5

MATERIALS AND FIELDS

In this chapter we extend our study of fields in free space of the preceding three chapters to fields in the presence of materials. Materials contain charged particles which act as sources of electromagnetic fields. Under the application of external fields, these charged particles respond, giving rise to secondary fields comparable to the applied fields. While the properties of materials that produce these effects are determined on the atomic or "microscopic" scale, it is possible to develop a consistent theory based on "macroscopic" scale observations, that is, observations averaged over volumes large compared with atomic dimensions. We will learn that these macroscopic scale phenomena are equivalent to charge and current distributions acting as though they were situated in free space, so that the secondary fields can be found by using the knowledge gained in the preceding chapters. In fact, we have an interesting situation in which the equivalent charge and current distributions are related to the total fields in the material comprising the applied and the secondary fields, whereas the secondary fields are related to the equivalent charge and current distributions. We are thus faced with the simultaneous solution of two sets of equations governing these two relationships. Following this logic, we will introduce new vector fields and develop a new set of Maxwell's equations with associated constitutive relations which eliminate the necessity for the simultaneous solution by taking into account implicitly the equivalent charge and current distributions.

5.1 Conduction and Nonmagnetic Materials

Depending upon their response to an applied electric field, materials may be classified as conductors, semiconductors, or dielectrics. According to the classical model, an atom consists of a tightly bound, positively charged nucleus surrounded by a diffuse electron cloud having an equal and opposite charge to the nucleus, as shown in Fig. 5.1. While the electrons for the most

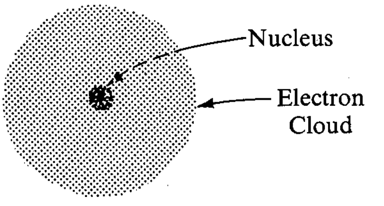


Fig. 5.1. Classical model of an atom.

part are less tightly bound, the majority of them are associated with the nucleus and are known as "bound" electrons. These bound electrons can be displaced but not removed from the influence of the nucleus upon application of an electric field. Not taking part in this bonding mechanism are the "free" or "conduction" electrons. These electrons are constantly under thermal agitation, being released from the parent atom at one point and recaptured at another point. In the absence of an applied electric field, their motion is completely random; that is, the average thermal velocity on a macroscopic scale is zero so that there is no net current and the electron cloud maintains a fixed position. When an electric field is applied, an additional velocity due to the Coulomb force is superimposed on the random velocities, thereby causing a "drift" of the average position of the electrons along the direction opposite to that of the electric field. This process is known as "conduction." In certain materials, a large number of electrons may take part in this process. These materials are known as "conductors." In certain other materials, only very few or a negligible number of electrons may participate in conduction. These materials are known as "dielectrics" or insulators. We will later learn that a characteristic called polarization is more important than conduction in dielectrics. A class of materials for which conduction occurs not only by electrons but also by another type of carriers known as "holes"—vacancies created by detachment of electrons due to breaking of covalent bonds with other atoms—is intermediate to that of conductors and dielectrics. These materials are called "semiconductors."

The quantum theory describes the motion of the current carriers in terms of energy levels. According to this theory, the electrons in an atom can have associated with them only certain discrete values of energy. When a large number of atoms are packed together, as in a crystalline solid, each energy level in the individual atom splits into a number of levels with slightly different energies, with the degree of splitting governed by the interatomic spacing, thereby giving rise to alternate allowed and forbidden bands of energy levels as shown in Fig. 5.2. Each allowed band can be thought of as an

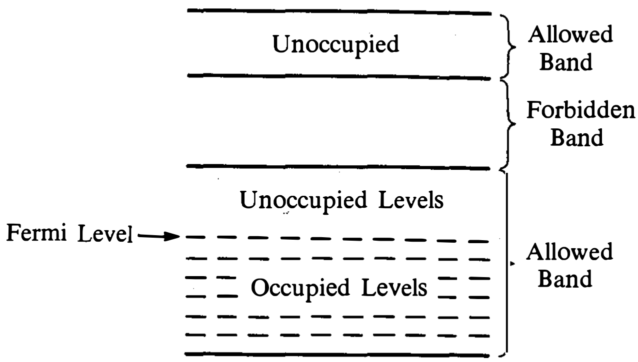


Fig. 5.2. Energy band structure for a crystalline solid.

almost continuous region of allowed energy levels. For example, for a typical solid having an atomic density of 10^{29} per m^3, there will be almost 10^{29} levels in each band. A forbidden band consists of energy levels which no electron in any atom of the solid can occupy. According to Pauli's exclusion principle, each allowed energy level may not be occupied by more than one electron. Electrons naturally tend to occupy the lowest energy levels; at a temperature of absolute zero, all the levels below a certain level known as the Fermi level are occupied and all the levels above the Fermi level are unoccupied. Hence, depending upon the location of the Fermi level, we can have different cases as shown in Fig. 5.3.

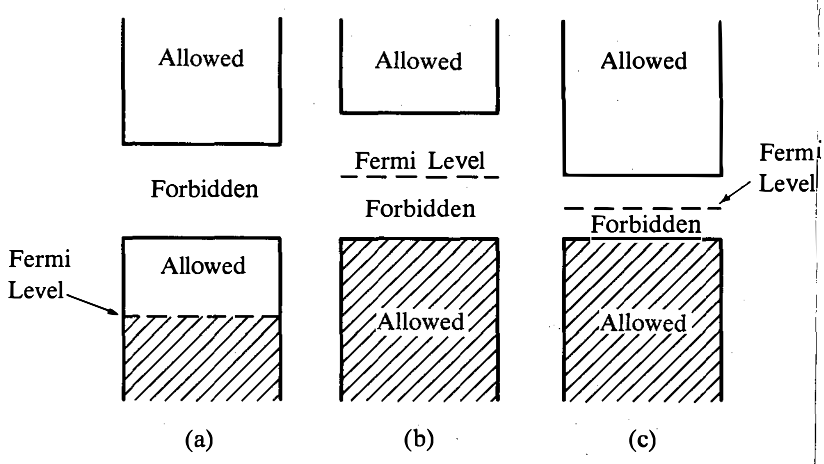


Fig. 5.3. Energy band diagrams for different cases: (a) Conductor. (b) Dielectric. (c) Semiconductor.

For case (a), the Fermi level lies within an allowed band. The band is therefore only partially filled at the temperature of absolute zero. At higher temperatures, the electron population in the band spreads out somewhat but only very few electrons reach above the Fermi level. Thus, since there are many unfilled levels in the same band, it is possible to increase the energy of the system by moving the electrons to these unoccupied levels very easily by the application of an electric field, thereby resulting in a drift velocity of the electrons in the direction opposite to that of the electric field. The material is then classified as a conductor. If the Fermi level is between two allowed bands as in (b) and (c) of Fig. 5.3, the lower band is completely filled whereas the next higher band is completely empty at the temperature of absolute zero. If the width of the forbidden band is very large as in (b), the situation at normal temperatures is essentially the same as at absolute zero and hence there are no neighboring empty energy levels for the electrons to move. The only way for conduction to take place is for the electrons in the filled band to get excited and move to the next higher band. But this is very difficult to achieve with reasonable electric fields and the material is then classified as a dielectric. Only by supplying a very large amount of energy can an electron be excited to move from the lower band to the higher band where it has available neighboring empty levels for causing conduction. The dielectric is said to break down under such conditions. If, on the other hand, the width of the forbidden band in which the Fermi level lies is not too large, as in (c), some of the electrons in the lower band move into the upper band at normal temperatures so that conduction can take place under the influence of an electric field, not only in the upper band but also in the lower band because of the vacancies (holes) left by the electrons which moved into the upper band. The material is then classified as a semiconductor. A semiconductor crystal in pure form is known as an intrinsic semiconductor. It is possible to alter the properties of an intrinsic crystal by introducing impurities into it. The crystal is then said to be an extrinsic semiconductor.

5.2 Conduction Current Density, Conductivity, and Ohm's Law

In Section 5.1 we classified materials on the basis of their ability to permit conduction of electrons under the application of an external electric field. For conductors, we are interested in knowing about the relationship between the "drift velocity" of the electrons and the applied electric field, since the predominant process is conduction. But for collisions with the atomic lattice, the electric field continuously accelerates the electrons in the direction opposite to it as they move about at random. Collisions with the atomic lattice, however, provide the frictional mechanism by means of which the electrons lose some of the momentum gained between collisions. The net effect is as though the electrons drift with an average drift velocity $\mathbf{v}_d$, under the influence of the Coulomb force exerted by the applied electric field and an opposing force due to the frictional mechanism. This opposing force is proportional to the momentum of the electron and inversely proportional to the average time τ between collisions. Thus the equation of motion of an electron is given by

$$m\frac{d\mathbf{v}_d}{dt} = e\mathbf{E} - \frac{m\mathbf{v}_d}{\tau} \tag{5-1}$$

where e and m are the charge and mass of an electron.

Rearranging (5-1), we have

$$m\frac{d\mathbf{v}_d}{dt} + \frac{m}{\tau}\mathbf{v}_d = e\mathbf{E} \tag{5-2}$$

For the sudden application of a constant electric field $\mathbf{E}_0$ at $t = 0$, the solution for (5-2) is given by

$$\mathbf{v}_d = \frac{e\tau}{m}\mathbf{E}_0 - \frac{e\tau}{m}\mathbf{E}_0 \exp(-t/\tau) \tag{5-3}$$

where we have evaluated the arbitrary constant of integration by using the initial condition that $\mathbf{v}_d = 0$ at $t = 0$. The values of τ for typical conductors such as copper are of the order of 10^{-14} sec so that the exponential term on the right side of (5-3) decays to negligible value in a time much shorter than that of practical interest. Thus, neglecting this term, we have

$$\mathbf{v}_d = \frac{e\tau}{m}\mathbf{E}_0 \tag{5-4}$$

and the drift velocity is proportional in magnitude and opposite in direction to the applied electric field since the value of e is negative.

In fact, since we can represent a time-varying field as a superposition of step functions starting at appropriate times, the exponential term in (5-3) may be neglected as long as the electric field varies slowly compared to τ. For fields varying sinusoidally with time, this means that as long as the period T of the sinusoidal variation is several times the value of τ, or the radian frequency $\omega \ll 2\pi/\tau$, the drift velocity follows the variations in the electric field. Since $1/\tau \approx 10^{14}$, this condition is satisfied even at frequencies up to several hundred gigahertz. Thus, for all practical purposes, we can assume that

$$\mathbf{v}_d = \frac{e\tau}{m}\mathbf{E} \tag{5-5}$$

Now, we define the "mobility," μ_e of the electron as the ratio of the magnitudes of the drift velocity and the applied electric field. Then we have

$$\mu_e = \frac{|\mathbf{v}_d|}{|\mathbf{E}|} = \frac{|e|\tau}{m} \tag{5-6}$$

and

$$\mathbf{v}_d = -\mu_e\mathbf{E} \qquad \text{for electrons} \tag{5-7a}$$

For values of τ typically of the order of 10^{-14} sec, we note by substituting for $|e|$ and m on the right side of (5-6) that the electron mobilities are of the order of 10^{-3} C-sec/kg. Alternative units for the mobility are square meters per volt-second. In semiconductors, conduction is due not only to the movement of electrons but also to the movement of holes. We can define the mobility μ_h of a hole similarly to μ_e as the ratio of the drift velocity of the hole to the applied electric field. Thus we have

$$\mathbf{v}_d = \mu_h \mathbf{E} \qquad \text{for holes} \tag{5-7b}$$

Note from (5-7b) that conduction of a hole takes place along the direction of the applied electric field since a hole is a vacancy created by the removal of an electron and hence a hole movement is equivalent to the movement of a positive charge of value equal to the magnitude of the charge of an electron. In general, the mobility of holes is lower than the mobility of electrons for a particular semiconductor. For example, for silicon, the values of μ_e and μ_h are

$$\mu_e = 0.125 \text{ m}^2/\text{volt-sec} \qquad \mu_h = 0.048 \text{ m}^2/\text{volt-sec}$$

The drift of electrons in a conductor and that of electrons and holes in a semiconductor is equivalent to a current flow. This current is known as the conduction current, in contrast to the convection current produced by the motion of charges in free space. The conduction current density may be obtained in the following manner. If there are N_e free electrons per cubic meter of the material, then the amount of charge ΔQ passing through an infinitesimal area ΔS at a point in the material in a time Δt is given by

$$\begin{aligned}\Delta Q &= N_e e(\Delta \mathbf{S} \cdot \mathbf{v}_d\, \Delta t) \\ &= N_e e(\Delta S\, \mathbf{i}_n \cdot \mathbf{v}_d)\, \Delta t\end{aligned} \tag{5-8}$$

where $\Delta \mathbf{S} = \Delta S\, \mathbf{i}_n$. The current ΔI flowing across ΔS is given by

$$\Delta I = \frac{\Delta Q}{\Delta t} = N_e e\, \Delta S\, \mathbf{i}_n \cdot \mathbf{v}_d \tag{5-9}$$

The magnitude of the current density at the point is the ratio of ΔI to ΔS for an orientation of ΔS which maximizes this ratio and as ΔS tends to zero. Obviously, the ratio is a maximum for an orientation of ΔS normal to $\mathbf{v}_d$ and is equal to $N_e|e|v_d$. Thus the conduction current density $\mathbf{J}_c$ resulting from the drift of electrons in the conductor is given by

$$\mathbf{J}_c = N_e e \mathbf{v}_d \tag{5-10}$$

Substituting for $\mathbf{v}_d$ from (5-7a), we have

$$\mathbf{J}_c = -\mu_e N_e e \mathbf{E} \tag{5-11}$$

Defining a quantity σ as

$$\sigma = -\mu_e N_e e = \mu_e N_e |e| \tag{5-12}$$

we obtain the simple and important relationship between $\mathbf{J}_c$ and $\mathbf{E}$

$$\mathbf{J}_c = \sigma \mathbf{E} \tag{5-13}$$

The quantity σ is known as the electrical conductivity of the material and Eq. (5-13) is known as Ohm's law valid at a point. Equation (5-13) indicates that $\mathbf{J}_c$ is proportional to $\mathbf{E}$. Materials for which this relationship holds, that is, σ is independent of the magnitude as well as the direction of $\mathbf{E}$ are known as linear isotropic conductors. For certain conductors, each component of $\mathbf{J}_c$ can be dependent on all components of $\mathbf{E}$. In such cases, $\mathbf{J}_c$ is not parallel to $\mathbf{E}$ and the conductors are not isotropic. Such conductors are known as anisotropic conductors.

In a semiconductor we have two types of current carriers: electrons and holes. Accordingly, the current density in a semiconductor is the sum of the contributions due to the drifts of electrons and holes. If the densities of holes and electrons are N_h and N_e, respectively, the conduction current density is given by

$$\mathbf{J}_c = (\mu_h N_h|e| + \mu_e N_e|e|)\mathbf{E} \tag{5-14}$$

Thus the conductivity of a semiconducting material is given by

$$\sigma = \mu_h N_h|e| + \mu_e N_e|e| \tag{5-15a}$$

For an intrinsic semiconductor, $N_h = N_e$ so that (5-15a) reduces to

$$\sigma = (\mu_h + \mu_e)N_e|e| \tag{5-15b}$$

The units of conductivity are (meter²/volt-second)(coulomb/meter³) or ampere/volt-meter, also commonly known as mhos per meter, where a mho ("ohm" spelled in reverse and having the symbol ℧) is an ampere per volt. The ranges of conductivities for conductors, semiconductors, and dielectrics are shown in Fig. 5.4. Values of conductivities for a few materials are listed in Table 5.1. The constant values of conductivities do not imply that the conduction current density is proportional to the applied electric

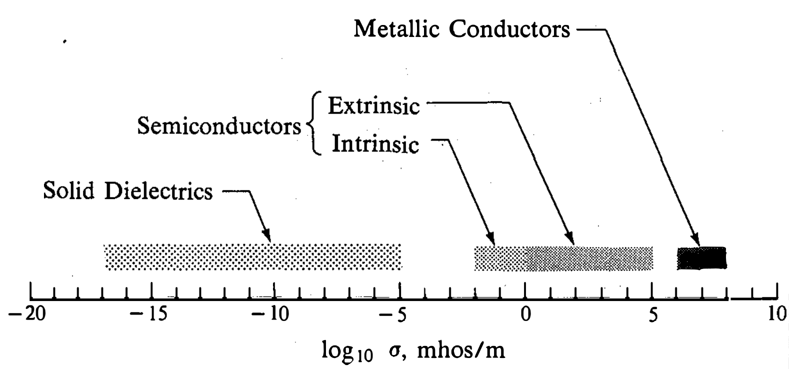


Fig. 5.4. Ranges of conductivities for conductors, semiconductors, and dielectrics.

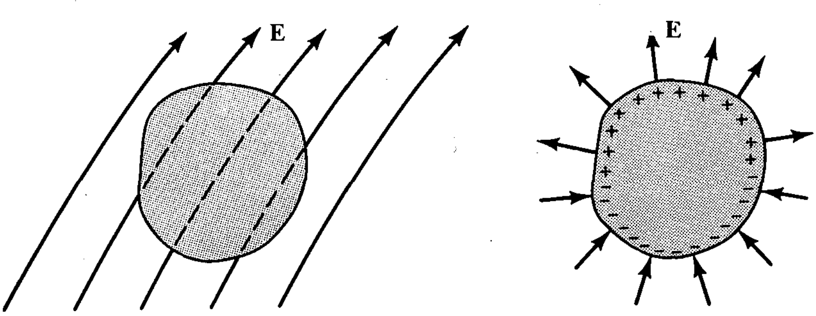
TABLE 5.1. Conductivities of Some Materials

Material	*Conductivity, mhos/m*	*Material*	*Conductivity, mhos/m*
Silver	6.1×10^7	Sea water	4
Copper	5.8×10^7	Intrinsic germanium	2.2
Gold	4.1×10^7	Intrinsic silicon	1.6×10^{-3}
Aluminum	3.5×10^7	Fresh water	10^{-3}
Tungsten	1.8×10^7	Distilled water	2×10^{-4}
Brass	1.5×10^7	Dry earth	10^{-5}
Nickel	1.3×10^7	Wood	10^{-8}–10^{-11}
Solder	7.0×10^6	Bakelite	10^{-9}
Lead	4.8×10^6	Glass	10^{-10}–10^{-14}
Constantin	2.0×10^6	Porcelain	2×10^{-13}
Mercury	1.0×10^6	Mica	10^{-11}–10^{-15}
Nichrome	8.9×10^5	Fused quartz	0.4×10^{-17}

field intensity for all values of current density and field intensity. However, the range of current densities for which the material is linear, that is, for which the conductivity is a constant, is very large for conductors.

5.3 Conductors in Electric Fields

In Sections 5.1 and 5.2 we learned that the free electrons in a conductor drift under the influence of an electric field. Let us now consider an arbitrary-shaped conductor of uniform conductivity σ placed in a static electric field as shown in Fig. 5.5(a). The free electrons in the conductor move opposite to the direction lines of the electric field. If there is a way by means of which the flow of electrons can be continued to form a closed circuit, then a continuous flow of current takes place. In this section we will consider the conductor to be bounded by free space, in which case the electrons are held



(a) (b)

Fig. 5.5. For illustrating the surface charge formation at the boundary of a conductor placed in an electric field.

at the boundary from moving further by the atomic forces within the conductor and by the insulating property of free space. Thus a negative surface charge forms on that part of the boundary through which the electric field lines enter the conductor originally, as shown in Fig. 5.5(b). Now, since the conductor as a whole is neutral, an amount of positive charge equal in magnitude to the negative surface charge must exist somewhere in the conductor. Where in the conductor may this charge or, for that matter, any charge placed inside the conductor reside? We will answer this question in the following example.

EXAMPLE 5-1. Assume that, at $t = 0$, a charge distribution of density ρ_0 is created in a portion of a conductor of uniform conductivity σ. In the remaining portion of the conductor, the charge density is zero. It is desired to show that the charge density in the conductor decays exponentially to zero and appears as a surface charge at the boundary of the conductor.

Denoting the charge density and the electric field intensity at any time t in the interior of the conductor to be ρ and $\mathbf{E}$, respectively, we have, from Maxwell's divergence equation for the electric field,

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{2-82}$$

The time variation of charge density is governed by the continuity equation

$$\nabla \cdot \mathbf{J}_c + \frac{\partial \rho}{\partial t} = 0 \tag{5-16}$$

where $\mathbf{J}_c$ is the conduction current density due to the flow of charges in the conductor under the influence of $\mathbf{E}$. Equation (5-16) stated in integral form tells us that the total current leaving a volume of the conducting material is equal to the time rate of decrease of charge inside that volume. Substituting $\mathbf{J}_c = \sigma\mathbf{E}$ in (5-16), we have

$$\nabla \cdot \sigma\mathbf{E} + \frac{\partial \rho}{\partial t} = 0 \tag{5-17}$$

Since σ is uniform, we can take it outside the divergence operation in (5-17) to obtain

$$\sigma\nabla \cdot \mathbf{E} + \frac{\partial \rho}{\partial t} = 0 \tag{5-18}$$

Now, combining (5-18) and (2-82), we obtain a differential equation for ρ as given by

$$\frac{\partial \rho}{\partial t} + \frac{\sigma}{\epsilon_0}\rho = 0 \tag{5-19}$$

The solution to (5-19) is obtained by rearranging it and integrating as follows:

$$\int \frac{d\rho}{\rho} = -\int \frac{\sigma}{\epsilon_0}\,dt$$
$$\ln \rho = -\frac{\sigma}{\epsilon_0}t + \ln A \tag{5-20}$$

where $\ln A$ is the arbitrary constant of integration. Substituting the initial condition $\rho = \rho_0$ at $t = 0$ in (5-20) and rearranging, we obtain finally

$$\rho = \rho_0\, e^{-(\sigma/\epsilon_0)t} = \rho_0\, e^{-t/T} \tag{5-21}$$

where we define

$$T = \frac{\epsilon_0}{\sigma} \tag{5-22}$$

Thus the charge density inside the conductor decays exponentially with a time constant equal to ϵ_0/σ. In particular, if the charge density at any point is initially zero, it remains at zero. Hence no portion of the charge which decays in one region within the conductor can reappear in any other region within the conductor. On the other hand, the charge must be conserved. Thus the decaying charge can appear only as a surface charge at the boundary of the conductor. To see how fast the charge density at an interior point decays and appears simultaneously as a surface charge, let us consider the example of copper. For copper, $\sigma = 5.80 \times 10^7$ mhos/m so that

$$T = \frac{\epsilon_0}{\sigma} = \frac{10^{-9}}{36\pi \times 5.80 \times 10^7} = 1.5 \times 10^{-19}\text{ sec}$$

Thus, in a time equal to 1.5×10^{-19} sec, the charge density decays to e^{-1} times or about 37% of its initial value. We note that this time constant is extremely short so that we can assume that any charge density in the interior of a conductor disappears to the surface almost instantaneously. (Furthermore, we can assume that the surface charge formation follows any time variation in the electric field causing it so long as this time variation is slow compared to the time constant.) On the other hand, the time constant can be up to several days for dielectric materials. ▮

Returning now to the case of Fig. 5.5, we conclude that the positive charge equal in magnitude to the negative surface charge appears as a surface charge on that part of the boundary through which the electric field lines leave the conductor originally, as shown in Fig. 5.5(b). The surface charge distribution formed in this manner produces a secondary electric field which opposes the applied field inside the conductor. The secondary field should, in fact, cancel the applied field inside the conductor completely. If it does not, there will be further movement of charges to the surface until a distribution is achieved which produces a secondary field inside the conductor that cancels the applied field completely. All this adjustment should be governed by the time constant so that we can assume that a surface charge distribution which reduces the field inside the conductor to zero is formed almost instantaneously. The surface charge distribution will, in general, produce a secondary field outside the conductor which modifies the applied field.

Let us now investigate the properties of the electric field at the surface of a conductor. To do this, let us assume that the electric field intensity $\mathbf{E}$ on the free-space side of the boundary has a component E_t tangential to the boundary and a component E_n normal to the boundary. The electric field intensity inside the conductor is, of course, equal to zero. We now consider a rectangular path *abcda* of infinitesimal area in the plane normal to the boundary and with its sides *bc* and *ad* parallel to E_t and on either side of the boundary as shown in Fig. 5.6(a). Since the sides of the rectangle

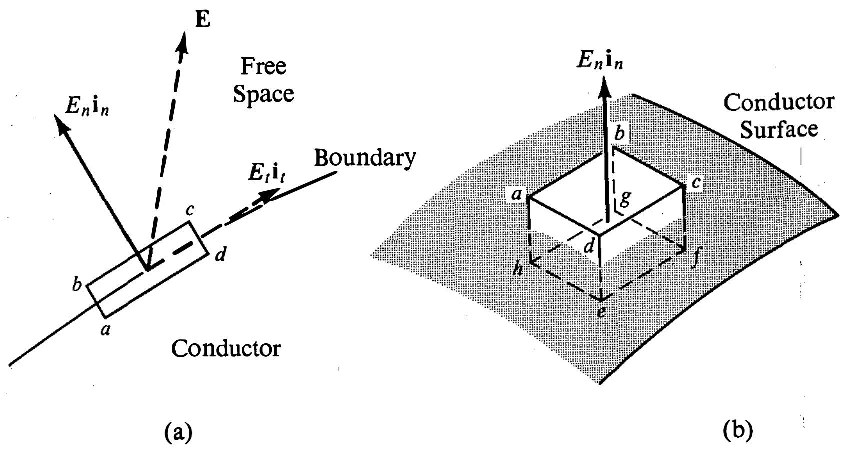


Fig. 5.6. For investigating the properties of the electric field intensity vector at the surface of a conductor.

are infinitesimally small, we can assume that E_t and E_n are constants along them. Applying $\oint \mathbf{E} \cdot d\mathbf{l} = 0$ to the path *abcda*, we have

$$\int_a^b \mathbf{E} \cdot d\mathbf{l} + \int_b^c \mathbf{E} \cdot d\mathbf{l} + \int_c^d \mathbf{E} \cdot d\mathbf{l} + \int_d^a \mathbf{E} \cdot d\mathbf{l} = 0 \qquad (5\text{-}23)$$

The second integral in (5-23) is equal to $E_t(bc)$ and the fourth integral is zero. Now, if we let *ab* and *cd* tend to zero, shrinking the rectangle to the surface but still enclosing it, the first and third integrals in (5-23) go to zero, giving us

$$E_t(bc) = 0$$

or

$$E_t = 0 \qquad (5\text{-}24)$$

Thus the tangential component of the electric field intensity at the boundary of a conductor placed in an electric field is zero. The electric field at the boundary is entirely normal to the surface. Note that we have not considered any time-varying magnetic flux enclosed by the rectangular path *abcda* since we are using static field laws. However, even if we do consider the time-varying magnetic flux, it will go to zero as *abcda* is shrunk to the surface, yielding the same result as (5-24).

We now suspect that the normal electric field at the boundary is related to the surface charge density. To investigate this, let us consider a rectangular box *abcdefgh* of infinitesimal volume enclosing an infinitesimal area of the boundary and parallel to it as shown in Fig. 5.6(b). Applying Gauss' law in integral form given by

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\epsilon_0}(\text{charge enclosed by } S)$$

to the surface area of the box, we have

$$\int_{\substack{\text{top}\\\text{surface}\\abcd}} \mathbf{E} \cdot d\mathbf{S} + \int_{\substack{\text{bottom}\\\text{surface}\\efgh}} \mathbf{E} \cdot d\mathbf{S} + \int_{\substack{\text{side}\\\text{surfaces}}} \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\epsilon_0}\begin{pmatrix}\text{charge enclosed in the}\\\text{volume of the box}\end{pmatrix} \tag{5-25}$$

The second integral in (5-25) is zero since **E** is zero inside the conductor. Since the area *abcd* is infinitesimal, we assume **E** to be constant on it so that the first integral is equal to $E_n(abcd)$. Now, if we let the side surfaces tend to zero, shrinking the box to the surface but still enclosing it, the third integral goes to zero and the charge enclosed by the box tends to the surface charge density ρ_s times the area *abcd*, giving us

$$E_n(abcd) = \frac{1}{\epsilon_0}\rho_s(abcd)$$

or

$$E_n = \frac{\rho_s}{\epsilon_0} \tag{5-26}$$

Thus the electric field intensity at a point on the surface of a conductor placed in an electric field is entirely normal to the surface and equal to $1/\epsilon_0$ times the surface charge density at that point.

Finally, since the electric field on the conductor surface is entirely normal to it, we note that no work is required to move an imaginary test charge on the conductor surface or, for that matter, inside the conductor (since $\mathbf{E} = 0$). Thus the conductor surface as well as the interior of the conductor are equipotentials. We now summarize the properties associated with conductors in electric fields as follows:

(a) The charge density at any point in the interior of a conductor is zero. Any charge must reside on the surface only with an appropriate density to produce a secondary electric field inside the conductor which is exactly opposite to the applied electric field so that property (b) below is satisfied.

(b) The electric field intensity inside the conductor is zero.

