4

THE ELECTROMAGNETIC FIELD

In Chapter 2 we studied the static or time-independent electric field in free space. We introduced Maxwell's equations for the static electric field gradually from the experimental law of Coulomb concerning the force between two charges. In Chapter 3 we studied the static or time-independent magnetic field in free space. We introduced Maxwell's equations for the static magnetic field gradually from the experimental law of Ampere concerning the force between two current loops. In this chapter we will study time-varying electric and magnetic fields. We will learn that Maxwell's curl equations for the static electric and magnetic fields have to be modified for time-varying fields in accordance with an experimental law of Faraday and a purely mathematical contribution of Maxwell. When these modifications are made, we will find that the time-varying electric and magnetic fields are coupled; that is, they are interdependent and hence the name "electromagnetic field." As in the case of Chapters 2 and 3, we will in this chapter be concerned with the electromagnetic field in free space only.

4.1 The Lorentz Force Equation

In Section 2.1 we introduced the electric field concept in terms of a force field acting upon charges, whereas in Section 3.1 we introduced the magnetic field concept, also in terms of a force field acting upon charges but only when they are in motion. If an electric field **E** as well as a magnetic field **B** exist in a region, then the force **F** experienced by a test charge q moving with velocity $\mathbf{v}$ is simply the sum of the electric and magnetic forces given by (2-2) and (3-1), respectively. Thus

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{4-1}$$

Equation (4-1) is known as the Lorentz force equation, and the force given by it is known as the Lorentz force. For a continuous charge distribution of density ρ moving with a velocity $\mathbf{v}$, we can define a force per unit volume, $\mathbf{f}$. Considering an infinitesimal volume dv, we then have

$$\mathbf{f}\,dv = \rho\,dv\,(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = (\rho\mathbf{E} + \mathbf{J} \times \mathbf{B})\,dv$$

or

$$\mathbf{f} = \rho\mathbf{E} + \mathbf{J} \times \mathbf{B} \tag{4-2}$$

where $\mathbf{J} = \rho\mathbf{v}$ is the volume current density.

EXAMPLE 4-1. A test charge q C, moving with a velocity $\mathbf{v} = (\mathbf{i}_x + \mathbf{i}_y)$ m/sec, experiences no force in a region of electric and magnetic fields. If the magnetic flux density $\mathbf{B} = (\mathbf{i}_x - 2\mathbf{i}_z)$ Wb/m², find $\mathbf{E}$.

From (4-1), the electric field intensity $\mathbf{E}$ must be equal to $-\mathbf{v} \times \mathbf{B}$ for the charge to experience no force. Thus

$$\begin{aligned}\mathbf{E} &= -(\mathbf{i}_x + \mathbf{i}_y) \times (\mathbf{i}_x - 2\mathbf{i}_z) \\ &= (2\mathbf{i}_x - 2\mathbf{i}_y + \mathbf{i}_z) \text{ volts/m} \quad \blacksquare\end{aligned}$$

EXAMPLE 4-2. A region is characterized by crossed electric and magnetic fields, $\mathbf{E} = E_0\mathbf{i}_y$ and $\mathbf{B} = B_0\mathbf{i}_z$ as shown in Fig. 4.1, where E_0 and B_0 are constants. A small test charge q having a mass m starts from rest at the origin at $t = 0$. We wish to obtain the parametric equations of motion of the test charge.

The force exerted by the crossed electric and magnetic fields on the test charge is

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = q[E_0\mathbf{i}_y + (v_x\mathbf{i}_x + v_y\mathbf{i}_y + v_z\mathbf{i}_z) \times (B_0\mathbf{i}_z)] \tag{4-3}$$

The equations of motion of the test charge can therefore be written as

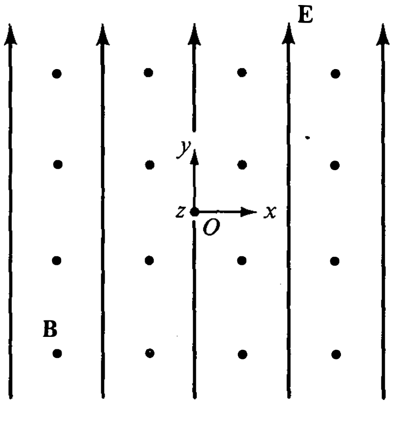

Fig. 4.1. A region of crossed electric and magnetic fields.

$$\frac{dv_x}{dt} = \frac{qB_0}{m} v_y \tag{4-4a}$$

$$\frac{dv_y}{dt} = -\frac{qB_0}{m} v_x + \frac{q}{m} E_0 \tag{4-4b}$$

$$\frac{dv_z}{dt} = 0 \tag{4-4c}$$

Eliminating v_y from (4-4a) and (4-4b), we have

$$\frac{d^2 v_x}{dt^2} + \left(\frac{qB_0}{m}\right)^2 v_x = \left(\frac{q}{m}\right)^2 B_0 E_0 \tag{4-5}$$

