6. TRANSMISSION LINES

In Chap. 4 we studied the principles of uniform plane wave propagation in free space. In Chap. 5 we extended the study of wave propagation to material media. In both chapters we were concerned with propagation in unbounded media. In this and the next chapters we shall consider guided wave propagation, that is, propagation of waves between boundaries. The boundaries are generally provided by conductors, whereas the media between the boundaries are generally dielectrics. There are two kinds of waveguiding systems. These are transmission lines and waveguides. A transmission line consists of two or more parallel conductors, whereas a waveguide is generally made up of one conductor. Our goal in particular in this chapter is to learn the principles of transmission lines.

We shall introduce the transmission line by considering a uniform plane wave and placing two parallel plane, perfect conductors such that the fields remain unaltered by satisfying the "boundary conditions" on the perfect conductor surfaces, which we will derive at the outset. The wave is then guided between and parallel to the conductors, thus leading to the parallel-plate line. We shall learn to represent a line by the "distributed" parameter equivalent circuit and discuss wave propagation on the line in terms of voltage and current. We shall learn to compute the circuit parameters for the parallel-plate line and then extend the computation to the general case of a line of arbitrary cross section. We shall discuss the "standing wave" phenomenon by considering the short-circuited line and reflection and transmission of waves at the junction between two lines in cascade.

6.1 BOUNDARY CONDITIONS ON A PERFECT CONDUCTOR SURFACE

In Sec. 5.6 we learned that the fields inside a perfect conductor are zero, as illustrated in Fig. 6.1. In this section we shall use this property to derive the "boundary conditions" for the fields on the surface of a perfect conductor.

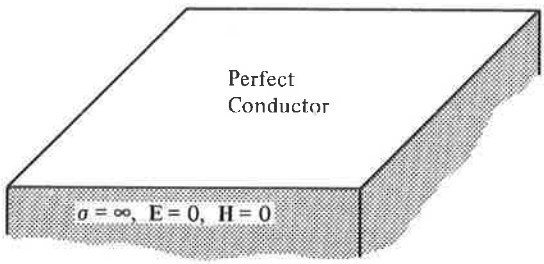

Figure 6.1. Showing that the fields inside a perfect conductor are zero.

Boundary conditions are simply a set of relationships relating the field components at a point adjacent to and on one side of the boundary between two different media to the field components at a corresponding point adjacent to and on the other side of the boundary. These relationships arise from the fact that Maxwell's equations in integral form involve closed paths and surfaces and they must be satisfied for all possible closed paths and surfaces whether they lie entirely in one medium or encompass a portion of the boundary between two different media. In the latter case, Maxwell's equations in integral form must be satisfied collectively by the fields on either side of the boundary, thereby resulting in the boundary conditions. To derive these boundary conditions, we recall that Maxwell's equations in integral form are given by

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S} \tag{6.1a}$$

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt}\int \mathbf{D} \cdot d\mathbf{S} \tag{6.1b}$$

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho\, dv \tag{6.1c}$$

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0 \tag{6.1d}$$

We shall apply these equations, one at a time, to a closed path or a closed surface encompassing the surface of a perfect conductor and derive the corresponding boundary conditions.

Considering Faraday's law in integral form, that is, (6.1a) first and applying it to an infinitesimal rectangular closed path *abcda* chosen such that *ab* and *cd* are very close to and on either side of the perfect conductor surface as shown in Fig. 6.2, we have

$$\oint_{abcda} \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_{abcd} \mathbf{B} \cdot d\mathbf{S}$$

or

$$\int_a^b \mathbf{E} \cdot d\mathbf{l} + \int_b^c \mathbf{E} \cdot d\mathbf{l} + \int_c^d \mathbf{E} \cdot d\mathbf{l} + \int_d^a \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_{abcd} \mathbf{B} \cdot d\mathbf{S} \quad (6.2)$$

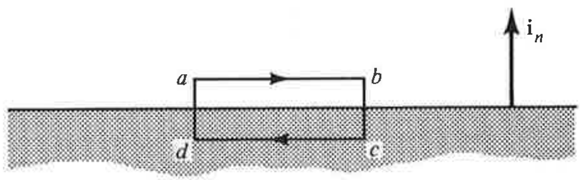

Figure 6.2. For deriving the boundary condition for the tangential component of **E** on a perfect conductor surface.

But $\int_c^d \mathbf{E} \cdot d\mathbf{l} = 0$ since **E** is zero inside the perfect conductor. If we now let *ad* and *bc* $\longrightarrow 0$ by making *ab* and *cd* almost touch each other but remaining on either side of the boundary, the quantities $\int_b^c \mathbf{E} \cdot d\mathbf{l}$, $\int_d^a \mathbf{E} \cdot d\mathbf{l}$, and $\int_{abcd} \mathbf{B} \cdot d\mathbf{S}$ all tend to zero, leaving us

$$\int_a^b \mathbf{E} \cdot d\mathbf{l} = 0 \quad (6.3)$$

Since *ab* is infinitesimal in size, we can write (6.3) as

$$E_{ab}(ab) = 0 \quad (6.4)$$

where E_{ab} is the component of **E** on the perfect conductor surface along the line *ab*. Thus we obtain

$$E_{ab} = 0 \quad (6.5)$$

Since we can choose the rectangle *abcda* with any orientation, it follows that E_{ab} is zero for any orientation of *ab*. Hence we obtain the first boundary condition that "the tangential component of **E** at a point on a perfect conductor surface is equal to zero." We can express this statement concisely in vector form as

$$\mathbf{i}_n \times \mathbf{E} = 0 \tag{6.6}$$

on the perfect conductor surface where $\mathbf{i}_n$ is the unit normal vector to the conductor surface, as shown in Fig. 6.2.

Considering next Ampere's circuital law in integral form, that is, (6.1b), and applying it to the rectangular path *abcda* of Fig. 6.2, we have

$$\oint_{abcda} \mathbf{H} \cdot d\mathbf{l} = \int_{abcd} \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt}\int_{abcd} \mathbf{D} \cdot d\mathbf{S}$$

or

$$\int_a^b \mathbf{H} \cdot d\mathbf{l} + \int_b^c \mathbf{H} \cdot d\mathbf{l} + \int_c^d \mathbf{H} \cdot d\mathbf{l} + \int_d^a \mathbf{H} \cdot d\mathbf{l} = \int_{abcd} \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt}\int_{abcd} \mathbf{D} \cdot d\mathbf{S} \tag{6.7}$$

But $\int_c^d \mathbf{H} \cdot d\mathbf{l} = 0$ since $\mathbf{H}$ is zero inside the perfect conductor. If we now let *ad* and *bc* $\longrightarrow 0$ as before, the quantities $\int_b^c \mathbf{H} \cdot d\mathbf{l}$, $\int_d^a \mathbf{H} \cdot d\mathbf{l}$, and $\int_{abcd} \mathbf{D} \cdot d\mathbf{S}$ all tend to zero, but $\int_{abcd} \mathbf{J} \cdot d\mathbf{S}$ does not necessarily tend to zero since there can be a surface current enclosed by the area *abcd* although the area *abcd* tends to zero, as shown in Fig. 6.3(a). If α is the angle between the surface

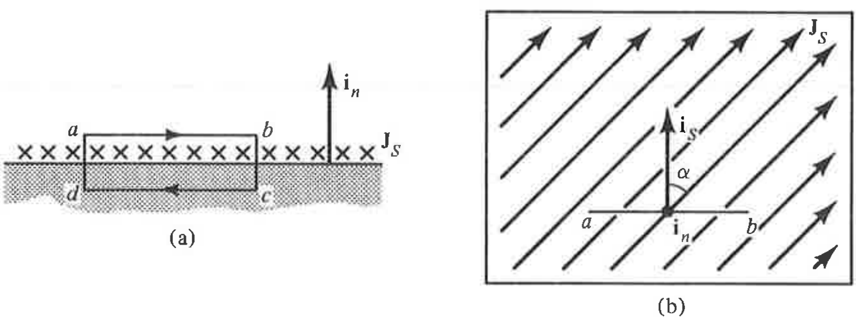

Figure 6.3. For deriving the boundary condition for the tangential component of **H** on a perfect conductor surface.

current density vector $\mathbf{J}_S$ and the unit normal vector $\mathbf{i}_S$ to the area *abcd*, directed in the right-hand sense, as shown in Fig. 6.3(b), then

$$\int_{abcd} \mathbf{J} \cdot d\mathbf{S} = J_S(ab \cos \alpha) \tag{6.8}$$

Thus we obtain

$$\int_a^b \mathbf{H} \cdot d\mathbf{l} = J_S(ab \cos \alpha)$$

or

$$H_{ab}(ab) = J_S(ab \cos \alpha)$$

$$H_{ab} = J_S \cos \alpha \tag{6.9}$$

The maximum value of H_{ab}, that is, the tangential component H_t of $\mathbf{H}$ on the conductor surface is obtained for α equal to zero, that is, when ab is oriented perpendicular to $\mathbf{J}_S$ and then

$$H_t = J_S \tag{6.10}$$

Hence we obtain the second boundary condition that "the tangential component of $\mathbf{H}$ at a point on a perfect conductor surface is perpendicular (in the right-hand sense) to the surface current density at that point and is equal in magnitude to the surface current density." We can express this statement concisely in vector form as

$$\mathbf{i}_n \times \mathbf{H} = \mathbf{J}_S \tag{6.11}$$

on the perfect conductor surface where $\mathbf{i}_n$ is again the unit normal vector to the conductor surface pointing out of the conductor, as shown in Fig. 6.3(a).

Considering now Gauss' law for the electric field in integral form, that is, (6.1c), and applying it to an infinitesimal rectangular box $abcdefgh$ chosen such that the surfaces $abcd$ and $efgh$ are very close to and on either side of the perfect conductor surface, as shown in Fig. 6.4, we have

$$\oint_{\substack{\text{surface}\\\text{of the}\\\text{box}}} \mathbf{D} \cdot d\mathbf{S} = \int_{\substack{\text{volume}\\\text{of the}\\\text{box}}} \rho \, dv$$

or

$$\int_{abcd} \mathbf{D} \cdot d\mathbf{S} + \int_{\substack{\text{side}\\\text{surfaces}}} \mathbf{D} \cdot d\mathbf{S} + \int_{efgh} \mathbf{D} \cdot d\mathbf{S} = \int_{\substack{\text{volume}\\\text{of the}\\\text{box}}} \rho \, dv \tag{6.12}$$

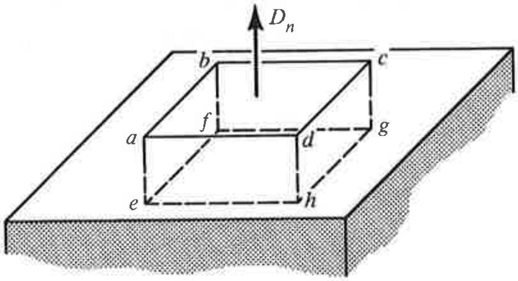

Figure 6.4. For deriving the boundary conditions for the normal components of **D** and **B** on a perfect conductor surface.

But $\int_{efgh} \mathbf{D} \cdot d\mathbf{S} = 0$ since $\mathbf{D}$ is zero inside the perfect conductor. If we now let the side surfaces $\longrightarrow 0$ by making *abcd* and *efgh* almost touch each other but remaining on either side of the boundary, $\int_{\substack{\text{side}\\\text{surfaces}}} \mathbf{D} \cdot d\mathbf{S}$ tends to zero and $\int_{\substack{\text{volume}\\\text{of the box}}} \rho\, dv$ tends to the surface charge enclosed by the box. If the surface charge density is ρ_S, then the surface charge enclosed by the box is $\rho_S(abcd)$. Thus we obtain

$$\int_{abcd} \mathbf{D} \cdot d\mathbf{S} = \rho_S(abcd)$$

or

$$D_n(abcd) = \rho_S$$

$$D_n = \rho_S \tag{6.13}$$

where D_n is the normal component of $\mathbf{D}$. Hence we obtain the third boundary condition that "the normal component of $\mathbf{D}$ at a point on a perfect conductor surface is equal to the surface charge density at that point." We can express this statement concisely in vector form as

$$\mathbf{i}_n \cdot \mathbf{D} = \rho_S \tag{6.14}$$

on the perfect conductor surface.

Considering finally Gauss' law for the magnetic field in integral form, that is, (6.1d), and applying it to the rectangular box *abcdefgh* of Fig. 6.4, we have

$$\oint_{\substack{\text{surface}\\\text{of the}\\\text{box}}} \mathbf{B} \cdot d\mathbf{S} = 0$$

or

$$\int_{abcd} \mathbf{B} \cdot d\mathbf{S} + \int_{\substack{\text{side}\\\text{surfaces}}} \mathbf{B} \cdot d\mathbf{S} + \int_{efgh} \mathbf{B} \cdot d\mathbf{S} = 0 \tag{6.15}$$

But $\int_{efgh} \mathbf{B} \cdot d\mathbf{S} = 0$ since $\mathbf{B}$ is zero inside the perfect conductor surface. If we now let the side surfaces $\longrightarrow 0$ as before, $\int_{\substack{\text{side}\\\text{surfaces}}} \mathbf{B} \cdot d\mathbf{S}$ tends to zero. Thus we obtain

$$\int_{abcd} \mathbf{B} \cdot d\mathbf{S} = 0$$

or

$$B_n(abcd) = 0$$

$$B_n = 0 \tag{6.16}$$

where B_n is the normal component of $\mathbf{B}$. Hence we obtain the fourth boundary condition that "the normal component of **B** at a point on a perfect conductor surface is equal to zero." We can express this statement concisely in vector form as

$$\mathbf{i}_n \cdot \mathbf{B} = 0 \tag{6.17}$$

on the perfect conductor surface.

Summarizing the four boundary conditions for the field components on a perfect conductor surface, we have

$$\mathbf{i}_n \times \mathbf{E} = 0$$
$$\mathbf{i}_n \times \mathbf{H} = \mathbf{J}_S$$
$$\mathbf{i}_n \cdot \mathbf{D} = \rho_S$$
$$\mathbf{i}_n \cdot \mathbf{B} = 0$$

where $\mathbf{i}_n$ is the unit normal vector pointing out of the conductor, $\mathbf{J}_S$ is the surface current density, and ρ_S is the surface charge density on the conductor surface.

Example 6.1. Let us consider a perfect dielectric medium $z < 0$ bounded by a perfect conductor $z > 0$, as shown in Fig. 6.5. Let the fields in the dielectric medium be given by the superposition of (+) and (−) uniform plane waves propagating normal to the conductor surface, that is,

$$\mathbf{E} = E_1 \cos(\omega t - \beta z)\,\mathbf{i}_x + E_2 \cos(\omega t + \beta z)\,\mathbf{i}_x$$
$$\mathbf{H} = \frac{E_1}{\eta} \cos(\omega t - \beta z)\,\mathbf{i}_y - \frac{E_2}{\eta} \cos(\omega t + \beta z)\,\mathbf{i}_y$$

where $\beta = \omega\sqrt{\mu\epsilon}$ and $\eta = \sqrt{\mu/\epsilon}$. We wish to investigate the relationship between E_2 and E_1.

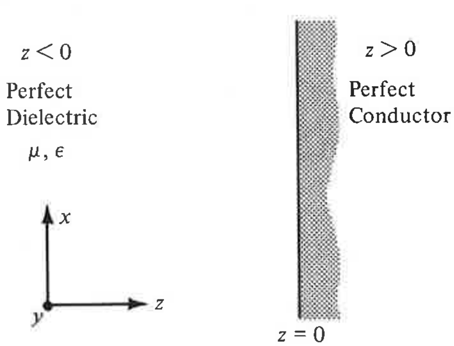

Figure 6.5. A perfect dielectric medium bounded by a perfect conductor.