(c) The electric field intensity at any point on the surface of the conductor is entirely normal to it and equal to $1/\epsilon_0$ times the surface charge density at that point.

(d) The conductor, including its surface, is an equipotential region.

EXAMPLE 5-2. An infinite plane conducting slab of thickness d occupies the region between $z = 0$ and $z = d$ as shown in Fig. 5.7(a). A uniform electric field $\mathbf{E} = E_0\mathbf{i}_z$, where E_0 is a constant is applied. It is desired to find the charge densities induced on the surfaces of the slab.

Since the applied electric field is uniform and is directed along the z direction, a negative charge of uniform density forms on the surface $z = 0$ due to the accumulation of free electrons at that surface. A positive charge

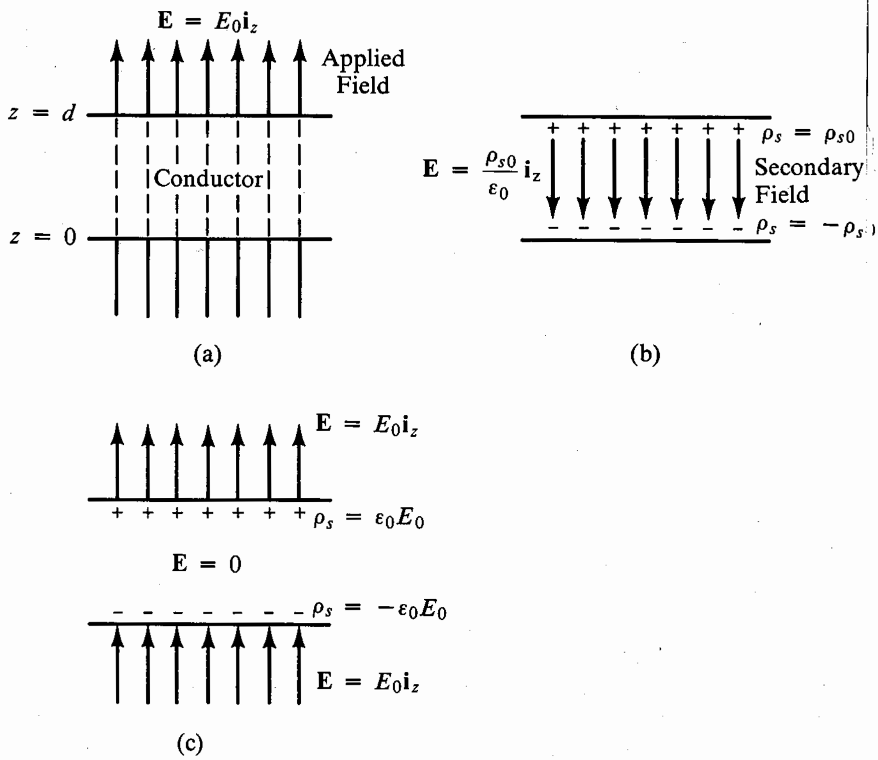


Fig. 5.7. (a) Infinite plane slab conductor in a uniform applied field. (b) Induced surface charge at the boundaries of the conductor and the secondary field. (c) Sum of the applied and the secondary fields.

of equal and opposite uniform density forms on the surface $z = d$ due to a deficiency of electrons at that surface. Let these surface charge densities be $-\rho_{s0}$ and ρ_{s0}, respectively. To satisfy the property that the field in the interior of the conductor is zero, the secondary field produced by the surface charges must be equal and opposite to the applied field; that is, it must be equal to $-E_0\mathbf{i}_z$. Now, each sheet of uniform charge density produces a field intensity directed normally away from it and having a magnitude $1/2\epsilon_0$ times the charge density so that the field due to the two surface charges together is equal to $-(\rho_{s0}/\epsilon_0)\mathbf{i}_z$ inside the conductor and zero outside the conductor as shown in Fig. 5.7(b). Thus, for zero field inside the conductor,

$$-\frac{\rho_{s0}}{\epsilon_0}\mathbf{i}_z = -E_0\mathbf{i}_z$$

or

$$\rho_{s0} = \epsilon_0 E_0 \tag{5-27}$$

The field outside the conductor remains the same as the applied field since the secondary field in that region due to the surface charges is zero. The induced surface charge distribution and the fields inside and outside the conductor are shown in Fig. 5.7(c). Note that the property that the field intensity at a point on the surface of the conductor is normal to it and equal to $1/\epsilon_0$ times the surface charge density at that point is satisfied on both surfaces $z = 0$ and $z = d$. ▮

5.4 Polarization in Dielectric Materials

We stated at the beginning of Section 5.1 that the bound electrons in an atom can be displaced but not removed from the influence of the parent nucleus upon application of an external electric field. When the centroids of the electron clouds surrounding the nucleii are displaced from the centroids of the nucleii, as shown in Fig. 5.8(a), to create a charge separation and hence form microscopic electric dipoles, the atoms are said to be "polarized." The schematic representation of an electric dipole formed in this manner is shown in Fig. 5.8(b). Such "polarization" may exist in the molecular structure of certain dielectric materials even under the application of no external electric field. The molecules are then said to be polar molecules. However, the polarization of individual atoms and molecules is randomly oriented and hence the material is not polarized on a macroscopic scale. In certain other dielectric materials, no polarization exists initially in the molecular structure. The molecules are then said to be nonpolar molecules.

Upon the application of an external electric field, the centroids of the electron clouds in the nonpolar molecules may become displaced from the centroids of the nucleii due to the Coulomb forces acting on the charges. This kind of polarization is known as electronic polarization. In the case of polar molecules, the electric field has the influence of exerting torques on

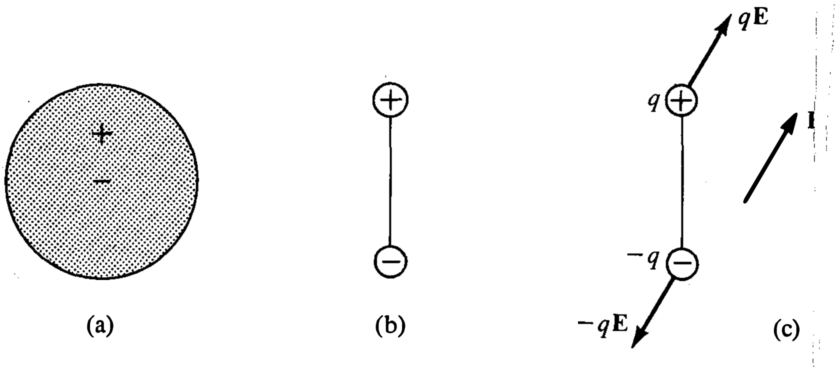


Fig. 5.8. (a) Polarization of bound charge in an atom under the influence of an electric field. (b) Schematic representation of electric dipole created due to polarization. (c) Torque acting on an electric dipole under the influence of an electric field.

the microscopic dipoles as shown in Fig. 5.8(c), to convert the initially random polarization into a partially coherent one along the field, on a macroscopic scale. This kind of polarization is known as orientational polarization. Certain materials, called "electrets," when allowed to solidify in the applied electric field, become permanently polarized in the direction of the field, that is, retain the polarization even after removal of the field. Certain other materials, known as "ferroelectric" materials, exhibit spontaneous, permanent polarization. A third kind of polarization, known as ionic polarization, results from the separation of positive and negative ions in molecules held together by ionic bonds formed by the transfer of electrons from one atom to another in the molecule. All three polarizations may occur simultaneously in a material.

The net dipole moment created due to polarization in a dielectric material will produce a field which opposes the applied electric field and changes its distribution both inside and outside the dielectric material, in general, from the one that existed in the absence of the material. This will be the topic of discussion in Section 5.5. In the remainder of this section, we will first derive the relationship between the dipole moments of the individual microscopic dipoles and the electric field responsible for the polarization by considering electronic polarization by means of an example. We will then define a new vector **P** which represents polarization on a macroscopic scale and relate it to the average macroscopic electric field.

EXAMPLE 5-3. Assume that the nucleus of an atom is a point charge and that the electron cloud has originally a spherically symmetric, radially uniform charge distribution which is retained as it is displaced relative to the nucleus under the influence of a polarizing electric field. (This assumption is justified if the displacement between the centroids of the electron cloud and the nucleus is negligible compared to the radius of the electron cloud.) It is desired to find the dipole moment resulting from the polarizing field.

Let the electric field causing the displacement between the two centroids be $\mathbf{E}_p = E_0\mathbf{i}_z$, so that the displacement is along the z axis as shown in Fig. 5.9. Let this displacement be equal to d. The two forces which are acting on the nucleus are (a) the Coulomb force $\mathbf{F}_1$ due to the electric field $\mathbf{E}_p$ and (b) the restoring force $\mathbf{F}_2$ due to the electric field produced at the nucleus by the electron cloud.

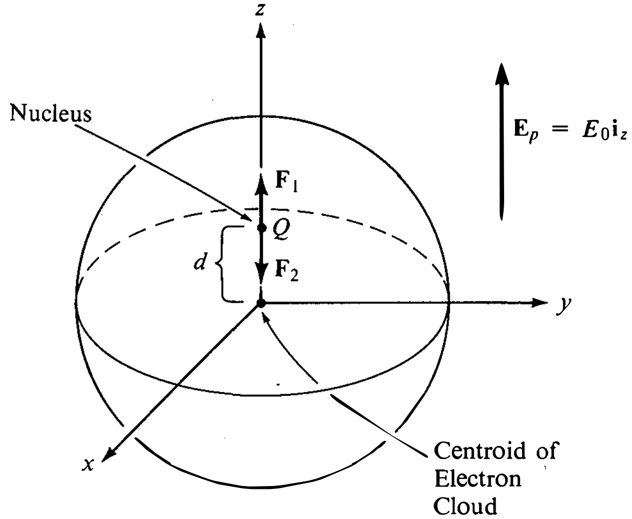


Fig. 5.9 For obtaining the dipole moment due to electronic polarization of an atom.

The force $\mathbf{F}_1$ is given by

$$\mathbf{F}_1 = Q\mathbf{E}_p = QE_0\mathbf{i}_z \tag{5-28}$$

where Q is the charge of the nucleus. To find the restoring force $\mathbf{F}_2$, we take advantage of the spherical symmetry of the charge distribution in the electron cloud about its center and apply Gauss' law to a sphere of radius d centered at the origin to obtain the electric field $\mathbf{E}_2$ at the nucleus due to the electron cloud as

$$\mathbf{E}_2 = \frac{1}{\epsilon_0}\frac{\text{charge enclosed by spherical surface of radius } d}{\text{area of the spherical surface}}\mathbf{i}_z \tag{5-29}$$

Now, since the total charge in the electron cloud is $-Q$ and since the charge density is uniform, the charge enclosed by the spherical surface of radius

d is $-Qd^3/a^3$, where a is the radius of the electron cloud. Thus we obtain

$$\mathbf{E}_2 = \frac{-Qd^3/a^3}{4\pi\epsilon_0 d^2}\mathbf{i}_z = -\frac{Qd}{4\pi\epsilon_0 a^3}\mathbf{i}_z \tag{5-30}$$

Hence the restoring force on the nucleus is given by

$$\mathbf{F}_2 = Q\mathbf{E}_2 = -\frac{Q^2 d}{4\pi\epsilon_0 a^3}\mathbf{i}_z \tag{5-31}$$

For equilibrium displacement d of the nucleus relative to the center of the electron cloud, the two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ must add to zero, giving us

$$d = \frac{4\pi\epsilon_0 a^3}{Q} E_0 \tag{5-32}$$

Thus the equilibrium displacement d is proportional to the electric field intensity E_0. The dipole moment $\mathbf{p}_e$ formed by the charge separation is then given by

$$\mathbf{p}_e = Qd\mathbf{i}_z = Q\frac{4\pi\epsilon_0 a^3}{Q}E_0\mathbf{i}_z = 4\pi\epsilon_0 a^3\mathbf{E}_p \tag{5-33}$$

Equation (5-33) indicates that the dipole moment $\mathbf{p}_e$ is proportional to the field $\mathbf{E}_p$ causing it. Defining a proportionality constant α_e as

$$\alpha_e = 4\pi\epsilon_0 a^3 \tag{5-34}$$

we have

$$\mathbf{p}_e = \alpha_e\mathbf{E}_p \tag{5-35}$$

The proportionality constant α_e is known as the "electronic polarizability" of the atom. ∎

It is found that the dipole moments due to orientational and ionic polarizations are also proportional to the polarizing field $\mathbf{E}_p$. The average dipole moment $\mathbf{p}$ per molecule is then given by

$$\mathbf{p} = \alpha\mathbf{E}_p \tag{5-36}$$

where α is known as the molecular polarizability. Let us now consider a small volume Δv of the dielectric material. If N denotes the number of molecules per unit volume of the material, then there are $N\,\Delta v$ molecules in the volume Δv. We define a vector $\mathbf{P}$, called the "polarization vector," as

$$\mathbf{P} = \frac{1}{\Delta v}\sum_{j=1}^{N\,\Delta v}\mathbf{p}_j = N\mathbf{p} \tag{5-37}$$

which has the meaning of "dipole moment per unit volume" or the "dipole moment density" in the material. Substituting (5-36) into (5-37), we have

$$\mathbf{P} = N\alpha\mathbf{E}_p \tag{5-38}$$

The units of $\mathbf{P}$ are coulombs per square meter.

The field $\mathbf{E}_p$ in (5-36) and hence in (5-38) is the average electric field acting to polarize the individual molecule and is generally called the polarizing field or the local field. It is the average field that would exist in an imaginary cavity created by removing the molecule in question, keeping all the other molecules polarized in their locations. It is not the same as the average macroscopic field **E** at the molecule with all the molecules including the one in question remaining polarized in their locations. It is equal to the field **E** minus the average field produced by the dipole in the imaginary cavity. We have to find this average field to express $\mathbf{E}_p$ in terms of **E** so that **P** can be related to **E**. To determine this field rigorously, we need detailed information about the shape and charge distribution of the molecule. However, we will consider a simple special case of a spherical cavity and obtain the required field in the following example.

EXAMPLE 5-4. Two equal and opposite point charges Q and $-Q$ are situated at $(0, 0, d/2)$ and $(0, 0 - d/2)$, respectively, in cartesian coordinates as shown in Fig. 5.10, forming a dipole of moment $\mathbf{p} = Qd\mathbf{i}_z$. Obtain the average electric field intensity due to the dipole in a spherical volume of radius $a > d/2$ and centered at the origin.

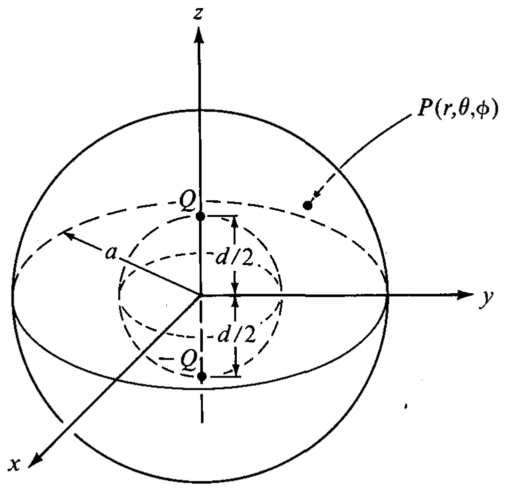


Fig. 5.10. For obtaining the average electric field intensity due to an electric dipole in a spherical volume.

Let us consider the fields due to the positive and negative point charges independently. Considering first the positive charge Q located at $(0, 0, d/2)$, we note that its electric field at an arbitrary point $P(r, \theta, \phi)$ is given by

$$\mathbf{E}_+ = \frac{Q}{4\pi\epsilon_0}\frac{1}{(r^2 + d^2/4 - rd\cos\theta)}\mathbf{i}_{QP} \tag{5-39}$$

where $\mathbf{i}_{QP}$ is the unit vector along the line from the point charge Q to the point P. The volume integral of this field evaluated in the spherical volume V of radius a is given by

$$\int_V \mathbf{E}_+ \, dv = -Q\left[\int_V \frac{1}{4\pi\epsilon_0(r^2 + d^2/4 - rd\cos\theta)} \mathbf{i}_{PQ} \, dv\right] \tag{5-40}$$

where $\mathbf{i}_{PQ} = -\mathbf{i}_{QP}$. The quantity inside the brackets on the right side of (5-40) can be recognized as the electric field intensity produced at the location of the point charge by a volume charge distribution of uniform density 1 C/m³ in the spherical volume V. From Gauss' law, this electric field intensity is equal to

$$\frac{1}{\epsilon_0}\left(\frac{\text{charge enclosed within the sphere of radius } d/2}{\text{surface area of the sphere of radius } d/2}\right)\mathbf{i}_z$$

or $(d/6\epsilon_0)\mathbf{i}_z$.

Thus we obtain

$$\int_V \mathbf{E}_+ \, dv = -\frac{Qd}{6\epsilon_0}\mathbf{i}_z \tag{5-41a}$$

Similarly, the volume integral of the electric field due to the negative charge $-Q$ located at $(0, 0, -d/2)$ evaluated in the spherical volume V of radius a can be obtained as

$$\int_V \mathbf{E}_- \, dv = -\frac{Qd}{6\epsilon_0}\mathbf{i}_z \tag{5-41b}$$

The volume integral of the electric field due to the dipole is then given by

$$\int_V (\mathbf{E}_+ + \mathbf{E}_-) \, dv = -\frac{Qd}{3\epsilon_0}\mathbf{i}_z \tag{5-42}$$

Finally, the average field due to the dipole in the spherical volume is given by

$$\begin{aligned} \mathbf{E}_{av} &= \frac{1}{V}\int_V (\mathbf{E}_+ + \mathbf{E}_-) \, dv \\ &= \frac{1}{\frac{4}{3}\pi a^3}\left(-\frac{Qd}{3\epsilon_0}\mathbf{i}_z\right) = -\frac{\mathbf{p}}{4\pi\epsilon_0 a^3} \end{aligned} \tag{5-43}$$

It is left as an exercise (Problem 5.16) for the student to show that (5-43) is true for any arbitrary charge distribution of dipole moment $\mathbf{p}$ situated in the spherical volume of radius a. ▌

From the result (5-43) of Example 5-4, we now relate the polarizing field $\mathbf{E}_p$ with the average macroscopic field $\mathbf{E}$ as

$$\mathbf{E}_p = \mathbf{E} - \mathbf{E}_{av} = \mathbf{E} - \left(\frac{-\mathbf{p}}{4\pi\epsilon_0 a^3}\right) = \mathbf{E} + \frac{\mathbf{P}}{3(\frac{4}{3}\pi a^3)N\epsilon_0} \tag{5-44}$$

where we have substituted $\mathbf{p} = \mathbf{P}/N$ from (5-37). Now, if we assume that the molecular volume is equal to the volume of the spherical cavity, then $(\frac{4}{3}\pi a^3)N$ is equal to 1 since N is the number of molecules per unit volume.

Equation (5-44) then reduces to

$$\mathbf{E}_p = \mathbf{E} + \frac{\mathbf{P}}{3\epsilon_0} \tag{5-45}$$

Although we have obtained (5-45) by making certain simplifying assumptions, it is found that the experimentally observed behavior of many dielectric materials agrees remarkably well with that following from (5-45). Substituting (5-45) into (5-38), we obtain

$$\mathbf{P} = N\alpha\left(\mathbf{E} + \frac{\mathbf{P}}{3\epsilon_0}\right) \tag{5-46}$$

Rearranging (5-46), we obtain the relationship between **P** and **E** as

$$\mathbf{P} = \frac{3\alpha N}{3\epsilon_0 - \alpha N}\epsilon_0\mathbf{E} \tag{5-47}$$

Defining a dimensionless parameter χ_e, known as the "electric susceptibility," as

$$\chi_e = \frac{3\alpha N}{3\epsilon_0 - \alpha N} \tag{5-48}$$

Eq. (5-47) can be written as

$$\mathbf{P} = \epsilon_0\chi_e\mathbf{E} \tag{5-49}$$

This simple relationship between the polarization vector **P** and the average macroscopic electric field **E** in the dielectric indicates that **P** is proportional to **E**. Materials for which this relationship holds, that is, χ_e is independent of the magnitude as well as the direction of **E** are known as linear isotropic dielectric materials. For certain dielectric materials, each component of **P** can be dependent on all components of **E**. In such cases, **P** is not parallel to **E** and the materials are not isotropic. Such materials are known as anisotropic dielectric materials.

5.5 Dielectrics in Electric Fields; Polarization Charge and Current

In Section 5.4 we learned that polarization occurs in dielectric materials under the influence of an applied electric field. We defined polarization by means of a polarization vector **P**, which is the electric dipole moment per unit volume. The polarization vector is related to the electric field responsible for producing it, through Eq. (5-49). When a dielectric material is placed in an electric field, the induced polarization produces a secondary electric field, which reduces the applied field, which in turn causes a change in the polarization vector, and so on. When this adjustment process is complete, that is, when a steady state is reached, the sum of the originally applied field and the secondary field must be such that it produces a polarization which results in the secondary field. The situation is like a feedback loop as shown in Fig. 5.11. We will assume that the adjustment takes place instantaneously

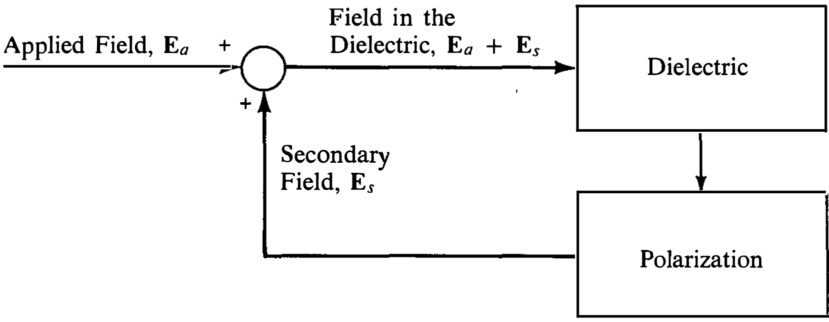


Fig. 5.11. Feedback loop illustrating the adjustment of polarization in a dielectric material to correspond to the sum of the applied field and the secondary field due to the polarization.

with the application of the field and investigate the different effects arising from the polarization. We do this by first considering some specific examples.

EXAMPLE 5-5. An infinite plane dielectric slab of uniform electric susceptibility χ_{e0} and of thickness d occupies the region $0 < z < d$ as shown in Fig. 5.12(a). A uniform electric field $\mathbf{E}_a = E_0\mathbf{i}_z$ is applied. It is desired to investigate the effect of polarization induced in the dielectric.

The applied electric field induces dipole moments in the dielectric with the negative charges separated from the positive charges and pulled away

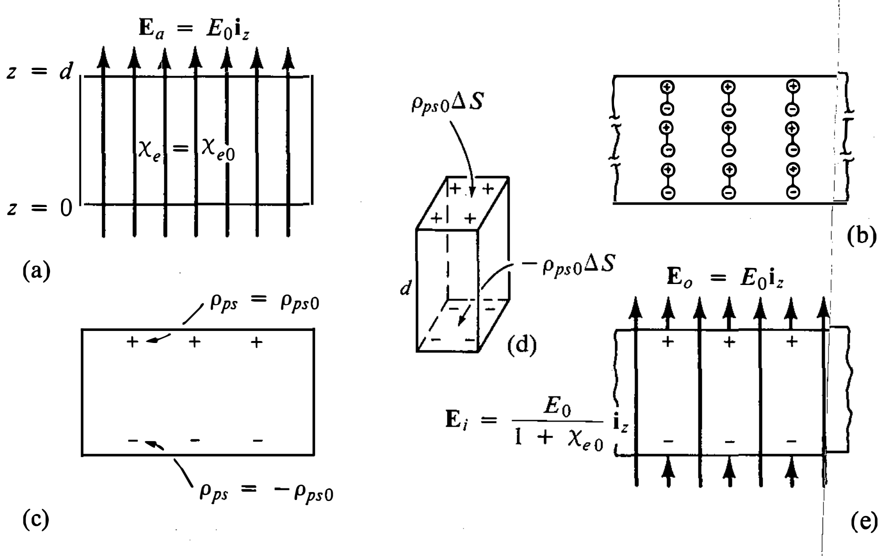


Fig. 5.12. For investigating the effects of polarization induced in a dielectric material of uniform susceptibility for a uniform applied electric field.

from the direction of the field. Since the electric field and the electric susceptibility are uniform, the density of the induced dipole moments, that is, the polarization vector **P**, is uniform as shown in Fig. 5.12(b). Such a distribution results in exact neutralization of all the charges except at the boundaries of the dielectric since, for each positive (or negative) charge not on the surface, there is the same amount of negative (or positive) charge associated with the dipole adjacent to it, thereby cancelling its effect. On the other hand, since the medium changes abruptly from dielectric to free space at the boundaries, no such neutralization of charges at the boundaries takes place. Thus the net result is the formation of a positive surface charge at the boundary $z = d$ and a negative surface charge at the boundary $z = 0$ as shown in Fig. 5.12(c). These surface charges are known as polarization surface charges since they are due to the polarization in the dielectric. In view of the uniform density of the dipole moments, the surface charge densities are uniform. Also, in the absence of a net charge in the interior of the dielectric, the surface charge densities must be equal in magnitude to preserve the charge neutrality of the dielectric.

Let us therefore denote the surface charge densities as

$$\rho_{ps} = \begin{cases} \rho_{ps0} & z = d \\ -\rho_{ps0} & z = 0 \end{cases} \tag{5-50}$$

where the subscript p in addition to the other subscripts stands for polarization. If we now consider a vertical column of infinitesimal rectangular cross-sectional area ΔS cut out from the dielectric as shown in Fig. 5.12(d), the equal and opposite surface charges make the column appear as a dipole of moment $(\rho_{ps0}\,\Delta S)\,d\mathbf{i}_z$. On the other hand, writing

$$\mathbf{P} = P_0\mathbf{i}_z \tag{5-51}$$

where P_0 is a constant in view of the uniformity of the induced polarization, the dipole moment of the column is equal to **P** times the volume of the column, or $P_0(d\,\Delta S)\,\mathbf{i}_z$. Equating the dipole moments computed in the two different ways, we have

$$\rho_{ps0} = P_0 \tag{5-52}$$

Thus we have related the surface charge density to the magnitude of the polarization vector. Now, the surface charge distribution produces a secondary field $\mathbf{E}_s$ given by

$$\mathbf{E}_s = \begin{cases} -\dfrac{\rho_{ps0}}{\epsilon_0}\mathbf{i}_z = -\dfrac{P_0}{\epsilon_0}\mathbf{i}_z & \text{for } 0 < z < d \\ 0 & \text{otherwise} \end{cases} \tag{5-53}$$

When the secondary field $\mathbf{E}_s$ is superimposed on the applied field the net result is a reduction of the field inside the dielectric. Denoting the total field inside the dielectric as $\mathbf{E}_i$, we have

$$\mathbf{E}_i = \mathbf{E}_a + \mathbf{E}_s = E_0\mathbf{i}_z - \frac{P_0}{\epsilon_0}\mathbf{i}_z = \left(E_0 - \frac{P_0}{\epsilon_0}\right)\mathbf{i}_z \tag{5-54}$$

But, from (5-49),

$$\mathbf{P} = \epsilon_0 \chi_{e0} \mathbf{E}_i \tag{5-55}$$

Substituting (5-51) and (5-54) into (5-55), we have

$$P_0 = \epsilon_0 \chi_{e0}\left(E_0 - \frac{P_0}{\epsilon_0}\right)$$

or

$$P_0 = \frac{\epsilon_0 \chi_{e0} E_0}{1 + \chi_{e0}} \tag{5-56}$$

Thus the polarization surface charge densities are given by

$$\rho_{ps} = \begin{cases} \dfrac{\epsilon_0 \chi_{e0} E_0}{1 + \chi_{e0}} & z = d \\ -\dfrac{\epsilon_0 \chi_{e0} E_0}{1 + \chi_{e0}} & z = 0 \end{cases} \tag{5-57}$$

and the electric field intensity inside the dielectric is

$$\mathbf{E}_i = \frac{E_0}{1 + \chi_{e0}}\mathbf{i}_z \tag{5-58}$$

Since the secondary field produced outside the dielectric by the surface charge distribution is zero, the total field $\mathbf{E}_o$ outside the dielectric remains the same as the applied field. The field distribution both inside and outside the dielectric is shown in Fig. 5.12(e). Although we have demonstrated only the formation of a polarization surface charge in this example, it is easy to visualize that a nonuniform applied electric field or a nonuniform electric susceptibility of the material will result in the formation of a polarization volume charge in the dielectric due to imperfect cancellation of the charges associated with the dipoles. ▮

EXAMPLE 5-6. An infinite plane dielectric slab of uniform electric susceptibility χ_{e0} and of thickness d occupies the region $0 < z < d$. A spatially uniform but time-varying electric field $\mathbf{E} = E_0 \cos \omega t\, \mathbf{i}_z$ is applied. It is desired to investigate the effect of polarization induced in the dielectric. Assume that the induced polarization follows exactly the time variations of the applied field.