The solution for (4-5) is

$$v_x = \frac{E_0}{B_0} + C_1 \cos \omega_c t + C_2 \sin \omega_c t \tag{4-6}$$

where C_1 and C_2 are arbitrary constants and $\omega_c = qB_0/m$. Substituting (4-6) into (4-4a), we obtain

$$v_y = -C_1 \sin \omega_c t + C_2 \cos \omega_c t \tag{4-7}$$

Using initial conditions given by

$$v_x = v_y = 0 \qquad \text{at } t = 0$$

to evaluate C_1 and C_2 in (4-6) and (4-7), we obtain

$$v_x = \frac{E_0}{B_0} - \frac{E_0}{B_0} \cos \omega_c t \tag{4-8}$$

$$v_y = \frac{E_0}{B_0} \sin \omega_c t \tag{4-9}$$

Integrating (4-8) and (4-9) with respect to t, we have

$$x = \frac{E_0}{B_0} t - \frac{E_0}{\omega_c B_0} \sin \omega_c t + C_3 \tag{4-10}$$

$$y = -\frac{E_0}{\omega_c B_0} \cos \omega_c t + C_4 \tag{4-11}$$

Using initial conditions given by

$$x = y = 0 \qquad \text{at } t = 0$$

to evaluate C_3 and C_4 in (4-10) and (4-11), we obtain

$$x = \frac{E_0}{B_0} t - \frac{E_0}{\omega_c B_0} \sin \omega_c t = \frac{E_0}{\omega_c B_0}(\omega_c t - \sin \omega_c t) \tag{4-12}$$

$$y = -\frac{E_0}{\omega_c B_0} \cos \omega_c t + \frac{E_0}{\omega_c B_0} = \frac{E_0}{\omega_c B_0}(1 - \cos \omega_c t) \tag{4-13}$$

Equation (4-4c), together with the initial conditions $v_z = 0$ and $z = 0$ at $t = 0$, yields a solution

$$z = 0 \tag{4-14}$$

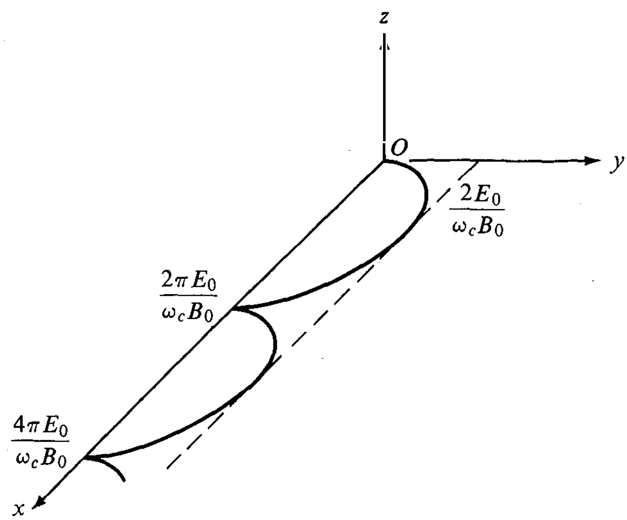

Fig. 4.2. Path of a test charge q in crossed electric and magnetic fields $\mathbf{E} = E_0\mathbf{i}_y$ and $\mathbf{B} = B_0\mathbf{i}_z$.

The equations of motion of the test charge in the crossed electric and magnetic field region are thus given by (4-12), (4-13), and (4-14). These equations represent a cycloid in the $z = 0$ plane, as shown in Fig. 4.2. ▮

4.2 Faraday's Law in Integral Form

We learned in Section 2.2 that Coulomb's experiments demonstrated that charges at rest experience forces as given by Coulomb's law, leading to the interpretation of an electric field set up by charges at rest. Similarly, we learned in Section 3.3 that Ampere's experiments showed that current loops experience forces as given by Ampere's law, leading to the interpretation of a magnetic field being set up by currents, that is, charges in motion. In this section we present the results of experiments by yet another scientist, Michael Faraday. Faraday demonstrated that a magnetic field changing with time results in a flow of current in a loop of wire placed in the magnetic field region. When the magnetic field does not change with time, there is no current flow in the wire. This implies that a time-varying magnetic field exerts electric-type forces on charges. Thus Faraday's experiments demonstrate that a time-varying magnetic field produces an electric field.

