution to a final one consistent with Laplace's equation in the approximate sense given by (9.49). There are several variations of the iteration technique. For example, by employing an initial guess other than zeros, a faster convergence may be achieved. The end result will, however, still be only to within the specified accuracy.

The values of potentials obtained by the iteration technique for a specified maximum allowable value of 0.1 V for the error

$$\Delta = \left[V_0 - \frac{1}{4}(V_1 + V_2 + V_3 + V_4)\right]$$

are shown by the second rows of numbers beside the grid points in Fig. 9.10. When the specified maximum error is decreased to 0.01 V, thereby demanding a more accurate solution, the values of potentials obtained are shown by the third rows of numbers beside the grid points in Fig. 9.10. When these two rows are compared with the upper rows, it appears that the specification of a greater required accuracy in the iteration leads to a less accurate end result. This is, however, not the case since the iteration method can only converge to a solution that is consistent with (9.49) and not to the analytical solution.

Hence let us find the exact values of the unknown potentials consistent with (9.49). To do this, we write a set of simultaneous equations for these potentials by applying (9.49) at each grid point inside the slot. Thus denoting the unknown potentials to be V_{ij}, $i, j = 1, 2, 3$, as shown in Fig. 9.10, we obtain a set of nine equations given in matrix form by

$$\left[\begin{array}{ccc|ccc|ccc} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ \hline -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ \hline 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{array}\right] \left[\begin{array}{c} V_{11} \\ V_{12} \\ V_{13} \\ \hline V_{21} \\ V_{22} \\ V_{23} \\ \hline V_{31} \\ V_{32} \\ V_{33} \end{array}\right] = \left[\begin{array}{c} 70.71 \\ 100.00 \\ 70.71 \\ \hline 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \end{array}\right] \tag{9.50}$$

The matrix equation (9.50) can be inverted directly, since it involves only a 9×9 matrix. Imagine, however, the situation if the number of grid points is large. For example, even for a 16×16 grid of squares, it will be necessary to invert a 225×225 matrix! Fortunately, however, it is not necessary to directly invert the matrix. To illustrate this we see from the partitionings in (9.50) that it can be written in compact form as

$$\begin{bmatrix} M & -U & 0 \\ -U & M & -U \\ 0 & -U & M \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} V_g \\ 0 \\ 0 \end{bmatrix} \tag{9.51}$$

where

$$M = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix} \tag{9.52a}$$

$$U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{9.52b}$$

$$V_i = \begin{bmatrix} V_{i1} \\ V_{i2} \\ V_{i3} \end{bmatrix} \qquad i = 1, 2, 3 \tag{9.52c}$$

$$V_g = \begin{bmatrix} 70.71 \\ 100.00 \\ 70.71 \end{bmatrix} \tag{9.52d}$$

From (9.51), we can write the following equations successively:

$$-UV_2 + MV_3 = 0$$

$$V_2 = MV_3 \tag{9.53a}$$

$$-UV_1 + MV_2 - UV_3 = 0$$

$$V_1 = MV_2 - UV_3 = (M^2 - U)V_3 \tag{9.53b}$$

$$MV_1 - UV_2 = V_g$$

$$(M^3 - 2M)V_3 = V_g$$

$$V_3 = (M^3 - 2M)^{-1}V_g \tag{9.53c}$$

Substituting for M and V_g in (9.53c) from (9.52a) and (9.52d), respectively, and simplifying, we get

$$\begin{bmatrix} V_{31} \\ V_{32} \\ V_{33} \end{bmatrix} = \begin{bmatrix} 68 & -48 & 12 \\ -48 & 80 & -48 \\ 12 & -48 & 68 \end{bmatrix}^{-1} \begin{bmatrix} 70.71 \\ 100.00 \\ 70.71 \end{bmatrix} \tag{9.54}$$

Thus, we have simplified the problem into one of inversion of a 3 × 3 matrix. Inverting the 3 × 3 matrix and performing the matrix multiplication on the right side of (9.54), we obtain the values of V_{31}, V_{32}, and V_{33}. The remaining values can then be found from (9.53a) and (9.53b). The results are shown by the fourth rows of numbers beside the grid points in Fig. 9.10.

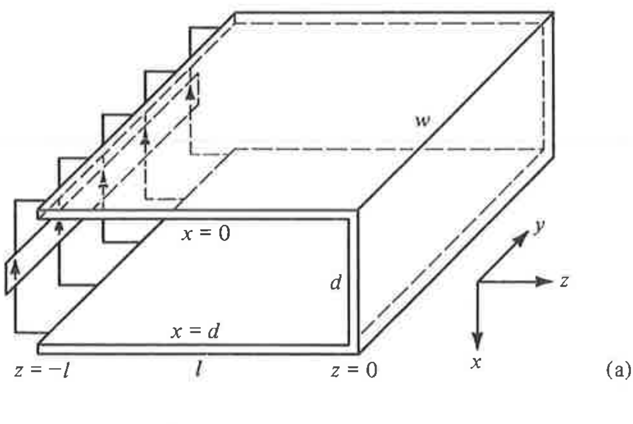
It can now be seen by comparing the second and third rows of values with the fourth rows of values that the iteration method does converge closer to the exact solution consistent with (9.49) as the specified allowable error is decreased. To obtain a solution closer to the exact analytical solution, we must decrease the spacing between the grid points. For example, for an 8 × 8 grid of squares, the solution obtained by the iteration method for a specified maximum error of 0.01 V is shown by the set of numbers in the last rows in Fig. 9.10.

9.5 LOW-FREQUENCY BEHAVIOR VIA QUASISTATICS

In Example 6.4 in Sec. 6.4 we illustrated the determination of the low-frequency behavior of a physical structure from its input impedance by considering the example of the short-circuited line. We expressed the input impedance of the short-circuited line as an infinite series involving powers of the frequency ω and by considering the term proportional to ω we found that for a line of length l, the input impedance is equivalent to that of a single inductor for frequencies low enough such that $l \ll \lambda$, the wavelength corresponding to the frequency. In this section we shall illustrate the determination of the low-frequency behavior by a quasistatic extension of the static field existing in the structure when the frequency of the source driving the structure is zero. The quasistatic extension consists of starting with a time-varying field having the same spatial characteristics as that of the static field and obtaining the field solutions containing terms up to and including the first power in ω.