Since E_x is tangential to the perfect conductor surface, the boundary condition for the tangential component of **E** given by (6.6) requires that

$$[E_x]_{z=0} = 0$$

or

$$[E_1 \cos(\omega t - \beta z) + E_2 \cos(\omega t + \beta z)]_{z=0} = 0$$
$$E_1 \cos \omega t + E_2 \cos \omega t = 0 \qquad \text{for all } t$$

Thus we obtain the required relationship to be

$$E_2 = -E_1$$

Proceeding further, we obtain the total electric field in the dielectric as given by

$$\begin{aligned}\mathbf{E} &= E_1 \cos(\omega t - \beta z)\,\mathbf{i}_x - E_1 \cos(\omega t + \beta z)\,\mathbf{i}_x \\ &= 2E_1 \sin \omega t \sin \beta z\,\mathbf{i}_x\end{aligned}$$

and the total magnetic field in the dielectric as given by

$$\begin{aligned}\mathbf{H} &= \frac{E_1}{\eta} \cos(\omega t - \beta z)\,\mathbf{i}_y + \frac{E_1}{\eta} \cos(\omega t + \beta z)\,\mathbf{i}_y \\ &= \frac{2E_1}{\eta} \cos \omega t \cos \beta z\,\mathbf{i}_y\end{aligned}$$

These expressions for **E** and **H** correspond to standing waves. We shall discuss the standing wave phenomenon in Sec. 6.4.

Now, from the boundary condition for the tangential component of **H** given by (6.11), we obtain

$$\begin{aligned}[\mathbf{J}_S]_{z=0} &= \mathbf{i}_n \times [\mathbf{H}]_{z=0} = -\mathbf{i}_z \times [\mathbf{H}]_{z=0} \\ &= -\mathbf{i}_z \times \frac{2E_1}{\eta} \cos \omega t\,\mathbf{i}_y \\ &= \frac{2E_1}{\eta} \cos \omega t\,\mathbf{i}_x\end{aligned}$$

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6.2 PARALLEL-PLATE TRANSMISSION LINE

In the previous section we introduced the boundary conditions for the field components on the surface of a perfect conductor. We learned that the tangential component of the electric field intensity and the normal component of the magnetic field intensity are zero on the perfect conductor surface. Let us now consider the uniform plane electromagnetic wave propagating in the z direction and having an x component only of the electric field and a y component only of the magnetic field, that is,

$$\mathbf{E} = E_x(z, t)\,\mathbf{i}_x$$
$$\mathbf{H} = H_y(z, t)\,\mathbf{i}_y$$

and place perfectly conducting sheets in two planes $x = 0$ and $x = d$, as shown in Fig. 6.6. Since the electric field is completely normal and the magnetic field is completely tangential to the sheets, the two boundary conditions referred to above are satisfied, and hence the wave will simply propagate, as though the sheets were not present, being guided by the sheets. We then have a simple case of transmission line, namely, the parallel-plate transmission line.

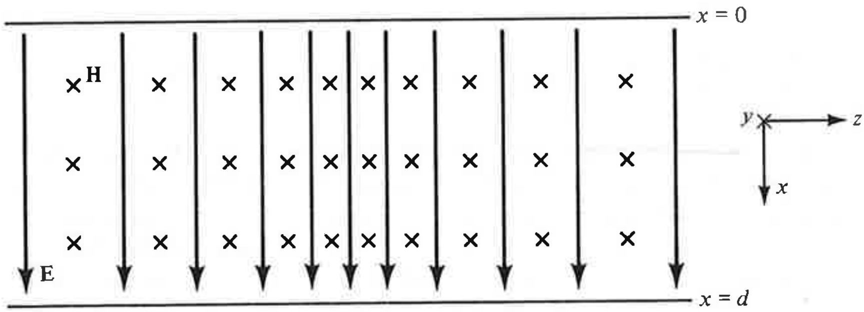

Figure 6.6. Uniform plane electromagnetic wave propagating between two perfectly conducting sheets.

According to the remaining two boundary conditions, there must be charges and currents on the conductors. The charge densities on the two plates are

$$[\rho_S]_{x=0} = [\mathbf{i}_n \cdot \mathbf{D}]_{x=0} = \mathbf{i}_x \cdot \epsilon E_x \mathbf{i}_x = \epsilon E_x \tag{6.18a}$$
$$[\rho_S]_{x=d} = [\mathbf{i}_n \cdot \mathbf{D}]_{x=d} = -\mathbf{i}_x \cdot \epsilon E_x \mathbf{i}_x = -\epsilon E_x \tag{6.18b}$$

where ϵ is the permittivity of the medium between the two plates. The current densities on the two plates are

$$[\mathbf{J}_S]_{x=0} = [\mathbf{i}_n \times \mathbf{H}]_{x=0} = \mathbf{i}_x \times H_y \mathbf{i}_y = H_y \mathbf{i}_z \tag{6.19a}$$
$$[\mathbf{J}_S]_{x=d} = [\mathbf{i}_n \times \mathbf{H}]_{x=d} = -\mathbf{i}_x \times H_y \mathbf{i}_y = -H_y \mathbf{i}_z \tag{6.19b}$$

In addition, there is conduction current in the medium between the plates flowing from one plate to the other with density given by

$$\mathbf{J}_c = \sigma E = \sigma E_x \mathbf{i}_x \tag{6.20}$$

where σ is the conductivity of the medium. In (6.18)–(6.20) it is understood that the charge and current densities are functions of z and t as E_x and H_y are. Thus the wave propagation along the transmission line is supported by charges and currents on the plates, varying with time and distance along the line, as shown in Fig. 6.7.

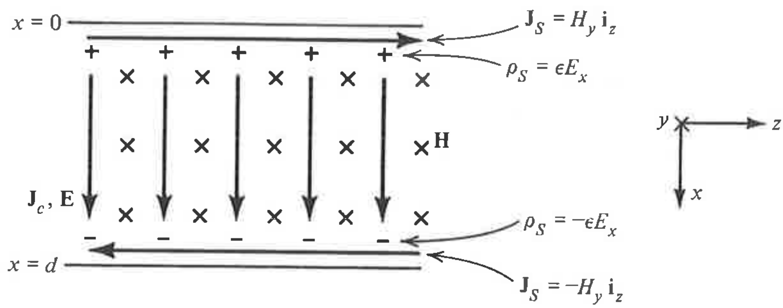

Figure 6.7. Charges and currents on the plates of a parallel-plate transmission line.

Let us now consider finitely sized plates having width w in the y direction, as shown in Fig. 6.8(a), and neglect fringing of the fields at the edges or assume that the structure is part of a much larger-sized configuration. By considering a constant z plane, that is, a plane "transverse" to the direction of propagation of the wave, as shown in Fig. 6.8(b), we can find the voltage between the two conductors in terms of the line integral of the electric field intensity evaluated along any path in that plane between the two conductors. Since the electric field is directed in the x direction and since it is uniform in that plane, this voltage is given by

$$V(z, t) = \int_{x=0}^{d} E_x(z, t)\,dx = E_x(z, t)\int_{x=0}^{d} dx = dE_x(z, t) \qquad (6.21a)$$

Thus each transverse plane is characterized by a voltage between the two conductors which is related simply to the electric field as given by (6.21a). Each transverse plane is also characterized by a current I flowing in the positive z direction on the upper conductor and in the negative z direction on the lower conductor. From Fig. 6.8(b), we can see that this current is given by

$$\begin{aligned} I(z, t) &= \int_{y=0}^{w} J_S(z, t)\,dy = \int_{y=0}^{w} H_y(z, t)\,dy = H_y(z, t)\int_{y=0}^{w} dy \\ &= wH_y(z, t) \end{aligned} \qquad (6.21b)$$

since H_y is uniform in the cross-sectional plane. Thus the current crossing

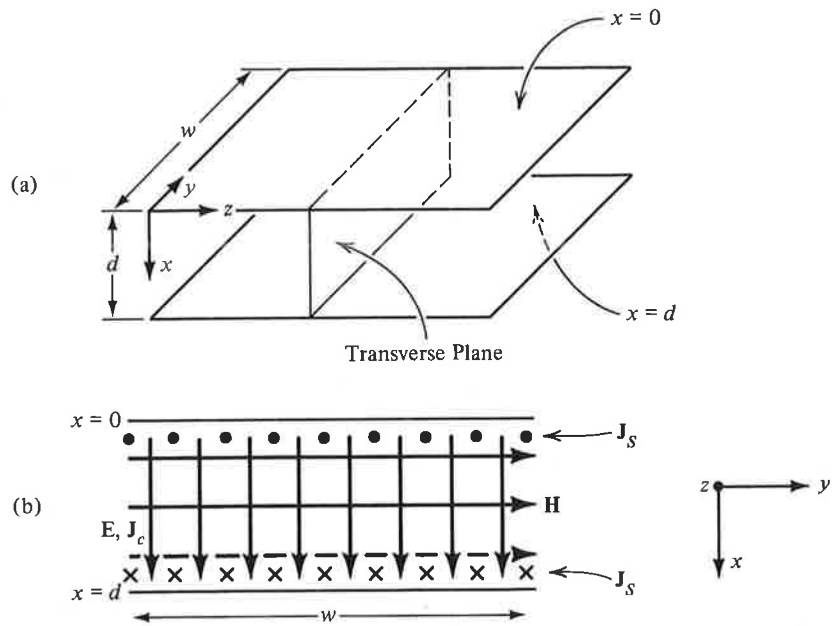

Figure 6.8. (a) Parallel-plate transmission line. (b) A transverse plane of the parallel-plate transmission line.

a given transverse plane is related simply to the magnetic field in that plane as given by (6.21b).

Proceeding further, we can find the power flow down the line by evaluating the surface integral of the Poynting vector over a given transverse plane. Thus

$$\begin{aligned} P(z, t) &= \int_{\substack{\text{transverse}\\ \text{plane}}} (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S} \\ &= \int_{x=0}^{d} \int_{y=0}^{w} E_x(z, t) H_y(z, t)\, \mathbf{i}_z \cdot dx\, dy\, \mathbf{i}_z \\ &= \int_{x=0}^{d} \int_{y=0}^{w} \frac{V(z, t)}{d} \frac{I(z, t)}{w}\, dx\, dy \\ &= V(z, t) I(z, t) \end{aligned} \tag{6.22}$$

which is the familiar relationship employed in circuit theory.

We now recall from Sec. 5.4 that E_x and H_y satisfy the two differential equations

$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t} = -\mu \frac{\partial H_y}{\partial t} \tag{6.23a}$$

$$\frac{\partial H_y}{\partial z} = -J_{cx} - \frac{\partial D_x}{\partial t} = -\sigma E_x - \epsilon \frac{\partial E_x}{\partial t} \tag{6.23b}$$

From (6.21a) and (6.21b), however, we have

$$E_x = \frac{V}{d} \tag{6.24a}$$

$$H_y = \frac{I}{w} \tag{6.24b}$$

Substituting for E_x and H_y in (6.23a) and (6.23b) from (6.24a) and (6.24b), respectively, we now obtain two differential equations for voltage and current along the line as

$$\frac{\partial}{\partial z}\left(\frac{V}{d}\right) = -\mu \frac{\partial}{\partial t}\left(\frac{I}{w}\right) \tag{6.25a}$$

$$\frac{\partial}{\partial z}\left(\frac{I}{w}\right) = -\sigma\left(\frac{V}{d}\right) - \epsilon \frac{\partial}{\partial t}\left(\frac{V}{d}\right) \tag{6.25b}$$

or

$$\frac{\partial V}{\partial z} = -\left(\frac{\mu d}{w}\right) \frac{\partial I}{\partial t} \tag{6.26a}$$

$$\frac{\partial I}{\partial z} = -\left(\frac{\sigma w}{d}\right) V - \left(\frac{\epsilon w}{d}\right) \frac{\partial V}{\partial t} \tag{6.26b}$$

These equations are known as the "transmission-line equations." They characterize the wave propagation along the line in terms of line voltage and line current instead of in terms of the fields.

We now define three quantities familiarly known as the "circuit parameters." These are the inductance, the capacitance, and the conductance (reciprocal of resistance) per unit length of the transmission line in the z direction and are denoted by the symbols $\mathcal{L}$, $\mathcal{C}$, and $\mathcal{G}$, respectively. The inductance per unit length, having the units henries per meter (H/m), is defined as the ratio of the magnetic flux per unit length at any value of z to the line current at that value of z. Noting from Fig. 6.8 that the cross-sectional area normal to the magnetic field lines and per unit length in the z direction is $(d)(1)$ or d, we find the magnetic flux per unit length to be $B_y d$ or $\mu H_y d$. Since the line current is $H_y w$, we then have

$$\mathcal{L} = \frac{\mu H_y d}{H_y w} = \frac{\mu d}{w} \tag{6.27a}$$

The capacitance per unit length, having the units farads per meter (F/m), is defined as the ratio of the charge per unit length on either plate at any value of z to the line voltage at that value of z. Noting from Fig. 6.8 that the cross-sectional area normal to the electric field lines and per unit length in the z direction is $(w)(1)$ or w, we find the charge per unit length to be $\rho_S w$ or

$\epsilon E_x w$. Since the line voltage is $E_x d$, we then have

$$\mathcal{C} = \frac{\epsilon E_x w}{E_x d} = \frac{\epsilon w}{d} \tag{6.27b}$$

The conductance per unit length, having the units mhos per meter (℧/m), is defined as the ratio of the conduction current per unit length flowing from one plate to the other at any value of z to the line voltage at that value of z. Noting from Fig. 6.8 that the cross-sectional area normal to the conduction current flow and per unit length in the z direction is $(w)(1)$ or w, we find the conduction current per unit length to be $J_{cx}w$ or $\sigma E_x w$. We then have

$$\mathcal{G} = \frac{\sigma E_x w}{E_x d} = \frac{\sigma w}{d} \tag{6.27c}$$

We note that $\mathcal{L}$, $\mathcal{C}$, and $\mathcal{G}$ are purely dependent on the dimensions of the line and are independent of E_x and H_y. We further note that

$$\mathcal{L}\mathcal{C} = \mu\epsilon \tag{6.28a}$$

$$\frac{\mathcal{G}}{\mathcal{C}} = \frac{\sigma}{\epsilon} \tag{6.28b}$$

We now recognize the quantities in parentheses in (6.26a) and (6.26b) to be $\mathcal{L}$, $\mathcal{G}$, and $\mathcal{C}$, respectively, of the line. Thus we obtain the transmission-line equations in terms of these parameters as

$$\frac{\partial V}{\partial z} = -\mathcal{L}\frac{\partial I}{\partial t} \tag{6.29a}$$

$$\frac{\partial I}{\partial z} = -\mathcal{G}V - \mathcal{C}\frac{\partial V}{\partial t} \tag{6.29b}$$

These equations permit us to discuss wave propagation along the line in terms of circuit quantities instead of in terms of field quantities. It should, however, not be forgotten that the actual phenomenon is one of electromagnetic waves guided by the conductors of the line.

It is customary to represent a transmission line by means of its circuit equivalent, derived from the transmission-line equations (6.29a) and (6.29b). To do this, let us consider a section of infinitesimal length Δz along the line between z and $z + \Delta z$. From (6.29a), we then have

$$\lim_{\Delta z \to 0} \frac{V(z + \Delta z, t) - V(z, t)}{\Delta z} = -\mathcal{L}\frac{\partial I(z, t)}{\partial t}$$

or, for $\Delta z \longrightarrow 0$,

$$V(z + \Delta z, t) - V(z, t) = -\mathcal{L}\,\Delta z\,\frac{\partial I(z, t)}{\partial t} \tag{6.30}$$

This equation can be represented by the circuit equivalent shown in Fig. 6.9(a) since it satisfies Kirchhoff's voltage law written around the loop *abcda*. Similarly, from (6.29b), we have

$$\lim_{\Delta z \to 0} \frac{I(z + \Delta z, t) - I(z, t)}{\Delta z} = \lim_{\Delta z \to 0} \left[-\mathcal{G} V(z + \Delta z, t) - \mathcal{C}\,\frac{\partial V(z + \Delta z, t)}{\partial t} \right]$$

or, for $\Delta z \longrightarrow 0$,

$$I(z + \Delta z, t) - I(z, t) = -\mathcal{G}\,\Delta z\, V(z + \Delta z, t) - \mathcal{C}\,\Delta z\,\frac{\partial V(z + \Delta z, t)}{\partial t} \tag{6.31}$$

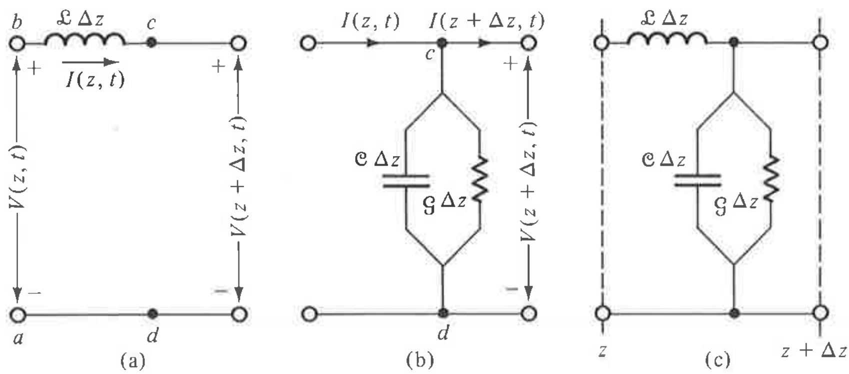

Figure 6.9. Development of circuit equivalent for an infinitesimal length Δz of a transmission line.