Since the applied field and the electric susceptibility of the dielectric are spatially uniform, the induced polarization is such that only surface charges of equal and opposite density are formed at the boundaries of the dielectric, and no volume charge is formed inside the dielectric. At any particular time, the surface charge densities are given by (5-57), with the value of the applied field at that time substituted for E_0. Thus the time-varying surface charge densities are

$$\rho_{ps}(t) = \begin{cases} \dfrac{\epsilon_0 \chi_{e0} E_0}{1+\chi_{e0}} \cos \omega t & z = d \\ -\dfrac{\epsilon_0 \chi_{e0} E_0}{1+\chi_{e0}} \cos \omega t & z = 0 \end{cases} \tag{5-59}$$

But if the charge in a volume is varying with time, there must be a current flow out of or into that volume in accordance with the continuity equation, given in integral form by

$$\oint_S \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt} \int_V \rho \, dv = 0 \tag{4-103}$$

where S is the surface bounding the volume V. Obviously, in the present case the current flow must be inside the dielectric from one boundary to the other. This current is known as the polarization current since it is due to the polarization in the dielectric. For this example, the polarization current density must be entirely z-directed because of the uniformity of the polarization surface charge distributions and it must be uniform since the polarization volume charge density inside the dielectric is zero.

Let us therefore denote the polarization current density as

$$\mathbf{J}_p = J_{p0} \mathbf{i}_z \qquad 0 < z < d \tag{5-60}$$

where the subscript p stands for polarization. To find J_{p0} we apply (4-103) to a rectangular box enclosing an infinitesimal area ΔS of the surface $z = 0$ and parallel to it as shown in Fig. 5.13. Noting that the current outside the dielectric slab and the volume charge inside the slab are zero, we obtain

$$J_{p0} \, \Delta S + \frac{d}{dt}\{[\rho_{ps}]_{z=0} \Delta S\} = 0$$

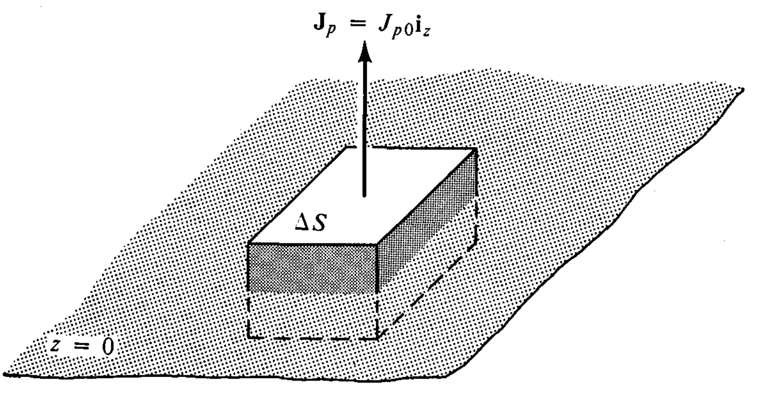


Fig. 5.13. For the determination of the polarization current density resulting from the time variation of the polarization charges induced in a dielectric material.

Thus

$$J_{p0} = -\frac{d}{dt}[\rho_{ps}]_{z=0} = -\frac{d}{dt}\left(-\frac{\epsilon_0 \chi_{e0} E_0}{1+\chi_{e0}} \cos \omega t\right) = -\frac{\epsilon_0 \chi_{e0} E_0 \omega}{1+\chi_{e0}} \sin \omega t$$

and

$$\mathbf{J}_p = -\frac{\epsilon_0 \chi_{e0} E_0 \omega}{1+\chi_{e0}} \sin \omega t \, \mathbf{i}_z \qquad 0 < z < d \tag{5-61}$$

It is left as an exercise for the student to verify that the same result is obtained for $\mathbf{J}_p$ by applying (4-103) to a rectangular box enclosing an infinitesimal area ΔS of the surface $z = d$ and parallel to it. Note that the polarization current density is out of phase by 90° with the applied electric field. ▮

We now derive general expressions for polarization surface and volume charge densities and polarization current density in terms of the polarization vector. To do this, let us consider a dielectric material of volume V' in which the polarization vector $\mathbf{P}$ is an arbitrary function of position as shown in Fig. 5.14. We divide the volume V' into a number of infinitesimal volumes dv'_i,

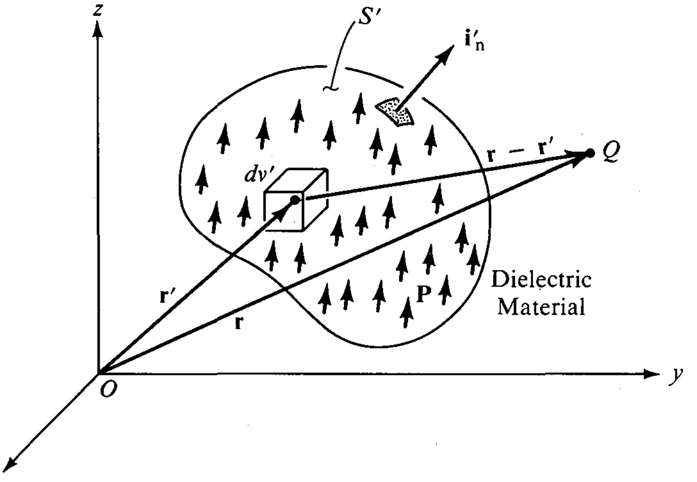


Fig. 5.14. For evaluating the electric potential due to induced polarization in a dielectric material.

$i = 1, 2, 3, \ldots, n$ defined by position vectors $\mathbf{r}'_i$, $i = 1, 2, 3, \ldots, n$, respectively. In each infinitesimal volume, we can consider $\mathbf{P}$ to be a constant so that the dipole moment in the ith volume is $\mathbf{P}_i \, dv'_i$. From (2-109), the scalar potential dV_i at a point $Q(\mathbf{r})$ due to the dipole moment in the ith volume is given by

$$dV_i = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{P}_i \, dv'_i \cdot (\mathbf{r} - \mathbf{r}'_i)}{|\mathbf{r} - \mathbf{r}'_i|^3}$$

The total potential at $Q(\mathbf{r})$ due to the dipole moments in all the n infinitesimal volumes is then given by

$$V = \sum_{i=1}^{n} dV_i = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \frac{\mathbf{P}_i \, dv_i' \cdot (\mathbf{r} - \mathbf{r}_i')}{|\mathbf{r} - \mathbf{r}_i'|^3} \tag{5-62}$$

Equation (5-62) is good only for $|\mathbf{r}| \gg |\mathbf{r}_i'|$, where $\mathbf{i} = 1, 2, 3, \ldots, n$ since each dv_i' has a finite although infinitesimal volume. However, in the limit that $n \longrightarrow \infty$, all the infinitesimal volumes tend to zero; the right side of (5-62) becomes an integral and the expression is valid for any $\mathbf{r}$. Thus

$$\begin{aligned} V(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \int_{\text{volume } V'} \frac{\mathbf{P} \, dv' \cdot (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \\ &= \frac{1}{4\pi\epsilon_0} \int_{\text{volume } V'} \mathbf{P} \cdot \nabla' \frac{1}{|\mathbf{r} - \mathbf{r}'|} \, dv' \end{aligned} \tag{5-63}$$

Substituting the vector identity

$$\nabla' \cdot \frac{\mathbf{P}}{|\mathbf{r} - \mathbf{r}'|} = \mathbf{P} \cdot \nabla' \frac{1}{|\mathbf{r} - \mathbf{r}'|} + \frac{1}{|\mathbf{r} - \mathbf{r}'|} \nabla' \cdot \mathbf{P}$$

in (5-63), we obtain

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{\text{volume } V'} \nabla' \cdot \frac{\mathbf{P}}{|\mathbf{r} - \mathbf{r}'|} \, dv' - \frac{1}{4\pi\epsilon_0} \int_{\text{volume } V'} \frac{1}{|\mathbf{r} - \mathbf{r}'|} \nabla' \cdot \mathbf{P} \, dv' \tag{5-64}$$

Applying the divergence theorem to the first integral on the right side of (5-64), we get

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{\text{surface } S'} \frac{\mathbf{P} \cdot \mathbf{i}_n'}{|\mathbf{r} - \mathbf{r}'|} \, dS' + \frac{1}{4\pi\epsilon_0} \int_{\text{volume } V'} \frac{-\nabla' \cdot \mathbf{P}}{|\mathbf{r} - \mathbf{r}'|} \, dv' \tag{5-65}$$

where S' is the surface bounding the volume V' and $\mathbf{i}_n'$ is the unit normal vector to dS'.

The first integral on the right side of (5-65) represents the potential at $Q(\mathbf{r})$ due to a surface charge of density $\mathbf{P} \cdot \mathbf{i}_n'$ on the surface S' and the second integral is the potential at $Q(\mathbf{r})$ due to a volume charge of density $(-\nabla' \cdot \mathbf{P})$ in the volume V'. Thus the potential at $Q(\mathbf{r})$ due to the polarization in the dielectric is the same as the sum of the potentials at $Q(\mathbf{r})$ due to a polarization surface charge of density

$$\rho_{ps}(\mathbf{r}') = \mathbf{P}(\mathbf{r}') \cdot \mathbf{i}_n' \qquad \text{on } S' \tag{5-66a}$$

and due to a polarization volume charge of density

$$\rho_p(\mathbf{r}') = -\nabla' \cdot \mathbf{P}(\mathbf{r}') \qquad \text{in } V' \tag{5-66b}$$

We note that the total charge in V' is

$$\oint_{S'} \rho_{ps} \, dS' + \int_{V'} \rho_p \, dv' = \oint_{S'} (\mathbf{P} \cdot \mathbf{i}_n') \, dS' - \int_{V'} (\nabla' \cdot \mathbf{P}) \, dv' = 0$$

according to the divergence theorem, so that the charge neutrality of the dielectric is satisfied. Thus the total polarization volume charge in V' is equal to the negative of the total polarization surface charge on S'. Omitting the primes in (5-66a) and (5-66b), we have

$$\rho_{ps} = \mathbf{P} \cdot \mathbf{i}_n \tag{5-67}$$

$$\rho_p = -\nabla \cdot \mathbf{P} \tag{5-68}$$

Now, the polarization current density $\mathbf{J}_p$ in the dielectric due to the time variation of the polarization charge density should satisfy the continuity equation

$$\nabla \cdot \mathbf{J}_p + \frac{\partial \rho_p}{\partial t} = 0 \tag{5-69}$$

Substituting for ρ_p in (5-69) from (5-68), we have

$$\nabla \cdot \mathbf{J}_p - \frac{\partial}{\partial t}(\nabla \cdot \mathbf{P}) = 0$$

or

$$\nabla \cdot \left(\mathbf{J}_p - \frac{\partial \mathbf{P}}{\partial t}\right) = 0$$

or

$$\mathbf{J}_p - \frac{\partial \mathbf{P}}{\partial t} = \text{constant with time} \tag{5-70}$$

The constant must, however, be zero since we know that $\mathbf{J}_p$ is zero when $\partial \mathbf{P}/\partial t$ is zero. Thus

$$\mathbf{J}_p = \frac{\partial \mathbf{P}}{\partial t} \tag{5-71}$$

Summarizing what we have learned in this section, the induced dipole moments due to polarization in a dielectric material placed in an electric field have the effect of creating in general the following:

(a) polarization surface charges, having densities given by (5-67), at the boundaries of the dielectric,

(b) polarization volume charge of density given by (5-68) in the dielectric and such that the total volume charge is exactly the negative of the total surface charge so as to preserve the charge neutrality of the material, and

(c) polarization current of density given by (5-71) resulting from the time variation of the polarization charges.

We have also shown that the polarization charges and currents alter the applied electric field in the material. Such a modification of the applied field occurs outside the material as well in the general case. The magnetic field associated with the applied electric field is also altered by the addition of the secondary magnetic field due to the polarization current and the time-variation of the secondary electric field.

5.6 Displacement Flux Density and Relative Permittivity

In Section 5.5 we learned that the electric field in a dielectric material is the superposition of an applied field $\mathbf{E}_a$ and a secondary field $\mathbf{E}_s$ which results from the polarization $\mathbf{P}$, which in turn is induced by the total field $(\mathbf{E}_a + \mathbf{E}_s)$, as shown in Fig. 5.11. Thus, from Fig. 5.11 and Eq. (5-49), we have

$$\mathbf{P} = \epsilon_0 \chi_e (\mathbf{E}_a + \mathbf{E}_s) \tag{5-72}$$

$$\mathbf{E}_s = f(\mathbf{P}) \tag{5-73}$$

where $f(\mathbf{P})$ denotes a function of $\mathbf{P}$. Determination of the secondary field $\mathbf{E}_s$ and hence the total field $(\mathbf{E}_a + \mathbf{E}_s)$ for a given applied field $\mathbf{E}_a$ requires a simultaneous solution of (5-72) and (5-73) which, in general, is very inconvenient. To circumvent this problem, we make use of the results of Section 5.5, in which we found that the induced polarization is equivalent to a polarization surface charge of density ρ_{ps}, a polarization volume charge of density ρ_p, and a polarization current of density $\mathbf{J}_p$, as given by (5-67), (5-68), and (5-71), respectively. The secondary electric and magnetic fields are the fields produced by these charges and current as if they were situated in free space, in the same way as the charges and currents responsible for the applied electric field and its associated magnetic field.

Thus the secondary electromagnetic field satisfies Maxwell's equations

$$\nabla \cdot \mathbf{E}_s = \frac{\rho_p}{\epsilon_0} \tag{5-74a}$$

$$\nabla \cdot \mathbf{B}_s = 0 \tag{5-74b}$$

$$\nabla \times \mathbf{E}_s = -\frac{\partial \mathbf{B}_s}{\partial t} \tag{5-74c}$$

$$\nabla \times \mathbf{B}_s = \mu_0 \left[\mathbf{J}_p + \frac{\partial}{\partial t}(\epsilon_0 \mathbf{E}_s) \right] \tag{5-74d}$$

where $\mathbf{B}_s$ is the secondary magnetic field. On the other hand, if the "true" charge and current densities responsible for the applied field $\mathbf{E}_a$ with its associated magnetic field $\mathbf{B}_a$ are ρ and $\mathbf{J}$, respectively, we have

$$\nabla \cdot \mathbf{E}_a = \frac{\rho}{\epsilon_0} \tag{5-75a}$$

$$\nabla \cdot \mathbf{B}_a = 0 \tag{5-75b}$$

$$\nabla \times \mathbf{E}_a = -\frac{\partial \mathbf{B}_a}{\partial t} \tag{5-75c}$$

$$\nabla \times \mathbf{B}_a = \mu_0 \left[\mathbf{J} + \frac{\partial}{\partial t}(\epsilon_0 \mathbf{E}_a) \right] \tag{5-75d}$$

Now, adding (5-74a)–(5-74d) to (5-75a)–(5-75d), respectively, we obtain

$$\nabla \cdot (\mathbf{E}_a + \mathbf{E}_s) = \frac{\rho + \rho_p}{\epsilon_0} \tag{5-76a}$$

$$\nabla \cdot (\mathbf{B}_a + \mathbf{B}_s) = 0 \tag{5-76b}$$

$$\nabla \times (\mathbf{E}_a + \mathbf{E}_s) = -\frac{\partial}{\partial t}(\mathbf{B}_a + \mathbf{B}_s) \tag{5-76c}$$

$$\nabla \times (\mathbf{B}_a + \mathbf{B}_s) = \mu_0 \left\{ \mathbf{J} + \mathbf{J}_p + \frac{\partial}{\partial t}[\epsilon_0(\mathbf{E}_a + \mathbf{E}_s)] \right\} \tag{5-76d}$$

Substituting

$$\mathbf{E} = \mathbf{E}_a + \mathbf{E}_s \tag{5-77a}$$

$$\mathbf{B} = \mathbf{B}_a + \mathbf{B}_s \tag{5-77b}$$

$$\rho_p = -\nabla \cdot \mathbf{P} \tag{5-68}$$

$$\mathbf{J}_p = \frac{\partial \mathbf{P}}{\partial t} \tag{5-71}$$

in (5-76a)–(5-76d), and rearranging, we obtain

$$\nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho \tag{5-78a}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{5-78b}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{5-78c}$$

$$\nabla \times \mathbf{B} = \mu_0 \left[\mathbf{J} + \frac{\partial}{\partial t}(\epsilon_0 \mathbf{E} + \mathbf{P}) \right] \tag{5-78d}$$

where **E** and **B** are the total fields.

We now define a vector **D**, known as the displacement flux density vector, and given by

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \tag{5-79}$$

Note that the units of **D** are the same as those of $\epsilon_0 \mathbf{E}$ and **P**, that is, coulombs per square meter, and hence it is a flux density vector. Substituting (5-79) into (5-78a)–(5-78d), we obtain

$$\nabla \cdot \mathbf{D} = \rho \tag{5-80}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{5-81}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{5-82}$$

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \tag{5-83}$$

Thus the new field **D** results in a set of equations which does not explicitly contain the polarization charge and current densities, unlike the equations (5-76a)–(5-76d).

Substituting for **P** in (5-79) from (5-49), we have

$$\mathbf{D} = \epsilon_0\mathbf{E} + \epsilon_0\chi_e\mathbf{E} = \epsilon_0(1 + \chi_e)\mathbf{E} = \epsilon_0\epsilon_r\mathbf{E} = \epsilon\mathbf{E} \tag{5-84}$$

where we define

$$\epsilon_r = 1 + \chi_e \tag{5-85}$$

and

$$\epsilon = \epsilon_0\epsilon_r \tag{5-86}$$

The quantity ϵ_r is known as the relative permittivity or dielectric constant of the dielectric and ϵ is the permittivity of the dielectric. Note that ϵ_r is dimensionless and that (5-84) is true only for linear dielectrics if ϵ is to be treated as a constant for a particular dielectric, whereas (5-79) holds in general. Substituting (5-84) into (5-80)–(5-83), we obtain

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon} \tag{5-87a}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{5-87b}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{5-87c}$$

$$\nabla \times \mathbf{B} = \mu_0\left[\mathbf{J} + \frac{\partial}{\partial t}(\epsilon\mathbf{E})\right] \tag{5-87d}$$

Equations (5-87a)–(5-87d) are the same as Maxwell's equations for free space except that ϵ_0 is replaced by ϵ. Thus the electric and magnetic fields in the presence of a dielectric can be computed in exactly the same manner as for free space except that we have to use ϵ instead of ϵ_0 for permittivity. In fact, if $\chi_e = 0$, $\epsilon_r = 1$ and $\epsilon = \epsilon_0$ so that free space can be considered as a dielectric with $\epsilon = \epsilon_0$, and hence, Eqs. (5-87a)–(5-87d) can be used for free space as well. The permittivity ϵ takes into account the effects of polarization and there is no need to consider them when we use ϵ for ϵ_0, thereby eliminating the necessity for the simultaneous solution of (5-72) and (5-73). In the case of a boundary between two different dielectrics, the appropriate boundary conditions for **D** take into account implicitly the polarization surface charge. We will consider these boundary conditions in Section 5.12. The relative permittivity is an experimentally measurable parameter and its values for several dielectric materials are listed in Table 5.2.

EXAMPLE 5-7. For the dielectric slab of Example 5-5, find and sketch the direction lines of the displacement flux density and the electric field intensity vectors both inside and outside the dielectric.

From Example 5-5, the electric field intensity inside the dielectric is given by

$$\mathbf{E}_i = \frac{E_0}{1 + \chi_{e0}}\mathbf{i}_z \tag{5-58}$$

TABLE 5.2. Relative Permittivities of Some Materials

Material	*Relative Permittivity*	*Material*	*Relative Permittivity*
Air	1.0006	Dry earth	5
Paper	2–3	Glass	5–10
Rubber	2–3.5	Mica	6
Teflon	2.1	Porcelain	6
Polyethylene	2.26	Neoprene	6.7
Polystyrene	2.56	Wet earth	10
Plexiglass	2.6–3.5	Ethyl alchohol	24.3
Nylon	3.5	Glycerol	42.5
Fused quartz	3.8	Distilled water	81
Bakelite	4.9	Titanium dioxide	100

The relative permittivity of the dielectric is $1 + \chi_{e0}$. Thus the displacement flux density inside the dielectric is

$$\mathbf{D}_i = \epsilon_0(1 + \chi_{e0})\mathbf{E}_i = \frac{\epsilon_0(1 + \chi_{e0})}{1 + \chi_{e0}} E_0 \mathbf{i}_z = \epsilon_0 E_0 \mathbf{i}_z$$

Outside the dielectric, the electric field intensity is the same as the applied value so that the displacement flux density is

$$\mathbf{D}_o = \epsilon_0 \mathbf{E}_a = \epsilon_0 E_0 \mathbf{i}_z$$

Thus, for this example, the displacement flux density vectors inside and outside the dielectric are the same and equal to the displacement flux density associated with the applied electric field intensity. Both **D** and **E** fields inside and outside the dielectric are shown in Fig. 5.15. We note that the direction lines of **D** do not begin or end on the polarization charges whereas the direction lines of **E** begin and end on them. The direction lines of **D** begin and

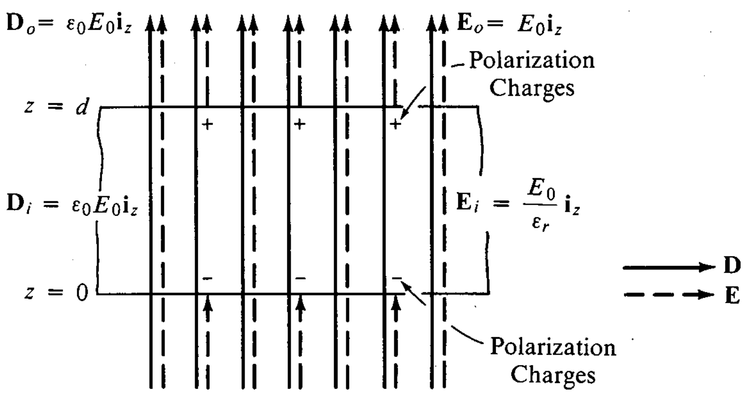


Fig. 5.15. Displacement flux density and electric field intensity vectors for the dielectric slab of Example 5-5.

end only on the charges other than the polarization charges whereas the direction lines of **E** begin and end on both kinds of charges. ▮

EXAMPLE 5-8. A point charge Q is situated at the center of a spherical dielectric shell of uniform permittivity ϵ and having inner and outer radii a and b, respectively, as shown in Fig. 5.16. The entire arrangement is enclosed by a grounded conducting shell of inner radius c and concentric with the dielectric shell. Find and sketch the **D** and **E** fields in three different regions: $0 < r < a$, $a < r < b$, and $b < r < c$. Also find and sketch the **P** field and the polarization charges in the dielectric and the charge induced on the conductor surface.

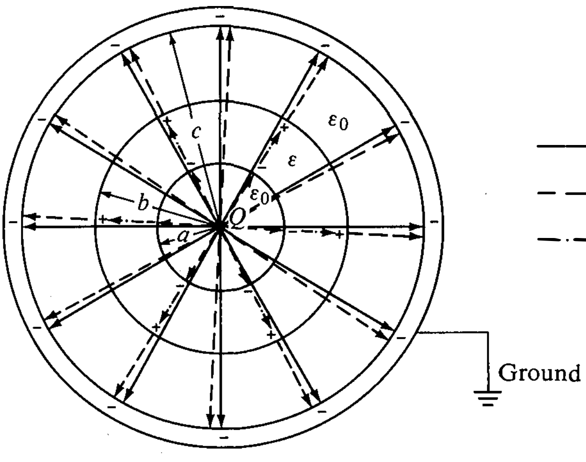


Fig. 5.16. Displacement flux density, electric field intensity, and polarization vectors for the arrangement of a point charge at the center of a spherical dielectric shell enclosed by a grounded spherical conductor concentric with the dielectric shell.

We make use of the spherical symmetry associated with the problem and apply the integral form of (5-87a) given by

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\epsilon} \int_V \rho \, dv = \frac{1}{\epsilon} (\text{true charge enclosed by } S) \qquad (5\text{-}88)$$

to three different spherical surfaces centered at the point charge and lying in the three different regions. Thus we obtain

$$\mathbf{E} = \begin{cases} \dfrac{Q}{4\pi\epsilon_0 r^2} \mathbf{i}_r & 0 < r < a \\ \dfrac{Q}{4\pi\epsilon r^2} \mathbf{i}_r & a < r < b \\ \dfrac{Q}{4\pi\epsilon_0 r^2} \mathbf{i}_r & b < r < c \end{cases} \qquad (5\text{-}89)$$

The corresponding **D** field is given by

$$\mathbf{D} = \begin{cases} \epsilon_0 \mathbf{E} = \dfrac{Q}{4\pi r^2}\mathbf{i}_r & 0 < r < a \\[2ex] \epsilon \mathbf{E} = \dfrac{Q}{4\pi r^2}\mathbf{i}_r & a < r < b \\[2ex] \epsilon_0 \mathbf{E} = \dfrac{Q}{4\pi r^2}\mathbf{i}_r & b < r < c \end{cases}$$
$$= \frac{Q}{4\pi r^2}\mathbf{i}_r \qquad 0 < r < c \tag{5-90}$$

Alternatively and more elegantly, we can use the integral form of (5-80) given by

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho \, dv = (\text{true charge enclosed by } S) \tag{5-91}$$

and apply it to a spherical surface centered at the point charge and having any radius r, where $0 < r < c$. Since the right side of (5-91) does not depend upon the permittivity of the medium, we then obtain the result given by (5-90). Having obtained this, we can then find **E** in the three different regions by dividing **D** by the corresponding permittivity.

Now, from (5-79), the polarization vector **P** inside the dielectric is given by

$$\begin{aligned} \mathbf{P} &= \mathbf{D} - \epsilon_0[\mathbf{E}]_{a<r<b} \\ &= \frac{Q}{4\pi r^2}\mathbf{i}_r - \epsilon_0 \frac{Q}{4\pi\epsilon r^2}\mathbf{i}_r = \frac{Q}{4\pi r^2}\left(1 - \frac{\epsilon_0}{\epsilon}\right)\mathbf{i}_r \end{aligned} \tag{5-92}$$

The polarization volume and surface charge densities are

$$\begin{aligned} \rho_p &= -\nabla \cdot \mathbf{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 P_r) \\ &= -\frac{1}{r^2}\frac{\partial}{\partial r}\left[\frac{Q}{4\pi}\left(1 - \frac{\epsilon_0}{\epsilon}\right)\right] = 0 \end{aligned} \tag{5-93a}$$

$$\begin{aligned} [\rho_{ps}]_{r=a} &= [\mathbf{P}]_{r=a} \cdot (-\mathbf{i}_r) \\ &= \left[\frac{Q}{4\pi a^2}\left(1 - \frac{\epsilon_0}{\epsilon}\right)\mathbf{i}_r\right] \cdot (-\mathbf{i}_r) = -\frac{Q}{4\pi a^2}\left(1 - \frac{\epsilon_0}{\epsilon}\right) \end{aligned} \tag{5-93b}$$

$$\begin{aligned} [\rho_{ps}]_{r=b} &= [\mathbf{P}]_{r=b} \cdot (\mathbf{i}_r) \\ &= \left[\frac{Q}{4\pi b^2}\left(1 - \frac{\epsilon_0}{\epsilon}\right)\mathbf{i}_r\right] \cdot (\mathbf{i}_r) = \frac{Q}{4\pi b^2}\left(1 - \frac{\epsilon_0}{\epsilon}\right) \end{aligned} \tag{5-93c}$$

The **D** and **E** fields in the three regions, the **P** field in the dielectric, and the polarization surface charge densities are shown in Fig. 5.16. From (5-26), the surface charge density induced on the conductor surface $r = c$ is given by

$$[\rho_s]_{r=c} = \epsilon_0[\mathbf{E}]_{r=c} \cdot (-\mathbf{i}_r) = -\epsilon_0[E_r]_{r=c} = -\frac{Q}{4\pi c^2} \tag{5-94}$$

so that the total charge induced on the conductor surface is $-Q$. These charges are shown in Fig. 5.16. We can obtain this result alternatively by recalling that **E** inside the conductor is zero. Then $\oint_S \mathbf{E} \cdot d\mathbf{S}$ for any surface S entirely within the conductor must be zero. For this to be true, an amount of charge equal and opposite to the sum of all kinds of charges (polarization or otherwise) enclosed by the conductor must be induced on the conductor surface. Since the sum of all kinds of charges enclosed by the conductor is

$$Q + [\rho_{ps}]_{r=a}4\pi a^2 + [\rho_{ps}]_{r=b}4\pi b^2 = Q$$

the induced charge on the conductor surface must be $-Q$. Alternatively and more elegantly, we note that $\mathbf{D} = \epsilon_0\mathbf{E}$ is zero inside the conductor. Hence $\oint_S \mathbf{D} \cdot d\mathbf{S}$ for any surface S entirely within the conductor must be zero. For this to be true, an amount of charge equal and opposite to all charges other than polarization charges, enclosed by the conductor must be induced on the conductor surface. Since the charge, other than polarization charge, enclosed by the conductor is the point charge Q, the induced charge on the conductor surface must be $-Q$. This induced charge required to make the field inside the conductor equal to zero is acquired from the ground.