To introduce the quasistatic field approach, we shall first consider the same physical structure as a short-circuited parallel-plate line, that is, an arrangement of two parallel, plane, perfect conductors joined at one end by another perfectly conducting sheet, as shown in Fig. 9.11(a). We shall neglect fringing of the fields by assuming that the spacing d between the plates is very small compared to the dimensions of the plates or that the structure is part of a structure of much larger extent in the y and z directions. For a constant current source of value I_0 driving the structure at the end $z = -l$, as shown in the figure, such that the surface current densities on the two plates are given by

$$\mathbf{J}_S = \begin{cases} \dfrac{I_0}{w}\mathbf{i}_z & \text{for } x = 0 \\ -\dfrac{I_0}{w}\mathbf{i}_z & \text{for } x = d \end{cases} \tag{9.55}$$



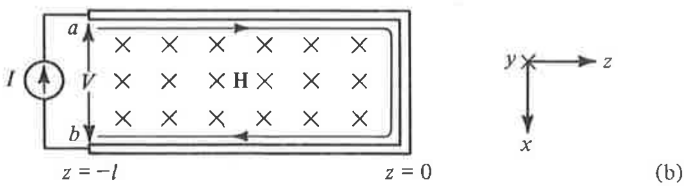


Figure 9.11. (a) A parallel-plate structure short-circuited at one end and driven by a current source at the other end. (b) Magnetic field between the plates for a constant current source.

the medium between the plates is characterized by a uniform y-directed magnetic field as shown by the cross-sectional view in Fig. 9.11(b). The field is zero outside the plates. From the boundary condition for the tangential magnetic field intensity at the surface of a perfect conductor, the magnitude of this field is I_0/w. Thus we obtain the static magnetic field intensity between the plates to be

$$\mathbf{H} = \frac{I_0}{w}\mathbf{i}_y \qquad \text{for } 0 < x < d \tag{9.56}$$

The corresponding magnetic flux density is given by

$$\mathbf{B} = \mu\mathbf{H} = \frac{\mu I_0}{w}\mathbf{i}_y \qquad \text{for } 0 < x < d \tag{9.57}$$

The magnetic flux ψ linking the current is simply the flux crossing the cross-sectional plane of the structure. Since $\mathbf{B}$ is uniform in the cross-sectional plane and normal to it,

$$\psi = B_y(dl) = \frac{\mu dl}{w} I_0 \tag{9.58}$$

The ratio of this magnetic flux to the current, that is, the inductance of the structure, is

$$L = \frac{\psi}{I_0} = \frac{\mu dl}{w} \tag{9.59}$$

To discuss the quasistatic behavior of the structure, we now let the current source be varying sinusoidally with time at a frequency ω and assume that the magnetic field between the plates varies accordingly. Thus for

$$I(t) = I_0 \cos \omega t \tag{9.60}$$

we have

$$\mathbf{H}_0 = \frac{I_0}{w} \cos \omega t \, \mathbf{i}_y \tag{9.61}$$

where the subscript 0 denotes that the field is of the zeroth power in ω. In terms of phasor notation, we have for

$$\bar{I} = I_0 \tag{9.62}$$

$$\bar{H}_{y0} = \frac{I_0}{w} \tag{9.63}$$

The time-varying magnetic field (9.61) gives rise to an electric field in accordance with Maxwell's curl equation for $\mathbf{E}$. Expansion of the curl equation for the case under consideration gives

$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_{y0}}{\partial t} = -\mu\frac{\partial H_{y0}}{\partial t}$$

or, in phasor form

$$\frac{\partial \bar{E}_x}{\partial z} = -j\omega\mu\bar{H}_{y0} \tag{9.64}$$

Substituting for $\bar{H}_{y0}$ from (9.63), we have

$$\frac{\partial \bar{E}_x}{\partial z} = -j\omega\mu\frac{\bar{I}_0}{w}$$

or

$$\bar{E}_x = -j\omega\mu\frac{\bar{I}_0}{w}z + \bar{C} \tag{9.65}$$

The constant $\bar{C}$ is, however, equal to zero since $[\bar{E}_x]_{z=0} = 0$ to satisfy the boundary condition of zero tangential electric field on the perfect conductor surface. Thus we obtain the quasistatic electric field in the structure to be

$$\bar{E}_{x1} = -j\omega\frac{\mu z}{w}\bar{I}_0 \tag{9.66}$$

where the subscript 1 denotes that the field is of the first power in ω.

The voltage developed across the current source is now given by

$$\begin{aligned}\bar{V} &= \int_a^b [\bar{E}_{x1}]_{z=-l}\,dx \\ &= j\omega\frac{\mu dl}{w}\bar{I}_0 \\ &= j\omega L\bar{I}_0 \end{aligned} \tag{9.67}$$

Thus the quasistatic extension of the static field in the structure of Fig. 9.11 illustrates that its input behavior for low frequencies is equivalent to that of a single inductor as we found in Example 6.4.

Example 9.6. Let us consider the case of two parallel perfectly conducting plates separated by a lossy medium characterized by conductivity σ, permittivity ϵ, and permeability μ and driven by a voltage source at one end, as shown in Fig. 9.12(a). We wish to determine its low-frequency behavior by using the quasistatic field approach.

Assuming the voltage source to be a constant voltage source, we first obtain the static electric field in the medium between the plates to be

$$\mathbf{E} = \frac{V_0}{d}\mathbf{i}_x$$

following the procedure of Example 9.3. The conduction current density in

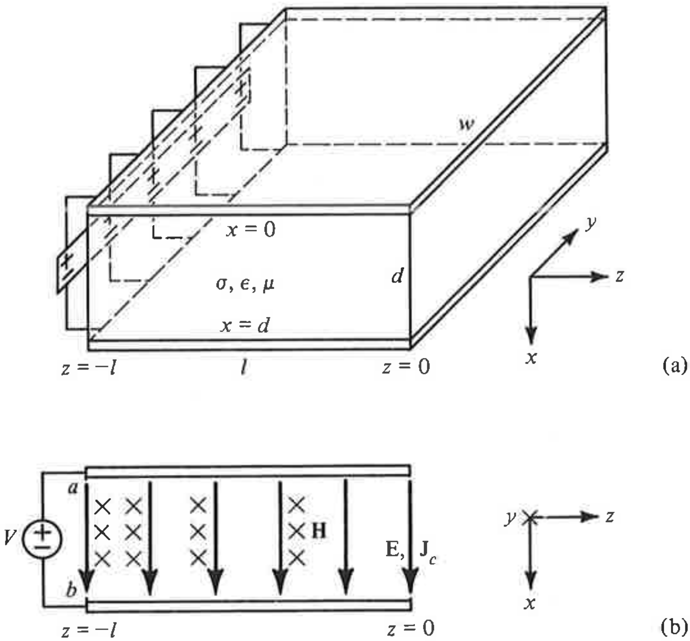


Figure 9.12. (a) A parallel-plate structure with lossy medium between the plates and driven by a voltage source. (b) Electric and magnetic fields between the plates for a constant voltage source.