This equation can be represented by the circuit equivalent shown in Fig. 6.9(b) since it satisfies Kirchhoff's current law written for node *c*. Combining the two equations, we then obtain the equivalent circuit shown in Fig. 6.9(c) for a section Δz of the line. It then follows that the circuit representation for a portion of length l of the line consists of an infinite number of such sections in cascade, as shown in Fig. 6.10. Such a circuit is known as a "distributed circuit" as opposed to the "lumped circuits" that are familiar in circuit theory. The distributed circuit notion arises from the fact that the inductance, capacitance, and conductance are distributed uniformly and overlappingly along the line.

A more physical interpretation of the distributed circuit concept follows from energy considerations. We know that the uniform plane wave propagation between the conductors of the line is characterized by energy storage

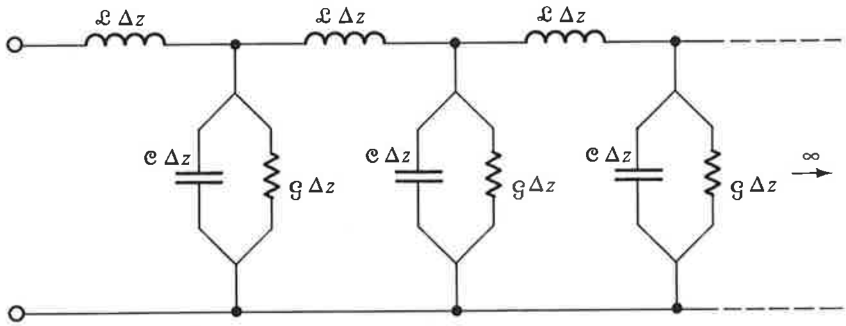

Figure 6.10. Distributed circuit representation of a transmission line.

in the electric and magnetic fields and power dissipation due to the conduction current flow. If we consider a section Δz of the line, the energy stored in the electric field in this section is given by

$$\begin{aligned} W_e &= \frac{1}{2}\epsilon E_x^2 \text{ (volume)} = \frac{1}{2}\epsilon E_x^2(dw\,\Delta z) \\ &= \frac{1}{2}\frac{\epsilon w}{d}(E_x d)^2\,\Delta z = \frac{1}{2}\mathcal{C}\,\Delta z\,V^2 \end{aligned} \tag{6.32}$$

The energy stored in the magnetic field in that section is given by

$$\begin{aligned} W_m &= \frac{1}{2}\mu H_y^2 \text{ (volume)} = \frac{1}{2}\mu H_y^2\,(dw\,\Delta z) \\ &= \frac{1}{2}\frac{\mu d}{w}(H_y w)^2\,\Delta z = \frac{1}{2}\mathcal{L}\,\Delta z\,I^2 \end{aligned} \tag{6.33}$$

The power dissipated due to conduction current flow in that section is given by

$$\begin{aligned} P_d &= \sigma E_x^2 \text{ (volume)} = \sigma E_x^2\,(dw\,\Delta z) \\ &= \frac{\sigma w}{d}(E_x d)^2\,\Delta z = \mathcal{G}\,\Delta z\,V^2 \end{aligned} \tag{6.34}$$

Thus we note that $\mathcal{L}$, $\mathcal{C}$, and $\mathcal{G}$ are elements associated with energy storage in the magnetic field, energy storage in the electric field, and power dissipation due to the conduction current flow in the dielectric, respectively, for a given infinitesimal section of the line. Since these phenomena occur continuously and since they overlap, the inductance, capacitance, and conductance must be distributed uniformly and overlappingly along the line. In actual practice, the conductors of the transmission line are imperfect, resulting in slight penetration of the fields into the conductors, in accordance with the skin effect phenomenon. This gives rise to power dissipation and magnetic field energy storage in the conductors, which are taken into account by including a resistance and additional inductance in the series branch of the transmission-line equivalent circuit (see Problem 6.9).

6.3 TRANSMISSION LINE WITH AN ARBITRARY CROSS SECTION

In the previous section we considered the parallel-plate transmission line made up of perfectly conducting sheets lying in the planes $x = 0$ and $x = d$ so that the boundary conditions of zero tangential component of the electric field and zero normal component of the magnetic field are satisfied by the uniform plane wave characterized by the fields

$$\mathbf{E} = E_x(z, t)\,\mathbf{i}_x$$
$$\mathbf{H} = H_y(z, t)\,\mathbf{i}_y$$

thereby leading to the situation in which the uniform plane wave is guided by the conductors of the transmission line. In the general case, however, the conductors of the transmission line have arbitrary cross sections and the fields consist of both x and y components and are dependent on x and y coordinates in addition to the z coordinate. Thus the fields between the conductors are given by

$$\mathbf{E} = E_x(x, y, z, t)\,\mathbf{i}_x + E_y(x, y, z, t)\,\mathbf{i}_y$$
$$\mathbf{H} = H_x(x, y, z, t)\,\mathbf{i}_x + H_y(x, y, z, t)\,\mathbf{i}_y$$

These fields are no longer uniform in x and y but are directed entirely transverse to the direction of propagation, that is, the z axis, which is the axis of the transmission line. Hence they are known as "transverse electromagnetic waves," or "TEM waves." The uniform plane waves are simply a special case of the transverse electromagnetic waves.

To extend the computation of the transmission line parameters $\mathcal{L}$, $\mathcal{C}$, and $\mathcal{G}$ to the general case, let us consider a transmission line made up of parallel, perfect conductors of arbitrary cross sections, as shown by the cross-sectional view in Fig. 6.11(a). Let us assume that the inner conductor is positive with respect to the outer conductor and that the current flows along the positive z direction (into the page) on the inner conductor and along the negative z direction (out of the page) on the outer conductor. We can then draw a "field map," that is, a graphical sketch of the direction lines of the fields between the conductors, from the following considerations: (a) The electric field lines must originate on the inner conductor and be normal to it and must terminate on the outer conductor and be normal to it since the tangential component of the electric field on a perfect conductor surface must be zero. (b) The magnetic field lines must be everywhere perpendicular to the electric field lines; although this can be shown by a rigorous mathematical proof, it is intuitively obvious since, first, the magnetic field lines must be tangential near the conductor surfaces and, second, at any arbitrary point the fields

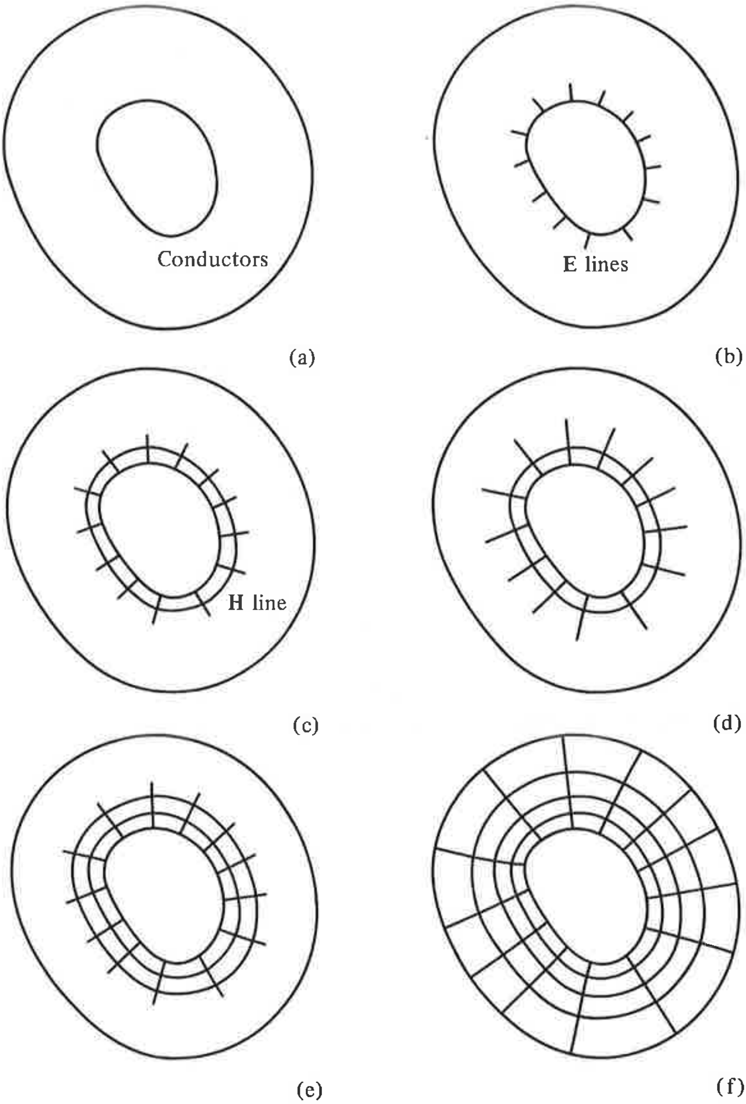

Figure 6.11. Construction of a transmission line field map consisting of curvilinear rectangles.

correspond to those of a locally uniform plane wave. Thus suppose that we start with the inner conductor and draw several lines normal to it at several points on the surface as shown in Fig. 6.11(b). We can then draw a curved line displaced from the conductor surface and such that it is perpendicular everywhere to the electric field lines of Fig. 6.11(b), as shown in Fig. 6.11(c). This contour represents a magnetic field line and forms the basis for further extension of the electric field lines, as shown in Fig. 6.11(d). A second magnetic field line can then be drawn so that it is everywhere perpendicular to the extended electric field lines, as shown in Fig. 6.11(e). This procedure is continued until the entire cross section between the conductors is filled with two sets of orthogonal contours, as shown in Fig. 6.11(f), thereby resulting in a field map made up of curvilinear rectangles.

By drawing the field lines with very small spacings, we can make the rectangles so small that each of them can be considered to be the cross section of a parallel-plate line. In fact, by choosing the spacings appropriately, we can even make them a set of squares. If we now replace the magnetic field lines by perfect conductors, since it does not violate any boundary condition, it can be seen that the arrangement can be viewed as the parallel combination, in the angular direction, of m number of series combinations of n number of parallel-plate lines in the radial direction, where m is the number of squares in the angular direction, that is, along a magnetic field line, and n is the number of squares in the radial direction, that is, along an electric field line. We can then find simple expressions for $\mathcal{L}$, $\mathcal{C}$, and $\mathcal{G}$ of the line in the following manner.

Let us for simplicity consider the field map of Fig. 6.12, consisting of eight segments 1, 2, . . . , 8 in the angular direction and two segments a and b in the radial direction. The arrangement is then a parallel combination, in the angular direction, of eight series combinations of two lines in the radial direction, each having a curvilinear rectangular cross section. Let $I_1, I_2, \ldots, I_8$ be the currents associated with the segments 1, 2, . . . , 8, respectively, and let ψ_a and ψ_b be the magnetic fluxes per unit length in the z direction associated with the segments a and b, respectively. Then the inductance per unit length of the transmission line is given by

$$\begin{aligned}
\mathcal{L} = \frac{\psi}{I} &= \frac{\psi_a + \psi_b}{I_1 + I_2 + \cdots + I_8} \\
&= \frac{1}{\dfrac{I_1}{\psi_a} + \dfrac{I_2}{\psi_a} + \cdots \dfrac{I_8}{\psi_a}} + \frac{1}{\dfrac{I_1}{\psi_b} + \dfrac{I_2}{\psi_b} \cdots + \dfrac{I_8}{\psi_b}} \\
&= \frac{1}{\dfrac{1}{\mathcal{L}_{1a}} + \dfrac{1}{\mathcal{L}_{2a}} + \cdots + \dfrac{1}{\mathcal{L}_{8a}}} + \frac{1}{\dfrac{1}{\mathcal{L}_{1b}} + \dfrac{1}{\mathcal{L}_{2b}} + \cdots + \dfrac{1}{\mathcal{L}_{8b}}} \qquad (6.35a)
\end{aligned}$$

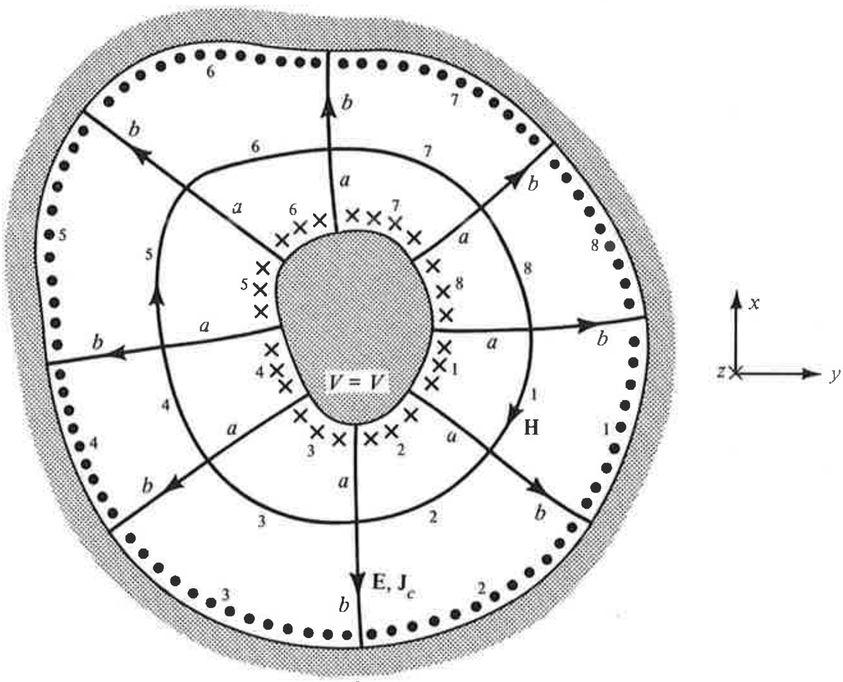

Figure 6.12. For deriving the expressions for the transmission-line parameters from the field map.