From Fig. 5.16, we once again note that the direction lines of **E** begin and end on all kinds of charges (polarization or otherwise) whereas the direction lines of **D** begin and end only on charges other than polarization charges. The gaps in the direction lines of **E** resulting from the polarization charges are filled by the direction lines of **P**. The flux of **E** through a spherical surface centered at the point charge varies from medium to medium, depending upon the permittivity of the medium in which the surface lies. On the other hand, the flux of **D** through that surface is always equal to only the true charges, that is, charges other than the polarization charges, enclosed by the surface, irrespective of the permittivities of the media bounded by the surface. Thus there is a displacement flux from the true charges which is independent of the medium as originally discovered by Faraday when he found experimentally that the induced charge on the conductor surface was independent of the medium. However, the vector **D** was introduced later by Maxwell, who called it the "displacement." This explains the name "displacement flux density" for **D**. In Section 4.4 we introduced the concept of displacement current as the time derivative of the flux of $\epsilon_0\mathbf{E}$. We now recognize that $\epsilon_0\mathbf{E}$ is simply the displacement flux density in free space and hence the name displacement current, again attributed to Maxwell, for the time derivative of the flux of $\epsilon_0\mathbf{E}$. It follows that the generalization of the displacement current density of Section 4.5 to dielectric media is $\frac{\partial \mathbf{D}}{\partial t} = \frac{\partial}{\partial t}(\epsilon_0\mathbf{E} + \mathbf{P})$, which reduces to $\frac{\partial}{\partial t}(\epsilon\mathbf{E})$ for linear dielectrics. ∎

5.7 Magnetization and Magnetic Materials

Thus far in this chapter, we have been concerned with the response of materials to electric fields. We now turn our attention to materials known as magnetic materials which, as the name implies, are classified according to their magnetic behavior. According to a simplified atomic model, the electrons associated with a particular nucleus orbit around the nucleus in circular paths while spinning about themselves. In addition, the nucleus itself has a spin motion associated with it. Since the movement of charge constitutes a current, these orbital and spin motions are equivalent to current loops of atomic dimensions. We learned in Chapter 3 that a circular current loop is the magnetic analog of the electric dipole. Thus each atom can be characterized by a superposition of magnetic dipole moments corresponding to the electron orbital motions, electron spin motions, and the nuclear spin. However, owing to the heavy mass of the nucleus, the angular velocity of the nuclear spin is much smaller than that of an electron spin and hence the equivalent current associated with the nuclear spin is much smaller than the equivalent current associated with an electron spin. The dipole moment due to the nuclear spin can therefore be neglected in comparison with the other two effects. The schematic representations of a magnetic dipole as seen from along its axis and from a point in its plane are shown in Figs. 5.17(a) and 5.17(b), respectively.

In many materials, the net magnetic moment of each atom is zero in the absence of an applied magnetic field. An applied magnetic field has the effect of inducing a net dipole moment or "magnetizing" the material by changing the angular velocities of the electron orbits. This induced "magnetization" is in opposition to the applied field so that there is a net reduction in the magnetic flux density in the material from the applied value. Such materials are said to be "diamagnetic." In fact, "diamagnetism," which is analogous to electronic polarization, is prevalent in all materials. We will illustrate the diamagnetic effect by means of the following example.

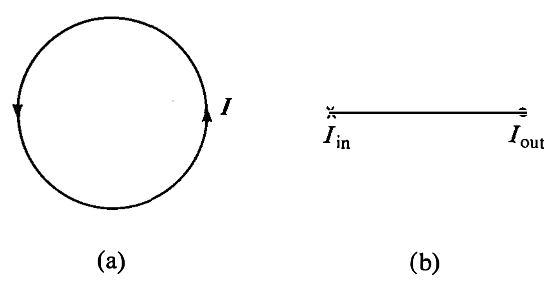


Fig. 5.17. Schematic representation of a magnetic dipole: (a) as seen from along its axis, and (b) as seen from a point in its plane.

EXAMPLE 5-9. Assume that the nucleus of an atom is a point charge equal to $|e|$, where e is the charge of an electron. Consider an electron of mass m_e in a circular orbit of radius a around the nucleus with an angular velocity ω_0 rad/sec. It is desired to find the change in the dipole moment of the orbiting electron due to the application of a uniform external magnetic field perpendicular to the orbital plane of the electron, assuming that the radius of the orbit remains equal to a.

Let the nucleus be at the origin and the electronic orbit be in the xy plane as shown in Fig. 5.18, so that the angular velocity in the absence of the external field is $\pm\omega_0\mathbf{i}_z$. Let the applied magnetic field be $\mathbf{B}_m = B_0\mathbf{i}_z$ and the resulting angular velocity be $\pm\omega\mathbf{i}_z$. Under equilibrium conditions, the centripetal force $-m_e\omega^2 a\mathbf{i}_r$ acting on the electron is equal to the sum of two forces: (a) the Coulomb force $\mathbf{F}_1$ due to the attraction of the electron by the nucleus and (b) the magnetic force $\mathbf{F}_2$ due to the applied field acting on the orbiting electron. These forces are given by

$$\mathbf{F}_1 = -\frac{e^2}{4\pi\epsilon_0 a^2}\mathbf{i}_r$$

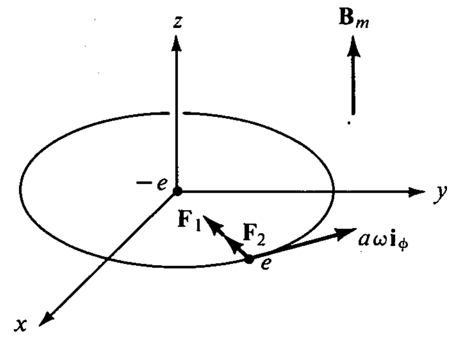


Fig. 5.18. For obtaining the change in the dipole moment of an electronic orbit around the nucleus due to an applied magnetic field.

and

$$\mathbf{F}_2 = \mp|e|\omega a\mathbf{i}_\phi \times B_0\mathbf{i}_z = \mp|e|\omega a B_0\mathbf{i}_r$$

Thus

$$-m_e\omega^2 a\mathbf{i}_r = -\frac{e^2}{4\pi\epsilon_0 a^2}\mathbf{i}_r \mp |e|\omega a B_0\mathbf{i}_r$$

or

$$\omega^2 = \frac{e^2}{4\pi m_e\epsilon_0 a^3} \pm \frac{|e|\omega B_0}{m_e} \tag{5-95}$$

In the absence of the external field, B_0 is zero, $\omega = \omega_0$ and we have

$$\omega_0^2 = \frac{e^2}{4\pi m_e\epsilon_0 a^3} \tag{5-96}$$

Substituting (5-96) into (5-95), we obtain

$$\omega^2 - \omega_0^2 = (\omega + \omega_0)(\omega - \omega_0) = \pm\frac{|e|\omega B_0}{m_e} \tag{5-97}$$

The perturbation in ω_0 by the external field is, however, so small that $\omega + \omega_0$ can be approximated by 2ω. Equation (5-97) then reduces to

$$\omega - \omega_0 \approx \pm\frac{|e|B_0}{2m_e} \tag{5-98}$$

Now, the equivalent current due to an orbiting electron is equal to the amount of charge passing through any point on the orbit in 1 sec, or e times the number of times that the electron passes through the point in 1 sec. For an angular velocity of $\omega\mathbf{i}_z$, the number of times is $\omega/2\pi$ so that the equivalent current is $|e|\omega/2\pi$. This current circulates in the sense opposite to that of the electron orbit since the electronic charge is negative. Thus the magnetic dipole moment due to the orbiting electron is given by

$$\mathbf{m} = \mp\frac{|e|\omega}{2\pi}\pi a^2\mathbf{i}_z = \mp\frac{|e|\omega a^2}{2}\mathbf{i}_z \tag{5-99}$$

The dipole moment in the absence of the external field is

$$\mathbf{m}_0 = \mp\frac{|e|\omega_0 a^2}{2}\mathbf{i}_z \tag{5-100}$$

The change in the dipole moment due to application of $\mathbf{B}_m$ is

$$\Delta\mathbf{m} = \mathbf{m} - \mathbf{m}_0 = \mp\frac{|e|a^2}{2}(\omega - \omega_0)\mathbf{i}_z \tag{5-101}$$

Substituting (5-98) into (5-101), we obtain

$$\Delta\mathbf{m} = \mp\frac{|e|a^2}{2}\left(\pm\frac{|e|B_0}{2m_e}\right)\mathbf{i}_z = -\frac{e^2a^2}{4m_e}\mathbf{B}_m \tag{5-102}$$

Thus the change in the dipole moment and hence the magnetic field resulting from the change is in opposition to the applied magnetic field and independent of the sense of the electron orbit. This is consistent with Lenz' law, discussed in Section 4.2, which states that the change in magnetic flux enclosed by a loop induces a current in the loop which opposes the change in the flux. In the present case, the application of the external magnetic field causes the change in flux enclosed by the electron orbit and the induced current is the current corresponding to the change in the angular velocity of the electron. ∎

The result of Example 5-9 illustrates the principle behind the diamagnetic property of materials without going into great detail. The change in the magnetic moment of each electronic orbit brought about by the applied magnetic field results in a net magnetization of the material which otherwise has a zero net moment.

In certain materials, diamagnetism is dominated by other effects known as paramagnetism, ferromagnetism, antiferromagnetism, and ferrimagnetism. Paramagnetism is similar to orientational polarization in dielectric materials. In "paramagnetic" materials, the individual atoms possess net nonzero magnetic moments even in the absence of an applied magnetic field. However, these "permanent" magnetic moments of the individual atoms are randomly oriented so that the net magnetization on a macroscopic scale is zero. An applied magnetic field has the influence of exerting torques on the permanent atomic magnetic dipoles as shown in Figure 5.19, to convert the initially random alignment into a partially coherent one along the field thereby inducing a net magnetization which results in an enhancement of the applied field.

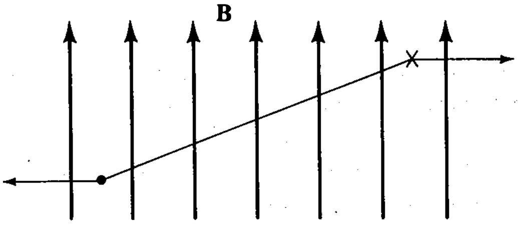


Fig. 5.19. Torque acting on a magnetic dipole under the influence of a magnetic field.

Ferromagnetism is the property by means of which a material can exhibit spontaneous magnetization, that is, magnetization even in the absence of an applied field, below a certain critical temperature known as the Curie temperature. Above the Curie temperature, the spontaneous magnetization vanishes and the ordinary paramagnetic behavior results. Ferromagnetic materials possess strong dipole moments owing to the predominance of the electron spin moments over the electron orbital moments. The theory of ferromagnetism is based on the concept of magnetic "domains," as formulated by Weiss in 1907. A magnetic domain is a small region in the material in which the atomic dipole moments are all aligned in one direction, due to strong interaction fields arising from the neighboring dipoles. In the absence of an external magnetic field, although each domain is magnetized to saturation, the magnetizations in various domains are randomly oriented as shown in Fig. 5.20(a) for a single crystal specimen. The random orientation results from minimization of the associated energy. The net magnetization is therefore zero on a macroscopic scale.

With the application of a weak external magnetic field, the volumes of the domains in which the original magnetizations are favorably oriented relative to the applied field grow at the expense of the volumes of the other domains, as shown in Fig. 5.20(b). This feature is known as domain wall

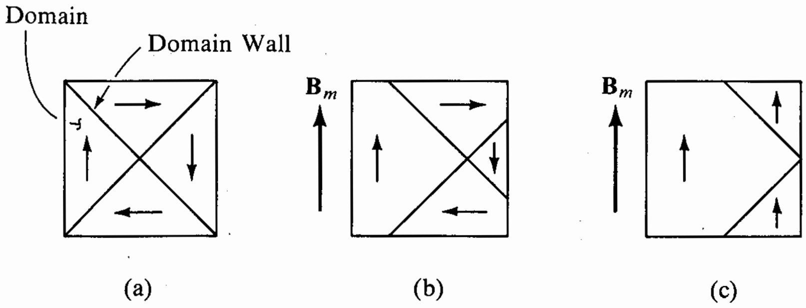
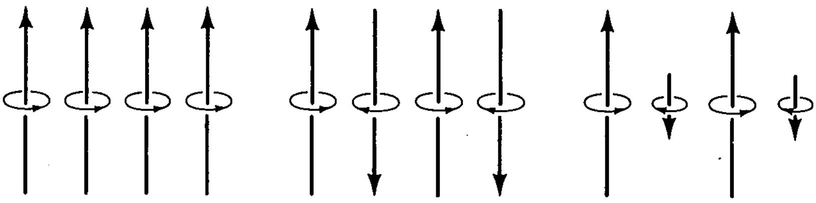


Fig. 5.20. For illustrating the different steps in the magnetization of a ferromagnetic specimen: (a) Unmagnetized state. (b) Domain wall motion. (c) Domain rotation.

motion. Upon removal of the applied field, the domain wall motion reverses, bringing the material close to its original state of magnetization. With the application of stronger external fields, the domain wall motion continues to such an extent that it becomes irreversible; that is, the material does not return to its original unmagnetized state on a macroscopic scale upon removal of the field. With the application of still stronger fields, the domain wall motion is accompanied by domain rotation, that is, alignment of the magnetizations in the individual domains with the applied field as shown in Fig. 5.20(c), thereby magnetizing the material to saturation. The material retains some magnetization along the direction of the applied field even after removal of the field. In fact, an external field opposite to the original direction has to be applied to bring the net magnetization back to zero. The phenomenon by means of which the present state of magnetization of the given material is dependent on its previous magnetic history is known as "hysteresis." We will discuss this topic further in Section 5.9. Unlike in the case of diamagnetic and paramagnetic materials, the magnetization in ferromagnetic materials is nonlinearly related to the applied field.

Antiferromagnetism and ferrimagnetism are modifications of ferromagnetism in materials which contain two interlocking sets of atoms. If the spin moments associated with these two sets of atoms are aligned parallel to each other, as shown in Fig. 5.21(a), the material behaves ferromagnetically. On the other hand, if the spin moments are aligned antiparallel to each other and are equal in magnitude as shown in Fig. 5.21(b), so that the net magnetic moment is zero even under the application of an external field, the material is said to be antiferromagnetic. If the antiparallel moments are unequal in magnitude as shown in Fig. 5.21(c), the net magnetic moment is not zero and the material is said to be ferrimagnetic. A subgroup of ferrimagnetic materials known as "ferrites" is of considerable importance technically because these materials have much lower conductivities than ferromagnetic



(a) (b) (c)

Fig. 5.21. Spin moments associated with interlocking sets of atoms for (a) ferromagnetic, (b) antiferromagnetic, and (c) ferrimagnetic materials.

materials while possessing comparable magnetization properties as ferromagnetic materials.

The net magnetic dipole moment created due to the magnetization of a material by an applied magnetic field produces a field which adds to the applied field (except in the case of materials for which the diamagnetic effect is the only one present) and changes its distribution both inside and outside the material in general from the one that exists in the absence of the material. This will be the topic of discussion in Section 5.8. In the remainder of this section, we will define a new vector **M**, which represents the magnetization on a macroscopic scale, and relate it to the magnetic flux density. To do this let us consider a small volume Δv of a magnetic material. If N denotes the number of molecules per unit volume of the material, then there are $N\,\Delta v$ molecules in the volume Δv. We define a vector **M**, called the "magnetization vector" as

$$\mathbf{M} = \frac{1}{\Delta v}\sum_{j=1}^{N\Delta v} \mathbf{m}_j = N\mathbf{m} \qquad (5\text{-}103)$$

where **m** is the average magnetic dipole moment per molecule. The magnetization vector **M** has the meaning of magnetic "dipole moment per unit volume" analogous to **P** in the case of dielectric materials. The units of **M** are ampere-meter²/meter³ or amperes per meter. We may relate the average dipole moment **m** to the magnetizing field $\mathbf{B}_m$ as given by

$$\mathbf{m} = \alpha_m \mathbf{B}_m \qquad (5\text{-}104)$$

where α_m, which may be called the magnetic polarizability, is a constant for linear magnetic materials but may be a function of $\mathbf{B}_m$ for nonlinear magnetic materials. Substituting (5-104) into (5-103), we have

$$\mathbf{M} = N\alpha_m \mathbf{B}_m \qquad (5\text{-}105)$$

The field $\mathbf{B}_m$ is the average magnetic field acting to magnetize the individual molecule and is generally called the local field, analogous to $\mathbf{E}_p$ in the case of dielectric polarization. It is the average field that would exist in an imaginary cavity created by removing the molecule under question while keeping all the other molecules magnetized in their locations. Thus it is not the same as the average macroscopic field **B** at the molecule with all the molecules including the one in question remaining magnetized in their locations. It is equal to the field **B** minus the average field produced by the dipole moment in the imaginary cavity. We have to find this average field to express $\mathbf{B}_m$ in terms of **B** so that **M** can be related to **B**. To do this, we once again consider a simple special case of a spherical cavity and obtain the required field in the following example.

EXAMPLE 5-10. A circular loop of radius a and centered at the origin lies in the xy plane, as shown in Fig. 5.22. It carries a current I amp in the ϕ direction, thus forming a dipole of moment $\mathbf{m} = I\pi a^2 \mathbf{i}_z$. Obtain the average magnetic flux density due to the dipole in a spherical volume of radius $b > a$ and centered at the origin.

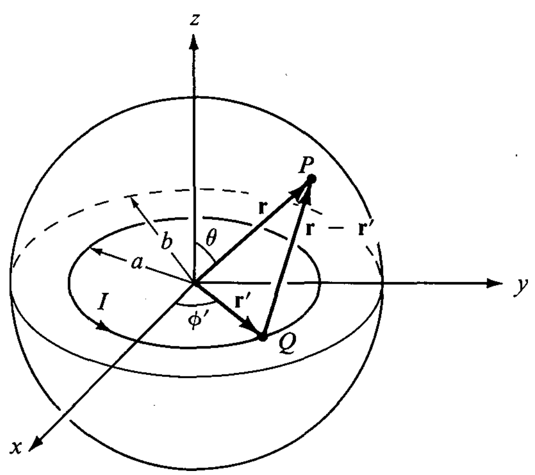


Fig. 5.22. For obtaining the average magnetic flux density due to a magnetic dipole in a spherical volume.

Let us consider an infinitesimal current element $Ia\, d\phi'\, \mathbf{i}_{\phi'}$ at the point $Q(a, \pi/2, \phi')$ on the current loop. The magnetic flux density $d\mathbf{B}$ at a point $P(r, \theta, \phi)$ due to this current element is given by

$$d\mathbf{B} = \frac{\mu_0}{4\pi} \frac{Ia\, d\phi'\, \mathbf{i}_{\phi'} \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \tag{5-106}$$

where $\mathbf{r}$ and $\mathbf{r}'$ are position vectors corresponding to P and Q, respectively. The integral of $d\mathbf{B}$ evaluated in the spherical volume V of radius b can be written as

$$\int_V (d\mathbf{B})\,dv = -\frac{\mu_0 I a\,d\phi'\,\mathbf{i}_{\phi'}}{4\pi} \times \left(-\int_V \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\,dv\right) \tag{5-107}$$

since the integration is with respect to the coordinates of the field point P. Now, the integral on the right side of (5-107) can be recognized as the electric field intensity at $(a, \pi/2, \phi')$ due to a volume charge distribution of uniform density $4\pi\epsilon_0$ C/m^3 in the spherical volume V. From Gauss' law, this electric field intensity is $(4\pi a/3)(\mathbf{r}'/|\mathbf{r}'|)$. Substituting this result in (5-107), we have

$$\begin{aligned}\int_V (d\mathbf{B})\,dv &= -\frac{\mu_0 I a\,d\phi'\,\mathbf{i}_{\phi'}}{4\pi} \times \frac{4\pi a}{3}\frac{\mathbf{r}'}{|\mathbf{r}'|} \\ &= \frac{\mu_0 I a^2}{3}\,d\phi'\,\mathbf{i}_z\end{aligned} \tag{5-108}$$

The volume integral of $\mathbf{B}$ in the volume V due to the entire current loop is then given by

$$\begin{aligned}\int_V \mathbf{B}\,dv &= \int_{\phi'=0}^{2\pi}\int_V (d\mathbf{B})\,dv \\ &= \frac{\mu_0 I a^2}{3}\int_{\phi'=0}^{2\pi} d\phi'\,\mathbf{i}_z = \frac{2\mu_0 I\pi a^2}{3}\,\mathbf{i}_z\end{aligned} \tag{5-109}$$

Finally, the average field due to the dipole in the spherical volume is given by

$$\begin{aligned}\mathbf{B}_{av} &= \frac{1}{V}\int_V \mathbf{B}\,dv \\ &= \frac{1}{\frac{4}{3}\pi b^3}\left(\frac{2\mu_0 I\pi a^2}{3}\mathbf{i}_z\right) = \frac{\mu_0 \mathbf{m}}{2\pi b^3}\end{aligned} \tag{5-110}$$

It is left as an exercise (Problem 5.28) for the student to show that (5-110) is true for any arbitrary current distribution of dipole moment $\mathbf{m}$ situated in the spherical volume of radius b. ▌

From the result (5-110) of Example 5-10, we now relate the magnetizing field $\mathbf{B}_m$ with the average macroscopic field $\mathbf{B}$ as

$$\mathbf{B}_m = \mathbf{B} - \mathbf{B}_{av} = \mathbf{B} - \frac{\mu_0 \mathbf{m}}{2\pi b^3} = \mathbf{B} - \frac{\mu_0 \mathbf{M}}{\frac{3}{2}(\frac{4}{3}\pi b^3)N} \tag{5-111}$$

where we have substituted $\mathbf{m} = \mathbf{M}/N$ from (5-103). Now, if we assume that the molecular volume is equal to the volume of the spherical cavity, then $(\frac{4}{3}\pi b^3)N$ is equal to 1 so that (5-11) reduces to

$$\mathbf{B}_m = \mathbf{B} - \tfrac{2}{3}\mu_0 \mathbf{M} \tag{5-112}$$

Although we have obtained (5-112) by considering a spherical volume for the molecule, it is found that the general expression for $\mathbf{B}_m$ is of the form

$$\mathbf{B}_m = \mathbf{B} + (\gamma - 1)\mu_0 \mathbf{M} \tag{5-113}$$

However, γ may be larger than the value $\frac{1}{3}$ in (5-112) by several orders of magnitude for some materials. Substituting (5-105) into (5-113), we obtain

$$\frac{\mathbf{M}}{N\alpha_m} = \mathbf{B} + (\gamma - 1)\mu_0\mathbf{M} \tag{5-114}$$

Rearranging (5-114), we obtain the relationship between **M** and $\mathbf{B}_m$ as

$$\mathbf{M} = \frac{N\alpha_m}{1 - (\gamma - 1)\mu_0 N\alpha_m}\mathbf{B} \tag{5-115}$$

Defining a dimensionless parameter χ_m, known as the "magnetic susceptibility," as

$$\chi_m = \frac{\mu_0 N\alpha_m}{1 - \gamma\mu_0 N\alpha_m} \tag{5-116}$$

Eq. (5-115) can be written as

$$\mathbf{M} = \frac{\chi_m}{1 + \chi_m}\frac{\mathbf{B}}{\mu_0} \tag{5-117}$$

We have thus established a simple relationship between the magnetization vector **M** and the average macroscopic magnetic field **B** in a magnetic material through the parameter χ_m. The parameter χ_m is, however, constant only for diamagnetic and paramagnetic materials and is dependent on **B** for ferromagnetic materials. Values of χ_m for some diamagnetic and paramagnetic materials are listed in Table 5.3. Also, comparing (5-117) with (5-49), we observe that whereas **M** and **B** are analogous to **P** and **E**, respectively, χ_m is not analogous to χ_e owing to the manner in which χ_m is defined. We will discover the reason for this in Section 5.9. Equation (5-117) indicates that **M** is parallel to **B**. Materials for which this relationship holds, that is, χ_m is independent of the direction of **B** are known as isotropic magnetic materials. For certain magnetic materials, each component of **M** can be dependent on all components of **B**. In such cases, **M** is not parallel to **B** and the materials are not isotropic. Such materials are known as anisotropic magnetic materials.

TABLE 5.3. Magnetic Susceptibilities of Some Diamagnetic and Paramagnetic Materials

Diamagnetic Material	χ_m	*Paramagnetic Material*	χ_m
Nitrogen	-0.50×10^{-8}	Air	3.6×10^{-7}
Hydrogen	-0.21×10^{-8}	Oxygen	2.1×10^{-6}
Gold	-3.60×10^{-5}	Magnesium	1.2×10^{-5}
Mercury	-3.20×10^{-5}	Aluminum	2.3×10^{-5}
Silver	-2.60×10^{-5}	Tungsten	6.8×10^{-5}
Copper	-0.98×10^{-5}	Platinum	2.9×10^{-4}
Sodium	-0.24×10^{-5}	Palladium	8.2×10^{-4}
Bismuth	-1.66×10^{-4}	Liquid oxygen	3.5×10^{-3}

5.8 Magnetic Materials in Magnetic Fields; Magnetization Current

In Section 5.7 we learned that magnetization occurs in magnetic materials under the influence of an applied magnetic field. We defined magnetization by means of a magnetization vector **M**, which is the magnetic dipole moment per unit volume. The magnetization vector is related to the magnetic field responsible for producing it through Eq. (5-117). When a magnetic material is placed in a magnetic field, the resulting magnetization produces a secondary magnetic field, which increases the applied field, which in turn causes a change in the magnetization vector, and so on. When this adjustment process is complete, that is, when a steady state is reached, the sum of the originally applied field and the secondary field must be such that it produces a magnetization which results in the secondary field. The situation is like a feedback loop as shown in Fig. 5.23. We will assume that the adjustment takes place instantaneously with the application of the field and investigate the different effects arising from the magnetization. We do this by first considering an example.

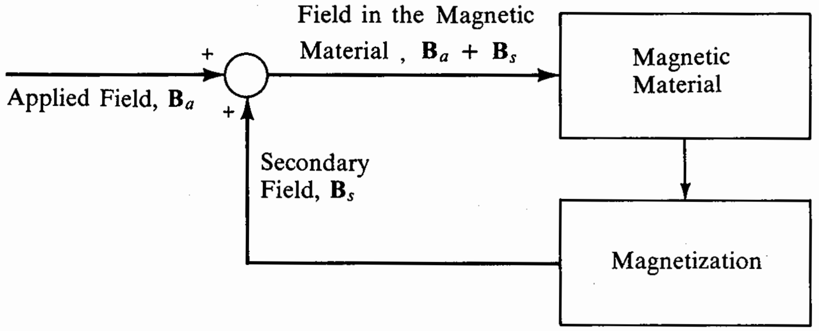


Fig. 5.23. Feedback loop illustrating the adjustment of magnetization in a magnetic material to correspond to the sum of the applied field and the secondary field due to the magnetization.

EXAMPLE 5-11. An infinite plane slab of magnetic material of uniform magnetic susceptibility χ_{m0} and of thickness d occupies the region $0 < z < d$, as shown in Fig. 5.24(a). A uniform magnetic field $\mathbf{B}_a = B_0\mathbf{i}_x$ is applied. It is desired to investigate the effect of magnetization in the material.

The applied magnetic field results in magnetic dipole moments in the material which are oriented along the field. Since the magnetic field and the magnetic susceptibility are uniform, the density of the dipole moments, that is, the magnetization vector **M**, is uniform as shown in Fig. 5.24(b). Such a distribution results in exact cancellation of currents everywhere except at the boundaries of the material since, for each current segment not on the surface, there is a current segment associated with the dipole adjacent to it

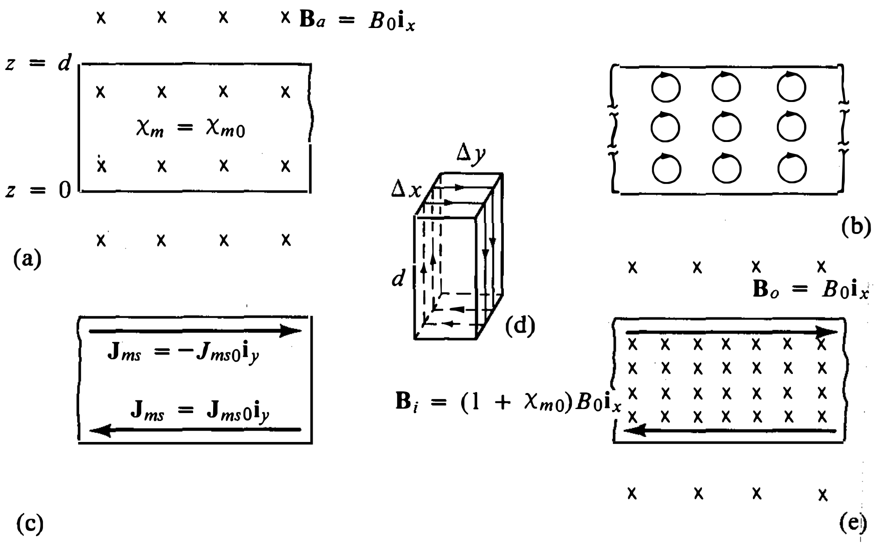


Fig. 5.24. For investigating the effects of magnetization induced in a magnetic material of uniform susceptibility for a uniform applied magnetic field.

and carrying the same amount of current in the opposite direction, thereby cancelling its effect. On the other hand, since the medium changes abruptly from magnetic material to free space at the boundaries, no such cancellation of currents at the boundaries takes place. Thus the net result is the formation of a negative y-directed surface current at the boundary $z = d$ and a positive y-directed surface current at the boundary $z = 0$ as shown in Fig. 5.24(c). These surface currents are known as magnetization surface currents since they are due to the magnetization in the material. In view of the uniform density of the dipole moments, the surface current densities are uniform. Also, in the absence of a net current in the interior of the magnetic material, the surface current densities must be equal in magnitude so that whatever current flows on one surface returns via the other surface.