the medium is then given by

$$\mathbf{J}_c = \sigma\mathbf{E} = \frac{\sigma V_0}{d}\mathbf{i}_x$$

The conduction current gives rise to a static magnetic field in accordance with Maxwell's curl equation for **H** given for static fields by

$$\nabla \times \mathbf{H} = \mathbf{J}_c = \sigma\mathbf{E}$$

For the case under consideration, this reduces to

$$\frac{\partial H_y}{\partial z} = -\sigma E_x = -\frac{\sigma V_0}{d}$$

giving us

$$H_y = -\frac{\sigma V_0 z}{d} + C_1$$

The constant C_1 is, however, equal to zero since $[H_y]_{z=0} = 0$ in view of the boundary condition that the surface current density on the plates must be zero at $z = 0$. Thus the static magnetic field in the medium between the plates is given by

$$\mathbf{H} = -\frac{\sigma V_0 z}{d}\mathbf{i}_y$$

The static electric and magnetic field distributions are shown by the cross-sectional view of the structure in Fig. 9.12(b).

To determine the quasistatic behavior of the structure, we now let the voltage source be varying sinusoidally with time at a frequency ω and assume that the electric and magnetic fields vary with time accordingly. Thus for

$$V = V_0 \cos \omega t$$

we have

$$\mathbf{E}_0 = \frac{V_0}{d} \cos \omega t \, \mathbf{i}_x \tag{9.68a}$$

$$\mathbf{H}_0 = -\frac{\sigma V_0 z}{d} \cos \omega t \, \mathbf{i}_y \tag{9.68b}$$

where the subscript 0 denotes that the fields are of the zeroth power in ω. In terms of phasor notation, we have for $\bar{V} = V_0$,

$$\bar{E}_{x0} = \frac{V_0}{d} \tag{9.69a}$$

$$\bar{H}_{y0} = -\frac{\sigma V_0 z}{d} \tag{9.69b}$$

The time-varying electric field (9.68a) gives rise to a magnetic field in accordance with

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}_0}{\partial t} = \epsilon \frac{\partial \mathbf{E}_0}{\partial t}$$

and the time-varying magnetic field (9.68b) gives rise to an electric field in accordance with

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}_0}{\partial t} = -\mu \frac{\partial \mathbf{H}_0}{\partial t}$$

For the case under consideration and using phasor notation, these equations reduce to

$$\frac{\partial \bar{H}_y}{\partial z} = -j\omega\epsilon \bar{E}_{x0} = -j\omega \frac{\epsilon V_0}{d}$$

$$\frac{\partial \bar{E}_x}{\partial z} = -j\omega\mu \bar{H}_{y0} = j\omega \frac{\mu\sigma V_0 z}{d}$$

giving us

$$\bar{H}_{y1} = -j\omega\frac{\epsilon V_0 z}{d} + \bar{C}_2$$

$$\bar{E}_{x1} = j\omega\frac{\mu\sigma V_0 z^2}{2d} + \bar{C}_3$$

where the subscript 1 denotes that the fields are of the first power in ω. The constant $\bar{C}_2$ is, however, equal to zero in view of the boundary condition that the surface current density on the plates must be zero at $z = 0$. To evaluate the constant $\bar{C}_3$, we note that $[\bar{E}_{x1}]_{z=-l} = 0$ since the boundary condition at the source end, that is,

$$\bar{V} = \int_a^b [\bar{E}_x]_{z=-l}\, dx$$

is satisfied by $\bar{E}_{x0}$ alone. Thus we have

$$j\omega\frac{\mu\sigma V_0(-l)^2}{2d} + \bar{C}_3 = 0$$

or

$$\bar{C}_3 = -j\omega\frac{\mu\sigma V_0 l^2}{2d}$$

Substituting for $\bar{C}_3$ and $\bar{C}_2$ in the expressions for $\bar{E}_{x1}$ and $\bar{H}_{y1}$, respectively, we get

$$\bar{E}_{x1} = j\omega\frac{\mu\sigma V_0(z^2 - l^2)}{2d} \tag{9.70a}$$

$$\bar{H}_{y1} = -j\omega\frac{\epsilon V_0 z}{d}$$

The result for $\bar{H}_{y1}$ is, however, not complete since $\bar{E}_{x1}$ gives rise to a conduction current of density proportional to ω which in turn provides an additional contribution to $\bar{H}_{y1}$. Denoting this contribution to be H_{y1}^c, we have

$$\frac{\partial \bar{H}_{y1}^c}{\partial z} = -\sigma\bar{E}_{x1} = -j\omega\frac{\mu\sigma^2 V_0(z^2 - l^2)}{2d}$$

$$\bar{H}_{y1}^c = -j\omega\frac{\mu\sigma^2 V_0(z^3 - 3zl^2)}{6d} + \bar{C}_4$$

The constant $\bar{C}_4$ is zero for the same reason that $\bar{C}_2$ is zero. Hence setting $\bar{C}_4$ equal to zero and adding the resulting expression for $\bar{H}_{y1}^c$ to the right side of the expression for $\bar{H}_{y1}$, we obtain the complete expression for $\bar{H}_{y1}$ as

$$\bar{H}_{y1} = -j\omega\frac{\epsilon V_0 z}{d} - j\omega\frac{\mu\sigma^2 V_0(z^3 - 3zl^2)}{6d} \tag{9.70b}$$