Let $Q_1, Q_2, \ldots, Q_8$ be the charges per unit length in the z direction associated with the segments $1, 2, \ldots, 8$, respectively, and let V_a and V_b be the voltages associated with the segments a and b, respectively. Then the capacitance per unit length of the transmission line is given by

$$\begin{aligned}
\mathcal{C} &= \frac{Q}{V} = \frac{Q_1 + Q_2 + \ldots + Q_8}{V_a + V_b} \\
&= \frac{1}{\frac{V_a}{Q_1} + \frac{V_b}{Q_1}} + \frac{1}{\frac{V_a}{Q_2} + \frac{V_b}{Q_2}} + \ldots + \frac{1}{\frac{V_a}{Q_8} + \frac{V_b}{Q_8}} \\
&= \frac{1}{\frac{1}{\mathcal{C}_{1a}} + \frac{1}{\mathcal{C}_{1b}}} + \frac{1}{\frac{1}{\mathcal{C}_{2a}} + \frac{1}{\mathcal{C}_{2b}}} + \ldots + \frac{1}{\frac{1}{\mathcal{C}_{8a}} + \frac{1}{\mathcal{C}_{8b}}}
\end{aligned} \tag{6.35b}$$

Let $I_{c1}, I_{c2}, \ldots, I_{c8}$ be the conduction currents per unit length in the z direction associated with the segments $1, 2, \ldots, 8$, respectively. Then the conductance per unit length of the transmission line is given by

$$\mathcal{G} = \frac{I_c}{V} = \frac{I_{c1} + I_{c2} + \ldots + I_{c8}}{V_a + V_b}$$
$$= \frac{1}{\dfrac{V_a}{I_{c1}} + \dfrac{V_b}{I_{c1}}} + \frac{1}{\dfrac{V_a}{I_{c2}} + \dfrac{V_b}{I_{c2}}} + \ldots + \frac{1}{\dfrac{V_a}{I_{c8}} + \dfrac{V_b}{I_{c8}}}$$
$$= \frac{1}{\dfrac{1}{\mathcal{G}_{1a}} + \dfrac{1}{\mathcal{G}_{1b}}} + \frac{1}{\dfrac{1}{\mathcal{G}_{2a}} + \dfrac{1}{\mathcal{G}_{2b}}} + \ldots + \frac{1}{\dfrac{1}{\mathcal{G}_{8a}} + \dfrac{1}{\mathcal{G}_{8b}}} \tag{6.35c}$$

Generalizing the expressions (6.35a), (6.35b), and (6.35c) to m segments in the angular direction and n segments in the radial direction, we obtain

$$\mathcal{L} = \sum_{j=1}^{n} \frac{1}{\sum_{i=1}^{m} \dfrac{1}{\mathcal{L}_{ij}}} \tag{6.36a}$$

$$\mathcal{C} = \sum_{i=1}^{m} \frac{1}{\sum_{j=1}^{n} \dfrac{1}{\mathcal{C}_{ij}}} \tag{6.36b}$$

$$\mathcal{G} = \sum_{i=1}^{m} \frac{1}{\sum_{j=1}^{n} \dfrac{1}{\mathcal{G}_{ij}}} \tag{6.36c}$$

where $\mathcal{L}_{ij}$, $\mathcal{C}_{ij}$, and $\mathcal{G}_{ij}$ are the inductance, capacitance, and conductance per unit length corresponding to the rectangle ij. If the map consists of curvilinear squares, then $\mathcal{L}_{ij}$, $\mathcal{C}_{ij}$, and $\mathcal{G}_{ij}$ are equal to μ, ϵ, and σ, respectively, according to (6.27a), (6.27b), and (6.27c), respectively, since the width w of the plates is equal to the spacing d of the plates for each square. Thus we obtain simple expressions for $\mathcal{L}$, $\mathcal{C}$, and $\mathcal{G}$ as given by

$$\mathcal{L} = \mu \frac{n}{m} \tag{6.37a}$$

$$\mathcal{C} = \epsilon \frac{m}{n} \tag{6.37b}$$

$$\mathcal{G} = \sigma \frac{m}{n} \tag{6.37c}$$

The computation of $\mathcal{L}$, $\mathcal{C}$, and $\mathcal{G}$ then consists of sketching a field map consisting of curvilinear squares, counting the number of squares in each direction, and substituting these values in (6.37a), (6.37b), and (6.37c). Note that once again

$$\mathcal{L}\mathcal{C} = \mu\epsilon \tag{6.38a}$$

$$\frac{\mathcal{G}}{\mathcal{C}} = \frac{\sigma}{\epsilon} \tag{6.38b}$$

We shall now consider an example of the application of the curvilinear squares technique.

Example 6.2. The coaxial cable is a transmission line made up of parallel, coaxial, cylindrical conductors. Let the radius of the inner conductor be a and that of the outer conductor be b. We wish to find expressions for $\mathcal{L}$, $\mathcal{C}$, and $\mathcal{G}$ of the coaxial cable by using the curvilinear squares technique.

Figure 6.13 shows the cross-sectional view of the coaxial cable and the field map. In view of the symmetry associated with the conductor configuration, the construction of the field map is simplified in this case. The electric field lines are radial lines from one conductor to the other, and the magnetic field lines are circles concentric with the conductors, as shown in the figure.

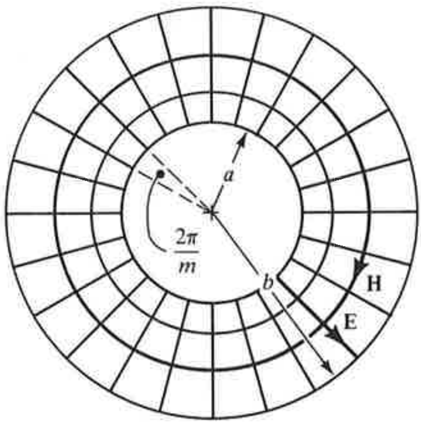

Figure 6.13. Field map consisting of curvilinear squares for a coaxial cable.

Let the number of curvilinear squares in the angular direction be m. Then to find the number of curvilinear squares in the radial direction, we note that the angle subtended at the center of the conductors by adjacent pairs of electric field lines is equal to $2\pi/m$. Hence at any arbitrary radius r between the two conductors, the side of the curvilinear square is equal to $r(2\pi/m)$. The number of squares in an infinitesimal distance dr in the radial direction is then equal to $\dfrac{dr}{r(2\pi/m)}$ or $\dfrac{m}{2\pi}\dfrac{dr}{r}$. The total number of squares in the radial direction from the inner to the outer conductor is given by

$$n = \int_{r=a}^{b} \frac{m}{2\pi}\frac{dr}{r} = \frac{m}{2\pi}\ln\frac{b}{a}$$

The required expressions for $\mathcal{L}$, $\mathcal{C}$, and $\mathcal{G}$ are then given by

$$\mathcal{L} = \mu\frac{n}{m} = \frac{\mu}{2\pi}\ln\frac{b}{a} \tag{6.39a}$$

$$\mathcal{C} = \epsilon \frac{m}{n} = \frac{2\pi\epsilon}{\ln (b/a)} \tag{6.39b}$$

$$\mathcal{G} = \sigma \frac{m}{n} = \frac{2\pi\sigma}{\ln (b/a)} \tag{6.39c}$$

These expressions are exact. We have been able to obtain exact expressions in this case because of the geometry involved. When the geometry is not so simple, we can only obtain approximate values for $\mathcal{L}$, $\mathcal{C}$, and $\mathcal{G}$. ■

We have just discussed an example of the determination of the transmission-line parameters $\mathcal{L}$, $\mathcal{C}$, and $\mathcal{G}$ for a coaxial cable. There are other configurations having different cross sections for which one can obtain the parameters either by the curvilinear squares technique or by other analytical or experimental techniques. We shall, however, not pursue the discussion of these techniques any further. With the understanding that different transimssion lines are characterized by different values of $\mathcal{L}$, $\mathcal{C}$, and $\mathcal{G}$, which can be computed from the formulas, we now recall that the voltage and current on the line are governed by the transmission-line equations

$$\frac{\partial V}{\partial z} = -\mathcal{L}\frac{\partial I}{\partial t} \tag{6.40a}$$

$$\frac{\partial I}{\partial z} = -\mathcal{G}V - \mathcal{C}\frac{\partial V}{\partial t} \tag{6.40b}$$

For the sinusoidally time-varying case, the corresponding differential equations for the phasor voltage $\bar{V}$ and phasor current $\bar{I}$ are given by

$$\frac{\partial \bar{V}}{\partial z} = -j\omega\mathcal{L}\bar{I} \tag{6.41a}$$

$$\frac{\partial \bar{I}}{\partial z} = -\mathcal{G}\bar{V} - j\omega\mathcal{C}\bar{V} = -(\mathcal{G} + j\omega\mathcal{C})\bar{V} \tag{6.41b}$$

Combining (6.41a) and (6.41b) by eliminating $\bar{I}$, we obtain the wave equation for $\bar{V}$ as

$$\begin{aligned}\frac{\partial^2 \bar{V}}{\partial z^2} &= -j\omega\mathcal{L}\frac{\partial \bar{I}}{\partial z} = j\omega\mathcal{L}(\mathcal{G} + j\omega\mathcal{C})\bar{V} \\ &= \bar{\gamma}^2\bar{V}\end{aligned} \tag{6.42}$$

where

$$\bar{\gamma} = \sqrt{j\omega\mathcal{L}(\mathcal{G} + j\omega\mathcal{C})} \tag{6.43}$$

is the propagation constant associated with the wave propagation on the line. The solution for $\bar{V}$ is given by

$$\bar{V}(z) = \bar{A}e^{-\bar{\gamma}z} + \bar{B}e^{\bar{\gamma}z} \tag{6.44}$$

where $\bar{A}$ and $\bar{B}$ are arbitrary constants to be determined by the boundary conditions. The corresponding solution for $\bar{I}$ is then given by

$$\begin{aligned}\bar{I}(z) &= -\frac{1}{j\omega\mathcal{L}}\frac{\partial \bar{V}}{\partial z} = -\frac{1}{j\omega\mathcal{L}}(-\bar{\gamma}\bar{A}e^{-\bar{\gamma}z} + \bar{\gamma}\bar{B}e^{\bar{\gamma}z}) \\ &= \sqrt{\frac{\mathcal{G} + j\omega\mathcal{C}}{j\omega\mathcal{L}}}(\bar{A}e^{-\bar{\gamma}z} - \bar{B}e^{\bar{\gamma}z}) \\ &= \frac{1}{\bar{Z}_0}(\bar{A}e^{-\bar{\gamma}z} - \bar{B}e^{\bar{\gamma}z}) \end{aligned} \tag{6.45}$$

where

$$\bar{Z}_0 = \sqrt{\frac{j\omega\mathcal{L}}{\mathcal{G} + j\omega\mathcal{C}}} \tag{6.46}$$

is known as the "characteristic impedance" of the transmission line.

The solutions for the line voltage and line current given by (6.44) and (6.45), respectively, represent the superposition of (+) and (−) waves, that is, waves propagating in the positive z and negative z directions, respectively. They are completely analogous to the solutions for the electric and magnetic fields in the medium between the conductors of the line. In fact, the propagation constant given by (6.43) is the same as the propagation constant $\sqrt{j\omega\mu(\sigma + j\omega\epsilon)}$, as it should be. The characteristic impedance of the line is analogous to (but not equal to) the intrinsic impedance of the material medium between the conductors of the line. We note that for a perfect dielectric medium between the conductors, that is, for $\sigma = 0$, $\mathcal{G} = 0$ and

$$\bar{Z}_0 = Z_0 = \sqrt{\frac{\mathcal{L}}{\mathcal{C}}} \tag{6.47}$$

is purely real. For example, for the coaxial cable of Example 6.2, with a perfect dielectric between the conductors,

$$\begin{aligned} Z_0 &= \sqrt{\frac{\mathcal{L}}{\mathcal{C}}} = \sqrt{\frac{\mu}{2\pi}\ln\frac{b}{a}\bigg/\frac{2\pi\epsilon}{\ln(b/a)}} \\ &= \frac{1}{2\pi}\sqrt{\frac{\mu}{\epsilon}}\ln\frac{b}{a} \end{aligned} \tag{6.48}$$

For $\mu = \mu_0$, $\epsilon = 2.25\epsilon_0$, and $b/a = 3.67$, the characteristic impedance of the coaxial cable is approximately 52 ohms.

6.4 SHORT-CIRCUITED TRANSMISSION LINE

In the previous section we found the general solutions for the complex voltage and complex current $\bar{V}$ and $\bar{I}$, respectively, on a transmission line. For a "lossless line," that is, for a line consisting of a perfect dielectric medium between the conductors, $\mathcal{G} = 0$, and

$$\bar{\gamma} = \alpha + j\beta = \sqrt{j\omega\mathcal{L}\cdot j\omega\mathcal{C}} = j\omega\sqrt{\mathcal{L}\mathcal{C}} \tag{6.49}$$

Thus the attenuation constant α is equal to zero, which is to be expected, and the phase constant β is equal to $\omega\sqrt{\mathcal{L}\mathcal{C}}$. We can then write the solutions for $\bar{V}$ and $\bar{I}$ as

$$\bar{V}(z) = \bar{A}e^{-j\beta z} + \bar{B}e^{j\beta z} \tag{6.50a}$$

$$\bar{I}(z) = \frac{1}{Z_0}(\bar{A}e^{-j\beta z} - \bar{B}e^{j\beta z}) \tag{6.50b}$$

where $Z_0 = \sqrt{\mathcal{L}/\mathcal{C}}$ as given by (6.47).

Let us now consider a lossless line short circuited at the far end $z = 0$, as shown in Fig. 6.14(a), in which the double-ruled lines represent the conductors of the transmission line. In actuality, the arrangement may consist, for example, of a perfectly conducting rectangular sheet joining the two conductors of a parallel-plate line as in Fig. 6.14(b) or a perfectly conducting ring-shaped sheet joining the two conductors of a coaxial cable as in Fig. 6.14(c). We shall assume that the line is driven by a voltage generator of frequency ω at the left end $z = -l$ so that waves are set up on the line. The short circuit at $z = 0$ requires that the tangential electric field on the surface of the conductor comprising the short circuit be zero. Since the voltage between the conductors of the line is proportional to this electric field which is transverse to them, it follows that the voltage across the short circuit has to be zero. Thus we have

$$\bar{V}(0) = 0 \tag{6.51}$$

Applying the boundary condition given by (6.51) to the general solution for $\bar{V}$ given by (6.50a), we have

$$\bar{V}(0) = \bar{A}e^{-j\beta(0)} + \bar{B}e^{j\beta(0)} = 0$$

or

$$\bar{B} = -\bar{A} \tag{6.52}$$

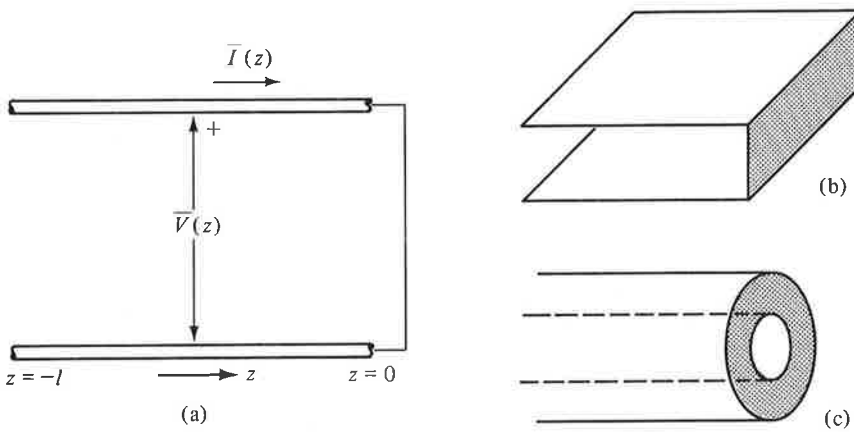

Figure 6.14. Transmission line short-circuited at the far end.