Let us therefore denote the surface current densities as

$$\mathbf{J}_{ms} = \begin{cases} J_{ms0}\mathbf{i}_y & z = 0 \\ -J_{ms0}\mathbf{i}_y & z = d \end{cases} \tag{5-118}$$

where the subscript m in addition to the other subscripts stands for magnetization. If we now consider a vertical column of infinitesimal rectangular cross-sectional area $\Delta S = (\Delta x)(\Delta y)$ cut out from the magnetic material as shown in Fig. 5-24(d), the rectangular current loop of width Δx makes the column appear as a dipole of moment $(J_{ms0}\,\Delta x)(d\,\Delta y)\mathbf{i}_x$. On the other hand, writing

$$\mathbf{M} = M_0\mathbf{i}_x \tag{5-119}$$

where M_0 is a constant in view of the uniformity of the magnetization, the dipole moment of the column is equal to $\mathbf{M}$ times the volume of the column, or $M_0(d\,\Delta x\,\Delta y)\mathbf{i}_x$. Equating the dipole moments computed in the two different ways, we have

$$J_{ms0} = M_0 \tag{5-120}$$

Thus we have related the surface current density to the magnitude of the magnetization vector. Now, the surface current distribution produces a secondary field $\mathbf{B}_s$ given by

$$\mathbf{B}_s = \begin{cases} \mu_0 J_{ms0}\mathbf{i}_x = \mu_0 M_0 \mathbf{i}_x & \text{for } 0 < z < d \\ 0 & \text{otherwise} \end{cases} \tag{5-121}$$

When the secondary field $\mathbf{B}_s$ is superimposed on the applied field, the net result is an increase in the field inside the material. Denoting the total field inside the material by $\mathbf{B}_i$, we have

$$\mathbf{B}_i = \mathbf{B}_a + \mathbf{B}_s = B_0\mathbf{i}_x + \mu_0 M_0 \mathbf{i}_x = (B_0 + \mu_0 M_0)\mathbf{i}_x \tag{5-122}$$

But, from (5-117),

$$\mathbf{M} = \frac{\chi_{m0}}{1 + \chi_{m0}} \frac{\mathbf{B}_i}{\mu_0} \tag{5-123}$$

Substituting (5-119) and (5-122) into (5-123), we have

$$M_0 = \frac{\chi_{m0}}{1 + \chi_{m0}} \frac{B_0 + \mu_0 M_0}{\mu_0}$$

or

$$M_0 = \frac{\chi_{m0} B_0}{\mu_0} \tag{5-124}$$

Thus the magnetization surface current densities are given by

$$\mathbf{J}_{ms} = \begin{cases} \dfrac{\chi_{m0} B_0}{\mu_0}\mathbf{i}_y & z = 0 \\ -\dfrac{\chi_{m0} B_0}{\mu_0}\mathbf{i}_y & z = d \end{cases} \tag{5-125}$$

and the magnetic flux density inside the material is

$$\mathbf{B}_i = (1 + \chi_{m0})B_0\mathbf{i}_x \tag{5-126}$$

Since the secondary field produced outside the material by the surface current distribution is zero, the total field B_0 outside the material remains the same as the applied field. The field distribution both inside and outside the magnetic material is shown in Fig. 5-24(e). Although we have demonstrated only the formation of a magnetization surface current in this example, it is easy to visualize that a nonuniform applied magnetic field or a nonuniform magnetic susceptibility of the material will result in the formation of a magnetization volume current in the magnetic material due to imperfect cancellation of the currents associated with the dipoles. ∎

We now derive general expressions for magnetization surface and volume current densities in terms of the magnetization vector. To do this, let us consider a magnetic material of volume V' in which the magnetization vector $\mathbf{M}$ is an arbitrary function of position, as shown in Fig. 5.25. We divide the

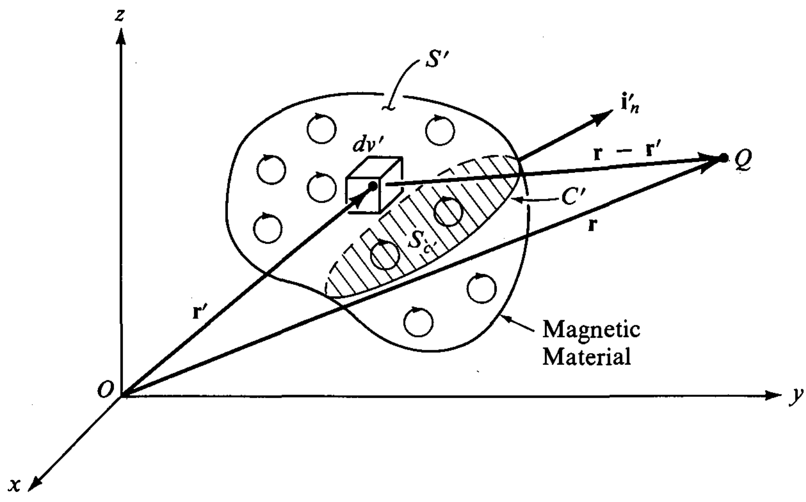


Fig. 5.25. For evaluating the magnetic vector potential due to induced magnetization in a magnetic material.

volume V' into a number of infinitesimal volumes dv'_i, $i = 1, 2, 3, \ldots, n$ defined by position vectors $\mathbf{r}'_i$, $i = 1, 2, 3, \ldots, n$, respectively. In each infinitesimal volume, we can consider $\mathbf{M}$ to be a constant so that the dipole moment in the ith volume is $\mathbf{M}_i\, dv'_i$. From (3-96), the magnetic vector potential $d\mathbf{A}_i$ at a point $Q(\mathbf{r})$ due to the dipole moment in the ith volume is given by

$$d\mathbf{A}_i = \frac{\mu_0}{4\pi} \frac{\mathbf{M}_i\, dv'_i \times (\mathbf{r} - \mathbf{r}'_i)}{|\mathbf{r} - \mathbf{r}'_i|^3}$$

The total vector potential at $Q(\mathbf{r})$ due to the dipole moments in all the n infinitesimal volumes is then given by

$$\mathbf{A}_i = \sum_{i=1}^{n} d\mathbf{A}_i = \frac{\mu_0}{4\pi} \sum_{i=1}^{n} \frac{\mathbf{M}_i\, dv'_i \times (\mathbf{r} - \mathbf{r}'_i)}{|\mathbf{r} - \mathbf{r}'_i|^3} \tag{5-127}$$

Equation (5-127) is good only for $|\mathbf{r}| \gg |\mathbf{r}'_i|$, where $i = 1, 2, 3, \ldots, n$ since each dv'_i has a finite although infinitesimal volume. However, in the limit that $n \longrightarrow \infty$, all the infinitesimal volumes tend to zero; the right side of (5-127) becomes an integral and the expression is valid for any $\mathbf{r}$. Thus

$$\begin{aligned} \mathbf{A}(\mathbf{r}) &= \frac{\mu_0}{4\pi} \int_{V'} \frac{\mathbf{M}\, dv' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \\ &= \frac{\mu_0}{4\pi} \int_{V'} \mathbf{M} \times \nabla' \frac{1}{|\mathbf{r} - \mathbf{r}'|}\, dv' \end{aligned} \tag{5-128}$$

Substituting the vector identity

$$\nabla' \times \frac{\mathbf{M}}{|\mathbf{r}-\mathbf{r}'|} = \nabla' \frac{1}{|\mathbf{r}-\mathbf{r}'|} \times \mathbf{M} + \frac{1}{|\mathbf{r}-\mathbf{r}'|} \nabla' \times \mathbf{M}$$

in (5-128), we obtain

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{V'} \frac{\nabla' \times \mathbf{M}}{|\mathbf{r}-\mathbf{r}'|} dv' - \frac{\mu_0}{4\pi} \int_{V'} \nabla' \times \frac{\mathbf{M}}{|\mathbf{r}-\mathbf{r}'|} dv' \qquad (5\text{-}129)$$

Taking the dot product of the second integral on the right side of (5-129) with the unit vector $\mathbf{i}_x$ and using the divergence theorem, we have

$$\begin{aligned} \mathbf{i}_x \cdot \int_{V'} \nabla' \times \frac{\mathbf{M}}{|\mathbf{r}-\mathbf{r}'|} dv' &= \int_{V'} \mathbf{i}_x \cdot \nabla' \times \frac{\mathbf{M}}{|\mathbf{r}-\mathbf{r}'|} dv' \\ &= -\int_{V'} \nabla' \cdot \left(\mathbf{i}_x \times \frac{\mathbf{M}}{|\mathbf{r}-\mathbf{r}'|}\right) dv' \qquad (5\text{-}130) \\ &= -\oint_{S'} \mathbf{i}_x \times \frac{\mathbf{M}}{|\mathbf{r}-\mathbf{r}'|} \cdot \mathbf{i}'_n \, dS' \end{aligned}$$

where S' is the surface bounding the volume V' and $\mathbf{i}'_n$ is the unit normal vector to dS'. Proceeding further, we obtain

$$\begin{aligned} \mathbf{i}_x \cdot \int_{V'} \nabla' \times \frac{\mathbf{M}}{|\mathbf{r}-\mathbf{r}'|} dv' &= -\oint_{S'} \mathbf{i}_x \times \frac{\mathbf{M}}{|\mathbf{r}-\mathbf{r}'|} \cdot \mathbf{i}'_n \, dS' \\ &= -\oint_{S'} \mathbf{i}_x \cdot \frac{\mathbf{M} \times \mathbf{i}'_n}{|\mathbf{r}-\mathbf{r}'|} dS' \qquad (5\text{-}131\text{a}) \\ &= -\mathbf{i}_x \cdot \oint_{S'} \frac{\mathbf{M} \times \mathbf{i}'_n}{|\mathbf{r}-\mathbf{r}'|} dS' \end{aligned}$$

Similarly, we can show that

$$\mathbf{i}_y \cdot \int_{V'} \nabla' \times \frac{\mathbf{M}}{|\mathbf{r}-\mathbf{r}'|} dv' = -\mathbf{i}_y \cdot \oint_{S'} \frac{\mathbf{M} \times \mathbf{i}'_n}{|\mathbf{r}-\mathbf{r}'|} dS' \qquad (5\text{-}131\text{b})$$

and

$$\mathbf{i}_z \cdot \int_{V'} \nabla' \times \frac{\mathbf{M}}{|\mathbf{r}-\mathbf{r}'|} dv' = -\mathbf{i}_z \cdot \oint_{S'} \frac{\mathbf{M} \times \mathbf{i}'_n}{|\mathbf{r}-\mathbf{r}'|} dS' \qquad (5\text{-}131\text{c})$$

It then follows from (5-131a)–(5-131c) that

$$\int_{V'} \nabla' \times \frac{\mathbf{M}}{|\mathbf{r}-\mathbf{r}'|} dv' = -\oint_{S'} \frac{\mathbf{M} \times \mathbf{i}'_n}{|\mathbf{r}-\mathbf{r}'|} dS' \qquad (5\text{-}132)$$

Substituting (5-132) into (5-129), we get

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{V'} \frac{\nabla' \times \mathbf{M}}{|\mathbf{r}-\mathbf{r}'|} dv' + \frac{\mu_0}{4\pi} \oint_{S'} \frac{\mathbf{M} \times \mathbf{i}'_n}{|\mathbf{r}-\mathbf{r}'|} dS' \qquad (5\text{-}133)$$

The first integral on the right side of (5-133) represents the vector potential at $Q(\mathbf{r})$ due to a volume current of density $\nabla' \times \mathbf{M}$ in the volume V' and the second integral is the vector potential at $Q(\mathbf{r})$ due to a surface current of density $\mathbf{M} \times \mathbf{i}'_n$ on the surface S'. Thus the vector potential at $Q(\mathbf{r})$ due to the magnetization in the magnetic material is the same as the sum of the vector potentials at $Q(\mathbf{r})$ due to a magnetization volume current of density

$$\mathbf{J}_m(\mathbf{r}') = \nabla' \times \mathbf{M}(\mathbf{r}') \qquad \text{in } V' \tag{5-134}$$

and due to a magnetization surface current of density

$$\mathbf{J}_{ms}(\mathbf{r}') = \mathbf{M}(\mathbf{r}') \times \mathbf{i}'_n \qquad \text{on } S' \tag{5-135}$$

We note that the total volume current through any cross-sectional area $S_{c'}$ (of the volume V') bounded by the contour C' as shown in Fig. 5.25 is given by

$$\begin{aligned}\int_{S_{c'}} \mathbf{J}_m \cdot d\mathbf{S}_{c'} &= \int_{S_{c'}} (\nabla' \times \mathbf{M}) \cdot d\mathbf{S}_{c'} = \oint_{C'} \mathbf{M} \cdot d\mathbf{l}' \\ &= -\oint_{C'} (\mathbf{M} \times \mathbf{i}'_n) \cdot (\mathbf{i}'_n \times d\mathbf{l}') = -\oint_{C'} \mathbf{J}_{ms} \cdot (\mathbf{i}'_n \times d\mathbf{l}')\end{aligned} \tag{5-136}$$

where we have used Stokes' theorem to transform the surface integration to line integration. The right side of (5-136) is exactly the surface current crossing the contour C' in the opposite direction to the volume current. Omitting the primes in (5-134) and (5-136), we have

$$\mathbf{J}_m = \nabla \times \mathbf{M} \tag{5-137}$$

$$\mathbf{J}_{ms} = \mathbf{M} \times \mathbf{i}_n \tag{5-138}$$

Summarizing what we have learned in this section, the magnetic dipole moments due to magnetization in a magnetic material placed in a magnetic field have the effect of creating in general the following:

(a) Magnetization surface currents, having densities given by (5-138), at the boundaries of the magnetic material.

(b) Magnetization volume current of density given by (5-137) in the magnetic material and such that the total volume current flowing through any cross-sectional area of the material is exactly opposite to the total surface current crossing the contour bounding the area.

We have also shown that the magnetization currents alter the applied magnetic field in the material. Such a modification of the applied field occurs outside the material as well in the general case. In the time-varying case, the electric field associated with the applied magnetic field is also altered by the addition of the secondary electric field due to the time variation of the secondary magnetic field.

5.9 Magnetic Field Intensity, Relative Permeability, and Hysteresis

In Section 5.8 we learned that the magnetic field in a magnetic material is the superposition of an applied field $\mathbf{B}_a$ and a secondary field $\mathbf{B}_s$ which results from the magnetization $\mathbf{M}$, which in turn is produced by the total field $(\mathbf{B}_a + \mathbf{B}_s)$, as shown in Fig. 5-23. Thus, from Fig. 5-23 and Eq. (5-117), we have

$$\mathbf{M} = \frac{\chi_m}{1+\chi_m} \frac{\mathbf{B}_a + \mathbf{B}_s}{\mu_0} \tag{5-139}$$

$$\mathbf{B}_s = f(\mathbf{M}) \tag{5-140}$$

where $f(\mathbf{M})$ denotes a function of $\mathbf{M}$. Determination of the secondary field $\mathbf{B}_s$ and hence the total field $\mathbf{B}_a + \mathbf{B}_s$ for a given applied field $\mathbf{B}_a$ requires a simultaneous solution of (5-139) and (5-140) which, in general, is very inconvenient. To circumvent this problem, we make use of the results of Section 5.8, in which we found that the magnetization is equivalent to a magnetization surface current of density $\mathbf{J}_{ms}$ and a magnetization volume current of density $\mathbf{J}_m$ as given by (5-138) and (5-137), respectively. The secondary magnetic and electric fields are the fields produced by these currents as if they were situated in free space, in the same way as the currents responsible for the applied magnetic field and its associated electric field.

Thus the secondary electromagnetic field satisfies Maxwell's equations

$$\nabla \cdot \mathbf{D}_s = 0 \tag{5-141a}$$

$$\nabla \cdot \mathbf{B}_s = 0 \tag{5-141b}$$

$$\nabla \times \mathbf{E}_s = -\frac{\partial \mathbf{B}_s}{\partial t} \tag{5-141c}$$

$$\nabla \times \mathbf{B}_s = \mu_0 \left(\mathbf{J}_m + \frac{\partial \mathbf{D}_s}{\partial t} \right) \tag{5-141d}$$

where $\mathbf{E}_s$ is the secondary electric field intensity and $\mathbf{D}_s$ is its associated displacement flux density. On the other hand, if the "true" current and charge densities responsible for the applied field $\mathbf{B}_a$ with its associated electric field intensity $\mathbf{E}_a$ and displacement flux density $\mathbf{D}_a$ are $\mathbf{J}$ and ρ, respectively, we have

$$\nabla \cdot \mathbf{D}_a = \rho \tag{5-142a}$$

$$\nabla \cdot \mathbf{B}_a = 0 \tag{5-142b}$$

$$\nabla \times \mathbf{E}_a = -\frac{\partial \mathbf{B}_a}{\partial t} \tag{4-142c}$$

$$\nabla \times \mathbf{B}_a = \mu_0 \left(\mathbf{J} + \frac{\partial \mathbf{D}_a}{\partial t} \right) \tag{5-142d}$$

Now, adding (5-141a)–(5-141d) to (5-142a)–(5-142d), respectively, we obtain

$$\nabla \cdot (\mathbf{D}_a + \mathbf{D}_s) = \rho + 0 = \rho \tag{5-143a}$$

$$\nabla \cdot (\mathbf{B}_a + \mathbf{B}_s) = 0 \tag{5-143b}$$

$$\nabla \times (\mathbf{E}_a + \mathbf{E}_s) = -\frac{\partial}{\partial t}(\mathbf{B}_a + \mathbf{B}_s) \tag{5-143c}$$

$$\nabla \times (\mathbf{B}_a + \mathbf{B}_s) = \mu_0 \left[\mathbf{J} + \mathbf{J}_m + \frac{\partial}{\partial t}(\mathbf{D}_a + \mathbf{D}_s) \right] \tag{5-143d}$$

Substituting

$$\mathbf{B} = \mathbf{B}_a + \mathbf{B}_s \tag{5-144a}$$

$$\mathbf{E} = \mathbf{E}_a + \mathbf{E}_s \tag{5-144b}$$

$$\mathbf{D} = \mathbf{D}_a + \mathbf{D}_s \tag{5-145}$$

$$\mathbf{J}_m = \nabla \times \mathbf{M} \tag{5-137}$$

in (5-143a)–(5-143d) and rearranging, we have

$$\nabla \cdot \mathbf{D} = \rho \tag{5-146a}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{5-146b}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{5-146c}$$

$$\nabla \times \left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M}\right) = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{5-146d}$$

where **E**, **B**, and **D** are the total fields.

We now define a vector **H**, known as the magnetic field intensity vector and given by

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \tag{5-147}$$

Note that the units of **H** are the same as those of $\mathbf{B}/\mu_0$ and **M**, that is, amperes per meter. Comparing with the units of volts per meter for the electric field intensity, we see the reason for referring to **H** as the magnetic field intensity. Substituting (5-147) into (5-146a)–(5-146d), we obtain

$$\nabla \cdot \mathbf{D} = \rho \tag{5-148}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{5-149}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{5-150}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{5-151}$$

Thus the new field **H** results in a set of equations which do not explicitly contain the magnetization current density, unlike Eqs. (5-143a)–(5-143d). Substituting for **M** in (5-147) from (5-117), we have

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \frac{\chi_m}{1 + \chi_m}\frac{\mathbf{B}}{\mu_0} = \frac{\mathbf{B}}{\mu_0(1 + \chi_m)} = \frac{\mathbf{B}}{\mu_0 \mu_r} = \frac{\mathbf{B}}{\mu} \tag{5-152}$$

where we define

$$\mu_r = 1 + \chi_m \tag{5-153}$$

and

$$\mu = \mu_0 \mu_r \tag{5-154}$$

The quantity μ_r is known as the relative permeability of the magnetic material and μ is the permeability of the magnetic material. Note that μ_r is dimensionless and that (5-152) is true only for linear magnetic materials if μ is to be treated as a constant for a particular magnetic material, whereas (5-147) holds in general. Substituting (5-152) into (5-148)–(5-151), we obtain

$$\nabla \cdot \mathbf{D} = \rho \tag{5-155a}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{5-155b}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{5-155c}$$

$$\nabla \times \mathbf{B} = \mu\left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) \tag{5-155d}$$

Equations (5-155a)–(5-155d) are the same as Maxwell's equations for nonmagnetic materials as given by (5-80)–(5-83) except that μ_0 is replaced by μ. Thus the electric and magnetic fields in the presence of a magnetic material can be computed in exactly the same manner as for nonmagnetic materials except that we have to use μ instead of μ_0 for permeability. In fact, if $\chi_m = 0$, $\mu_r = 1$ and $\mu = \mu_0$ so that a nonmagnetic material can be considered as a magnetic material with $\mu = \mu_0$ and hence Eqs. (5-155a)–(5-155d) can be used for nonmagnetic materials as well. The permeability μ takes into account the effects of magnetization and there is no need to consider them when we use μ for μ_0, thereby eliminating the necessity for the simultaneous solution of (5-139) and (5-140). In the case of a boundary between two different magnetic materials, the appropriate boundary conditions for **H** take into account implicitly the magnetization surface current. We will consider these boundary conditions in Section 5.12. Substituting for **B** in (5-117) in terms of **H** by using (5-152), we obtain

$$\mathbf{M} = \frac{\chi_m}{1 + \chi_m}\frac{\mathbf{B}}{\mu_0} = \frac{\chi_m}{1 + \chi_m}\frac{\mu_0(1 + \chi_m)}{\mu_0}\mathbf{H} = \chi_m \mathbf{H} \tag{5-156}$$

Equation (5-156) represents the traditional definition for χ_m, because of which we defined χ_m in Section 5.7 in a manner which is not analogous to the definition of χ_e in Section 5.4.

EXAMPLE 5-12. For the slab of magnetic material in Example 5-11, find and sketch the magnetic field intensity and the magnetic flux density vectors both inside and outside the material.

From Example 5-11, the magnetic flux density inside the magnetic material is given by

$$\mathbf{B}_i = (1 + \chi_{m0})B_0\mathbf{i}_x \tag{5-126}$$

The relative permeability of the material is $1 + \chi_{m0}$. Thus the magnetic field intensity inside the material is

$$\mathbf{H}_i = \frac{\mathbf{B}_i}{\mu_0(1 + \chi_{m0})} = \frac{(1 + \chi_{m0})B_0}{\mu_0(1 + \chi_{m0})}\mathbf{i}_x = \frac{B_0}{\mu_0}\mathbf{i}_x$$

Outside the magnetic material, the magnetic flux density is the same as the applied value so that the magnetic field intensity is

$$\mathbf{H}_o = \frac{\mathbf{B}_a}{\mu_0} = \frac{B_0}{\mu_0}\mathbf{i}_x$$

Thus, for this example, the magnetic field intensity vectors inside and outside the magnetic material are the same and equal to the magnetic field intensity associated with the applied magnetic flux density. Both **H** and **B** fields inside and outside the material are shown in Fig. 5.26(a). Now, considering a rectangle *abcda* in the *xz* plane and with the sides *ab* and *cd* parallel to the boundary $z = 0$ as shown in Fig. 5.26(b), we note that **H** is uniform along the contour *abcda* since it has the same value both inside and outside the material. Thus

$$\oint_{abcda} \mathbf{H} \cdot d\mathbf{l} = \mathbf{H} \cdot \oint_{abcda} d\mathbf{l} = 0 \tag{5-157}$$

On the other hand, noting that **B** is parallel to *ab* and *cd* but having unequal

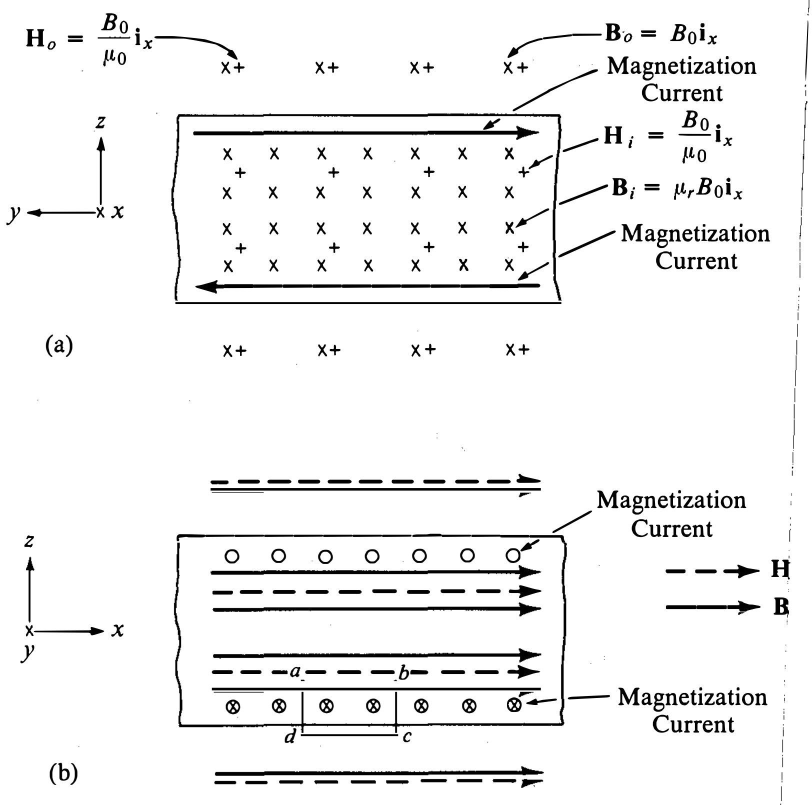


Fig. 5.26. Magnetic field intensity and magnetic flux density vectors for the magnetic material slab of Example 5-11.

magnitudes along them and perpendicular to *bc* and *da*, we obtain

$$\begin{aligned}
\oint_{abcda} \mathbf{B} \cdot d\mathbf{l} &= \int_a^b \mathbf{B}_i \cdot d\mathbf{l} + \int_c^d \mathbf{B}_o \cdot d\mathbf{l} \\
&= \int_a^b (1 + \chi_{m0}) B_0 \mathbf{i}_x \cdot dx\, \mathbf{i}_x + \int_c^d B_0 \mathbf{i}_x \cdot (-dx\, \mathbf{i}_x) \\
&= (1 + \chi_{m0}) B_0 (ab) - B_0 (cd) \\
&= \chi_m B_0 (ab) = \mu_0 \left(\frac{\chi_{m0} B_0}{\mu_0} \mathbf{i}_y \right) \cdot [(ab)\mathbf{i}_y] \\
&= \mu_0 \text{ (magnetization surface current enclosed by } abcda)
\end{aligned} \tag{5-158}$$

Comparing (5-157) and (5-158), we observe that the circulation of **H** is independent of magnetization currents whereas the circulation of **B** is not. The circulation of **H** depends only on those currents other than the magnetization currents, whereas the circulation of **B** depends on all kinds of currents. ▮

Returning now to Eq. (5-153), we note, from the values of χ_m for diamagnetic and paramagnetic materials listed in Table 5.3, that the relative permeabilities for these materials differ very little from unity. On the other hand, for ferromagnetic materials, the relative permeability can assume values of the order of several thousand. In fact, for these materials the relationship between **B** and **H** is nonlinear and is characterized by hysteresis so that there is no unique value of μ_r for a particular ferromagnetic material. The relationship between **B** and **H** is therefore presented in graphical form, as shown by a typical curve in Fig. 5.27. This curve is known as the hysteresis

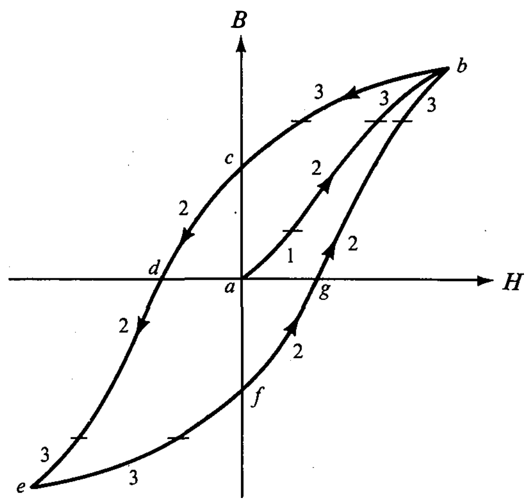


Fig. 5.27. Hysteresis curve for a ferromagnetic material.

curve or the **B-H** curve. To trace the development of the hysteresis effect, we start with an unmagnetized sample of ferromagnetic material in which both **B** and **H** are initially zero, corresponding to point *a* on the curve. As **H** is increased, the magnetization builds up, thereby increasing **B** gradually along the curve *ab* and finally to saturation at *b*, according to the following sequence of events as discussed in Section 5.7: (a) reversible motion of domain walls, (b) irreversible motion of domain walls, and (c) domain rotation. The regions corresponding to these events along the curve *ab* as well as other portions of the hysteresis curve are shown marked 1, 2, and 3, respectively, in Fig. 5-27. If the value of **H** is now decreased to zero, the value of **B** does not retrace the curve *ab* backwards but instead follows the curve *bc*, which indicates that a certain amount of magnetization remains in the material even after the magnetizing field is completly removed. In fact, it requires a magnetic field intensity in the opposite direction to bring **B** back to zero as shown by the portion *cd* of the curve. The value of **B** at the point *c* is known as the "remanence" or "retentivity," whereas the value of **H** at *d* is known as the "coercivity" of the material. Further increase in **H** in this direction results in the saturation of **B** in the direction opposite to that corresponding to *b* as shown by the portion *de* of the curve. If **H** is now decreased to zero, reversed in direction, and increased, the resulting variation of **B** occurs in accordance with the curve *efgb*, thereby completing the hysteresis loop. The characteristics of hysteresis curves for a few ferromagnetic materials are listed in Table 5.4. In view of the hysteresis effect, the incre-

TABLE 5.4. Characteristics of Hysteresis Curves for Some Ferromagnetic Materials

Material	*Remanence, Wb/m²*	*Coercivity, amp/m*	*Saturation Magnetization, Wb/m²*	*Maximum μ_r*
Cast iron	0.53	366	—	600
Permendur	1.4	160	2.4	5,000
Permalloy	—	24	1.6	25,000
Hypernik	0.73	3.2	1.65	70,000
Mumetal	—	4	0.65	100,000
Supermalloy	—	0.32	0.8	1,050,000

mental relative permeability defined by the slope of the hysteresis curve as given by

$$\mu_{ir} = \frac{1}{\mu_0}\frac{\Delta B}{\Delta H} \tag{5-159}$$

is a useful parameter in addition to the relative permeability given by

$$\mu_r = \frac{1}{\mu_0}\frac{B}{H} \tag{5-160}$$

for ferromagnetic materials.