The total field components correct to the first power in ω are then given by

$$\begin{aligned}\bar{E}_x &= \bar{E}_{x0} + \bar{E}_{x1} \\ &= \frac{V_0}{d} + j\omega\frac{\mu\sigma V_0(z^2 - l^2)}{2d} \qquad (9.71a)\\ \bar{H}_y &= \bar{H}_{y0} + \bar{H}_{y1} \\ &= -\frac{\sigma V_0 z}{d} - j\omega\frac{\epsilon V_0 z}{d} - j\omega\frac{\mu\sigma^2 V_0(z^3 - 3zl^2)}{6d} \qquad (9.71b)\end{aligned}$$

The current drawn from the voltage source is

$$\begin{aligned}\bar{I} &= w[\bar{H}_y]_{z=-l} \\ &= \left(\frac{\sigma wl}{d} + j\omega\frac{\epsilon wl}{d} - j\omega\frac{\mu\sigma^2 wl^3}{3d}\right)\bar{V} \qquad (9.72)\end{aligned}$$

Finally, the input admittance of the structure is given by

$$\begin{aligned}\bar{Y} = \frac{\bar{I}}{\bar{V}} &= j\omega\frac{\epsilon wl}{d} + \frac{\sigma wl}{d}\left(1 - j\omega\frac{\mu\sigma l^2}{3}\right) \\ &\approx j\omega\frac{\epsilon wl}{d} + \frac{1}{\dfrac{d}{\sigma wl}\left(1 + j\omega\dfrac{\mu\sigma l^2}{3}\right)} \\ &= j\omega\frac{\epsilon wl}{d} + \frac{1}{\dfrac{d}{\sigma wl} + j\omega\dfrac{\mu dl}{3w}} \\ &= j\omega C + \frac{1}{R + (j\omega L/3)} \qquad (9.73)\end{aligned}$$

where $C = \epsilon wl/d$ is the capacitance of the structure if the material is a perfect dielectric, $R = d/\sigma wl$ is the d.c. resistance (reciprocal of the conductance) of the structure, and $L = \mu dl/w$ is the inductance of the structure if the material is lossless and the two plates are short-circuited at $z = 0$. The equivalent circuit corresponding to (9.73) consists of capacitance C in parallel with the series combination of resistance R and inductance $L/3$, as shown in Fig. 9.13. ∎

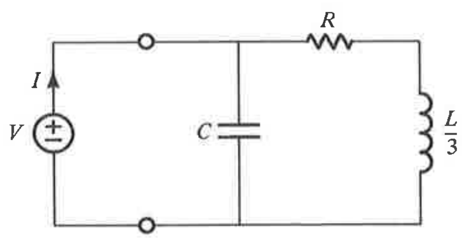


Figure 9.13. Equivalent circuit for the low-frequency input behavior of the structure of Fig. 9.12.

9.6 MAGNETIC CIRCUITS

In this section we shall introduce the principle of magnetic circuits. A simple example of magnetic circuit is the toroidal magnetic core of uniform permeability μ and having a uniform, circular cross-sectional area A and mean circumference l, as shown in Fig. 9.14. A current I_0 amp is passed through a filamentary wire of N turns wound around the toroid. Because of this current, a magnetic field is established in the core in the direction of advance of a right-hand screw as it is turned in the sense of the current, as shown in Fig. 9.14.

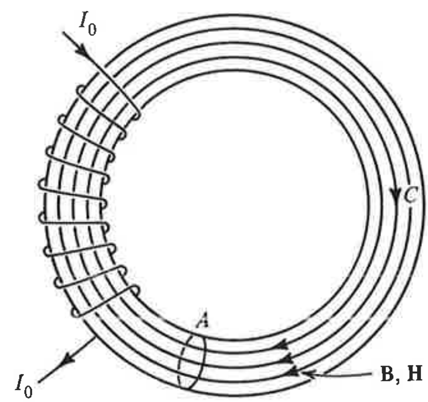


Figure 9.14. A toroidal magnetic circuit.

If the permeability of the core material is very large compared to the permeability of the surrounding medium, which is free space, the magnetic flux is confined almost entirely to the core in a manner similar to conduction current flow in wires or fluid flow in pipes. To illustrate this, let us consider lines of magnetic flux density on either side of a plane interface between a magnetic material of $\mu \gg \mu_0$ and free space, as shown in Fig. 9.15. Then from the boundary conditions for the magnetic field components, we have

$$B_1 \sin \alpha_1 = B_2 \sin \alpha_2 \tag{9.74a}$$

$$H_1 \cos \alpha_1 = H_2 \cos \alpha_2 \tag{9.74b}$$

Dividing (9.74a) by (9.74b), we get

$$\frac{B_1}{H_1} \tan \alpha_1 = \frac{B_2}{H_2} \tan \alpha_2$$

$$\mu_1 \tan \alpha_1 = \mu_2 \tan \alpha_2$$

$$\frac{\tan \alpha_1}{\tan \alpha_2} = \frac{\mu_2}{\mu_1} = \frac{\mu_0}{\mu_1} \ll 1$$

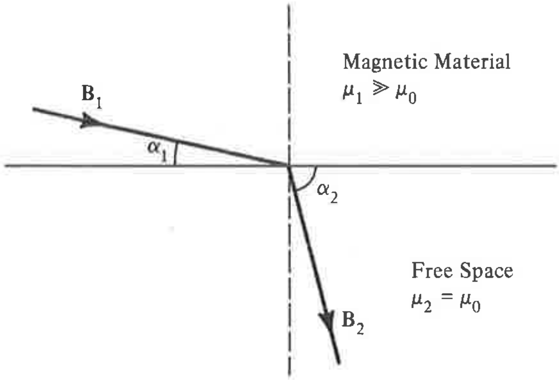


Figure 9.15. Lines of magnetic flux density at the boundary between free space and a magnetic material of $\mu \gg \mu_0$.

Thus $\alpha_1 \ll \alpha_2$, and

$$\frac{B_2}{B_1} = \frac{\sin \alpha_1}{\sin \alpha_2} \ll 1$$

For example, if the values of μ_1 and α_1 are 1000 μ_0 and 5°, respectively, then $\alpha_2 = 89.35°$ and $B_2/B_1 = 0.087$. For $\alpha_1 = 3°$, $\alpha_2 = 88.9°$ and $B_2/B_1 = 0.052$. The magnetic flux is for all practical purposes confined entirely to the core and very little flux appears as leakage flux outside the core.