Thus we find that the short circuit gives rise to a (—) or reflected wave whose voltage is exactly the negative of the (+) or incident wave voltage, at the short circuit. Substituting this result in (6.50a) and (6.50b), we get the particular solutions for the complex voltage and current on the short-circuited line to be

$$\bar{V}(z) = \bar{A}e^{-j\beta z} - \bar{A}e^{j\beta z} = -2j\bar{A}\sin\beta z \tag{6.53a}$$

$$\bar{I}(z) = \frac{1}{Z_0}(\bar{A}e^{-j\beta z} + \bar{A}e^{j\beta z}) = \frac{2\bar{A}}{Z_0}\cos\beta z \tag{6.53b}$$

The real voltage and current are then given by

$$\begin{aligned} V(z,t) &= \text{Re}[\bar{V}(z)e^{j\omega t}] = \text{Re}(2e^{-j\pi/2}Ae^{j\theta}\sin\beta z\, e^{j\omega t}) \\ &= 2A\sin\beta z\sin(\omega t+\theta) \end{aligned} \tag{6.54a}$$

$$\begin{aligned} I(z,t) &= \text{Re}[\bar{I}(z)e^{j\omega t}] = \text{Re}\left[\frac{2}{Z_0}Ae^{j\theta}\cos\beta z\, e^{j\omega t}\right] \\ &= \frac{2A}{Z_0}\cos\beta z\cos(\omega t+\theta) \end{aligned} \tag{6.54b}$$

where we have replaced $\bar{A}$ by $Ae^{j\theta}$ and $-j$ by $e^{-j\pi/2}$. The instantaneous power flow down the line is given by

$$\begin{aligned} P(z,t) &= V(z,t)I(z,t) \\ &= \frac{4A^2}{Z_0}\sin\beta z\cos\beta z\sin(\omega t+\theta)\cos(\omega t+\theta) \\ &= \frac{A^2}{Z_0}\sin 2\beta z\sin 2(\omega t+\theta) \end{aligned} \tag{6.54c}$$

These results for the voltage, current, and power flow on the short-circuited line given by (6.54a), (6.54b), and (6.54c), respectively, are illustrated in Fig. 6.15, which shows the variation of each of these quantities with distance from the short circuit for several values of time. The numbers 1, 2, 3, . . . , 9 beside the curves in Fig. 6.15 represent the order of the curves

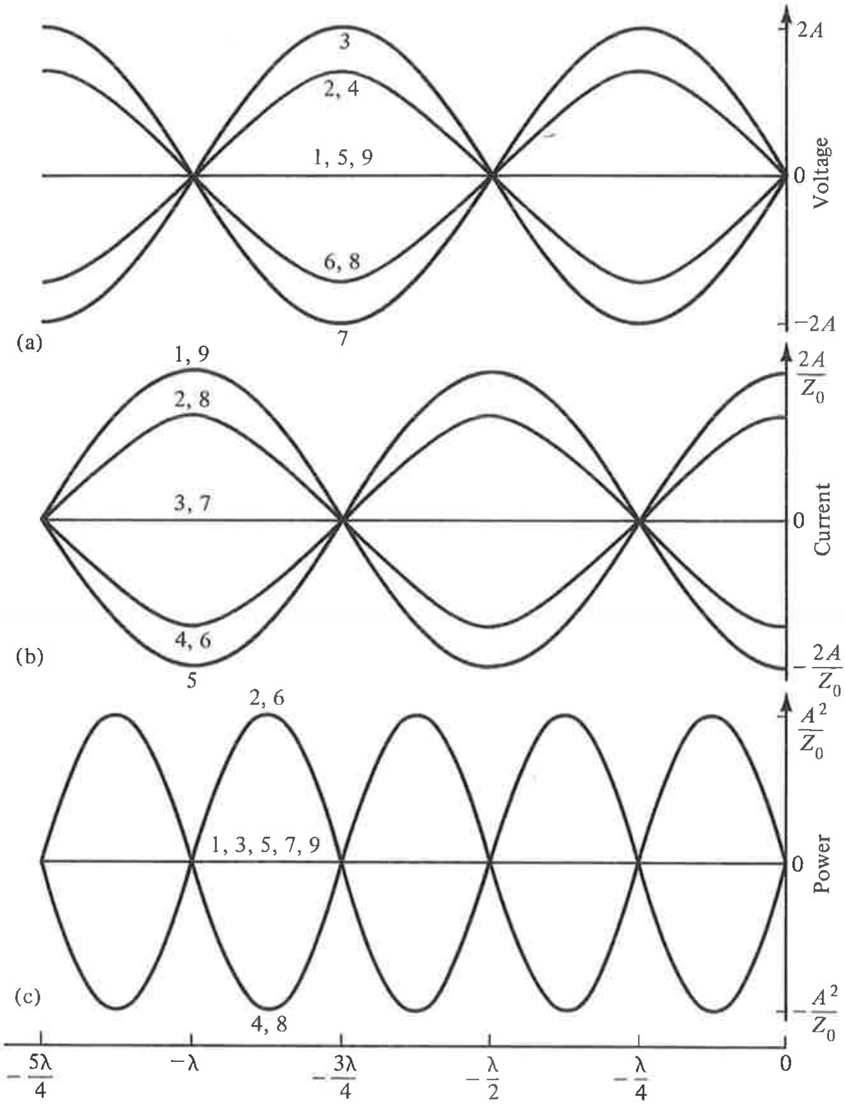

Figure 6.15. Time variations of voltage, current, and power flow associated with standing waves on a short-circuited transmission line.

corresponding to values of $(\omega t + \theta)$ equal to $0, \pi/4, \pi/2, \ldots, 2\pi$, It can be seen that the phenomenon is one in which the voltage, current, and power flow oscillate sinusoidally with time with different amplitudes at different locations on the line, unlike in the case of traveling waves in which a given point on the waveform progresses in distance with time. These waves are therefore known as "standing waves." In particular, they represent "complete standing waves" in view of the zero amplitudes of the voltage, current, and power flow at certain locations on the line, as shown by Fig. 6.15.

The line voltage amplitude is zero for values of z given by $\sin \beta z = 0$ or $\beta z = -m\pi$, $m = 1, 2, 3, \ldots$, or $z = -m\lambda/2$, $m = 1, 2, 3, \ldots$, that is at multiples of $\lambda/2$ from the short circuit. The line current amplitude is zero for values of z given by $\cos \beta z = 0$ or $\beta z = -(2m + 1)\pi/2$, $m = 0, 1, 2, 3, \ldots$, or $z = -(2m + 1)\lambda/4$, $m = 0, 1, 2, 3, \ldots$, that is, at odd multiples of $\lambda/4$ from the short circuit. The power flow amplitude is zero for values of z given by $\sin 2\beta z = 0$ or $\beta z = -m\pi/2$, $m = 1, 2, 3, \ldots$, or $z = -m\lambda/4$, $m = 1, 2, 3, \ldots$, that is, at multiples of $\lambda/4$ from the short circuit. Proceeding further, we find that the time-average power flow down the line, that is, power flow averaged over one period of the source voltage, is

$$\begin{aligned}\langle P \rangle &= \frac{1}{T}\int_{t=0}^{T} P(z, t)\, dt = \frac{\omega}{2\pi}\int_{t=0}^{2\pi/\omega} P(z, t)\, dt \\ &= \frac{\omega}{2\pi}\frac{A^2}{Z_0} \sin 2\beta z \int_{t=0}^{2\pi/\omega} \sin 2(\omega t + \theta)\, dt = 0 \end{aligned} \tag{6.55}$$

Thus the time average power flow down the line is zero at all points on the line. This is characteristic of complete standing waves.

From (6.53a) and (6.53b) or (6.54a) and (6.54b), or from Figs. 6.15(a) and 6.15(b), we find that the amplitudes of the sinusoidal time-variations of the line voltage and line current as functions of distance along the line are

$$|\bar{V}(z)| = 2A|\sin \beta z| = 2A\left|\sin \frac{2\pi}{\lambda} z\right| \tag{6.56a}$$

$$|\bar{I}(z)| = \frac{2A}{Z_0}|\cos \beta z| = \frac{2A}{Z_0}\left|\cos \frac{2\pi}{\lambda} z\right| \tag{6.56b}$$

Sketches of these quantities versus z are shown in Fig. 6.16. These are known as the "standing wave patterns." They are the patterns of line voltage and line current one would obtain by connecting an a.c. voltmeter between the conductors of the line and an a.c. ammeter in series with one of the conductors of the line and observing their readings at various points along the line. Alternatively, one can sample the electric and magnetic fields by means of probes.

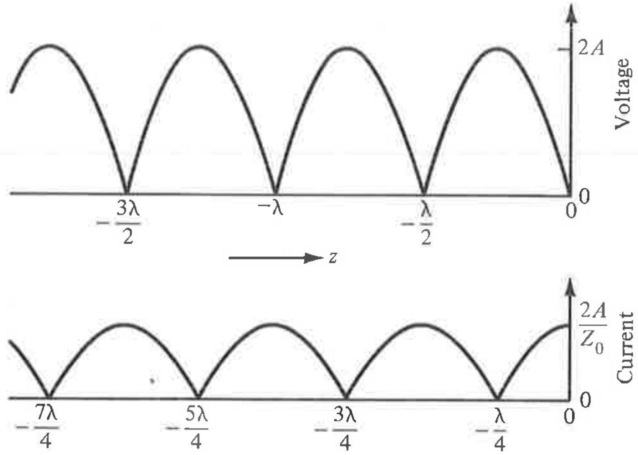

Figure 6.16. Standing wave patterns for voltage and current on a short-circuited line.

Returning now to the solutions for $\bar{V}(z)$ and $\bar{I}(z)$ given by (6.53a) and (6.53b), respectively, we can find the input impedance of the short-circuited line of length l by taking the ratio of the complex line voltage to the complex line current at the input $z = -l$. Thus

$$\bar{Z}_{\text{in}} = \frac{\bar{V}(-l)}{\bar{I}(-l)} = \frac{-2j\bar{A}\sin\beta(-l)}{\dfrac{2\bar{A}}{Z_0}\cos\beta(-l)}$$

$$= jZ_0 \tan\beta l = jZ_0 \tan\frac{2\pi}{\lambda}l$$

$$= jZ_0 \tan\frac{2\pi f}{v_p}l \tag{6.57}$$

We note from (6.57) that the input impedance of the short-circuited line is purely reactive. As the frequency is varied from a low value upward, the input reactance changes from inductive to capacitive and back to inductive, and so on, as illustrated in Fig. 6.17. The input reactance is zero for values of frequency equal to multiples of $v_p/2l$. These are the frequencies for which l is equal to multiples of $\lambda/2$ so that the line voltage is zero at the input and hence the input sees a short circuit. The input reactance is infinity for values of frequency equal to odd multiples of $v_p/4l$. These are the frequencies for which l is equal to odd multiples of $\lambda/4$ so that the line current is zero at the input and hence the input sees an open circuit.

Example 6.3. From the foregoing discussion of the input reactance of the short-circuited line, we note that as the frequency of the generator is varied

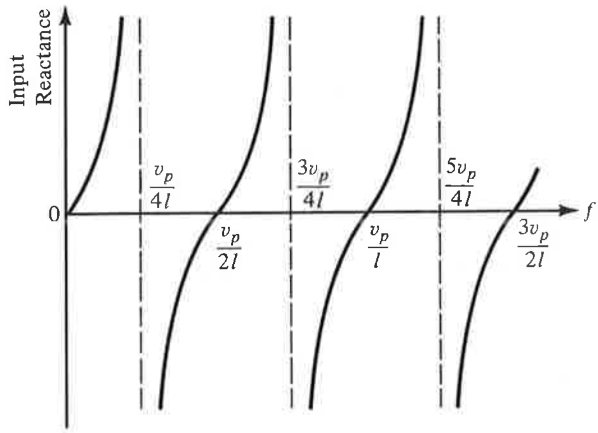

Figure 6.17. Variation of the input reactance of a short-circuited transmission line with frequency.

continuously upward, the current drawn from it undergoes alternatively maxima and minima corresponding to zero input reactance and infinite input reactance conditions, respectively. This behavior can be utilized for determining the location of a short circuit in the line.

Since the difference between a pair of consecutive frequencies for which the input reactance values are zero and infinity is $v_p/4l$, as can be seen from Fig. 6.17, it follows that the difference between successive frequencies for which the currents drawn from the generator are maxima and minima is $v_p/4l$. As a numerical example, if for an air dielectric line, it is found that as the frequency is varied from 50 MHz upward, the current reaches a minimum for 50.01 MHz and then a maximum for 50.04 MHz, then the distance l of the short circuit from the generator is given by

$$\frac{v_p}{4l} = (50.04 - 50.01) \times 10^6 = 0.03 \times 10^6 = 3 \times 10^4$$

Since $v_p = 3 \times 10^8$ m/s, it follows that

$$l = \frac{3 \times 10^8}{4 \times 3 \times 10^4} = 2500 \text{ m} = 2.5 \text{ km}$$

■

Example 6.4. We found that the input impedance of a short-circuited line of length l is given by

$$\bar{Z}_{\text{in}} = jZ_0 \tan \beta l$$

Let us investigate the low-frequency behavior of this input impedance.

First, we note that for any arbitrary value of βl,

$$\tan \beta l = \beta l + \frac{1}{3}(\beta l)^3 + \frac{2}{15}(\beta l)^5 + \dots$$

For $\beta l \ll 1$, i.e., $\frac{2\pi}{\lambda} l \ll 1$ or $l \ll \frac{\lambda}{2\pi}$ or $f \ll \frac{v_p}{2\pi l}$,

$$\tan \beta l \approx \beta l$$

$$\bar{Z}_{\text{in}} \approx jZ_0 \beta l = j\sqrt{\frac{\mathcal{L}}{\mathcal{C}}}\,\omega\sqrt{\mathcal{L}\mathcal{C}}\,l = j\omega\mathcal{L}l$$

Thus for frequencies $f \ll v_p/2\pi l$, the short-circuited line as seen from its input behaves essentially like a single inductor of value $\mathcal{L}l$, the total inductance of the line, as shown in Fig. 6.18(a).

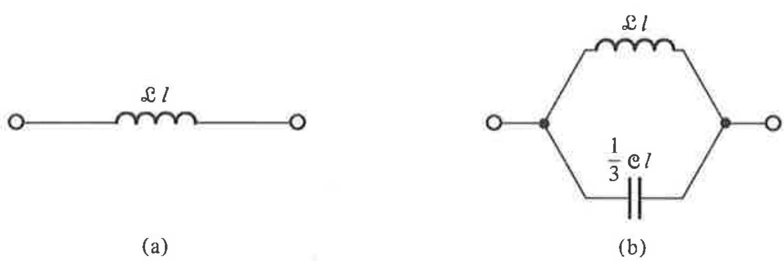

Figure 6.18. Equivalent circuits for the input behavior of a short-circuited transmission line.

Proceeding further, we observe that if the frequency is slightly beyond the range for which the above approximation is valid, then

$$\tan \beta l \approx \beta l + \frac{1}{3}(\beta l)^3$$

$$\begin{aligned}
\bar{Z}_{\text{in}} &\approx jZ_0\left(\beta l + \frac{1}{3}\beta^3 l^3\right) \\
&= j\sqrt{\frac{\mathcal{L}}{\mathcal{C}}}\left(\omega\sqrt{\mathcal{L}\mathcal{C}}\,l + \frac{1}{3}\omega^3 \mathcal{L}^{3/2}\mathcal{C}^{3/2} l^3\right) \\
&= j\omega\mathcal{L}l\left(1 + \frac{1}{3}\omega^2\mathcal{L}\mathcal{C}l^2\right)
\end{aligned}$$

$$\begin{aligned}
\bar{Y}_{\text{in}} &= \frac{1}{\bar{Z}_{\text{in}}} = \frac{1}{j\omega\mathcal{L}l}\left(1 + \frac{1}{3}\omega^2\mathcal{L}\mathcal{C}l^2\right)^{-1} \\
&\approx \frac{1}{j\omega\mathcal{L}l}\left(1 - \frac{1}{3}\omega^2\mathcal{L}\mathcal{C}l^2\right) \\
&= \frac{1}{j\omega\mathcal{L}l} + j\frac{1}{3}\omega\mathcal{C}l
\end{aligned}$$

Thus for frequencies somewhat above those for which the approximation $f \ll v_p/2\pi l$ is valid, the short-circuited line as seen from its input behaves like an inductor of value $\mathcal{L}l$ in parallel with a capacitance of value $\frac{1}{3}\mathcal{C}l$, as shown in Fig. 6.18(b).

These findings illustrate that a physical structure that can be considered as an inductor at low frequencies $f \ll v_p/2\pi l$ no longer behaves like an inductor if the frequency is increased beyond that range. In fact, it has a "stray" capacitance associated with it. As the frequency is still increased, the equivalent circuit becomes further complicated. Thus conventional circuit theory considerations of physical structures are strictly valid only for $f \ll v_p/2\pi l$, or $l \ll \lambda/2\pi$. ■

6.5 BOUNDARY CONDITIONS AT A DIELECTRIC DISCONTINUITY

In Sec. 6.1 we derived the boundary conditions for the field components at a perfect conductor surface by applying Maxwell's equations in integral form to infinitesimal closed paths and closed surfaces encompassing the boundary and by using the fact that the fields inside the perfect conductor are zero. In this section we shall derive the boundary conditions at an interface between two different perfect dielectric media by similarly considering the Maxwell's equations in integral form one at a time. We shall note, however, that fields exist on either side of the boundary and that there cannot be any surface charge or surface current on the boundary in view of the perfect dielectric nature of the two media. We shall then use these boundary conditions in the following section to study reflection and transmission at the junction of two transmission lines having different dielectrics.

Thus let us consider a plane boundary between two different dielectric media 1 and 2 characterized by ϵ_1, μ_1 and ϵ_2, μ_2, respectively, as shown in Fig. 6.19. Then, applying Faraday's law in integral form (6.1a) to the infinitesimal rectangular path *abcda* as shown in Fig. 6.19, we have

$$\int_a^b \mathbf{E} \cdot d\mathbf{l} + \int_b^c \mathbf{E} \cdot d\mathbf{l} + \int_c^d \mathbf{E} \cdot d\mathbf{l} + \int_d^a \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_{abcd} \mathbf{B} \cdot d\mathbf{S} \tag{6.58}$$

In the limit that *ad* and $bc \longrightarrow 0$, we obtain

$$\begin{aligned} E_{ab}(ab) + E_{cd}(cd) &= 0 \\ E_{ab}(ab) - E_{dc}(cd) &= 0 \quad \text{or} \quad E_{ab} = E_{dc} \end{aligned} \tag{6.59}$$

Since this is true for any orientation of the rectangle, it follows that "the tangential component of **E** is continuous at the dielectric interface." Thus

$$E_{t1} = E_{t2} \quad \text{or} \quad \mathbf{i}_n \times (\mathbf{E}_1 - \mathbf{E}_2) = 0 \tag{6.60}$$

where the subscript t denotes "tangential" and $\mathbf{i}_n$ is the unit normal vector to the boundary, as shown in Fig. 6.19.