5.10 Summary of Maxwell's Equations and Constitutive Relations

In the previous sections we introduced successively conductors, dielectrics, and magnetic materials. We discussed the various phenomena occurring in these materials in the presence of electric and magnetic fields. We learned several new concepts in this process. Important among these are the introduction of two new field vectors **D** and **H**. With the aid of these two new vectors, we developed a set of Maxwell's equations which permit us to solve field problems involving material media without explicitly taking into account the various phenomena occurring in them. These Maxwell's equations are given by

$$\nabla \cdot \mathbf{D} = \rho \tag{5-161}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{5-162}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{5-163}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{5-164}$$

where ρ and **J** are the volume densities of the true charges and currents responsible for the fields characterized by the field intensity vectors **E** and **H** and the corresponding flux density vectors **D** and **B**. Equations (5-161)–(5-164) can as well be used for free space since they reduce to those of free space when the pertinent quantities are allowed to approach their free-space values.

The true charges are those which are free to move and not bound to their respective nucleii, as polarization charges are. Conduction charges in materials and space charges in vacuum tubes are examples of true charges. The true currents are those constituted by the movement of the free charges, as compared to polarization and magnetization currents which are due to the movement of charges bound to their respective nucleii. Conduction currents in materials and convection currents due to movement of space charge in vacuum tubes are examples of true currents. Thus **J** in (5-164) can represent conduction currents as well as convection currents. The charge and current densities are related via the continuity equation given by

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0 \tag{5-165}$$

The four Maxwell's equations given by (5-161)–(5-164) are not all independent; Eq. (5-162) follows from Eq. (5-163) whereas Eq. (5-161) follows from Eq. (5-164) with the aid of the continuity equation.

The vectors **E** and **B** are the fundamental field vectors which define the force **F** acting on a charge q moving with a velocity **v** in an electromagnetic field in accordance with the Lorentz force equation given by

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{5-166}$$

The vectors **D** and **H** are mixed vectors which take into account the dielectric and magnetic properties of materials, respectively. They are related to **E** and **B**, respectively, via the equations

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \tag{5-167}$$

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \tag{5-168}$$

where **P** and **M** are the polarization and magnetization vectors, which define the state of polarization and magnetization, respectively, in the material. However, the relations which are more useful than (5-167) and (5-168) are

$$\mathbf{D} = \epsilon \mathbf{E} \tag{5-169}$$

$$\mathbf{H} = \frac{\mathbf{B}}{\mu} \tag{5-170}$$

where ϵ and μ are the permittivity and permeability, respectively, of the material which take into account implicitly the effects of **P** and **M**, respectively. Furthermore, for a material medium, the current density **J** is related to the electric field intensity **E** by

$$\mathbf{J} = \mathbf{J}_c = \sigma \mathbf{E} \tag{5-171}$$

where σ is the conductivity which takes into account the conductive property of the material. Equations (5-169), (5-170), and (5-171) are known as the constitutive relations. Together with the constitutive relations, Maxwell's equations form a sufficient set of equations to determine uniquely the electromagnetic field for a given ρ and **J** and in a medium for which ϵ, μ, and σ are specified.

For static fields, the time variations of all quantities are zero so that Maxwell's equations (5-161)–(5-164) reduce to

$$\nabla \cdot \mathbf{D} = \rho \tag{5-172}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{5-173}$$

$$\nabla \times \mathbf{E} = 0 \tag{5-174}$$

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{5-175}$$

whereas the continuity equation is given by

$$\nabla \cdot \mathbf{J} = 0 \tag{5-176}$$

In this case, we note that all four equations (5-172)–(5-175) are independent. For $\mathbf{J} = \mathbf{J}_c = \sigma \mathbf{E}$, Eq. (5-175) indicates coupling between electric and magnetic fields which is not present in the case of static fields in free space. We note, however, that this is a one-way coupling unlike the two-way coupling in the case of time-varying fields since the magnetic field depends upon the electric field through (5-175) but the electric field is independent of the magnetic field.

Returning to Maxwell's equations for arbitrarily time-varying fields given by (5-161)–(5-164), we obtain Maxwell's equations for sinusoidally time-varying fields by substituting the complex vectors $\bar{\mathbf{E}}$, $\bar{\mathbf{B}}$, $\bar{\mathbf{D}}$, and $\bar{\mathbf{H}}$ for the real vectors **E**, **B**, **D**, and **H** and by replacing $\partial/\partial t$ by $j\omega$. Thus we have

$$\nabla \cdot \bar{\mathbf{D}} = \rho \tag{5-177}$$

$$\nabla \cdot \bar{\mathbf{B}} = 0 \tag{5-178}$$

$$\nabla \times \bar{\mathbf{E}} = -j\omega\bar{\mathbf{B}} \tag{5-179}$$

$$\nabla \times \bar{\mathbf{H}} = \bar{\mathbf{J}} + j\omega\bar{\mathbf{D}} \tag{5-180}$$

Writing (5-180) for a material medium as

$$\nabla \times \bar{\mathbf{H}} = \sigma\bar{\mathbf{E}} + j\omega\epsilon\bar{\mathbf{E}} \tag{5-181}$$

we observe that for $\sigma \gg \omega\epsilon$, the magnitude of the conduction current density term is greater than the magnitude of the displacement current density term so that $\nabla \times \bar{\mathbf{H}} \approx \sigma\bar{\mathbf{E}}$. The material is then classified as a good conductor. On the other hand, for $\sigma \ll \omega\epsilon$, the magnitude of the displacement current density term is greater than the magnitude of the conduction current density term so that $\nabla \times \bar{\mathbf{H}} \approx j\omega\epsilon\bar{\mathbf{E}}$. The material is then classified as a good dielectric. The critical frequency for which the two terms are equal is given by $\sigma = \omega\epsilon$ or $\omega = \sigma/\epsilon$. Thus, depending upon whether $\omega \ll \sigma/\epsilon$ or $\omega \gg \sigma/\epsilon$, the material can be regarded as a good conductor or a good dielectric. The situation, however, is not so simple because both σ and ϵ are in general functions of frequency.

With the understanding that σ and ϵ are frequency dependent, we now classify nonmagnetic materials as follows for the purpose of writing simplified sets of Maxwell's equations:

(a) Perfect dielectrics: These are idealizations of good dielectrics. These contain no true charges and currents. The corresponding Maxwell's equations are obtained by setting $\rho = 0$ and $\mathbf{J} = 0$.

(b) Good conductors: The magnitude of the conduction current density $\sigma\mathbf{E}$ is much greater than the magnitude of the displacement current density $\partial\mathbf{D}/\partial t$. To obtain the special set of Maxwell's equations, we set $\partial\mathbf{D}/\partial t = 0$. We also set $\rho = 0$ since, in accordance with the finding in Section 5.3, any charge density inside the conductor decays exponentially with a time constant equal to ϵ/σ, where we have replaced ϵ_0 in Section 5.3 by ϵ. For good conductors, $\sigma/\epsilon \gg \omega$ so that any initial charge density decays to a negligible fraction of its value in a fraction of a period.

(c) Perfect conductors: These are idealizations of good conductors obtained by letting $\sigma \longrightarrow \infty$. The electric field inside a perfect conductor must be zero since otherwise, $\mathbf{J}_c = \sigma\mathbf{E}$ becomes infinite. Furthermore, for the time-varying case, the zero electric field results in $\partial\mathbf{B}/\partial t$ equal to zero or **B** equal to a constant with time. Thus a time-varying magnetic field cannot exist inside a perfect conductor. Hence we conclude that all fields inside a perfect conductor are zero for the time-varying case and the electric field is zero also for the static case. There remains only the possibility of a static magnetic field inside a perfect conductor.

We now list, in Table 5.5, Maxwell's equations and the continuity equation for the general case and for the special cases discussed above for both time-varying and static fields. Also listed are the corresponding integral forms of the equations. We note that, in certain cases, although certain terms on the right sides of the differential equations are set equal to zero, the corresponding terms on the right sides of the corresponding integral equations are not set equal to zero. This is because a differential equation is applicable at a point whereas the corresponding integral equation is appli-

TABLE 5.5. Summary of Maxwell's Equations and the Continuity Equation for Various Cases

Description	*Differential Form*	*Integral Form*
Time-varying fields; general case	$\nabla \cdot \mathbf{D} = \rho$	$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho\, dv$
	$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$
	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S}$
	$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$	$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt}\int_S \mathbf{D} \cdot d\mathbf{S}$
	$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0$	$\oint_S \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt}\int_V \rho\, dv = 0$
Static fields; general case	$\nabla \cdot \mathbf{D} = \rho$	$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho\, dv$
	$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$
	$\nabla \times \mathbf{E} = 0$	$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$
	$\nabla \times \mathbf{H} = \mathbf{J}$	$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S}$
	$\nabla \cdot \mathbf{J} = 0$	$\oint_S \mathbf{J} \cdot d\mathbf{S} = 0$
Time-varying fields; perfect dielectrics $\rho = 0, \mathbf{J} = 0$	$\nabla \cdot \mathbf{D} = 0$	$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho\, dv$
	$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$
	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S}$
	$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}$	$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt}\int_S \mathbf{D} \cdot d\mathbf{S}$

TABLE 5.5 (Cont'd.)

Description	*Differential Form*	*Integral Form*
Static fields; perfect dielectrics $\rho = 0, \mathbf{J} = 0$	$\nabla \cdot \mathbf{D} = 0$	$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho \, dv$
	$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$
	$\nabla \times \mathbf{E} = 0$	$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$
	$\nabla \times \mathbf{H} = 0$	$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S}$
Time-varying fields; good conductors $\lvert \sigma \mathbf{E} \rvert \gg \left\lvert \frac{\partial \mathbf{D}}{\partial t} \right\rvert$ uniform σ	$\nabla \cdot \mathbf{D} = 0$	$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho \, dv$
	$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$
	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{S}$
	$\nabla \times \mathbf{H} = \mathbf{J}_c = \sigma \mathbf{E}$	$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt} \int_S \mathbf{D} \cdot d\mathbf{S}$
	$\nabla \cdot \mathbf{J}_c = 0$	$\oint_S \mathbf{J}_c \cdot d\mathbf{S} + \frac{d}{dt} \int_V \rho \, dv = 0$
Static fields; conductors, uniform σ	$\nabla \cdot \mathbf{D} = 0$	$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho \, dv$
	$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$
	$\nabla \times \mathbf{E} = 0$	$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$
	$\nabla \times \mathbf{H} = \mathbf{J}_c = \sigma \mathbf{E}$	$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S}$
	$\nabla \cdot \mathbf{J}_c = 0$	$\oint_S \mathbf{J}_c \cdot d\mathbf{S} = 0$
Perfect conductors	All fields are zero for the time-varying case; electric field is zero for the static case	

cable over a region. For example, although there is no true charge density associated with any point in a perfect dielectric medium, it is possible that a closed surface situated entirely within such a medium of finite extent can enclose a charge contained in that part of the volume bounded by the surface but lying outside the medium. Hence, although $\nabla \cdot \mathbf{D} = 0$ in the medium, we have to write the corresponding integral form as $\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho \, dv$.

5.11 Power and Energy Considerations for Material Media

In Section 5.2 we learned that conductors are characterized by conduction current due to the movement of free charges under the influence of an applied electric field. In Section 4.8 we showed that the power expended by an electric field due to charges moving under its influence in a volume V is given by

$$P_d = \int_V \mathbf{E} \cdot \mathbf{J}\, dv \tag{5-182}$$

where $\mathbf{J}$ is the current density resulting from the movement of the charges. If the motion of the charges is in free space, they are accelerated by the electric field and hence the power expended by the electric field is converted into kinetic energy. On the other hand, the free charges in a conductor drift with an average drift velocity because of the frictional mechanism provided by their collisions with the atomic lattice. Hence the power expended by the electric field is dissipated in the conductor in the form of heat. Replacing $\mathbf{J}$ in (5-182) by $\sigma\mathbf{E}$, we obtain the expression for the power dissipated in a volume V of a conductor as

$$P_d = \int_V \mathbf{E} \cdot \sigma\mathbf{E}\, dv \tag{5-183}$$

It follows that the power dissipation density in a conductor is

$$p_d = \mathbf{E} \cdot \sigma\mathbf{E} = \sigma E^2 \tag{5-184}$$

For sinusoidally time-varying fields, the time-average power dissipation density is

$$\langle p_d \rangle = \tfrac{1}{2}\sigma\bar{\mathbf{E}} \cdot \bar{\mathbf{E}}^* \tag{5-185}$$

In Section 5.5 we learned that dielectrics in electric fields are characterized by induced polarization charges. From Section 4.6, the stored energy density associated with an electric field $\mathbf{E}$ in free space is given by

$$w_e = \tfrac{1}{2}\epsilon_0 E^2 = \tfrac{1}{2}\epsilon_0 \mathbf{E} \cdot \mathbf{E} \tag{5-186}$$

This result was obtained by finding the work required to be done by an external agent to bring together a set of point charges from infinity and then extending the result to a volume charge distribution. We can do the same for a dielectric medium provided we take into account the polarization charges in finding the work required for assembling the charge distribution. The effect of the polarization charges is to neutralize partially the true charges. Hence, as we bring together charges from infinity, they are neutralized partially by the polarization charges. Thus, for the same electric field intensity in the dielectric as in free space, we have to actually assemble a true-charge distribution of greater density than in the free-space case. This requires more work to be done in the dielectric case so that more energy is stored in the dielectric case than in the free-space case for the same electric field intensity. From

$$\nabla \cdot \mathbf{D} = \nabla \cdot \epsilon\mathbf{E} = \rho \tag{5-187}$$

the true-charge density which gives the same $\mathbf{E}$ in a dielectric medium of permittivity ϵ as in free space is ϵ/ϵ_0 times the charge density in the free-space case. From (4-118a), the work required to assemble a charge distribution is proportional to the charge density for a constant potential V and hence for a constant electric field intensity **E**. The energy density associated with the electric field in the dielectric is therefore given by

$$w_e = \frac{\epsilon}{\epsilon_0}\left(\frac{1}{2}\epsilon_0 E^2\right) = \frac{1}{2}\epsilon E^2 = \frac{1}{2}\epsilon \mathbf{E} \cdot \mathbf{E} = \frac{1}{2}\mathbf{D} \cdot \mathbf{E} \tag{5-188}$$

For sinusoidally time-varying fields, the time-average energy density is

$$\langle w_e \rangle = \tfrac{1}{4}\epsilon \bar{\mathbf{E}} \cdot \bar{\mathbf{E}}^* = \tfrac{1}{4}\bar{\mathbf{D}} \cdot \bar{\mathbf{E}}^* \tag{5-189}$$

Substituting $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$ in (5-188), we have

$$w_e = \tfrac{1}{2}(\epsilon_0 \mathbf{E} + \mathbf{P}) \cdot \mathbf{E} = \tfrac{1}{2}\epsilon_0 \mathbf{E} \cdot \mathbf{E} + \tfrac{1}{2}\mathbf{P} \cdot \mathbf{E} \tag{5-190}$$

However,

$$\begin{aligned} \tfrac{1}{2}\mathbf{P} \cdot \mathbf{E} &= \tfrac{1}{2}\int_0^{\mathbf{P},\mathbf{E}} d(\mathbf{P} \cdot \mathbf{E}) = \tfrac{1}{2}\int_0^{\mathbf{P},\mathbf{E}} (\mathbf{P} \cdot d\mathbf{E} + \mathbf{E} \cdot d\mathbf{P}) \\ &= \int_0^{\mathbf{P}} \mathbf{E} \cdot d\mathbf{P} \end{aligned} \tag{5-191}$$

where we have used the substitution $\mathbf{P} \cdot d\mathbf{E} = \mathbf{E} \cdot d\mathbf{P}$ in view of the linear relationship between **P** and **E**. Substituting (5-191) into (5-190), we get

$$w_e = \tfrac{1}{2}\epsilon_0 \mathbf{E} \cdot \mathbf{E} + \int_0^{\mathbf{P}} \mathbf{E} \cdot d\mathbf{P} \tag{5-192}$$

We note that the first term on the right side of (5-192) is the energy density in the electric field if the medium is free space. The second term on the right side of (5-192) is the work done per unit volume by the **E** field in the dielectric as the polarization is built up gradually from zero to the final value **P**. This is known as the polarization energy density.

In Section 5.8 we learned that magnetic materials in magnetic fields are characterized by magnetization currents. From Section 4.7, the stored energy density associated with a magnetic field **B** in free space is given by

$$w_m = \frac{1}{2}\frac{B^2}{\mu_0} = \frac{1}{2}\frac{\mathbf{B}}{\mu_0} \cdot \mathbf{B} = \frac{1}{2}\mu_0 H^2 = \frac{1}{2}\mu_0 \mathbf{H} \cdot \mathbf{H} \tag{5-193}$$

This result was obtained by finding the work required to be done for building up a volume current distribution. We can do the same for a magnetic material medium, provided we take into account the magnetization currents in finding the work required for building up the current distribution. The effect of the magnetization currents is to aid the true currents (for $\mu > \mu_0$). Hence, as the current is built up, it is aided by the magnetization current. Thus, for the same magnetic flux density in the magnetic material as in free space, it is sufficient if we actually build up a true current distribution of lesser density than in the free-space case. This requires less work to be done in the magnetic material case so that less energy is stored in the magnetic material case than in the free-space case for the same magnetic flux density. From

$$\nabla \times \mathbf{H} = \nabla \times \frac{\mathbf{B}}{\mu} = \mathbf{J} \tag{5-194}$$

the true current density which gives the same **B** in a magnetic material medium of permeability μ as in free space is μ_0/μ times the current density in the free-space case. From (4-130a), the work required to build up a current distribution is proportional to the current density for a constant vector potential **A** and hence for a constant magnetic flux density **B**. The energy density associated with the magnetic field in the magnetic material is therefore given by

$$w_m = \frac{\mu_0}{\mu}\left(\frac{1}{2}\frac{B^2}{\mu_0}\right) = \frac{1}{2}\frac{B^2}{\mu} = \frac{1}{2}\frac{\mathbf{B}}{\mu}\cdot\mathbf{B} = \frac{1}{2}\mathbf{H}\cdot\mathbf{B} \qquad (5\text{-}195)$$

For sinusoidally time-varying fields, the time-average energy density is

$$\langle w_m \rangle = \frac{1}{4}\frac{\bar{\mathbf{B}}}{\mu}\cdot\bar{\mathbf{B}}^* = \frac{1}{4}\bar{\mathbf{H}}\cdot\bar{\mathbf{B}}^* \qquad (5\text{-}196)$$

Substituting $\mathbf{B} = \mu_0\mathbf{H} + \mu_0\mathbf{M}$ in (5-195), we have

$$w_m = \tfrac{1}{2}\mathbf{H}\cdot(\mu_0\mathbf{H} + \mu_0\mathbf{M}) = \tfrac{1}{2}\mu_0\mathbf{H}\cdot\mathbf{H} + \tfrac{1}{2}\mu_0\mathbf{H}\cdot\mathbf{M} \qquad (5\text{-}197)$$

However,

$$\begin{aligned}\tfrac{1}{2}\mu_0\mathbf{H}\cdot\mathbf{M} &= \tfrac{1}{2}\int_0^{\mathbf{M},\mathbf{H}} d(\mu_0\mathbf{H}\cdot\mathbf{M}) = \tfrac{1}{2}\int_0^{\mathbf{M},\mathbf{H}} (\mu_0\mathbf{H}\cdot d\mathbf{M} + \mu_0\mathbf{M}\cdot d\mathbf{H}) \\ &= \int_0^{\mathbf{M}} \mu_0\mathbf{H}\cdot d\mathbf{M} \qquad (5\text{-}198)\end{aligned}$$

where we have used the substitution $\mathbf{H}\cdot d\mathbf{M} = \mathbf{M}\cdot d\mathbf{H}$ in view of the linear relationship between **M** and **H**. Substituting (5-198) into (5-197), we get

$$w_m = \tfrac{1}{2}\mu_0\mathbf{H}\cdot\mathbf{H} + \int_0^{\mathbf{M}} \mu_0\mathbf{H}\cdot d\mathbf{M} \qquad (5\text{-}199)$$

We note that the first term on the right side of (5-199) is the energy density in the **H** field if the medium is free space. The second term on the right side of (5-199) is the work done by the **H** field in the magnetic material as the magnetization is built up from zero to the final value **M**. This is known as the magnetization energy density. Note that for the same magnetic field intensity in the magnetic material as in free space, we have to actually build up a true current distribution of greater density than in the free-space case.

For nonlinear magnetic materials, we cannot use the result $\frac{1}{2}\mathbf{H}\cdot\mathbf{B}$ given by (5-195) for finding the magnetic energy stored in the material since μ is not constant for a particular material but is dependent on **H**. To obtain the correct expression, we write (5-199) as

$$\begin{aligned}w_m &= \int_0^{\mathbf{H}} d(\tfrac{1}{2}\mu_0\mathbf{H}\cdot\mathbf{H}) + \int_0^{\mathbf{M}} \mu_0\mathbf{H}\cdot d\mathbf{M} \\ &= \int_0^{\mu_0\mathbf{H}} \mathbf{H}\cdot d(\mu_0\mathbf{H}) + \int_0^{\mu_0\mathbf{M}} \mathbf{H}\cdot d(\mu_0\mathbf{M}) \qquad (5\text{-}200) \\ &= \int_0^{\mu_0\mathbf{H}+\mu_0\mathbf{M}} \mathbf{H}\cdot d(\mu_0\mathbf{H} + \mu_0\mathbf{M}) = \int_0^{\mathbf{B}} \mathbf{H}\cdot d\mathbf{B}\end{aligned}$$

It follows from (5-200) that the increase in stored energy density corresponding to an infinitesimal increment $d\mathbf{B}$ is $\mathbf{H} \cdot d\mathbf{B}$, where $\mathbf{H}$ is the magnetic field intensity at which $d\mathbf{B}$ is achieved.

Let us now consider the vector identity given by

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H}) \tag{5-201}$$

Substituting for $\nabla \times \mathbf{E}$ and $\nabla \times \mathbf{H}$ on the right side of (5-201) from Maxwell's equations (5-163) and (5-164) respectively, we obtain

$$\begin{aligned} \nabla \cdot (\mathbf{E} \times \mathbf{H}) &= \mathbf{H} \cdot \left(-\frac{\partial \mathbf{B}}{\partial t}\right) - \mathbf{E} \cdot \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) \\ &= -\mathbf{E} \cdot \sigma \mathbf{E} - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} - \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} \\ &= -\sigma \mathbf{E} \cdot \mathbf{E} - \frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{D} \cdot \mathbf{E}\right) - \frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{H} \cdot \mathbf{B}\right) \end{aligned} \tag{5-202}$$

or

$$-\nabla \cdot (\mathbf{E} \times \mathbf{H}) = p_d + \frac{\partial}{\partial t}(w_e + w_m) \tag{5-203}$$

where p_d, w_e, and w_m are, respectively, the power dissipation density due to the conductivity of the medium, the electric stored energy density, and the magnetic stored energy density. Integrating both sides of (5-203) in a volume V of the material and applying the divergence theorem to the left-side integral, we get

$$-\oint_S (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S} = \int_V p_d \, dv + \frac{\partial}{\partial t} \int_V (w_e + w_m) \, dv \tag{5-204}$$

where S is the surface bounding the volume V and $d\mathbf{S}$ is directed out of the volume V. The right side of (5-204) represents the time rate of increase of energy stored in the electric and magnetic fields in the volume V plus the time rate of energy dissipated in V due to conduction current flow. Thus $\oint_S (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S}$ represents the power flow across S out of the volume V. It follows that the density of power flow or the Poynting vector associated with the electromagnetic field in a material medium is given by

$$\mathbf{P} = \mathbf{E} \times \mathbf{H} \tag{5-205}$$

For sinusoidally time-varying fields, the complex Poynting's theorem is

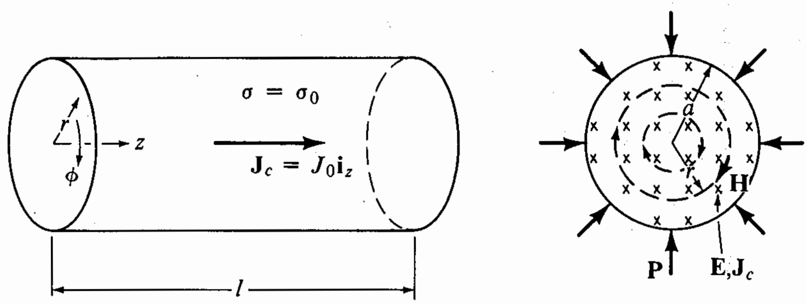
$$\oint_S \bar{\mathbf{P}} \cdot d\mathbf{S} = -\int_V \langle p_d \rangle \, dv - j2\omega \int_V (\langle w_m \rangle - \langle w_e \rangle) \, dv \tag{5-206}$$

where $\bar{\mathbf{P}}$ is the complex Poynting vector given by

$$\bar{\mathbf{P}} = \tfrac{1}{2}\bar{\mathbf{E}} \times \bar{\mathbf{H}}^* \tag{5-207}$$

and $\langle p_d \rangle$, $\langle w_m \rangle$, and $\langle w_e \rangle$ are given by (5-185), (5-196), and (5-189), respectively.

EXAMPLE 5-13. Current flows axially with uniform density $\mathbf{J}_c = J_0\mathbf{i}_z$ amp/m^2 along a cylindrical conductor of radius a and length l and having a uniform conductivity σ_0 mhos/m as shown in Fig. 5.28(a), by the application of a potential difference between the ends of the conductor. It is desired to verify that the total power dissipated in the conductor is correctly given by the surface integral of the Poynting vector over the surface of the conductor.



(a) (b)

Fig. 5.28. For showing that the power dissipated in a conductor due to conduction current flow is correctly given by the surface integral of the Poynting vector over the surface of the conductor.