If we assume that the magnetic flux ψ over the cross-sectional area of the toroid is equal to the flux density B_m at the mean radius of the toroid times the cross-sectional area of the toroid, we can then write

$$B_m = \frac{\psi}{A} \tag{9.75}$$

$$H_m = \frac{B_m}{\mu} = \frac{\psi}{\mu A} \tag{9.76}$$

From Ampere's circuital law, the magnetomotive force around the closed path C along the mean circumference of the toroid is equal to the current enclosed by that path. This current is equal to NI_0 since the filamentary wire penetrates the surface bounded by the path N times. Thus

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = NI_0$$

$$H_m l = NI_0 \tag{9.77}$$

Substituting for H_m from (9.76) and rearranging, we obtain

$$\psi = \frac{\mu N I_0 A}{l}$$

We now define the "reluctance" of the magnetic circuit, denoted by the symbol $\mathcal{R}$, as the ratio of the ampere turns NI_0 applied to the magnetic circuit to the magnetic flux ψ. Thus

$$\mathcal{R} = \frac{NI_0}{\psi} = \frac{l}{\mu A} \tag{9.78}$$

The reluctance of the magnetic circuit is analogous to the electric circuit quantity resistance and has the units of ampere turns per weber. We note from (9.78) that for a given magnetic material, the reluctance appears to be purely a function of the dimensions of the circuit. This is, however, not true since for the ferromagnetic materials used for the cores, μ is a function of the magnetic flux density in the material, as we learned in Sec. 5.3.

As a numerical example of computations involving the magnetic circuit of Fig. 9.14, let us consider a core of cross-sectional area 2 cm² and mean circumference 20 cm. Let the material of the core be annealed sheet steel for which the B versus H relationship is shown by the curve of Fig. 9.16. Then to establish a magnetic flux of 3×10^{-4} Wb in the core, the mean flux density must be $(3 \times 10^{-4})/(2 \times 10^{-4})$ or 1.5 Wb/m². From Fig. 9.16, the corresponding value of H is 1000 amp/m. The number of ampere turns required to establish the flux is then equal to $1000 \times 20 \times 10^{-2}$, or, 200, and the reluctance of the core is $200/(3 \times 10^{-4})$, or $(2/3) \times 10^6$ amp-turns/Wb. We shall now consider a more detailed example.

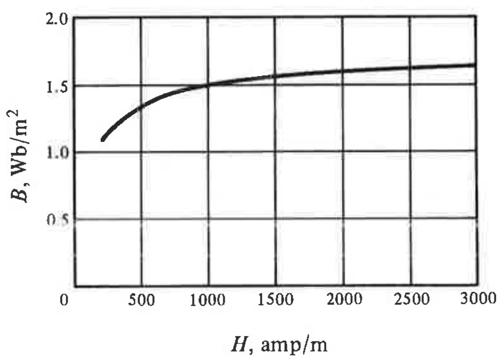


Figure 9.16. B versus H curve for annealed sheet steel.

Example 9.7. A magnetic circuit containing three legs and with an air gap in the right leg is shown in Fig. 9.17(a). A filamentary wire of N turns carrying current I is wound around the center leg. The core material is annealed sheet steel, for which the B versus H relationship is shown in Fig. 9.16. The dimensions of the magnetic circuit are

$$A_1 = A_3 = 3 \text{ cm}^2, \qquad A_2 = 6 \text{ cm}^2$$
$$l_1 = l_3 = 20 \text{ cm}, \qquad l_2 = 10 \text{ cm}, \qquad l_g = 0.2 \text{ mm}$$

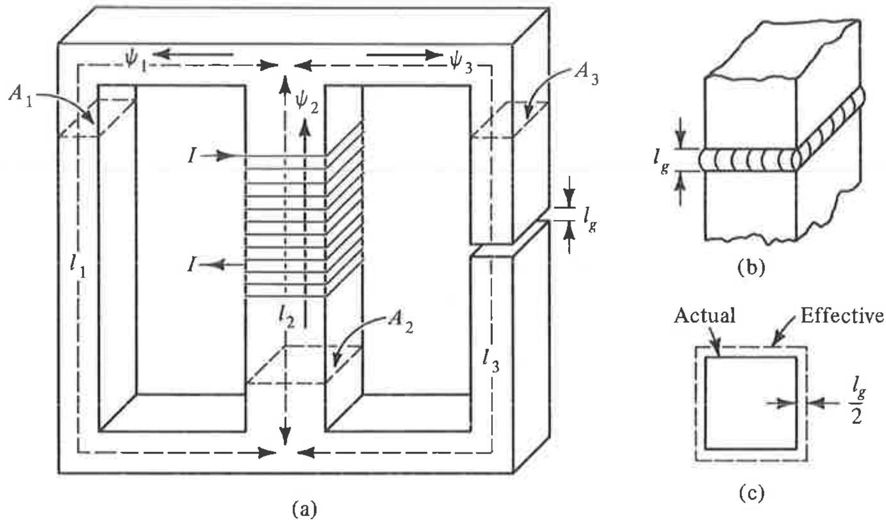


Figure 9.17. (a) A magnetic circuit. (b) Fringing of magnetic flux in the air gap of the magnetic circuit. (c) Effective and actual cross sections for the air gap.

Let us determine the value of NI required to establish a magnetic flux of 4×10^{-4} Wb in the air gap.

The current in the winding establishes a magnetic flux in the center leg which divides between the right and left legs. Fringing of the flux occurs in the air gap, as shown in Fig. 9.17(b). This is taken into account by using an effective cross section larger than the actual cross section, as shown in Fig. 9.17(c). Using subscripts 1, 2, 3, and g for the quantities associated with the left, center, and right legs, and the air gap, respectively, we can write

$$\psi_3 = \psi_g$$

$$\psi_2 = \psi_1 + \psi_3$$

Also, applying Ampere's circuital law to the right and left loops of the magnetic circuit, we obtain, respectively,

$$NI = H_2 l_2 + H_3 l_3 + H_g l_g$$

$$NI = H_2 l_2 + H_1 l_1$$

It follows from these two equations that

$$H_1 l_1 = H_3 l_3 + H_g l_g$$

which can also be written directly from a consideration of the outer loop of the magnetic circuit.