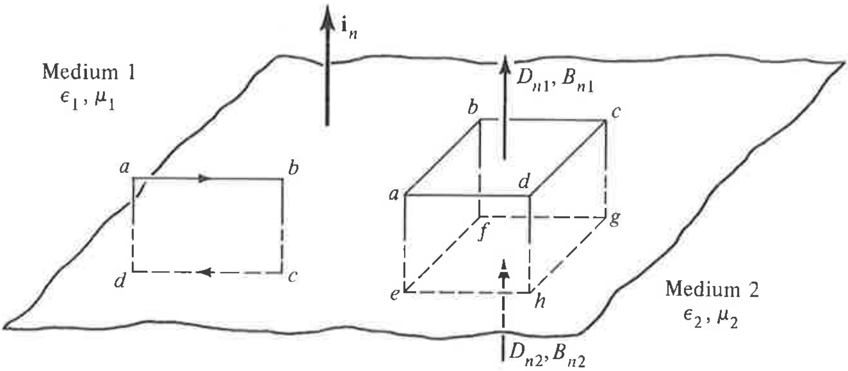

Figure 6.19. For deriving the boundary conditions at the interface between two perfect dielectric media.

Similarly, applying Ampere's circuital law in integral form (6.1b) to the rectangular path *abcda*, we have

$$\int_a^b \mathbf{H} \cdot d\mathbf{l} + \int_b^c \mathbf{H} \cdot d\mathbf{l} + \int_c^d \mathbf{H} \cdot d\mathbf{l} + \int_d^a \mathbf{H} \cdot d\mathbf{l}$$

$$= \int_{abcd} \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt}\int_{abcd} \mathbf{D} \cdot d\mathbf{S} \tag{6.61}$$

In the limit that ad and $bc \longrightarrow 0$ and noting that there is no current enclosed by *abcd*, we obtain

$$H_{ab}(ab) + H_{cd}(cd) = 0$$

$$H_{ab}(ab) - H_{dc}(cd) = 0 \quad \text{or} \quad H_{ab} = H_{dc} \tag{6.62}$$

Since this is true for any orientation of the rectangle, it follows that "the tangential component of **H** is continuous at the dielectric interface." Thus

$$H_{t1} = H_{t2} \quad \text{or} \quad \mathbf{i}_n \times (\mathbf{H}_1 - \mathbf{H}_2) = 0 \tag{6.63}$$

Considering next Gauss' law for the electric field in integral form (6.1c) and applying it to the infinitesimal rectangular box *abcdefgh*, as shown in Fig. 6.19, we have

$$\int_{abcd} \mathbf{D} \cdot d\mathbf{S} + \int_{\substack{\text{side}\\ \text{surfaces}}} \mathbf{D} \cdot d\mathbf{S} + \int_{efgh} \mathbf{D} \cdot d\mathbf{S} = \int_{\substack{\text{volume}\\ \text{of the}\\ \text{box}}} \rho\, dv \qquad (6.64)$$

In the limit that the side surfaces $\longrightarrow 0$ and noting that there is no charge enclosed by the box, we obtain

$$D_{n1}(abcd) - D_{n2}(efgh) = 0 \quad \text{or} \quad D_{n1} = D_{n2} \qquad (6.65)$$

where the subscript n denotes "normal," and D_{n1} and D_{n2} are both directed into medium 1. Thus "the normal component of **D** is continuous at the dielectric interface." In vector form, we have

$$\mathbf{i}_n \cdot (\mathbf{D}_1 - \mathbf{D}_2) = 0 \qquad (6.66)$$

Similarly, applying Gauss' law for the magnetic field in integral form (6.1d) to the rectangular box *abcdefgh*, we have

$$\int_{abcd} \mathbf{B} \cdot d\mathbf{S} + \int_{\substack{\text{side}\\ \text{surfaces}}} \mathbf{B} \cdot d\mathbf{S} + \int_{efgh} \mathbf{B} \cdot d\mathbf{S} = 0 \qquad (6.67)$$

In the limit that the side surfaces $\longrightarrow 0$, we obtain

$$B_{n1}(abcd) - B_{n2}(efgh) = 0 \quad \text{or} \quad B_{n1} = B_{n2} \qquad (6.68)$$

Thus "the normal component of **B** is continuous at the dielectric interface." In vector form, we have

$$\mathbf{i}_n \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0 \qquad (6.69)$$

Summarizing the boundary conditions for the field components at a dielectric interface, we have

$$\mathbf{i}_n \times (\mathbf{E}_1 - \mathbf{E}_2) = 0$$
$$\mathbf{i}_n \times (\mathbf{H}_1 - \mathbf{H}_2) = 0$$
$$\mathbf{i}_n \cdot (\mathbf{D}_1 - \mathbf{D}_2) = 0$$
$$\mathbf{i}_n \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0$$

Example 6.5. At a particular instant of time the fields at point 1 in Fig. 6.20 are given by

$$\mathbf{E}_1 = E_0(3\mathbf{i}_x + \mathbf{i}_z)$$
$$\mathbf{H}_1 = H_0(2\mathbf{i}_y)$$

where E_0 and H_0 are constants. Let us find the fields at point 2, lying adjacent to point 1 and on the other side of the interface between media 1 and 2.

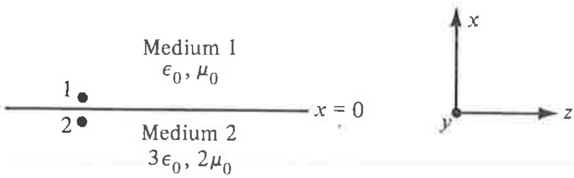

Figure 6.20. For illustrating the application of boundary conditions at the interface between two perfect dielectric media.

From (6.66), we have

$$D_{2x} = D_{1x} = \epsilon_0(3E_0) = 3\epsilon_0 E_0$$

$$E_{2x} = \frac{D_{2x}}{3\epsilon_0} = \frac{3\epsilon_0 E_0}{3\epsilon_0} = E_0$$

From (6.60), we get

$$E_{2y} = E_{1y} = 0$$
$$E_{2z} = E_{1z} = E_0$$

From (6.69), we obtain

$$B_{2x} = B_{1x} = \mu_0(0) = 0$$

$$H_{2x} = \frac{B_{2x}}{2\mu_0} = 0$$

From (6.63), we find

$$H_{2y} = H_{1y} = 2H_0$$
$$H_{2z} = H_{1z} = 0$$

Thus we obtain the rqeuired fields at point 2 to be

$$\mathbf{E}_2 = E_0(\mathbf{i}_x + \mathbf{i}_z)$$
$$\mathbf{H}_2 = H_0(2\mathbf{i}_y)$$

■

6.6 TRANSMISSION-LINE DISCONTINUITY

Let us now consider the case of two transmission lines 1 and 2 having different characteristic impedances Z_{01} and Z_{02}, respectively, and phase constants β_1 and β_2, respectively, connected in cascade and driven by a generator at the left end of line 1, as shown in Fig. 6.21(a). Physically, the arrangement may, for example, consist of two parallel-plate lines or two coaxial cables of

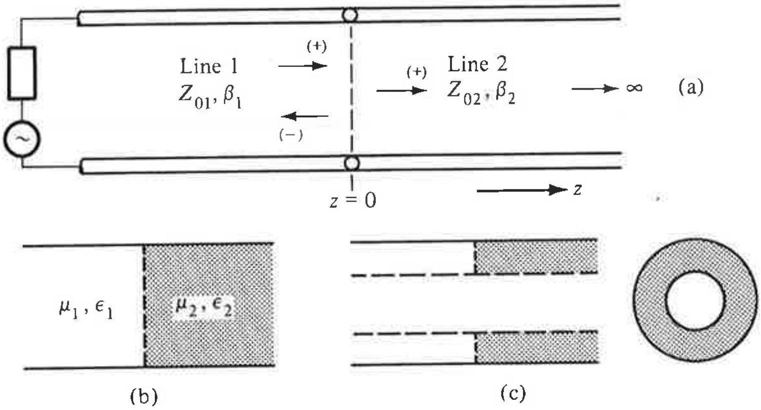

Figure 6.21. Two transmission lines connected in cascade.

different dielectrics in cascade, as shown in Figs. 6.21(b) and 6.21(c), respectively. In view of the discontinuity at the junction $z = 0$ between the two lines, the incident (+) wave on the junction sets up a reflected (−) wave in line 1 and a transmitted (+) wave in line 2. We shall assume that line 2 is infinitely long so that there is no (−) wave in that line.

We can now write the solutions for the complex voltage and complex current in line 1 as

$$\bar{V}_1(z) = \bar{V}_1^+ e^{-j\beta_1 z} + \bar{V}_1^- e^{j\beta_1 z} \tag{6.70a}$$

$$\bar{I}_1(z) = \bar{I}_1^+ e^{-j\beta_1 z} + \bar{I}_1^- e^{j\beta_1 z}$$
$$= \frac{1}{Z_{01}}(\bar{V}_1^+ e^{-j\beta_1 z} - \bar{V}_1^- e^{j\beta_1 z}) \tag{6.70b}$$

where $\bar{V}_1^+$, $\bar{V}_1^-$, $\bar{I}_1^+$, and $\bar{I}_1^-$ are the (+) and (−) wave voltages and currents at $z = 0-$ in line 1, that is, just to the left of the junction. The solutions for the complex voltage and current in line 2 are

$$\bar{V}_2(z) = \bar{V}_2^+ e^{-j\beta_2 z} \tag{6.71a}$$

$$\bar{I}_2(z) = \bar{I}_2^+ e^{-j\beta_2 z} = \frac{1}{Z_{02}} \bar{V}_2^+ e^{-j\beta_2 z} \tag{6.71b}$$

where $\bar{V}_2^+$ and $\bar{I}_2^+$ are the (+) wave voltage and current at $z = 0+$ in line 2, that is, just to the right of the junction.

At the junction the boundary conditions (6.60) and (6.63) require that the components of **E** and **H** tangential to the dielectric interface be continuous, as shown, for example, for the parallel-plate arrangement in Fig.

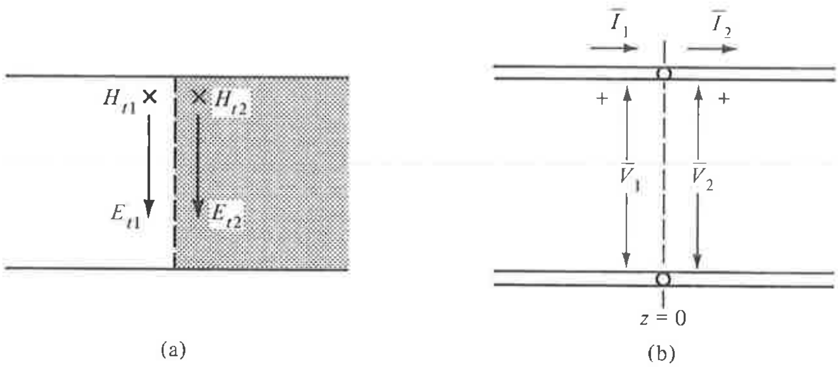

Figure 6.22. Application of boundary conditions at the junction between two transmission lines.

6.22(a). These are, in fact, the only components present since the transmission line fields are entirely transverse to the direction of propagation. Now, since the line voltage and current are related to these electric and magnetic fields, respectively, it then follows that the line voltage and line current be continuous at the junction, as shown in Fig. 6.22(b). Thus we obtain the boundary conditions at the junction in terms of line voltage and line current as

$$[\bar{V}_1]_{z=0-} = [\bar{V}_2]_{z=0+} \tag{6.72a}$$

$$[\bar{I}_1]_{z=0-} = [\bar{I}_2]_{z=0+} \tag{6.72b}$$

Applying these boundary conditions to the solutions given by (6.70a) and (6.70b), we obtain

$$\bar{V}_1^+ + \bar{V}_1^- = \bar{V}_2^+ \tag{6.73a}$$

$$\frac{1}{Z_{01}}(\bar{V}_1^+ - \bar{V}_1^-) = \frac{1}{Z_{02}}\bar{V}_2^+ \tag{6.73b}$$

Eliminating $\bar{V}_2^+$ from (6.73a) and (6.73b), we get

$$\bar{V}_1^+\left(\frac{1}{Z_{02}} - \frac{1}{Z_{01}}\right) + \bar{V}_1^-\left(\frac{1}{Z_{02}} + \frac{1}{Z_{01}}\right) = 0$$

or

$$\bar{V}_1^- = \bar{V}_1^+\frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}} \tag{6.74}$$

We now define the voltage reflection coefficient at the junction, Γ_V, as the ratio of the reflected wave voltage ($\bar{V}_1^-$) at the junction to the incident wave voltage ($\bar{V}_1^+$) at the junction. Thus

$$\Gamma_V = \frac{\bar{V}_1^-}{\bar{V}_1^+} = \frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}} \tag{6.75}$$

The current reflection coefficient at the junction, Γ_I, which is the ratio of the reflected wave current ($\bar{I}_1^-$) at the junction to the incident wave current ($\bar{I}_1^+$) at the junction is then given by

$$\Gamma_I = \frac{\bar{I}_1^-}{\bar{I}_1^+} = \frac{-\bar{V}_1^-/Z_{01}}{\bar{V}_1^+/Z_{01}} = -\frac{\bar{V}_1^-}{\bar{V}_1^+} = -\Gamma_V \tag{6.76}$$

We also define the voltage transmission coefficient at the junction, τ_V, as the ratio of the transmitted wave voltage ($\bar{V}_2^+$) at the junction to the incident wave voltage ($\bar{V}_1^+$) at the junction. Thus

$$\tau_V = \frac{\bar{V}_2^+}{\bar{V}_1^+} = \frac{\bar{V}_1^+ + \bar{V}_1^-}{\bar{V}_1^+} = 1 + \frac{\bar{V}_1^-}{\bar{V}_1^+} = 1 + \Gamma_V \tag{6.77}$$

The current transmission coefficient at the junction, τ_I, which is the ratio of the transmitted wave current ($\bar{I}_2^+$) at the junction to the incident wave current ($\bar{I}_1^+$) at the junction is given by

$$\tau_I = \frac{\bar{I}_2^+}{\bar{I}_1^+} = \frac{\bar{I}_1^+ + \bar{I}_1^-}{\bar{I}_1^+} = 1 + \frac{\bar{I}_1^-}{\bar{I}_1^+} = 1 - \Gamma_V \tag{6.78}$$

We note that for $Z_{02} = Z_{01}$, $\Gamma_V = 0$, $\Gamma_I = 0$, $\tau_V = 1$, and $\tau_I = 1$. Thus the incident wave is entirely transmitted as we may expect since there is no discontinuity at the junction.

Example 6.6. Let us consider the junction of two lines having characteristic impedances $Z_{01} = 50$ ohms and $Z_{02} = 75$ ohms, as shown in Fig. 6.23, and compute the various quantities.

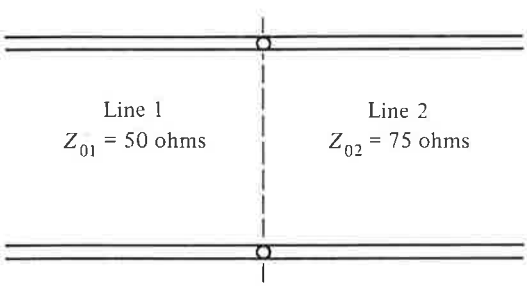

Figure 6.23. For the computation of several quantities pertinent to reflection and transmission at the junction between two transmission lines.