The power dissipation density inside the conductor is given by

$$p_d = \sigma E^2 = \sigma_0 \left(\frac{J_0}{\sigma_0}\right)^2 = \frac{J_0^2}{\sigma_0}$$

Since p_d is uniform over the volume of the conductor, the total power dissipated in the conductor is

$$P_d = p_d \text{ (volume of the conductor)} = \frac{J_0^2}{\sigma_0}(\pi a^2 l) = \frac{J_0^2 \pi a^2 l}{\sigma_0}$$

Applying $\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S}$ to a circular path of radius r in the cross-sectional plane of the conductor and centered at the axis of the conductor as shown in Fig. 5-28(b), we obtain, from symmetry considerations,

$$\mathbf{H} = H_\phi \mathbf{i}_\phi = \frac{J_0 \pi r^2}{2\pi r}\mathbf{i}_\phi = \frac{J_0 r}{2}\mathbf{i}_\phi \qquad \text{for } r \leq a$$

The Poynting vector is then given by

$$\mathbf{P} = \mathbf{E} \times \mathbf{H} = \frac{J_0}{\sigma_0}\mathbf{i}_z \times \frac{J_0 r}{2}\mathbf{i}_\phi = -\frac{J_0^2 r}{2\sigma_0}\mathbf{i}_r \qquad \text{for } r \leq a$$

We note that the Poynting vector is directed radially towards the axis of the conductor. Hence, to find the total power flow into the conductor, it is sufficient if we perform the surface integration of the Poynting vector over the cylindrical surface $r = a$. Over this surface, we have

$$[\mathbf{P}]_{r=a} = -\frac{J_0^2 a}{2\sigma_0}\mathbf{i}_r$$

Since the magnitude of $[\mathbf{P}]_{r=a}$ is uniform, the surface integral of the Poynting vector over $r = a$ is

$$P = \frac{J_0^2 a}{2\sigma_0}(\text{surface area}) = \frac{J_0^2 a}{2\sigma_0}(2\pi a l) = \frac{J_0^2 \pi a^2 l}{\sigma_0}$$

which is the same as the result obtained by volume integration of the power dissipation density. This merely shows that the surface integral of the Poynting vector gives the correct result for the power dissipated and does not mean that the power is entering through the cylindrical surface. The actual power must be supplied by the source which maintains the potential difference between the ends of the conductor. ∎

5.12 Boundary Conditions

In Section 5.10 we summarized Maxwell's equations for the general case of a medium characterized by arbitrary values of σ, ϵ, and μ and for different special cases. For electromagnetic field problems involving several different media, the fields in each medium must satisfy separately the Maxwell's equations in differential form for that medium. On the other hand, the integral forms of Maxwell's equations must be satisfied collectively by the fields in all the media associated with the contours, surfaces, and volumes over which the integrals are evaluated. Thus the sets of solutions for the fields in different media obtained by solving the corresponding Maxwell's equations in differential form are tied together by a set of conditions determined by the integral forms of Maxwell's equations. These conditions are known as the "boundary conditions" since they relate the fields on one side of a boundary between two media to the fields on the other side of that boundary. The boundary conditions take into account any surface charges and currents existing on the boundaries, which the differential equations ignore.

In fact, we already introduced certain boundary conditions in previous sections without mentioning the name. An example of this is in Section 5.3, where we found that the tangential component of the electric field intensity at the boundary between a conductor placed in an electric field is zero, whereas the normal component of the electric field intensity at a point on the boundary is equal to $1/\epsilon_0$ times the surface charge density at that point in order to satisfy the criterion of zero electric field inside the conductor in the absence of a mechanism to permit the flow of a current. In this section we will derive the boundary conditions for the most general case of time-varying fields in two media characterized by different sets of values of σ, ϵ, and μ by using the integral forms of Maxwell's equations and the continuity equation. From the experience gained in this process, we will then write simplified sets of boundary conditions for the different special cases. From Table 5.5, the integral forms of Maxwell's equations are

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho \, dv \tag{5-208}$$

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0 \tag{5-209}$$

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{S} \tag{5-210}$$

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt} \int_S \mathbf{D} \cdot d\mathbf{S} \tag{5-211}$$

and the integral form of the continuity equation is

$$\oint_S \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt} \int_V \rho \, dv = 0 \tag{5-212}$$

Let us consider two semiinfinite media separated by the plane boundary $z = 0$ as shown in Fig. 5.29. Let us denote all quantities pertinent to medium 1 by subscript 1 and all quantities pertinent to medium 2 by subscript 2. Let $\mathbf{i}_n$ be the unit normal vector to the surface $z = 0$ directed into medium

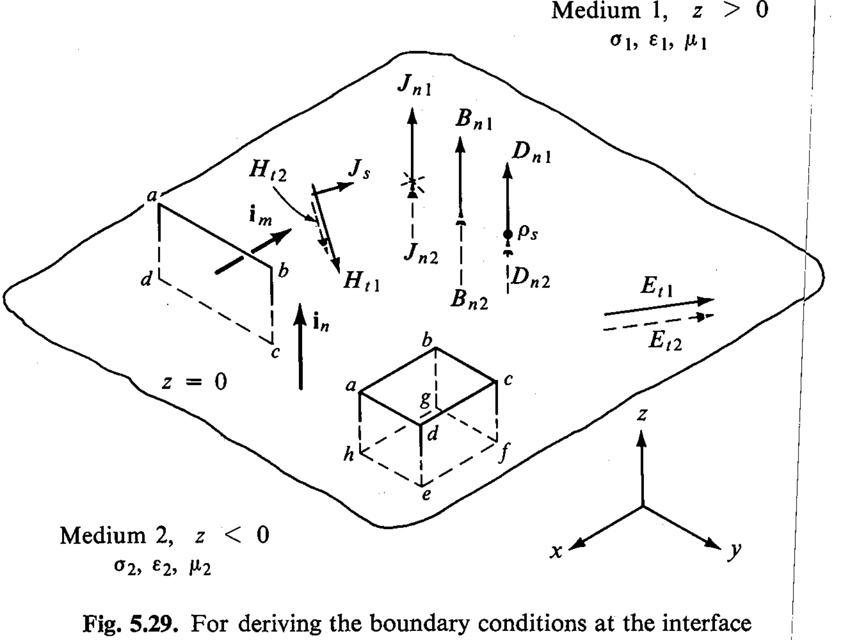


Fig. 5.29. For deriving the boundary conditions at the interface between two arbitrary media.

1 as shown in Fig. 5.29 and let all normal components of fields at the boundary in both media denoted by an additional subscript n be directed along $\mathbf{i}_n$. Let the surface charge density and the surface current density on $z = 0$ be ρ_s and $\mathbf{J}_s$, respectively. Note that the fields at the boundary in both media and the surface charge and current densities are, in general, functions of x, y, and t whereas the fields away from the boundary are, in general, functions of x, y, z, and t.

First we consider a rectangular box *abcdefgh* of infinitesimal volume enclosing an infinitesimal area of the boundary and parallel to it as shown in Fig. 5-29. Applying (5-208) to this box in the limit that the side surfaces (abbreviated ss) tend to zero, thereby shrinking the box to the surface, we have

$$\lim_{\text{ss}\to 0} \oint_{\Delta S} \mathbf{D} \cdot d\mathbf{S} = \lim_{\text{ss}\to 0} \int_{\Delta v} \rho \, dv \tag{5-213}$$

where ΔS and Δv are the surface area and the volume, respectively, of the box. In the limit that the box shrinks to the surface, the contribution from the side surfaces to the integral on the left side of (5-213) approaches zero. The sum of the contributions from the top and bottom surfaces becomes $[D_{n1}(abcd) - D_{n2}(efgh)]$ since *abcd* and *efgh* are infinitesimal areas. The quantity on the right side of (5-213) would be zero but for the surface charge on the boundary, since shrinking the box to the surface reduces only the volume charge enclosed by it to zero, keeping the surface charge still enclosed by it. This surface charge is equal to $\rho_s(abcd)$. Thus Eq. (5-213) gives

$$D_{n1}(abcd) - D_{n2}(efgh) = \rho_s(abcd)$$

or

$$D_{n1} - D_{n2} = \rho_s \tag{5-214}$$

since the two areas *abcd* and *efgh* are equal. In vector notation, (5-214) is written as

$$\mathbf{i}_n \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s \tag{5-215}$$

In words, Eqs. (5-214) and (5-215) state that, at any point on the boundary, the components of $\mathbf{D}_1$ and $\mathbf{D}_2$ normal to the boundary are discontinuous by the amount of the surface charge density at that point.

Similarly, applying (5-209) to the box *abcdefgh* in the limit that the box shrinks to the surface, we obtain

$$\lim_{\text{ss}\to 0} \oint_{\Delta S} \mathbf{B} \cdot d\mathbf{S} = 0 \tag{5-216}$$

Using the same argument as for the left side of (5-213), the quantity on the left side of (5-216) is equal to $[B_{n1}(abcd) - B_{n2}(efgh)]$. Thus Eq. (5-216) gives

$$B_{n1}(abcd) - B_{n2}(efgh) = 0$$

or

$$B_{n1} - B_{n2} = 0 \tag{5-217}$$

In vector notation, Eq. (5-217) is written as

$$\mathbf{i}_n \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0 \tag{5-218}$$

In words, Eqs. (5-217) and (5-218) state that, at any point on the boundary, the components of $\mathbf{B}_1$ and $\mathbf{B}_2$ normal to the boundary are equal.

Now, we consider a rectangular contour *abcda* of infinitesimal area in the plane normal to the boundary and with its sides *ab* and *cd* parallel to the boundary as shown in Fig. 5-29. Applying (5-210) to this contour in the limit that *ad* and *bc* $\longrightarrow 0$, thereby shrinking the rectangle to the surface, we have

$$\lim_{\substack{ad\to 0\\ bc\to 0}} \oint_{abcda} \mathbf{E} \cdot d\mathbf{l} = -\lim_{\substack{ad\to 0\\ bc\to 0}} \frac{d}{dt} \int_{\substack{\text{area}\\ abcd}} \mathbf{B} \cdot d\mathbf{S} \tag{5-219}$$

In the limit that the rectangle shrinks to the surface, the contribution from *ad* and *bc* to the integral on the left side of (5-219) approaches zero. Since *ab* and *cd* are infinitesimal, the sum of the contributions from *ab* and *cd* becomes $[E_{ab}(ab) + E_{cd}(cd)]$, where E_{ab} and E_{cd} are the components of $\mathbf{E}_1$ and $\mathbf{E}_2$ along *ab* and *cd*, respectively. The right side of (5-219) is equal to zero since the magnetic flux crossing the area *abcd* approaches zero as the area *abcd* tends to zero. Thus Eq. (5-219) gives

$$E_{ab}(ab) + E_{cd}(cd) = 0$$

or, since *ab* and *cd* are equal,

$$\mathbf{i}_{ab} \cdot (\mathbf{E}_1 - \mathbf{E}_2) = 0 \tag{5-220}$$

where $\mathbf{i}_{ab}$ is the unit vector along *ab*. Let us now define $\mathbf{i}_m$ to be the unit vector normal to the area *abcd* and in the direction of advance of a right-hand screw as it is turned in the sense of the path *abcda*. Note that $\mathbf{i}_m$ is tangential to the boundary. We then have

$$\mathbf{i}_{ab} = \mathbf{i}_m \times \mathbf{i}_n \tag{5-221}$$

Substituting (5-221) into (5-220) and rearranging the order of the scalar triple product, we obtain

$$\mathbf{i}_m \cdot \mathbf{i}_n \times (\mathbf{E}_1 - \mathbf{E}_2) = 0 \tag{5-222}$$

Since we can choose the rectangle *abcd* to be in any plane normal to the boundary, (5-222) must be true for all orientations of $\mathbf{i}_m$. It then follows that

$$\mathbf{i}_n \times (\mathbf{E}_1 - \mathbf{E}_2) = 0 \tag{5-223}$$

or, in scalar form,

$$E_{t1} - E_{t2} = 0 \tag{5-224}$$

where E_{t1} and E_{t2} are the tangential components of $\mathbf{E}_1$ and $\mathbf{E}_2$, respectively, at the boundary. In words, Eqs. (5-223) and (5-224) state that, at any point on the boundary, the components of $\mathbf{E}_1$ and $\mathbf{E}_2$ tangential to the boundary are equal.

Similarly, applying (5-211) to the contour *abcda* in the limit that *ad* and $bc \longrightarrow 0$, we have

$$\lim_{\substack{ad\to 0\\ bc\to 0}} \oint_{abcda} \mathbf{H} \cdot d\mathbf{l} = \lim_{\substack{ad\to 0\\ bc\to 0}} \int_{\substack{\text{area}\\ abcd}} \mathbf{J} \cdot d\mathbf{S} + \lim_{\substack{ad\to 0\\ bc\to 0}} \frac{d}{dt} \int_{\substack{\text{area}\\ abcd}} \mathbf{D} \cdot d\mathbf{S} \tag{5-225}$$

Using the same argument as for the left side of (5-219), the quantity on the left side of (5-225) is equal to $[H_{ab}(ab) + H_{cd}(cd)]$, where H_{ab} and H_{cd} are the components of $\mathbf{H}_1$ and $\mathbf{H}_2$ along *ab* and *cd*, respectively. The second integral on the right side of (5-225) is zero since the displacement flux crossing the area *abcd* approaches zero as the area *abcd* tends to zero. The first integral on the right side of (5-225) would also be equal to zero but for a contribution from the surface current on the boundary because shrinking the rectangle to the surface reduces only the volume current enclosed by it to zero, keeping the surface current still enclosed by it. This contribution is the surface current flowing normal to the line which *abcd* approaches when it shrinks to the surface, that is, the current crossing this line along the direction of $\mathbf{i}_m$. This quantity is equal to $[\mathbf{J}_s \cdot \mathbf{i}_m](ab)$. Thus Eq. (5-225) gives

$$H_{ab}(ab) + H_{cd}(cd) = (\mathbf{J}_s \cdot \mathbf{i}_m)(ab)$$

or, since *ab* and *cd* are equal,

$$\mathbf{i}_{ab} \cdot (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s \cdot \mathbf{i}_m \tag{5-226}$$

Substituting (5-221) into (5-226) and rearranging the order of the scalar triple product, we obtain

$$\mathbf{i}_n \times (\mathbf{H}_1 - \mathbf{H}_2) \cdot \mathbf{i}_m = \mathbf{J}_s \cdot \mathbf{i}_m \tag{5-227}$$

Since (5-227) must be true for all orientations of $\mathbf{i}_m$, that is, for a rectangle *abcd* in any plane normal to the boundary, it follows that

$$\mathbf{i}_n \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s \tag{5-228}$$

or, in scalar form,

$$H_{t1} - H_{t2} = J_s \tag{5-229}$$

where H_{t1} and H_{t2} are the tangential components of $\mathbf{H}_1$ and $\mathbf{H}_2$, respectively, at the boundary. In words, Eqs. (5-228) and (5-229) state that, at any point on the boundary, the components of $\mathbf{H}_1$ and $\mathbf{H}_2$ tangential to the boundary are discontinuous by the amount equal to the surface current density at that point.

Finally, applying (5-212) to the box *abcdefgh* in the limit that the box shrinks to the surface, we have

$$\lim_{ss\to 0} \oint_{\Delta S} \mathbf{J} \cdot d\mathbf{S} + \lim_{ss\to 0} \frac{d}{dt} \int_{\Delta v} \rho \, dv = 0 \tag{5-230}$$

In the limit that the box shrinks to the surface, the contribution to the first integral on the left side of (5-230) from the side surfaces of the box would be zero but for the surface current on the boundary, since although the volume current emanating from the side surfaces reduces to zero, there still remains the surface current emanating from them. This current is

$$\oint_{abcda} \mathbf{J}_s \cdot (d\mathbf{l} \times \mathbf{i}_n) = \int_{abcd} (\nabla_s \cdot \mathbf{J}_s)\, dS = (\nabla_s \cdot \mathbf{J}_s)(abcd)$$

where the subscript s in ∇_s denotes that the divergence is computed in the two dimensions tangential to the surface. The sum of the contributions from the top and bottom surfaces is equal to $[J_{n1}(abcd) - J_{n2}(efgh)]$ or $[\mathbf{i}_n \cdot (\mathbf{J}_1 - \mathbf{J}_2)](abcd)$. The second integral on the left side of (5-230) is equal to $(\partial \rho_s/\partial t)(abcd)$. Thus Eq. (5-230) gives

$$[\nabla_s \cdot \mathbf{J}_s + \mathbf{i}_n \cdot (\mathbf{J}_1 - \mathbf{J}_2)](abcd) + \frac{\partial \rho_s}{\partial t}(abcd) = 0$$

or

$$\mathbf{i}_n \cdot (\mathbf{J}_1 - \mathbf{J}_2) = -\nabla_s \cdot \mathbf{J}_s - \frac{\partial \rho_s}{\partial t} \tag{5-231}$$

In words, Eq. (5-231) states that, at any point on the boundary, the components of $\mathbf{J}_1$ and $\mathbf{J}_2$ normal to the boundary are discontinuous by the amount equal to the negative of the sum of the two-dimensional divergence of the surface current density and the time derivative of the surface charge density at that point.

Equations (5-215), (5-218), (5-223), (5-228), and (5-231) form the set of boundary conditions for the most general case of time-varying fields in two arbitrary media. Although we have derived these boundary conditions by considering a plane surface, it should be obvious that we can consider any arbitrary-shaped boundary and obtain the same results by letting the box and the rectangle shrink to points on the boundary. We can now write the boundary conditions for various special cases by inspection of the corresponding sets of Maxwell's equations and continuity equation listed in Table 5.5. These boundary conditions are listed in Table 5.6, together with the general boundary conditions. Depending upon the problem, only some of the boundary conditions need to be used in the determination of the fields, whereas some or all of the remaining boundary conditions are automatically satisfied and the rest determine the surface charge and current densities on the boundary. Before we consider some examples, let us investigate the boundary condition for the power flow normal to the boundary. Letting $\mathbf{P}_1$ and $\mathbf{P}_2$ be the Poynting vectors corresponding to the fields in media 1 and 2, respectively, we have

$$\begin{aligned}\mathbf{i}_n \cdot (\mathbf{P}_1 - \mathbf{P}_2) &= \mathbf{i}_n \cdot (\mathbf{E}_1 \times \mathbf{H}_1 - \mathbf{E}_2 \times \mathbf{H}_2) \\ &= (\mathbf{i}_n \times \mathbf{E}_1) \times (\mathbf{i}_n \times \mathbf{H}_1) \cdot \mathbf{i}_n - (\mathbf{i}_n \times \mathbf{E}_2) \times (\mathbf{i}_n \times \mathbf{H}_2) \cdot \mathbf{i}_n\end{aligned} \tag{5-232}$$

TABLE 5.6. Summary of Boundary Conditions for Various Cases ($\mathbf{i}_n$ is the unit vector normal to the boundary and drawn towards medium 1)

Description	*Boundary Conditions*
Time-varying fields Medium 1: arbitrary, $\sigma_1 \neq \infty$ Medium 2: arbitrary, $\sigma_2 \neq \infty$	$\mathbf{i}_n \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$ $\mathbf{i}_n \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0$ $\mathbf{i}_n \times (\mathbf{E}_1 - \mathbf{E}_2) = 0$ $\mathbf{i}_n \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s$ $\mathbf{i}_n \cdot (\mathbf{J}_1 - \mathbf{J}_2) = -\nabla_s \cdot \mathbf{J}_s - \frac{\partial \rho_s}{\partial t}$
Static fields Medium 1: arbitrary, $\sigma_1 \neq \infty$ Medium 2: arbitrary, $\sigma_2 \neq \infty$	$\mathbf{i}_n \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$ $\mathbf{i}_n \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0$ $\mathbf{i}_n \times (\mathbf{E}_1 - \mathbf{E}_2) = 0$ $\mathbf{i}_n \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s$ $\mathbf{i}_n \cdot (\mathbf{J}_1 - \mathbf{J}_2) = -\nabla_s \cdot \mathbf{J}_s$
Time-varying fields Medium 1: perfect dielectric, $\sigma_1 = 0$ Medium 2: perfect dielectric, $\sigma_2 = 0$	$\mathbf{i}_n \cdot (\mathbf{D}_1 - \mathbf{D}_2) = 0$ $\mathbf{i}_n \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0$ $\mathbf{i}_n \times (\mathbf{E}_1 - \mathbf{E}_2) = 0$ $\mathbf{i}_n \times (\mathbf{H}_1 - \mathbf{H}_2) = 0$
Static fields Medium 1: perfect dielectric, $\sigma_1 = 0$ Medium 2: perfect dielectric, $\sigma_2 = 0$	$\mathbf{i}_n \cdot (\mathbf{D}_1 - \mathbf{D}_2) = 0$ $\mathbf{i}_n \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0$ $\mathbf{i}_n \times (\mathbf{E}_1 - \mathbf{E}_2) = 0$ $\mathbf{i}_n \times (\mathbf{H}_1 - \mathbf{H}_2) = 0$
Time-varying fields Medium 1: perfect dielectric, $\sigma_1 = 0$ Medium 2: perfect conductor, $\sigma_2 = \infty$	$\mathbf{i}_n \cdot \mathbf{D}_1 = \rho_s$ $\mathbf{i}_n \cdot \mathbf{B}_1 = 0$ $\mathbf{i}_n \times \mathbf{E}_1 = 0$ $\mathbf{i}_n \times \mathbf{H}_1 = \mathbf{J}_s$
Static electric field Medium 1: perfect dielectric, $\sigma_1 = 0$ Medium 2: perfect conductor, $\sigma_2 = \infty$	$\mathbf{i}_n \cdot \mathbf{D}_1 = \rho_s$ $\mathbf{i}_n \times \mathbf{E}_1 = 0$

Substituting (5-223) and (5-228) into (5-232), we get

$$\begin{aligned}
\mathbf{i}_n \cdot (\mathbf{P}_1 - \mathbf{P}_2) &= (\mathbf{i}_n \times \mathbf{E}_2) \times [(\mathbf{i}_n \times \mathbf{H}_2) + \mathbf{J}_s] \cdot \mathbf{i}_n \\
&\quad - (\mathbf{i}_n \times \mathbf{E}_2) \times (\mathbf{i}_n \times \mathbf{H}_2) \cdot \mathbf{i}_n \\
&= [(\mathbf{i}_n \times \mathbf{E}_2) \times \mathbf{J}_s] \cdot \mathbf{i}_n \\
&= [(\mathbf{i}_n \times \mathbf{E}_1) \times \mathbf{J}_s] \cdot \mathbf{i}_n \\
&= [(\mathbf{i}_n \cdot \mathbf{J}_s)\mathbf{E}_1 - (\mathbf{J}_s \cdot \mathbf{E}_1)\mathbf{i}_n] \cdot \mathbf{i}_n \\
&= -\mathbf{J}_s \cdot \mathbf{E}_1
\end{aligned} \tag{5-233}$$

since $(\mathbf{i}_n \cdot \mathbf{J}_s)$ is equal to zero. Thus, at any point on the boundary, the components of the Poynting vector normal to the boundary are discontinuous by the amount equal to the power density associated with the surface current density at that point. In the absence of a surface current, the normal components of the Poynting vector are continuous.

EXAMPLE 5-14. In Fig. 5.30, medium 1 comprises the region $z > 0$ and medium 2 comprises the region $z < 0$. All fields are spatially uniform in both media

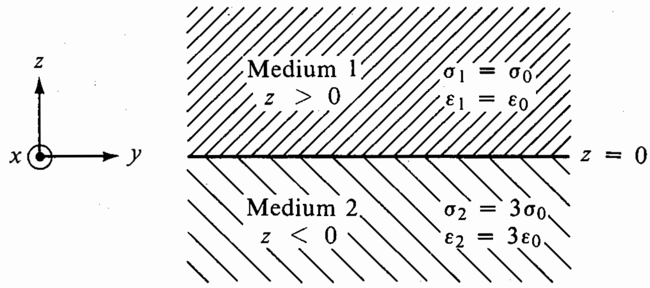


Fig. 5.30. For Example 5-14.

and independent of time. The quantities σ_0 and ϵ_0 are constants. If the current density in medium 1 is given by

$$\mathbf{J}_1 = J_0(\mathbf{i}_x + 2\mathbf{i}_y + 6\mathbf{i}_z)$$

where J_0 is a constant, find (a) the electric field intensity vector $\mathbf{E}_2$ in medium 2, and (b) the surface charge density ρ_s on the interface $z = 0$.

The electric field intensity $\mathbf{E}_1$ in medium 1 is given by

$$\mathbf{E}_1 = \frac{\mathbf{J}_1}{\sigma_1} = \frac{J_0}{\sigma_0}(\mathbf{i}_x + 2\mathbf{i}_y + 6\mathbf{i}_z)$$

From (5-223), the tangential component of $\mathbf{E}_2$ is equal to the tangental component of $\mathbf{E}_1$. Thus $E_{2x} = J_0/\sigma_0$ and $E_{2y} = 2J_0/\sigma_0$. Since all fields are spatially uniform and independent of time, $\nabla_s \cdot \mathbf{J}_s = 0$ and $\partial\rho_s/\partial t = 0$. Then, from (5-231) the normal component of $\mathbf{J}_2$ is equal to the normal component of $\mathbf{J}_1$. Thus $J_{2z} = 6J_0$ and $E_{2z} = J_{2z}/\sigma_2 = 6J_0/3\sigma_0 = 2J_0/\sigma_0$. The electric field intensity $\mathbf{E}_2$ in medium 2 is therefore given by

$$\mathbf{E}_2 = \frac{J_0}{\sigma_0}(\mathbf{i}_x + 2\mathbf{i}_y + 2\mathbf{i}_z)$$

From (5-215), the surface charge density ρ_s on the interface $z = 0$ is given by

$$\begin{aligned}\rho_s &= \mathbf{i}_z \cdot (\mathbf{D}_1 - \mathbf{D}_2)\\ &= D_{1z} - D_{2z} = \epsilon_1 E_{1z} - \epsilon_2 E_{2z}\\ &= \epsilon_0 \frac{6J_0}{\sigma_0} - 2\epsilon_0 \frac{2J_0}{\sigma_0} = 2\frac{\epsilon_0 J_0}{\sigma_0} \quad \blacksquare\end{aligned}$$

EXAMPLE 5-15. In Fig. 5.31, a perfect dielectric medium $x < 0$ is bounded by a perfect conductor ($x = 0$). The electric field intensity for $x < 0$ is given by

$$\begin{aligned}\mathbf{E}(x, y, z, t) = [&E_1 \cos(\omega t - \beta x \cos\theta - \beta z \sin\theta)\\ &+ E_2 \cos(\omega t + \beta x \cos\theta - \beta z \sin\theta)]\mathbf{i}_y\end{aligned}$$

where E_1, E_2, ω, β, and θ are constants. Find the relationship between E_1 and E_2. Then find the surface current density on the surface $z = 0$.

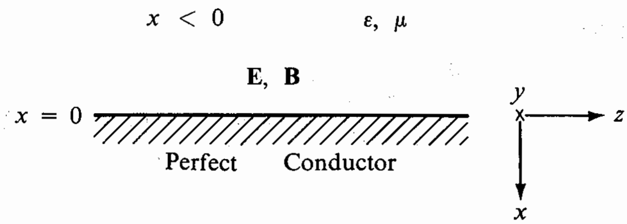


Fig. 5.31. For Example 5-15.

From the boundary conditions listed in Table 5.6, the tangential component of the electric field intensity at the surface of a perfect conductor must be zero. Thus

$$[E_y]_{x=0} = (E_1 + E_2)\cos(\omega t - \beta z \sin\theta) = 0$$

For this to be true for all values of z and t, $E_1 + E_2$ must be zero. Hence

$$E_2 = -E_1$$

The electric field intensity for $x < 0$ is then given by

$$\begin{aligned}\mathbf{E} &= [E_1\cos(\omega t - \beta x\cos\theta - \beta z\sin\theta) \\ &\quad - E_1\cos(\omega t + \beta x\cos\theta - \beta z\sin\theta)]\mathbf{i}_y \\ &= 2E_1\sin(\beta x\cos\theta)\sin(\omega t - \beta z\sin\theta)\,\mathbf{i}_y\end{aligned} \tag{5-234}$$

The corresponding magnetic flux density **B** can be obtained by using Maxwell's curl equation for **E**, given by

$$-\frac{\partial\mathbf{B}}{\partial t} = \nabla\times\mathbf{E} = -\frac{\partial E_y}{\partial z}\mathbf{i}_x + \frac{\partial E_y}{\partial x}\mathbf{i}_z \tag{5-235}$$

Substituting for E_y in (5-235) from (5-234) and integrating with respect to time, we obtain

$$\mathbf{B} = -\frac{2E_1\beta}{\omega}[\sin\theta\sin(\beta x\cos\theta)\sin(\omega t - \beta z\sin\theta)\,\mathbf{i}_x$$
$$-\cos\theta\cos(\beta x\cos\theta)\cos(\omega t - \beta z\sin\theta)\,\mathbf{i}_z]$$

The magnetic flux density at the surface of the perfect conductor is given by

$$[\mathbf{B}]_{x=0} = \frac{2E_1\beta}{\omega}\cos\theta\cos(\omega t - \beta z\sin\theta)\,\mathbf{i}_z$$

Note that the condition of zero normal component of **B** at the surface of the perfect conductor is automatically satisfied by the zero tangential component of **E**. This is because the boundary condition for the tangential component of **E** is obtained from the integral form of $\nabla\times\mathbf{E} = -\partial\mathbf{B}/\partial t$ whereas the boundary condition for the normal component of **B** is obtained from the integral form of $\nabla\cdot\mathbf{B} = 0$. However, $\nabla\cdot\mathbf{B} = 0$ follows from $\nabla\times\mathbf{E} = -\partial\mathbf{B}/\partial t$. Hence the two boundary conditions are not independent. Finally, the surface current density at the surface of the perfect conductor is given by

$$\mathbf{J}_s = -\mathbf{i}_x \times [\mathbf{H}]_{x=0}$$
$$= \frac{2E_1\beta}{\omega\mu} \cos\theta \cos(\omega t - \beta z \sin\theta)\, \mathbf{i}_y \quad \blacksquare$$

PROBLEMS

5.1. Consider two electrons moving under thermal agitation with equal and opposite velocities. A uniform electric field is applied along the direction of motion of one of the electrons. Show that the gain in kinetic energy by the accelerating electron is greater than the loss in kinetic energy by the decelerating electron.