Noting from Fig. 9.17(c) that the effective cross section of the air gap is $(\sqrt{3} + l_g)^2 = 3.07\ \text{cm}^2$, we find the required magnetic flux density in the air gap to be

$$B_g = \frac{\psi_g}{(A_g)_{\text{eff}}} = \frac{4 \times 10^{-4}}{3.07 \times 10^{-4}} = 1.303\ \text{Wb/m}^2$$

The magnetic field intensity in the air gap is

$$H_g = \frac{B_g}{\mu_0} = \frac{1.303}{4\pi \times 10^{-7}} = 0.1037 \times 10^7\ \text{amp/m}$$

The flux density in leg 3 is

$$B_3 = \frac{\psi_3}{A_3} = \frac{\psi_g}{A_3} = \frac{4 \times 10^{-4}}{3 \times 10^{-4}} = 1.333\ \text{Wb/m}^2$$

From Fig. 9.16, the value of H_3 is 475 amp/m.

Knowing the values of H_g and H_3, we then obtain

$$\begin{aligned} H_1 l_1 &= H_3 l_3 + H_g l_g \\ &= 475 \times 0.2 + 0.1037 \times 10^7 \times 0.2 \times 10^{-3} \\ &= 302.4\ \text{amp} \\ H_1 &= \frac{302.4}{0.2} = 1512\ \text{amp/m} \end{aligned}$$

From Fig. 9.16, the value of B_1 is 1.56 Wb/m² and hence the flux in leg 1 is

$$\psi_1 = B_1 A_1 = 1.56 \times 3 \times 10^{-4} = 4.68 \times 10^{-4}\ \text{Wb}$$

Thus

$$\begin{aligned} \psi_2 &= \psi_1 + \psi_3 \\ &= 4.68 \times 10^{-4} + 4 \times 10^{-4} = 8.68 \times 10^{-4}\ \text{Wb} \\ B_2 &= \frac{\psi_2}{A_2} = \frac{8.68 \times 10^{-4}}{6 \times 10^{-4}} = 1.447\ \text{Wb/m}^2 \end{aligned}$$

From Fig. 9.16, the value of H_2 is 750 amp/m. Finally, we obtain the required number of ampere turns to be

$$\begin{aligned} NI &= H_2 l_2 + H_1 l_1 \\ &= 750 \times 0.2 + 302.4 \\ &= 452.4 \end{aligned}$$

■

9.7 SUMMARY

In this chapter we learned that Maxwell's equations for static fields are given by

$$\nabla \times \mathbf{E} = 0 \tag{9.79a}$$

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{9.79b}$$

$$\nabla \cdot \mathbf{D} = \rho \tag{9.79c}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{9.79d}$$

whereas the continuity equation is

$$\nabla \cdot \mathbf{J} = 0 \tag{9.80}$$

These equations together with the constitutive relations

$$\mathbf{D} = \epsilon \mathbf{E} \tag{9.81a}$$

$$\mathbf{H} = \frac{\mathbf{B}}{\mu} \tag{9.81b}$$

$$\mathbf{J} = \mathbf{J}_c = \sigma \mathbf{E} \tag{9.81c}$$

govern the behavior of static fields.

First we learned from (9.79a) that, since the curl of the gradient of a scalar function is identically zero, **E** can be expressed as the gradient of a scalar function. The gradient of a scalar function Φ is given in Cartesian coordinates by

$$\nabla \Phi = \frac{\partial \Phi}{\partial x}\mathbf{i}_x + \frac{\partial \Phi}{\partial y}\mathbf{i}_y + \frac{\partial \Phi}{\partial z}\mathbf{i}_z$$

The magnitude of $\nabla \Phi$ at a given point is the maximum rate of increase of Φ at that point, and its direction is the direction in which the maximum rate of increase occurs, that is, normal to the constant Φ surface passing through that point.

From considerations of work associated with the movement of a test charge in the static electric field, we found that

$$\mathbf{E} = -\nabla V \tag{9.82}$$

where V is the electric potential. The electric potential V_A at a point A is the amount of work per unit charge done by the field in the movement of a test charge from the point A to a reference point O. It is the potential difference between A and O. Thus

$$V_A = [V]_A^O = \int_A^O \mathbf{E} \cdot d\mathbf{l} = -\int_O^A \mathbf{E} \cdot d\mathbf{l}$$

The potential difference between two points has the same physical meaning as the voltage between the two points. The voltage is, however, not a unique quantity since it depends on the path employed for evaluating it, whereas the potential difference, being independent of the path, has a unique value.

We considered the potential field of a point charge and found that for the point charge

$$V = \frac{Q}{4\pi\epsilon R}$$

where R is the radial distance away from the point charge. The equipotential surfaces for the point charge are thus spherical surfaces centered at the point charge. We illustrated the application of the potential concept in the determination of electric field due to charge distributions by considering the example of an electric dipole.

Substituting (9.82) into (9.79c), we derived Poisson's equation

$$\nabla^2 V = -\frac{\rho}{\epsilon} \tag{9.83}$$

which states that the Laplacian of the electric potential at a point is equal to $-1/\epsilon$ times the volume charge density at that point. In Cartesian coordinates,

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

For the one-dimensional case in which the charge density is a function of x only, (9.83) reduces to

$$\frac{\partial^2 V}{\partial x^2} = \frac{d^2 V}{dx^2} = -\frac{\rho}{\epsilon}$$

We illustrated the solution of this equation by considering the example of a p–n junction diode.

If $\rho = 0$, Poisson's equation reduces to Laplace's equation

$$\nabla^2 V = 0 \tag{9.84}$$

This equation is applicable for a charge-free dielectric region as well as for a conducting medium. We illustrated the solution of the one-dimensional Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} = \frac{d^2 V}{dx^2} = 0$$

by considering a parallel-plate arrangement. By using the separation of variables technique, we obtained the general solution to Laplace's equation in two dimensions

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

and illustrated its application by considering two examples. We also discussed the applicability of Laplace's equation for the determination of transmission-line parameters and field maps.

To illustrate the computer solution of Laplace's equation, we derived the approximate solution to Laplace's equation in two dimensions. This solution states that the potential V_0 at a point P is given by

$$V_0 \approx \frac{1}{4}(V_1 + V_2 + V_3 + V_4) \tag{9.85}$$

where V_1, V_2, V_3, and V_4 are the potentials at four equidistant points lying along mutually perpendicular axes through P. By means of an example, we discussed the iteration technique of computer solution based on the repeated application of (9.85) to a set of grid points in the region of interest until a solution that converges to within a specified error is obtained. We also discussed the direct solution for the potentials at the grid points consistent with (9.85) using matrix inversion techniques.