From (6.75)–(6.78), we have

$$\Gamma_V = \frac{75 - 50}{75 + 50} = \frac{25}{125} = \frac{1}{5}; \qquad \bar{V}_1^- = \frac{1}{5}\bar{V}_1^+$$

$$\Gamma_I = -\Gamma_V = -\frac{1}{5}; \qquad \bar{I}_1^- = -\frac{1}{5}\bar{I}_1^+$$

$$\tau_V = 1 + \Gamma_V = 1 + \frac{1}{5} = \frac{6}{5}; \qquad \bar{V}_2^+ = \frac{6}{5}\bar{V}_1^+$$

$$\tau_I = 1 - \Gamma_V = 1 - \frac{1}{5} = \frac{4}{5}; \qquad \bar{I}_2^+ = \frac{4}{5}\bar{I}_1^+$$

The fact that the transmitted wave voltage is greater than the incident wave voltage should not be of concern since it is the power balance that must be satisfied at the junction. We can verify this by noting that if the incident power on the junction is P_i, then

$$\text{reflected power, } P_r = \Gamma_V \Gamma_I P_i = -\frac{1}{25}P_i$$

$$\text{transmitted power, } P_t = \tau_V \tau_I P_i = \frac{24}{25}P_i$$

Recognizing that the minus sign for P_r signifies power flow in the negative z direction, we find that power balance is indeed satisfied at the junction. ■

Returning now to the solutions for the voltage and current in line 1 given by (6.70a) and (6.70b), respectively, we obtain by replacing $\bar{V}_1^-$ by $\Gamma_V \bar{V}_1^+$,

$$\begin{aligned} \bar{V}_1(z) &= \bar{V}_1^+ e^{-j\beta_1 z} + \Gamma_V \bar{V}_1^+ e^{j\beta_1 z} \\ &= \bar{V}_1^+ e^{-j\beta_1 z}(1 + \Gamma_V e^{j2\beta_1 z}) \end{aligned} \tag{6.79a}$$

$$\begin{aligned} \bar{I}_1(z) &= \frac{1}{Z_{01}}(\bar{V}_1^+ e^{-j\beta_1 z} - \Gamma_V \bar{V}_1^+ e^{j\beta_1 z}) \\ &= \frac{\bar{V}_1^+}{Z_{01}} e^{-j\beta_1 z}(1 - \Gamma_V e^{j2\beta_1 z}) \end{aligned} \tag{6.79b}$$

The amplitudes of the sinusoidal time-variations of the line voltage and line current as functions of distance along the line are then given by

$$\begin{aligned} |\bar{V}_1(z)| &= |\bar{V}_1^+||e^{-j\beta_1 z}||1 + \Gamma_V e^{j2\beta_1 z}| \\ &= |\bar{V}_1^+||1 + \Gamma_V \cos 2\beta_1 z + j\Gamma_V \sin 2\beta_1 z| \\ &= |\bar{V}_1^+|\sqrt{1 + \Gamma_V^2 + 2\Gamma_V \cos 2\beta_1 z} \end{aligned} \tag{6.80a}$$

$$|\bar{I}_1(z)| = \frac{|\bar{V}_1^+|}{Z_{01}}|e^{-j\beta_1 z}||1 - \Gamma_V e^{j2\beta_1 z}|$$

$$= \frac{|\bar{V}_1^+|}{Z_{01}}|1 - \Gamma_V \cos 2\beta_1 z - j\Gamma_V \sin 2\beta_1 z|$$

$$= \frac{|\bar{V}_1^+|}{Z_{01}}\sqrt{1 + \Gamma_V^2 - 2\Gamma_V \cos 2\beta_1 z} \tag{6.80b}$$

From (6.80a) and (6.80b), we note the following:

1. The line voltage amplitude undergoes alternate maxima and minima equal to $|\bar{V}_1^+|(1 + |\Gamma_V|)$ and $|\bar{V}_1^+|(1 - |\Gamma_V|)$, respectively. The line voltage amplitude at $z = 0$ is a maximum or minimum depending on whether Γ_V is positive or negative. The distance between a voltage maximum and the adjacent voltage minimum is $\pi/2\beta_1$ or $\lambda_1/4$.
2. The line current amplitude undergoes alternate maxima and minima equal to $\frac{|\bar{V}_1^+|}{Z_{01}}(1 + |\Gamma_V|)$ and $\frac{|\bar{V}_1^+|}{Z_{01}}(1 - |\Gamma_V|)$, respectively. The line current amplitude at $z = 0$ is a minimum or maximum depending on whether Γ_V is positive or negative. The distance between a current maximum and the adjacent current minimum is $\pi/2\beta_1$ or $\lambda_1/4$.

Knowing these properties of the line voltage and current amplitudes, we now sketch the voltage and current standing wave patterns, as shown in Fig. 6.24, assuming $\Gamma_V > 0$. Since these standing wave patterns do not contain perfect nulls, as in the case of the short-circuited line of Sec. 6.4, these are said to correspond to "partial standing waves."

We now define a quantity known as the "standing wave ratio" (SWR) as the ratio of the maximum voltage, V_{max}, to the minimum voltage, V_{min}, of the standing wave pattern. Thus we find that

$$\text{SWR} = \frac{V_{max}}{V_{min}} = \frac{|\bar{V}_1^+|(1 + |\Gamma_V|)}{|\bar{V}_1^+|(1 - |\Gamma_V|)} = \frac{1 + |\Gamma_V|}{1 - |\Gamma_V|} \tag{6.81}$$

The SWR is an important parameter in transmission-line matching. It is an indicator of the degree of the existence of standing waves on the line. We shall, however, not pursue the topic here any further. Finally, we note that for the case of Example 6.6, the SWR in line 1 is $\left(1 + \frac{1}{5}\right)/\left(1 - \frac{1}{5}\right)$ or 1.5. The SWR in line 2 is, of course, equal to 1 since there is no reflected wave in that line.

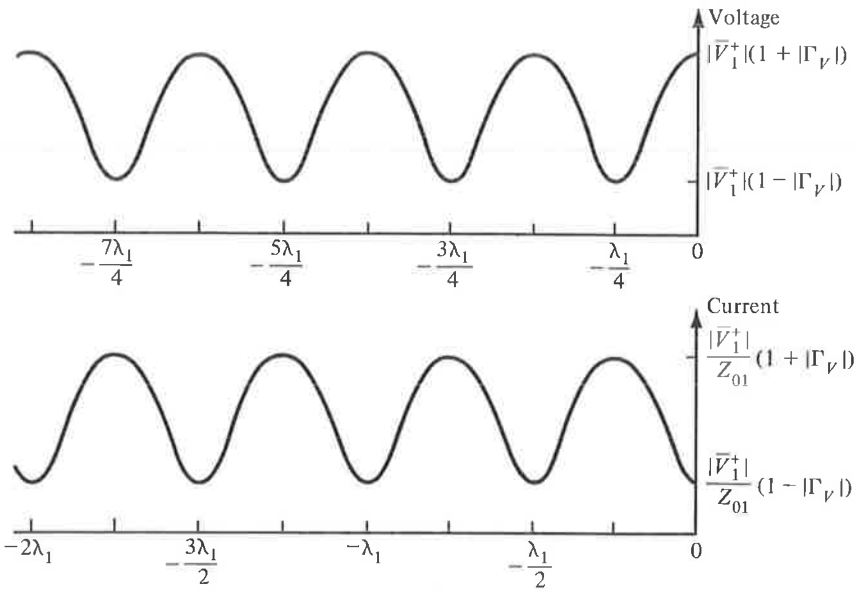

Figure 6.24. Standing wave patterns for voltage and current on a transmission line terminated by another transmission line.

6.7 SUMMARY

In this chapter we studied the principles of transmission lines by extending our knowledge of uniform plane wave propagation gained in the previous two chapters. To introduce the transmission line, we first derived the boundary conditions required to be satisfied by the field components at a perfect conductor surface. These boundary conditions, which follow from the application of Maxwell's equations in integral form to infinitesimal closed paths and surfaces straddling the boundary and from the property that the fields inside a perfect conductor are zero, are given in vector form by

$$\mathbf{i}_n \times \mathbf{E} = 0 \tag{6.82a}$$

$$\mathbf{i}_n \times \mathbf{H} = \mathbf{J}_S \tag{6.82b}$$

$$\mathbf{i}_n \cdot \mathbf{D} = \rho_S \tag{6.82c}$$

$$\mathbf{i}_n \cdot \mathbf{B} = 0 \tag{6.82d}$$

where $\mathbf{i}_n$ is the unit normal vector to the conductor surface and directed into the field region. Equations (6.82a) and (6.82d) state that the electric field be completely normal and that the magnetic field be completely tangential at a point on the conductor surface. The normal displacement flux density and the tangential magnetic field intensity are then related to the surface charge density and the surface current density as given by (6.82c) and (6.82b), respectively.

We used the boundary conditions (6.82a)–(6.82d) to illustrate that the placing of perfect conductors in planes normal to the electric field and hence tangential to the magnetic field of a uniform plane wave does not alter the field distribution and the wave is simply guided between and parallel to the conductors supported by the charges and currents on the conductors, as though they were not present, thereby constituting a parallel-plate transmission line. We then showed that wave propagation on a transmission line can be discussed in terms of voltage and current, which are related to the electric and magnetic fields, respectively, by deriving the "transmission-line equations"

$$\frac{\partial V}{\partial z} = -\mathcal{L}\frac{\partial I}{\partial t} \tag{6.83a}$$

$$\frac{\partial I}{\partial z} = -\mathcal{G}V - \mathcal{C}\frac{\partial V}{\partial t} \tag{6.83b}$$

which then led us to the concept of the distributed circuit.

The parameters $\mathcal{L}$, $\mathcal{C}$, and $\mathcal{G}$ in (6.83a) and (6.83b) are the inductance, capacitance, and conductance per unit length of line, which differ from one line to another. For the parallel-plate line having width w of the plates and spacing d between the plates, they are given by

$$\mathcal{L} = \frac{\mu d}{w}$$

$$\mathcal{C} = \frac{\epsilon w}{d}$$

$$\mathcal{G} = \frac{\sigma w}{d}$$

where μ, ϵ, and σ are the material parameters of the medium between the plates, and fringing of the fields is neglected. We learned how to compute $\mathcal{L}$, $\mathcal{C}$, and $\mathcal{G}$ for a line of arbitrary cross section by constructing a field map of the transverse electromagnetic wave fields, consisting of curvilinear squares in the cross-sectional plane of the line. If m is the number of squares tangential to the conductors and n is the number of squares normal to the conductors, then

$$\mathcal{L} = \mu\frac{n}{m}$$

$$\mathcal{C} = \epsilon\frac{m}{n}$$

$$\mathcal{G} = \sigma\frac{m}{n}$$

By applying this technique to the coaxial cable, we found that for a cable of inner radius a and outer radius b,

$$\mathcal{L} = \frac{\mu}{2\pi}\ln\frac{b}{a}$$

$$\mathcal{C} = \frac{2\pi\epsilon}{\ln(b/a)}$$

$$\mathcal{G} = \frac{2\pi\sigma}{\ln(b/a)}$$

The general solutions to the transmission-line equations (6.83a) and (6.83b), expressed in phasor form, that is,

$$\frac{\partial\bar{V}}{\partial z} = -j\omega\mathcal{L}I \qquad (6.84a)$$

$$\frac{\partial\bar{I}}{\partial z} = -\mathcal{G}\bar{V} - j\omega\mathcal{C}\bar{V} \qquad (6.84b)$$

are given by

$$\bar{V}(z) = \bar{A}e^{-\bar{\gamma}z} + \bar{B}e^{\bar{\gamma}z} \qquad (6.85a)$$

$$\bar{I}(z) = \frac{1}{\bar{Z}_0}(\bar{A}e^{-\bar{\gamma}z} - \bar{B}e^{\bar{\gamma}z}) \qquad (6.85b)$$

where

$$\bar{\gamma} = \sqrt{j\omega\mathcal{L}(\mathcal{G} + j\omega\mathcal{C})} \qquad [= \sqrt{j\omega\mu(\sigma + j\omega\epsilon)}]$$

$$\bar{Z}_0 = \sqrt{\frac{j\omega\mathcal{L}}{\mathcal{G} + j\omega\mathcal{C}}} \qquad \left[\neq \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}\right]$$

are the propagation constant and the characteristic impedance, respectively, of the line. For a lossless line ($\mathcal{G} = 0$), these reduce to

$$\bar{\gamma} = j\omega\sqrt{\mathcal{L}\mathcal{C}} \qquad (= j\omega\sqrt{\mu\epsilon})$$

$$\bar{Z}_0 = \sqrt{\frac{\mathcal{L}}{\mathcal{C}}} \qquad (\neq \sqrt{\mu/\epsilon})$$

The solutions given by (6.85a) and (6.85b) represent the superposition of (+) and (−) waves propagating in the medium between the conductors of the line, expressed in terms of the line voltage and current instead of in terms of the electric and magnetic fields. By applying these general solutions to the case of a lossless line short circuited at the far end and obtaining the particular solutions for that case, we discussed the standing wave phenomenon and the standing wave patterns resulting from the complete reflection of waves by the short circuit. We also examined the frequency behavior of the input impedance of a short-circuited line of length l, given by

$$\bar{Z}_{\text{in}} = jZ_0 \tan \beta l$$

and (a) illustrated its application in a technique for the location of short circuit in a line, and (b) learned that for a circuit element to behave as assumed by conventional (lumped) circuit theory, its dimensions must be a small fraction of the wavelength corresponding to the frequency of operation.

To extend the discussion of the reflection phenomenon to one of partial reflection and transmission, we first derived the boundary conditions at the interface between two dielectric media. These are given in vector form by

$$\mathbf{i}_n \times (\mathbf{E}_1 - \mathbf{E}_2) = 0 \tag{6.86a}$$

$$\mathbf{i}_n \times (\mathbf{H}_1 - \mathbf{H}_2) = 0 \tag{6.86b}$$

$$\mathbf{i}_n \cdot (\mathbf{D}_1 - \mathbf{D}_2) = 0 \tag{6.86c}$$

$$\mathbf{i}_n \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0 \tag{6.86d}$$

where $\mathbf{i}_n$ is the unit normal vector to the interface and directed into the medium having the subscript 1 for the fields. These boundary conditions point to the continuity of the tangential component of $\mathbf{E}$, the tangential component of $\mathbf{H}$, the normal component of $\mathbf{D}$, and the normal component of $\mathbf{B}$, at a point on the interface.

We used the boundary conditions (6.86a)–(6.86d) to investigate reflection and transmission of waves at a junction between two lossless lines. By applying them to the general solutions for the line voltage and current on either side of the junction, we deduced the ratio of the reflected wave voltage to the incident wave voltage, that is, the voltage reflection coefficient, to be

$$\Gamma_V = \frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}}$$

where Z_{01} is the characteristic impedance of the line from which the wave is incident and Z_{02} is the characteristic impedance of the line on which the wave is incident. The ratio of the transmitted wave voltage to the incident wave voltage, that is, the voltage transmission coefficient, is given by

$$\tau_V = 1 + \Gamma_V$$

The current reflection and transmission coefficients are given by

$$\Gamma_I = -\Gamma_V$$
$$\tau_I = 1 - \Gamma_V$$

Finally, we discussed the standing wave pattern resulting from the partial reflection of the wave at the junction and defined a quantity known as the standing wave ratio (SWR), which is a measure of the reflection phenomenon. In terms of Γ_V, it is given by

$$\text{SWR} = \frac{1 + |\Gamma_V|}{1 - |\Gamma_V|}$$

In retrospect, it can be seen that the discussion of the standing wave phenomenon and reflection and transmission at the junction of two lines is equally applicable to the solution of analogous uniform plane wave problems involving media unbounded in the two dimensions normal to the direction of propagation of the wave.

REVIEW QUESTIONS

6.1. What is a boundary condition? How do boundary conditions arise?

6.2. State the boundary conditions for the electric field components at the surface of a perfect conductor.

6.3. State the boundary conditions for the magnetic field components at the surface of a perfect conductor.

6.4. Summarize in vector form the boundary conditions at a perfect conductor surface, indicating correspondingly the Maxwell's equations in integral form from which they are derived.

6.5. Discuss the guiding of a uniform plane wave by a pair of parallel-plane, perfectly conducting sheets.

6.6. How is the voltage between the two conductors in a given cross-sectional plane of a parallel-plate transmission line related to the electric field in that plane?

6.7. How is the current flowing on the plates across a given cross-sectional plane of a parallel-plate transmission line related to the magnetic field in that plane?

6.8. What are transmission-line equations? How are they obtained from Maxwell's equations?

6.9. How is $\mathcal{L}$, the inductance per unit length of a transmission line, defined? What is it equal to for a parallel-plate transmission line?