The electric field produced by the time-varying magnetic field is such that the work done by it around a closed path C per unit charge in the limit that the charge tends to zero, that is, its circulation around the closed path C, is equal to the negative of the time rate of change of the magnetic flux ψ enclosed by the path C. In equation form we have

$$\text{circulation of } \mathbf{E} \text{ around } C = -\frac{d\psi}{dt} \tag{4-15}$$

The circulation of **E** around a closed path C is $\oint_C \mathbf{E} \cdot d\mathbf{l}$. The magnetic flux enclosed by C is given by the surface integral of the magnetic flux density evaluated over a surface S bounded by the contour C, that is, $\int_S \mathbf{B} \cdot d\mathbf{S}$. In evaluating $\int_S \mathbf{B} \cdot d\mathbf{S}$, we choose the normals to the infinitesimal surfaces comprising S to be pointing towards the side of advance of a right-hand screw as it is turned in the sense of C. Equation (4-15) is thus written as

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S} \tag{4-16}$$

The statement represented by (4-15) or (4-16) is known as Faraday's law. Note that the time derivative on the right side of (4-16) operates on the entire integral so that the circulation of **E** can be due to a change in **B** or a change in the surface S or both. Classically, the quantity $\oint \mathbf{E} \cdot d\mathbf{l}$ on the left side of (4-16) is known under different names, for example, induced electromotive force, induced electromotance, induced voltage. Certainly the word force is not appropriate, since **E** is force per unit charge and $\int \mathbf{E} \cdot d\mathbf{l}$ is work per unit charge. We shall simply refer to **E** as the induced electric field and to $\oint \mathbf{E} \cdot d\mathbf{l}$ as the circulation of **E**.

The minus sign on the right side of (4-16) needs an explanation. We know that the normal to a surface at a point on the surface can be directed towards either side of the surface. In formulating (4-16), we always direct the normal towards the side of advance of a right-hand screw as it is turned around C in the sense in which C is defined. For simplicity, let us consider the plane surface S bounded by a closed path C and let the magnetic flux density be uniform and directed normal to the surface, as shown in Fig. 4.3. If the flux density is increasing with time, $d\psi/dt$ is positive and $-d\psi/dt$ is negative so that $\oint_C \mathbf{E} \cdot d\mathbf{l}$ is negative. Hence the electric field produced by the increasing magnetic flux acts opposite to the sense of the contour C. If we place a test charge at a point on C, it will move opposite to C; if C is occupied by a wire, a current will flow in the sense opposite to

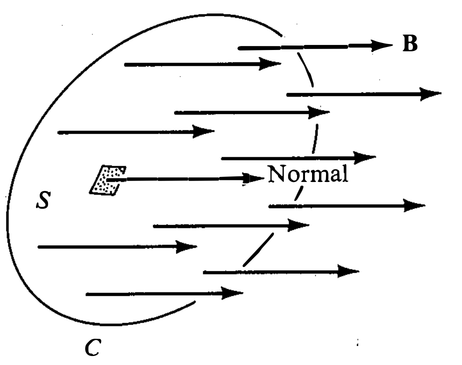

Fig. 4.3. Uniform magnetic field **B** directed normal to a plane surface S.

that of C. Such a current will produce a magnetic field directed to the side opposite to that of the normal since, if the wire is grabbed with the right hand and with the thumb pointing in the direction of the current, the fingers will be curled opposite to the normal as they penetrate the surface S. Thus the current will produce magnetic flux which opposes the increase in the original flux. Likewise, if the flux density is decreasing with time, $d\psi/dt$ is negative and $-d\psi/dt$ is positive so that $\oint_C \mathbf{E} \cdot d\mathbf{l}$ is positive. The electric field produced by the decreasing magnetic flux acts in the sense of the contour C so that, if C is occupied by a wire, a current will flow in the same sense as that of C. Such a current will produce a magnetic field directed to the side of the normal, thereby opposing the decrease in the flux. Thus the minus sign on the right side of (4-16) signifies that the induced electric field is such that it opposes the change in the magnetic flux producing it. This fact is known as Lenz' law. If the induced electric field is such that it aids the change in the magnetic flux instead of opposing it, any small change in the magnetic flux will set up a chain reaction by inducing an electric field, which will aid the change in the magnetic flux, which will increase the electric field, and so on, thereby violating the conservation of energy. Hence Lenz' law must be obeyed and the minus sign on the right side of (4-16) is very important.

EXAMPLE 4-3. The magnetic flux density is given by

$$\mathbf{B} = B_0 \cos \omega_1 t \, \mathbf{i}_x$$

where B_0 and ω_1 are constants. A rectangular loop of wire of area A is placed symmetrically with respect to the z axis and rotated about the z axis at a constant angular velocity ω_2 as shown in Fig. 4.4, such that the angle ϕ which the normal to the plane of the loop makes with the x axis is given by

$$\phi = \phi_0 + \omega_2 t$$

It is desired to find the circulation of the induced electric field around the contour C of the loop.