After having considered the solution of static field problems, we then turned to the quasistatic extension of the static field solution as a means of obtaining the low-frequency behavior of a physical structure. The quasistatic field approach involves starting with a time-varying field having the same spatial characteristics as the static field in the physical structure and then obtaining field solutions containing terms up to and including the first power in frequency by using Maxwell's curl equations for time-varying fields. We illustrated this approach by considering two examples, one of them involving a lossy medium.

Finally, we introduced the magnetic circuit, which is essentially an arrangement of closed paths for magnetic flux to flow around just as current in electric circuits. The closed paths are provided by ferromagnetic cores which because of their high permeability relative to that of the surrounding medium confine the flux almost entirely to within the core regions. We illustrated the analysis of magnetic circuits by considering two examples, one of them including an air gap in one of the legs.

REVIEW QUESTIONS

9.1. State Maxwell's curl equations for static fields.

9.2. What is the expansion for the gradient of a scalar in Cartesian coordinates? When can a vector be expressed as the gradient of a scalar?

9.3. Discuss the physical interpretation for the gradient of a scalar function.

9.4. Discuss the application of the gradient concept for the determination of unit vector normal to a surface.

9.5. How would you find the rate of increase of a scalar function along a specified direction by using the gradient concept?

9.6. Define electric potential. What is its relationship to the static electric field intensity?

9.7. Distinguish between voltage, as applied to time-varying fields, and potential difference.

9.8. What is a conservative field? Give two examples of conservative fields.

9.9. Describe the equipotential surfaces for a point charge.

9.10. Discuss the determination of the electric field intensity due to a charge distribution by using the potential concept.

9.11. What is the Laplacian of a scalar? What is its expansion in Cartesian coordinates?

9.12. State Poisson's equation.

9.13. Outline the solution of Poisson's equation for the potential in a region of known charge density varying in one dimension.

9.14. State Laplace's equation. In what regions is it valid?

9.15. Discuss the application of Laplace's equation for a conducting medium.

9.16. Outline the solution of Laplace's equation in one dimension.

9.17. Why is Laplace's equation applicable to the determination of transmission-line parameters and field maps?

9.18. Outline the solution of Laplace's equation in two dimensions by the separation of variables technique.

9.19. What is the principle behind the approximate solution of Laplace's equation in two dimensions?

9.20. Discuss the iteration technique for the computer solution of Laplace's equation in two dimensions.

9.21. By consulting appropriate reference books, discuss two variations of the iteration technique for the computer solution of Laplace's equation.

9.22. How would you apply the iteration technique for the computer solution of Laplace's equation in three dimensions?

9.23. What is meant by the quasistatic extension of the static field in a physical structure?

9.24. Outline the steps involved in the determination of the quasistatic electric field in a parallel-plate structure short circuited at one end.

9.25. Why must the surface current density on the plates of the structure of Fig. 9.12 be zero at $z = 0$?

9.26. Discuss the quasistatic behavior of the structure of Fig. 9.12 for $\sigma \approx 0$.

9.27. What is a magnetic circuit? Why is the magnetic flux in a magnetic circuit confined almost entirely to the core?

9.28. Define the reluctance of a magnetic circuit. What is the analogous electric circuit quantity?

9.29. Why is the reluctance for a given set of dimensions of a magnetic circuit not a constant?

9.30. How is the fringing of the magnetic flux in an air gap in a magnetic circuit taken into account?

PROBLEMS

9.1. Find the gradients of the following scalar functions: (a) $\sqrt{x^2 + y^2 + z^2}$; (b) xyz.

9.2. Determine which of the following vectors can be expressed as the gradient of a scalar function: (a) $y\mathbf{i}_x - x\mathbf{i}_y$; (b) $x\mathbf{i}_x + y\mathbf{i}_y + z\mathbf{i}_z$; (c) $2xy^3z\mathbf{i}_x + 3x^2y^2z\mathbf{i}_y + x^2y^3\mathbf{i}_z$.

9.3. Find the unit vector normal to the plane surface $5x + 2y + 4z = 20$.

9.4. Find a unit vector normal to the surface $x^2 - y^2 = 5$ at the point (3, 2, 1).

9.5. Find the rate of increase of the scalar function x^2y at the point (1, 2, 1) in the direction of the vector $\mathbf{i}_x - \mathbf{i}_y$.

9.6. For the static electric field given by $\mathbf{E} = y\mathbf{i}_x + x\mathbf{i}_y$, find the potential difference between points $A(1, 1, 1)$ and $B(2, 2, 2)$.

9.7. For a point charge Q situated at the point (1, 2, 0), find the potential difference between the point $A(3, 4, 1)$ and the point $B(5, 5, 0)$.

9.8. An arrangement of point charges known as the linear quadrupole consists of point charges Q, $-2Q$, and Q at the points $(0, 0, d)$, $(0, 0, 0)$, and $(0, 0, -d)$, respectively. Obtain the expression for the electric potential and hence for the electric field intensity at distances from the quadrupole large compared to d.

9.9. For a line charge of uniform density 10^{-3} C/m situated along the z axis between (0, 0, −1) and (0, 0, 1), obtain the series expression for the electric potential at the point $(0, y, 0)$ by dividing the line charge into 100 equal segments and considering the charge in each segment to be a point charge located at the center of the segment. Then find the series expression for the electric field intensity at the point (0, 1, 0).

9.10. Repeat Problem 9.9, assuming the line charge density to be $10^{-3}|z|$ C/m.

9.11. The potential distribution in a simplified model of a vacuum diode consisting of cathode in the plane $x = 0$ and anode in the plane $x = d$ and held at a potential V_0 relative to the cathode is given by

$$V = V_0\left(\frac{x}{d}\right)^{4/3} \qquad \text{for } 0 < x < d$$

(a) Find the space charge density distribution in the region $0 < x < d$. (b) Find the surface charge densities on the cathode and the anode.

9.12. Show that for the p–n junction diode of Fig. 9.5(a), the boundary condition of the continuity of the normal component of displacement flux density at $x = 0$ is automatically satisfied by Eq. (9.29).

9.13. Assume that the impurity concentration for the p–n junction diode of Fig. 9.5(a) is a linear function of distance across the junction. The space charge density distribution is then given by

$$\rho = kx \qquad \text{for } -d/2 < x < d/2$$

where d is the width of the space charge region and k is the proportionality constant. Find the solution for the potential in the space charge region.