6.10. How is $\mathcal{C}$, the capacitance per unit length of a transmission line, defined? What is it equal to for a parallel-plate transmission line?

6.11. How is $\mathcal{G}$, the conductance per unit length of a transmission line, defined? What is it equal to for a parallel-plate transmission line?

6.12. Are the three quantities $\mathcal{L}$, $\mathcal{C}$, and $\mathcal{G}$ independent? If not, how are they dependent on each other?

6.13. Draw the transmission-line equivalent circuit. How is it derived from the transmission-line equations?

6.14. Discuss the concept of the distributed circuit and compare it to a lumped circuit.

6.15. Discuss the physical phenomena associated with each of the elements in the transmission-line equivalent circuit.

6.16. What is a transverse electromagnetic wave?

6.17. What is a field map? Describe the procedure for drawing the field map for a transmission line of arbitrary cross section.

6.18. Draw a rough sketch of the field map for a line made up of two identical parallel cylindrical conductors with their axes separated by four times their radii.

6.19. Describe the procedure for computing the transmission line parameters $\mathcal{L}$, $\mathcal{C}$, and $\mathcal{G}$ from the field map.

6.20. How does a field map consisting of curvilinear squares simplify the computation of the line parameters?

6.21. Discuss the determination of $\mathcal{L}$, $\mathcal{C}$, and $\mathcal{G}$ for a coaxial cable by using the curvilinear squares technique.

6.22. By consulting an appropriate reference book, prepare a list of the expressions for $\mathcal{L}$, $\mathcal{C}$, and $\mathcal{G}$ for two or more transmission lines other than the parallel-plate and coaxial lines.

6.23. Discuss your understanding of the characteristic impedance of a transmission line. Why is it not equal to the intrinsic impedance of the medium between the conductors of the line?

6.24. What is the boundary condition to be satisfied at a short circuit on a line?

6.25. For an open-circuited line, what would be the boundary condition to be satisfied at the open circuit?

6.26. What is a standing wave? How do complete standing waves arise? Discuss their characteristics and give an example in mechanics.

6.27. What is a standing wave pattern? Discuss the voltage and current standing wave patterns for the short-circuited line.

6.28. What would be the voltage and current standing wave patterns for an open-circuited line?

6.29. Discuss the variation with frequency of the input reactance of a short-circuited line and its application in the determination of the location of a short circuit.

6.30. Can you suggest an alternative procedure to that described in Example 6.3 to locate a short circuit in a transmission line?

6.31. Under what condition do circuit elements behave as assumed by conventional (lumped) circuit theory?

6.32. State the boundary conditions for the electric field components at the interface between two dielectric media.

6.33. State the boundary conditions for the magnetic field components at the interface between two dielectric media.

6.34. Summarize in vector form the boundary conditions at the interface between two dielectric media, indicating correspondingly the Maxwell's equations in integral form from which they are derived.

6.35. What are the boundary conditions for the voltage and current at the junction between two transmission lines?

6.36. What is the voltage reflection coefficient at the junction between two transmission lines? How are the current reflection coefficient and the voltage and current transmission coefficients related to the voltage reflection coefficient?

6.37. What is the voltage reflection coefficient at the short circuit for a short-circuited line?

6.38. Can the transmitted wave current at the junction between two transmission lines be greater than the incident wave current? Explain.

6.39. What is a partial standing wave? Discuss the standing wave patterns corresponding to partial standing waves.

6.40. Define standing wave ratio (SWR). What are the standing wave ratios for (a) an infinitely long line, (b) a short-circuited line, (c) an open-circuited line, and (d) a line terminated by its characteristic impedance?

PROBLEMS

6.1. The plane $x + 2y + 3z = 5$ defines the surface of a perfect conductor. Find the possible direction(s) of the electric field intensity at a point on the conductor surface.

6.2. Given $\mathbf{E} = y\mathbf{i}_x + x\mathbf{i}_y$, determine if a perfect conductor can be placed in the surface $xy = 2$ without disturbing the field.

6.3. A perfect conductor occupies the region $x + 2y \leq 2$. Find the surface current density at a point on the conductor at which $\mathbf{H} = H_0\mathbf{i}_z$.

6.4. The displacement flux density at a point on the surface of a perfect conductor is given by $\mathbf{D} = D_0(\mathbf{i}_x + \sqrt{3}\,\mathbf{i}_y + 2\sqrt{3}\,\mathbf{i}_z)$. Find the magnitude of the surface charge density at that point.

6.5. It is known that at a point on the surface of a perfect conductor $\mathbf{D} = D_0(\mathbf{i}_x + 2\mathbf{i}_y + 2\mathbf{i}_z)$, $\mathbf{H} = H_0(2\mathbf{i}_x - 2\mathbf{i}_y + \mathbf{i}_z)$, and ρ_S is positive. Find ρ_S and $\mathbf{J}_S$ at that point.

6.6. Two infinite plane conducting sheets occupy the planes $x = 0$ and $x = 0.1$ m. An electric field given by

$$\mathbf{E} = E_0 \sin 10\pi x \cos 3\pi \times 10^9 t \, \mathbf{i}_z$$

where E_0 is a constant, exists in the region between the plates, which is free space. (a) Show that **E** satisfies the boundary condition on the sheets. (b) Obtain **H** associated with the given **E**. (c) Find the surface current densities on the two sheets.

6.7. A parallel-plate transmission line is made up of perfect conductors of width $w = 0.1$ m and lying in the planes $x = 0$ and $x = 0.02$ m. The medium between the conductors is a perfect dielectric of $\mu = \mu_0$. For a uniform plane wave having the electric field

$$\mathbf{E} = 100\pi \cos (2\pi \times 10^6 t - 0.02\pi z) \, \mathbf{i}_x \text{ V/m}$$

propagating between the conductors, find (a) the voltage between the conductors, (b) the current along the conductors, and (c) the power flow along the line.

6.8. A parallel-plate transmission line made up of perfect conductors has $\mathcal{L}$ equal to 10^{-7} H/m. If the medium between the plates is characterized by $\sigma = 10^{-11}$ mho/m, $\epsilon = 6\epsilon_0$, and $\mu = \mu_0$, find $\mathcal{C}$ and $\mathcal{G}$ of the line.

6.9. If the conductors of a transmission line are imperfect, then the transmission-line equivalent circuit contains a resistance and additional inductance in the series branch. Assuming that the thickness of the (imperfect) conductors of a parallel-plate line is several skin depths at the frequency of interest, show from considerations of skin effect phenomenon in a good conductor medium that the resistance and inductance per unit length along the conductors are $2/\sigma_c \delta w$ and $2/\omega\sigma_c\delta w$, respectively, where σ_c is the conductivity of the (imperfect) conductors, w is the width and δ is the skin depth. The factor 2 arises because of two conductors.

6.10. Show that for a transverse electromagnetic wave, the voltage between the conductors and the current along the conductors in a given transverse plane are uniquely defined in terms of the electric and magnetic fields, respectively, in that plane.

6.11. By constructing a field map consisting of curvilinear squares for a coaxial cable having $b/a = 3.5$, obtain the approximate values of the line parameters $\mathcal{L}$, $\mathcal{C}$, and $\mathcal{G}$ in terms of μ, ϵ, and σ of the dielectric. Compare the approximate values with the exact values given by expressions derived in Example 6.2.

6.12. Figure 6.25 shows the cross section of a parallel-wire line, that is, a line having two cylindrical conductors of radii a and with their axes separated by 2d. For $d/a = 2$, construct a field map consisting of curvilinear squares and obtain approximate values for the line parameters $\mathcal{L}$, $\mathcal{C}$, and $\mathcal{G}$. Compare the

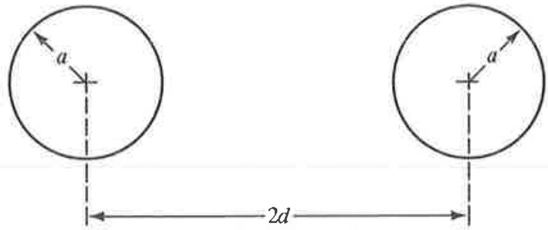

Figure 6.25. For Problem 6.12.

approximate values with the exact values given by expressions available from Sec. 10.6.

6.13. For a transmission line of arbitrary cross section and with the medium between the conductors characterized by $\sigma = 10^{-16}$ mho/m, $\epsilon = 2.5\epsilon_0$, and $\mu = \mu_0$, it is known that $\mathcal{C} = 10^{-10}$ F/m. (a) Find $\mathcal{L}$ and $\mathcal{G}$. (b) Find $\bar{Z}_0$ for $f = 10^6$ Hz.

6.14. For a coaxial cable employing air dielectric, find the ratio of the outer to the inner radii for which the characteristic impedance of the cable is 75 ohms.

6.15. Show that for the parallel-plate line, the characteristic impedance is d/w times the intrinsic impedance of the medium between the conductors of the line.

6.16. The strip line, employed in microwave integrated circuits, consists of a center conductor photoetched on the inner faces of two substrates sandwiched between two conductors, as shown by the cross-sectional view in Fig. 6.26. For the dimensions shown in the figure, construct a field map consisting of curvilinear squares and compute $\mathcal{L}$, $\mathcal{C}$, and Z_0, considering the substrate to be a perfect dielectric having $\epsilon = 9\epsilon_0$ and $\mu = \mu_0$. Assume for simplicity that the field is confined to the substrate region.

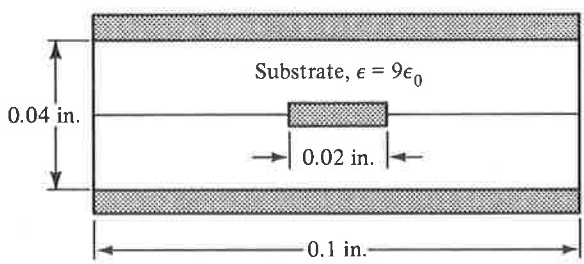

Figure 6.26. For Problem 6.16.

6.17. Consider a transmission-line equivalent circuit having impedance $\mathcal{Z}\,dz$ in the series branch and admittance $\mathcal{Y}\,dz$ in the shunt branch. (a) Write the transmission-line equations. (b) Show that $\bar{\gamma} = \sqrt{\mathcal{Z}\mathcal{Y}}$ and $\bar{Z}_0 = \sqrt{\mathcal{Z}/\mathcal{Y}}$. (c) If $\mathcal{Z}$ is the impedance of an inductor $\mathcal{L}_1$ and $\mathcal{Y}$ is the admittance of the parallel combination of an inductor $\mathcal{L}_2$ and a capacitor $\mathcal{C}$, find $\bar{\gamma}$ and discuss the propagation characteristics along the line.

6.18. Using the general solutions for the complex line voltage and current on a lossless line given by (6.50a) and (6.50b), respectively, obtain the particular solutions for the complex voltage and current on an open-circuited line. Then find the input impedance of an open-circuited line of length l.

6.19. Solve Example 6.3 by considering the standing wave patterns between the short circuit and the generator for the two frequencies of interest and by deducing the number of wavelengths at one of the two frequencies.

6.20. For an air dielectric short-circuited line of characteristic impedance 50 ohms, find the minimum values of the length for which its input impedance is equivalent to that of (a) an inductor of value 0.25×10^{-6} H at 100 MHz and (b) a capacitor of value 10^{-10} F at 100 MHz.

6.21. A transmission line of length 2 m having a nonmagnetic ($\mu = \mu_0$) perfect dielectric is short-circuited at the far end. A variable-frequency generator is connected at its input and the current drawn is monitored. It is found that the current reaches a maximum for $f = 500$ MHz and then a minimum for $f = 525$ MHz. Find the permittivity of the dielectric.

6.22. A voltage generator is connected to the input of a lossless line short circuited at the far end. The frequency of the generator is varied and the line voltage and line current at the input terminals are monitored. It is found that the voltage reaches a maximum value of 10 V at 405 MHz and the current reaches a maximum value of 0.2 amp at 410 MHz. (a) Find the characteristic impedance of the line. (b) Find the voltage and current values at 407 MHz.

6.23. Assuming that the criterion $f \ll v_p/2\pi l$ is satisfied for frequencies less than $0.1\, v_p/2\pi l$, compute the maximum length of an air dielectric short-circuited line for which the input impedance is approximately that of an inductor of value equal to the total inductance of the line for $f = 100$ MHz.

6.24. A lossless transmission line of length 2 m and having $\mathcal{L} = 0.5\mu_0$ and $\mathcal{C} = 18\epsilon_0$ is short circuited at the far end. (a) Find the phase velocity, v_p. (b) Find the wavelength, the length of the line in terms of the number of wavelengths, and the input impedance of the line for each of the following frequencies: 100 Hz; 100 MHz; and 12.5 MHz.

6.25. In Fig. 6.20, assume that medium 1 is characterized by $\epsilon = 12\epsilon_0$ and $\mu = 2\mu_0$ and that medium 2 is characterized by $\epsilon = 9\epsilon_0$ and $\mu = \mu_0$. If $\mathbf{E}_1 = E_0(3\mathbf{i}_x + 2\mathbf{i}_y - 6\mathbf{i}_z)$ and if $\mathbf{H}_1 = H_0(2\mathbf{i}_x - 3\mathbf{i}_y)$, find $\mathbf{E}_2$ and $\mathbf{H}_2$.

6.26. In Fig. 6.20, assume that medium 1 is characterized by $\epsilon = 4\epsilon_0$ and $\mu = 3\mu_0$ and that medium 2 is characterized by $\epsilon = 16\epsilon_0$ and $\mu = 9\mu_0$. If $\mathbf{D}_1 = D_0(\mathbf{i}_x - 2\mathbf{i}_y + \mathbf{i}_z)$ and if $\mathbf{B}_1 = B_0(\mathbf{i}_x + 2\mathbf{i}_y + 3\mathbf{i}_z)$, find $\mathbf{D}_2$ and $\mathbf{B}_2$.

6.27. Region 1 defined by $x + 2y < 2$ is free space and region 2 defined by $x + 2y > 2$ is a perfect dielectric medium having $\epsilon = 6\epsilon_0$ and $\mu = 2\mu_0$. Determine if the fields $\mathbf{E}_1 = E_0\mathbf{i}_y$ and $\mathbf{H}_1 = H_0\mathbf{i}_z$ and the fields $\mathbf{E}_2 = \frac{E_0}{3}(-\mathbf{i}_x + \mathbf{i}_y)$ and

$\mathbf{H}_2 = H_0\mathbf{i}_z$ at points 1 and 2, respectively, lying adjacent to and on either side of the boundary, satisfy the boundary conditions.

6.28. Repeat Example 6.6 with the values of Z_{01} and Z_{02} interchanged.

6.29. In the transmission-line system shown in Fig. 6.27, a power P_i is incident on the junction from line 1. Find (a) the power reflected into line 1, (b) the power transmitted into line 2, and (c) the power transmitted into line 3.

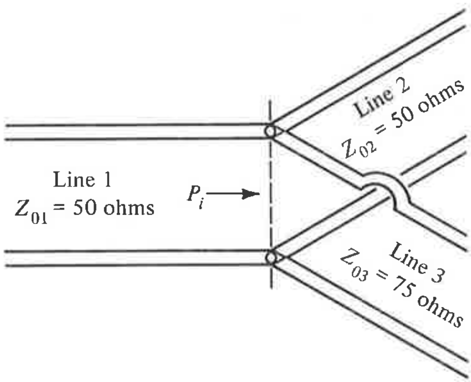

Figure 6.27. For Problem 6.29.

6.30. Show that the voltage minima of the standing wave pattern of Fig. 6.24 are sharper than the voltage maxima by computing the voltage amplitude halfway between the locations of voltage maxima and minima.

6.31. A line assumed to be infinitely long and of unknown characteristic impedance is connected to a line of characteristic impedance 50 ohms on which standing wave measurements are made. It is found that the standing wave ratio is 3 and that two consecutive voltage minima exist at 15 cm and 25 cm from the junction of the two lines. Find the unknown characteristic impedance.

6.32. A line assumed to be infinitely long and of unknown characteristic impedance when connected to a line of characteristic impedance 50 ohms produces a standing wave ratio of value 2 in the 50-ohm line. The same line when connected to a line of characteristic impedance 150 ohms produces a standing wave ratio of value 1.5 in the 150-ohm line. Find the unknown characteristic impedance.