5.2. (a) For a sinuosidally time-varying electric field $\mathbf{E} = \mathbf{E}_0 \cos \omega t$, where $\mathbf{E}_0$ is a constant, show that the steady-state solution for Eq. (5-2) is given by

$$\mathbf{v}_d = \frac{\tau e}{m\sqrt{1 + \omega^2\tau^2}} \mathbf{E}_0 \cos(\omega t - \tan^{-1} \omega\tau)$$

(b) Based on the assumption of one free electron per atom, the free electron density N_e in silver is $5.86 \times 10^{28}\ \text{m}^{-3}$. Using the conductivity for silver given in Table 5.1, find the frequency at which the drift velocity lags the applied field by $\pi/4$ rad. What is the ratio of the mobility at this frequency to the mobility at zero frequency?

5.3. The plane surfaces $x = 0, y > 0$ and $y = 0, x > 0$, and the curved surface $xy = 2$ form the boundaries of conductors extending away from the region between them. If the electrostatic potential in the region between the surfaces is given by $V = 50\, xy$ volts, find the surface charge densities on all three surfaces.

5.4. The region $z < -d$ is occupied by a conductor. An infinitely long line charge of uniform density ρ_{L0} C/m is situated along the x axis. From the secondary field required to make the total field inside the conductor equal to zero and from symmetry considerations as illustrated in Fig. 5.32, show that the induced charge

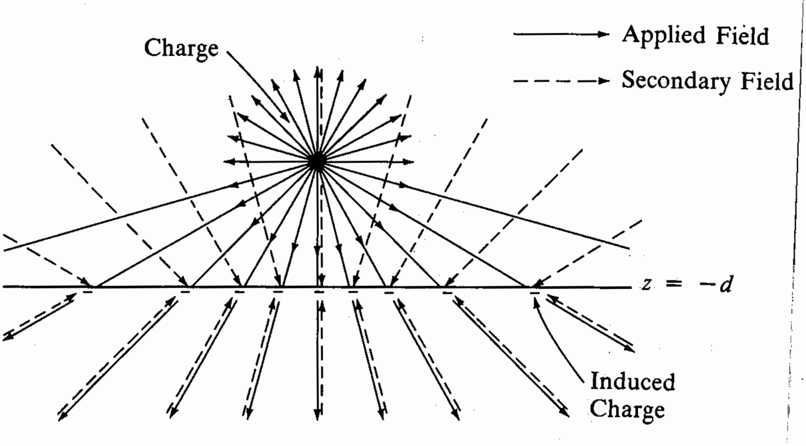


Fig. 5.32. For Problems 5.4 and 5.5. Charge is line charge for Problem 5.4 and point charge for Problem 5.5.

density on the surface of the conductor is given in cartesian coordinates by $-\rho_{L0}d/\pi(y^2 + d^2)$ C/m². Show that the induced surface charge per unit length along the x direction is equal to $-\rho_{L0}$. Show that the field outside the conductor is the same as the field due to the line charge along the x axis and an image line charge of uniform density $-\rho_{L0}$ C/m situated parallel to the actual line charge and passing through $(0, 0, -2d)$.

5.5. The region $z < -d$ is occupied by a conductor. A point charge Q C is situated at the origin. From the secondary field required to make the total field inside the conductor equal to zero and from symmetry considerations as illustrated in Fig. 5.32, show that the induced charge density on the surface of the conductor is given in cylindrical coordinates by $-Qd/2\pi(r_c^2 + d^2)^{3/2}$ C/m². Show that the total induced surface charge is $-Q$ C. Show that the field outside the conductor is the same as the field due to the point charge Q at the origin and an image point charge $-Q$ situated at $(0, 0, -2d)$.

5.6. (a) An infinite plane conducting slab carries uniformly distributed surface charges on both of its surfaces. If the net surface charge density, that is, the sum of the surface charge densities on the two surfaces, is ρ_{s0} C/m², find the surface charge densities on the two surfaces.

(b) Two infinite plane parallel conducting slabs carry uniformly distributed surface charges on all four of their surfaces. If the net surface charge densities are ρ_{s1} and ρ_{s2} C/m², respectively, for the two slabs, find the surface charge densities on all four surfaces.

5.7. Two infinitely long, coaxial, hollow cylindrical conductors of inner radii a and c, respectively, and outer radii $b\,(< c)$ and d, respectively, carry uniformly distributed surface charges on all four of their surfaces. If the net surface charges per unit length are ρ_{L1} and ρ_{L2} C for the inner and outer conductors, respectively, find the surface charge densities on all four surfaces.

5.8. Two concentric, spherical conducting shells of inner radii a and c, respectively, and outer radii $b(< c)$ and d, respectively, carry uniformly distributed surface charges on all four of their surfaces. If the net surface charges are Q_1 and Q_2 C for the inner and outer conductors, respectively, find the surface charge densities on all four surfaces.

5.9. Figure 5.33 shows the electric field intensities on either side of a point on a surface charge layer in free space.

(a) Using the integral form of Maxwell's curl equation for **E**, show that the tangential components E_{t1} and E_{t2} are equal.

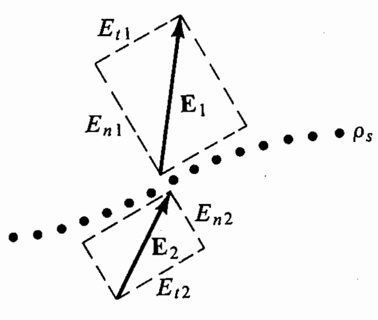


Fig. 5.33. For Problem 5.9.

(b) Using the integral form of Maxwell's divergence equation for **E**, show that the normal components E_{n1} and E_{n2} are related in the manner

$$E_{n1} - E_{n2} = \frac{\rho_s}{\epsilon_0}$$

where ρ_s is the surface charge density at the point.

5.10. Figure 5.34 shows the magnetic flux densities on either side of a point on a surface current layer in free space.

(a) Using the integral form of Maxwell's divergence equation for **B**, show that the normal components B_{n1} and B_{n2} are equal.

(b) Using the integral form of Maxwell's curl equation for **B**, show that the tangential components B_{t1} and B_{t2} are related in the manner

$$B_{t1} - B_{t2} = \mu_0 J_s$$

where $\mathbf{J}_s$ is the surface current density at the point. Note that $\mathbf{J}_s$ is directed into the paper whereas B_{t1} and B_{t2} are in the plane of the paper.

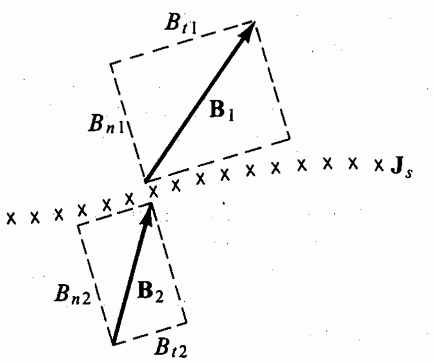


Fig. 5.34. For Problem 5.10.

5.11. The electric field intensity outside a conducting sphere of radius a and centered at the origin is given by

$$\mathbf{E} = E_0\left(1 + \frac{2a^3}{r^3}\right)\cos\theta\,\mathbf{i}_r - E_0\left(1 - \frac{a^3}{r^3}\right)\sin\theta\,\mathbf{i}_\theta$$

(a) Show that **E** satisfies Maxwell's equations.

(b) Show that the tangential component of **E** is zero at the conductor surface.

(c) Find the charge density on the conductor surface.

(d) Find the applied field by letting $a \longrightarrow 0$ and then find the secondary field both inside and outside $r = a$.

(e) Show that the secondary field on either side of the boundary satisfies the conditions (a) and (b) stated in Problem 5.9.

5.12. The radius of the electron cloud in a helium atom is approximately equal to 10^{-10} m. Compute the relative displacement between the centroids of the nucleus and the electron cloud under the influence of an electric field $E_0 = 5 \times 10^6$ volts/m. Compare your result with the radius of the electron cloud.

5.13. In Example 5-3, assume that the charge distribution in the electron cloud is a function of the radial distance from the centroid. If the relative displacement d between the centroids of the nucleus and the electron cloud is very small compared to the effective atomic radius, show that the electronic polarizability is approximately given by

$$\alpha_e \approx \frac{3\epsilon_0 Q}{|\rho(0)|}$$

where $\rho(0)$ is the charge density at the center of the electron cloud and Q is the magnitude of the total charge in the electron cloud. Verify the result for the uniformly charged cloud.

5.14. Show that the torque acting on an electric dipole of moment $\mathbf{p}$ in a uniform electric field $\mathbf{E}_p$ is equal to $\mathbf{p} \cdot \mathbf{E}_p$. Show that the torque tends to align the dipole moment with the field.

5.15. Two infinitely long line charges of uniform densities ρ_{L0} and $-\rho_{L0}$ are situated parallel to the z axis and pass through the points $(d/2, 0, 0)$ and $(-d/2, 0, 0)$, respectively. Show that the average electric field intensity in a cylindrical volume of radius $a > d/2$ and having the z axis as its axis is equal to $-(\rho_{L0} d/2\pi\epsilon_0 a^2)\mathbf{i}_x$.

5.16. Show that the average electric field intensity due to an arbitrary volume charge distribution of dipole moment $\mathbf{p}$ in a spherical volume of radius a is given by

$$\mathbf{E}_{av} = -\frac{\mathbf{p}}{4\pi\epsilon_0 a^3}$$

5.17. The region $a < r < b$ in spherical coordinates is filled with a dielectric material of uniform susceptibility χ_{e0}. A point charge Q is situated at the origin.

(a) Show that the polarization volume charge density is zero and that the polarization surface charge densities are given by

$$\rho_{ps} = \begin{cases} -\dfrac{\chi_{e0}}{1+\chi_{e0}} \dfrac{Q}{4\pi a^2} & r = a \\ \dfrac{\chi_{e0}}{1+\chi_{e0}} \dfrac{Q}{4\pi b^2} & r = b \end{cases}$$

(b) Find the electric field intensities in the three different regions $r < a$, $a < r < b$, and $r > b$.

(c) Discuss your results for the limiting case $a \longrightarrow 0$ and $b \longrightarrow \infty$.

5.18. The region $z < -d$ is occupied by a dielectric of uniform electric susceptibility χ_{e0}. A point charge Q is situated at the origin. Show that the polarization surface charge density is equal to

$$-\frac{Q\chi_{e0} d}{2\pi(2+\chi_{e0})(r_c^2 + d^2)^{3/2}}$$

and that the polarization volume charge density is zero. Show that the electric field intensity inside the dielectric is the same as that due to a point charge equal to $2Q/(2+\chi_{e0})$ at the origin. Show that the electric field intensity outside the dielectric is the same as that due to the point charge Q at the origin and an image point charge $-\chi_{e0}Q/(2+\chi_{e0})$ at $(0, 0, -2d)$.

5.19. A dielectric sphere of radius a and having uniform electric susceptibility χ_{e0} is centered at the origin. The electric field intensities outside and inside the sphere are given in spherical coordinates by

$$\mathbf{E}_o = \left(1 + \frac{2\chi_{e0}}{3+\chi_{e0}}\frac{a^3}{r^3}\right)E_0 \cos\theta\, \mathbf{i}_r - \left(1 - \frac{\chi_{e0}}{3+\chi_{e0}}\frac{a^3}{r^3}\right)E_0 \sin\theta\, \mathbf{i}_\theta$$

$$\mathbf{E}_i = \frac{3}{3+\chi_{e0}}(E_0 \cos\theta\, \mathbf{i}_r - E_0 \sin\theta\, \mathbf{i}_\theta)$$

where E_0 is a constant.

(a) Show that $\mathbf{E}_o$ and $\mathbf{E}_i$ satisfy Maxwell's equations.

(b) Find the applied field by letting $a \longrightarrow 0$ and then find the secondary field both inside and outside $r = a$.

(c) Show that the tangential components of the secondary field on either side of $r = a$ are equal.

(d) From the normal components of the secondary field on either side of $r = a$, obtain the polarization surface charge density at $r = a$, using condition (b) stated in Problem 5.9.

(e) Show that the surface charge density found in part (d) is consistent with the polarization vector corresponding to $\mathbf{E}_i$.

5.20. An infinite plane dielectric slab of thickness d and having a nonuniform electric susceptibility given by

$$\chi_e(z) = \frac{z}{4-z}$$

occupies the region $1 < z < 2$. A uniform electric field $\mathbf{E}_a = E_0\mathbf{i}_z$ is applied. Show that the induced polarization volume and surface charge densities are given by

$$\rho_p = -\tfrac{1}{4}\epsilon_0 E_0 \qquad 1 < z < 2$$

$$\rho_{ps} = \begin{cases} \tfrac{1}{2}\epsilon_0 E_0 & z = 2 \\ -\tfrac{1}{4}\epsilon_0 E_0 & z = 1 \end{cases}$$

Find the secondary and total electric fields both inside and outside the dielectric. Obtain the polarization current density in the dielectric if the applied electric field is time-varying in the manner $\mathbf{E}_a = E_0 \cos \omega t\, \mathbf{i}_z$.

5.21. Two perfectly conducting, infinite plane parallel sheets separated by a distance d carry uniformly distributed surface charges of equal and opposite densities ρ_{s0} and $-\rho_{s0}$, respectively. For each of the following cases, find the potential difference between the two plates:

(a) The medium between the two plates is free space.

(b) The medium between the two plates is a dielectric of uniform permittivity $\epsilon = 4\epsilon_0$.

(c) The medium between the two plates consists of two dielectric slabs of thicknesses t and $d - t$ and having permittivities $\epsilon_1 = 2\epsilon_0$ and $\epsilon_2 = 4\epsilon_0$, respectively.

(d) The medium between the two plates is a dielectric of nonuniform permittivity which varies linearly from a value of ϵ_1 near one plate to a value of ϵ_2 near the second plate.

5.22. Two perfectly conducting, infinite plane parallel sheets separated by a distance d carry uniformly distributed surface charges of equal and opposite densities. For each of the cases listed in Problem 5.21, find the required surface charge densities if the potential difference between the two plates is to be V_0.

5.23. An infinite plane dielectric slab of thickness d and having a nonuniform permittivity given by

$$\epsilon = \frac{4\epsilon_0}{(1 + z/d)^2}$$

occupies the region $0 < z < d$. A uniform electric field $\mathbf{E}_a = E_0\mathbf{i}_z$ is applied. Find the following quantities:

(a) $\mathbf{D}$ outside the dielectric.

(b) $\mathbf{D}$ inside the dielectric.

(c) $\mathbf{E}$ inside the dielectric.

(d) $\mathbf{P}$ inside the dielectric.

(e) ρ_{ps} on the surfaces $z = 0$ and $z = d$.

(f) ρ_p inside the dielectric.

5.24. The region $a < r < b$ in spherical coordinates is occupied by a dielectric material. A point charge Q is situated at the origin. It is found that the electric field intensity inside the dielectric is given by

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 b^2}\mathbf{i}_r \qquad a < r < b$$

Find the following quantities:

(a) The permittivity of the dielectric.

(b) ρ_{ps} on the surfaces $r = a$ and $r = b$.

(c) ρ_p inside the dielectric.

5.25. Show that the result given by (5-98) for the change in the angular velocity of an electron in a circular orbit of radius a under the influence of an applied magnetic field follows from the application of Faraday's law in integral form to the electronic orbit.

5.26. Show that the torque acting on an arbitrary current loop of dipole moment $\mathbf{m}$ in a uniform magnetic field $\mathbf{B}_m$ is equal to $\mathbf{m} \times \mathbf{B}_m$. Show that the torque tends to align the dipole moment with the field.

5.27. Two infinitely long filamentary wires situated parallel to the z axis and passing through the points $(d/2, 0, 0)$ and $(-d/2, 0, 0)$ carry currents I amp in the positive and negative z directions, respectively. Show that the average magnetic flux density in a cylindrical volume of radius $a > d/2$ and having the z axis as its axis is equal to $-(\mu_0 Id/2\pi a^2)\mathbf{i}_y$.

5.28. Show that the average magnetic flux density due to an arbitrary volume current distribution of dipole moment **m** in a spherical volume of radius b is given by

$$\mathbf{B}_{av} = \frac{\mu_0 \mathbf{m}}{2\pi b^3}$$

5.29. The region $a < r < b$ in cylindrical coordinates is filled with a magnetic material of uniform susceptibility χ_{m0}. A filamentary wire situated along the z axis carries current I amp in the z direction.

(a) Show that the magnetization volume current density is zero and that the magnetization surface current densities are given by

$$\mathbf{J}_{ms} = \begin{cases} \chi_{m0} \dfrac{I}{2\pi a} \mathbf{i}_z & r = a \\ -\chi_{m0} \dfrac{I}{2\pi b} \mathbf{i}_z & r = b \end{cases}$$

(b) Find the magnetic flux densities in the three different regions $r < a$, $a < r < b$, and $r > b$.

(c) Discuss your results for the limiting case $a \longrightarrow 0$ and $b \longrightarrow \infty$.

5.30. The region $z < -d$ is occupied by a magnetic material of uniform susceptibility χ_{m0}. An infinitely long filamentary wire carrying current I amp in the x direction is situated along the x axis. Show that the magnetization surface current density is equal to

$$\frac{\chi_{m0} dI}{\pi(2 + \chi_{m0})(y^2 + d^2)} \mathbf{i}_x \text{ amp/m}$$

and that the magnetization volume current density is equal to zero. Show that the magnetic flux density inside the magnetic material is the same as that due to a filamentary wire along the x axis carrying $[(2 + 2\chi_{m0})/(2 + \chi_{m0})]I$ amp in the x direction. Show that the magnetic flux density outside the magnetic material is the same as that due to the filamentary wire along the x axis carrying I amp in the x direction and an image filamentary wire parallel to the x axis and passing through $(0, 0, -2d)$ and carrying a current $\chi_{m0} I/(2 + \chi_{m0})$ amp in the x direction.

5.31. A sphere of magnetic material of radius a and having uniform susceptibility χ_{m0} is centered at the origin. The magnetic flux densities outside and inside the sphere are given in spherical coordinates by

$$\mathbf{B}_o = \left(1 + \frac{2\chi_{m0}}{3 + \chi_{m0}} \frac{a^3}{r^3}\right) B_0 \cos\theta \, \mathbf{i}_r - \left(1 - \frac{\chi_{m0}}{3 + \chi_{m0}} \frac{a^3}{r^3}\right) B_0 \sin\theta \, \mathbf{i}_\theta$$

$$\mathbf{B}_i = \frac{3(1 + \chi_{m0})}{3 + \chi_{m0}} (B_0 \cos\theta \, \mathbf{i}_r - B_0 \sin\theta \, \mathbf{i}_\theta)$$

where B_0 is a constant.

(a) Show that $\mathbf{B}_o$ and $\mathbf{B}_i$ satisfy Maxwell's equations.

(b) Find the applied field by letting $a \longrightarrow 0$ and then find the secondary field both inside and outside $r = a$.

(c) Show that the normal components of the secondary field on either side of $r = a$ are equal.

(d) From the tangential components of the secondary field on either side of $r = a$, obtain the magnetization surface current density at $r = a$, using condition (b) stated in Problem 5.10.

(e) Show that the surface current density found in part (d) is consistent with the magnetization vector corresponding to $\mathbf{B}_i$.

5.32. An infinite plane slab of magnetic material of thickness d and having a nonuniform magnetic susceptibility given by

$$\chi_m(z) = \frac{z}{4}$$

occupies the region $1 < z < 2$. A uniform magnetic field $\mathbf{B}_a = B_0\mathbf{i}_x$ is applied. Show that the induced magnetization volume and surface current densities are given by

$$\mathbf{J}_m = \frac{B_0}{4\mu_0}\mathbf{i}_y \qquad 1 < z < 2$$

$$\mathbf{J}_{ms} = \begin{cases} \dfrac{B_0}{4\mu_0}\mathbf{i}_y & z = 1 \\ -\dfrac{B_0}{2\mu_0}\mathbf{i}_y & z = 2 \end{cases}$$

Find the secondary and total magnetic fields both inside and outside the magnetic material.

5.33. Two perfectly conducting, infinite plane parallel sheets separated by a distance d carry uniformly distributed surface currents having equal and opposite densities $\mathbf{J}_{s0}$ and $-\mathbf{J}_{s0}$, respectively. For each of the following cases, find the magnetic flux between the current sheets per unit length along the direction of flow of the current.

(a) The medium between the two plates is free space.

(b) The medium between the two plates is a magnetic material of uniform permeability $\mu = 4\mu_0$.

(c) The medium between the two plates consists of two magnetic material slabs of thicknesses t and $d - t$ and having permeabilities $\mu_1 = 2\mu_0$ and $\mu_2 = 4\mu_0$, respectively.

(d) The medium between the two plates is a magnetic material of nonuniform permeability which varies linearly from a value of μ_1 near one plate to a value of μ_2 near the second plate.

5.34. Two perfectly conducting, infinite plane parallel sheets separated by a distance d carry uniformly distributed surface currents having equal and opposite densities. For each of the cases listed in Problem 5.33, find the required surface current densities if the magnetic flux between the current sheets per unit length along the direction of flow of the current is to be ψ_0.

5.35. An infinite plane magnetic material slab of thickness d and having a nonuniform permeability given by

$$\mu = \mu_0\left(1 + \frac{z}{d}\right)^2$$

occupies the region $0 < z < d$. A uniform magnetic field $\mathbf{B}_a = B_0\mathbf{i}_y$ is applied. Find the following quantities:

(a) $\mathbf{H}$ outside the magnetic material.

(b) $\mathbf{H}$ inside the magnetic material.

(c) $\mathbf{B}$ inside the magnetic material.

(d) $\mathbf{M}$ inside the magnetic material.

(e) $\mathbf{J}_{ms}$ on the surfaces $z = 0$ and $z = d$.

(f) $\mathbf{J}_m$ inside the magnetic material.

5.36. The region $a < r < b$ in cylindrical coordinates is occupied by a magnetic material. A filamentary wire situated along the z axis carries current I amp in the z direction. It is found that the magnetic flux density inside the magnetic material is given by

$$\mathbf{B} = \frac{\mu_0 I}{2\pi a}\mathbf{i}_\phi \qquad a < r < b$$

Find the following quantities:

(a) The permeability of the magnetic material.

(b) $\mathbf{J}_{ms}$ on the surfaces $r = a$ and $r = b$.

(c) $\mathbf{J}_m$ inside the magnetic material.

5.37. A portion of the **B-H** curve for a ferromagnetic material can be approximated by the analytical expression

$$\mathbf{B} = \mu_0 kH\mathbf{H}$$

where k is a constant having the units of meters per ampere. Find μ_r, μ_{ir}, χ_m, and $\mathbf{M}$.

5.38. Show that Eq. (5-162) follows from Eq. (5-163) whereas Eq. (5-161) follows from Eqs. (5-164) and (5-165).

5.39. Two infinite plane conducting sheets separated by a distance d carry uniformly distributed surface charges of densities ρ_{s0} and $-\rho_{s0}$, respectively. Find the electric stored energy per unit area of the plates if the medium between the plates is (a) free space, and (b) a dielectric of uniform permittivity $\epsilon = 4\epsilon_0$.

5.40. The region between two infinite plane conducting sheets separated by a distance d is characterized by a uniform electric field intensity E_0 directed normal to the plates. Find the electric stored energy per unit area of the plates if the medium between the plates is (a) free space, and (b) a dielectric of uniform permittivity $\epsilon = 4\epsilon_0$.

5.41. Two infinite plane conducting sheets separated by a distance d carry uniformly distributed surface currents of densities $\mathbf{J}_{s0}$ and $-\mathbf{J}_{s0}$, respectiveiy. Find the magnetic stored energy per unit area of the plates if the medium between the plates is (a) free space, and (b) a magnetic material of uniform permeability $\mu = 4\mu_0$.

5.42. The region between two infinite plane conducting sheets separated by a distance d is characterized by a uniform magnetic flux density B_0 directed tangential to the plates. Find the magnetic stored energy per unit area of the plates if the medium between the plates is (a) free space, and (b) a magnetic material of uniform permeability $\mu = 4\mu_0$.

5.43. For the **B-H** curve of Problem 5.37, find the work done per unit volume in magnetizing the material from zero to a certain value $B_0 = \mu_0 k H_0^2$.

5.44. The region $r \leq a$ in cylindrical coordinates is occupied by a magnetic material of uniform permeability μ. The magnetic field intensity is given by

$$\mathbf{H} = \begin{cases} H_0 \cos \omega t \, \mathbf{i}_z & r \leq a \\ 0 & \text{otherwise} \end{cases}$$

where H_0 is a constant. Show that the time rate of change of energy stored in the magnetic field per length l of the magnetic material is correctly given by the power flow into the material obtained by evaluating the surface integral of the Poynting vector over the surface of the cylindrical volume of length l and bounded by $r = a$.

5.45. The region $0 < z < d$ is occupied by a dielectric material of uniform permittivity ϵ. The electric field intensity is given by

$$\mathbf{E} = \begin{cases} E_0 \cos \omega t \, \mathbf{i}_z & 0 < z < d \\ 0 & \text{otherwise} \end{cases}$$

where E_0 is a constant. Assume cylindrical symmetry and show that the time rate of change of energy stored in the cylindrical volume $r < a$ of the dielectric material is correctly given by the power flow into the material obtained by evaluating the surface integral of the Poynting vector over the surface of that volume.

5.46. Medium 1, comprising the region $r < a$ in spherical coordinates, is a perfect dielectric of permittivity $\epsilon_1 = 2\epsilon_0$ whereas medium 2, comprising the region $r > a$, is a perfect dielectric of permittivity $\epsilon_2 = 4\epsilon_0$. The electric field intensity in medium 1 is given by $\mathbf{E}_1 = E_0 \, \mathbf{i}_z$, where E_0 is a constant. Find the electric field intensity at $r = a$ in medium 2.

5.47. Medium 1, comprising the region $z > 0$, is characterized by $\sigma_1 = 0$, $\epsilon_1 = \epsilon_0$, and $\mu_1 = 4\mu_0$ whereas medium 2, comprising the region $z < 0$, is characterized by $\sigma_2 = 0$, $\epsilon_2 = \epsilon_0$, and $\mu_2 = 2\mu_0$. All fields are spatially uniform in both media and independent of time. The magnetic flux density vector $\mathbf{B}_1$ in medium 1 is given by

$$\mathbf{B}_1 = B_0(2\mathbf{i}_x + 4\mathbf{i}_y + 5\mathbf{i}_z) \text{ Wb/m}^2$$

where B_0 is a constant. The boundary $z = 0$ between the two media carries a surface current of density $\mathbf{J}_s$ given by

$$\mathbf{J}_s = \frac{B_0}{\mu_0}(\mathbf{i}_x - 2\mathbf{i}_y) \text{ amp/m}$$

Determine the magnetic flux density vector $\mathbf{B}_2$ in medium 2.

5.48. Two infinite, perfectly conducting plates occupy the planes $x = 0$ and $x = a$. An electric field given by

$$\mathbf{E} = E_0 \sin \frac{\pi x}{a} \cos \frac{\pi t}{a\sqrt{\mu_0 \epsilon_0}} \mathbf{i}_z$$

where E_0 is a constant, exists in the medium between the plates, which is free space.

(a) Using one of Maxwell's curl equations, obtain the magnetic field associated with the given **E**.

(b) Determine the surface current densities on the two plates.

5.49. The region $z < 0$ is free space and the region $z > 0$ is a perfect dielectric of permittivity $\epsilon = 4\epsilon_0$. The electric field intensities $\mathbf{E}_1$ and $\mathbf{E}_2$ in the two media are given by

$$\mathbf{E}_1 = [E_i \cos \omega(t - \sqrt{\mu_0\epsilon_0}\, z) + E_r \cos \omega(t + \sqrt{\mu_0\epsilon_0}\, z)]\mathbf{i}_x \qquad \text{for } z < 0$$

$$\mathbf{E}_2 = E_t \cos \omega(t - 2\sqrt{\mu_0\epsilon_0}\, z)\, \mathbf{i}_x \qquad \text{for } z > 0$$

where E_i, E_r, and E_t are constants.

(a) Find $\mathbf{H}_1$ and $\mathbf{H}_2$ associated with $\mathbf{E}_1$ and $\mathbf{E}_2$, respectively.

(b) Find the relationships between E_r and E_i and between E_t and E_i.

5.50. Show that, for time-varying fields, the boundary condition for the normal co -ponent of **B** follows from the boundary condition for the tangential compone of **E**, whereas the boundary condition for the normal component of **D** follo s from the boundary conditions for the tangential component of **H** and the nor al component of **J**.