9.14. Two infinitely long cylindrical conductors having radii a and b and coaxial with the z axis are held at a potential difference of V_0. Using the cylindrical coordinate system, obtain the solution for the potential and hence for the electric field intensity in the charge-free dielectric region between the cylinders. Find the expression for the capacitance per unit length of the cylinders.

9.15. The region between the two plates of Fig. 9.6 is filled with two perfect dielectric media having permittivities ϵ_1 for $0 < x < t$ and ϵ_2 for $t < x < d$. (a) Find the solutions for the potentials in the two regions $0 < x < t$ and $t < x < d$. (b) Find the potential at the interface $x = t$.

9.16. Repeat Problem 9.15 if the two media are imperfect dielectrics having conductivities σ_1 and σ_2.

9.17. The potential distribution at the mouth of the slot of Fig. 9.7 is given by

$$V = V_0 \sin\frac{\pi y}{b} + \frac{1}{3} V_0 \sin\frac{3\pi y}{b}$$

(a) Find the solution for the potential distribution inside the slot. (b) Compute the value of the potential at the center of the slot, assuming the slot to be square.

9.18. Repeat Problem 9.17 for the potential distribution at the mouth of the slot given by

$$V = V_0 \sin^3 \frac{\pi y}{b}$$

9.19. Assume that the rectangular slot of Fig. 9.7 is covered at the mouth by conducting plates such that the potential distribution is given by

$$V = \begin{cases} 0 & \text{for } \ 0 < y < b/4 \\ V_0 & \text{for } \ b/4 < y < 3b/4 \\ 0 & \text{for } 3b/4 < y < b \end{cases}$$

Find the solution for the potential inside the slot.

9.20. For the rectangular slot of Example 9.4, (a) find the expression for the electric field intensity inside the slot and (b) find the electric field intensity at the center of the slot, assuming the slot to be square.

9.21. For the slot of Example 9.4, assume $a = b$ and the potential distribution at the mouth to be 100 $\sin^3 (\pi y/b)$ V. Compute the value of the potential at the center of the slot by (a) applying the iteration method to a 4×4 grid of squares, (b) using the 4×4 grid of squares to obtain the exact solution consistent with Eq. (9.49), and (c) applying the iteration method to an 8×8 grid of squares. Compare the results with the exact value given by the analytical solution found in Problem 9.18. Use a value of $\Delta = 0.01$ V for the iteration methods.

9.22. For the example of Fig. 9.10, divide the slot into a 16×16 grid of squares and by computing the potentials at the grid points surrounding the center of the slot by using the iteration technique and $\Delta = 0.01$ V, estimate the value of the electric field intensity at the center of the slot. Compare the estimated value with the exact value obtained in Problem 9.20.

9.23. The cross section of a structure that repeats endlessly in the plane of the paper is shown in Fig. 9.18. For the grid of points shown in the figure, compute the exact value of the potential at point A consistent with Eq. (9.49).

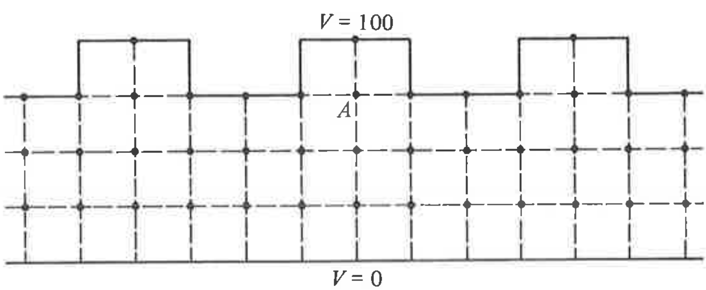


Figure 9.18. For Problem 9.23.

9.24. By considering Laplace's equation in three dimensions, show that the potential at a given point P in a charge-free region is approximately equal to the average of the potentials at the six equidistant points lying along mutually perpendicular axes through P. Then compute by the iteration method the potential at the center of the cubical box shown in Fig. 9.19 in which the top face is kept at 100 V relative to the other five faces. Use a $4 \times 4 \times 4$ grid of cubes and a value of 0.01 V for Δ.

9.25. For the structure of Fig. 9.11, assume that the medium between the plates is an imperfect dielectric of conductivity σ. (a) Show that the input impedance correct to the first power in ω is the same as if σ were zero. (b) Obtain the input impedance correct to the second power in ω and determine the equivalent circuit.

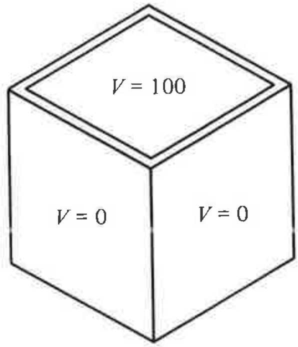


Figure 9.19. For problem 9.24.

9.26. For the structure of Fig. 9.11, continue the analysis beyond the quasistatic extension and obtain the input impedance correct to the third power in ω. Determine the equivalent circuit.

9.27. For the structure of Fig. 9.12, assume $\sigma = 0$ and continue the analysis beyond the quasistatic extension to obtain the input admittance correct to the third power in ω. Determine the equivalent circuit.

9.28. Find the condition(s) under which the quasistatic input behavior of the structure of Fig. 9.12 is essentially equivalent to (a) a capacitor in parallel with a resistor and (b) a resistor in series with an inductor.

9.29. For the toroidal magnetic circuit of Fig. 9.14, assume $A = 5\ \text{cm}^2$, $l = 20$ cm, and annealed sheet steel for the material of the core. Find the reluctance of the circuit for an applied NI equal to 400 amp-turns.

9.30. For the magnetic circuit of Fig. 9.17, assume the air gap to be in the center leg. Find the required NI to establish a magnetic flux of 9×10^{-4} Wb in the air gap.

9.31. For the magnetic circuit of Fig. 9.17, assume that there is no air gap. Find the magnetic flux established in the center leg for an applied NI equal to 180 amp-turns.

9.32. For the magnetic circuit of Fig. 9.17, assume no air gap and $A_1 = 5\ \text{cm}^2$ with all other dimensions remaining as specified in Example 9.7. Find the magnetic flux density in the center leg for an applied NI equal to 150 amp-turns.