The unit vector normal to the plane of the loop is

$$\mathbf{i}_n = \cos(\phi_0 + \omega_2 t)\, \mathbf{i}_x + \sin(\phi_0 + \omega_2 t)\, \mathbf{i}_y \tag{4-17}$$

The magnetic flux enclosed by the loop is

$$\begin{aligned} \psi &= \int_{\substack{\text{plane surface} \\ S \text{ bounded} \\ \text{by } C}} \mathbf{B} \cdot d\mathbf{S} \\ &= \int_S (B_0 \cos \omega_1 t\, \mathbf{i}_x) \cdot [\cos(\phi_0 + \omega_2 t)\, \mathbf{i}_x + \sin(\phi_0 + \omega_2 t)\, \mathbf{i}_y]\, dS \\ &= \int_S B_0 \cos \omega_1 t \cos(\phi_0 + \omega_2 t)\, dS = B_0 A \cos \omega_1 t \cos(\phi_0 + \omega_2 t) \end{aligned} \tag{4-18}$$

This is simply the flux enclosed at any time t by the projection of the loop at that time on to the yz plane, which is normal to the flux density. From

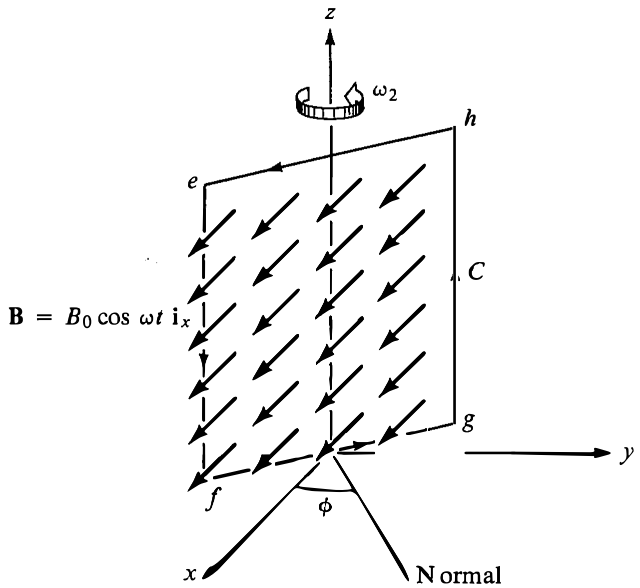

Fig. 4.4. A rectangular loop of wire rotating about the z axis with a constant angular velocity and situated in a time-varying magnetic field.

Faraday's law, we now have

$$\oint_C \mathbf{E}' \cdot d\mathbf{l} = -\frac{d\psi}{dt} = -\frac{d}{dt}[B_0 A \cos \omega_1 t \cos (\phi_0 + \omega_2 t)]$$
$$= B_0 A[\omega_1 \sin \omega_1 t \cos (\phi_0 + \omega_2 t) \qquad (4\text{-}19)$$
$$+ \omega_2 \cos \omega_1 t \sin (\phi_0 + \omega_2 t)]$$

where the prime in $\mathbf{E}'$ denotes that the electric field is associated with the contour of the moving loop. Note that the right side of (4-19) reduces to $B_0 A \omega_1 \cos \phi_0 \sin \omega_1 t$ for $\omega_2 = 0$, that is, for a stationary loop in a time-varying magnetic field and to $B_0 A \omega_2 \sin (\phi_0 + \omega_2 t)$ for $\omega_1 = 0$, that is, for a moving loop in a static magnetic field. ▮

EXAMPLE 4-4. The magnetic flux density is given in cylindrical coordinates by

$$\mathbf{B} = \begin{cases} B_0 \sin \omega t \, \mathbf{i}_z & \text{for } r < a \\ 0 & \text{for } r > a \end{cases}$$

where B_0 and ω are constants. It is desired to find the induced electric field everywhere.

We note that the time-varying magnetic field has circular symmetry about the z axis and is independent of z. Hence the induced electric field must also possess circular symmetry about the z axis and must be independent of z; that is, $\mathbf{E}$ can be a function of r only. Choosing a circular contour C of radius r and centered at the origin, as shown in Fig. 4.5, we note that the magnetic flux enclosed by the contour C is

$$\psi = \int_S \mathbf{B} \cdot d\mathbf{S} \tag{4-20}$$

where S is the plane surface bounded by the contour C. Substituting for $\mathbf{B}$ and $d\mathbf{S}$ in (4-20), we get, for $r < a$,

$$\psi = \int_S \mathbf{B} \cdot d\mathbf{S} = \int_S B_0 \sin \omega t \, \mathbf{i}_z \cdot dS \, \mathbf{i}_z$$
$$= B_0 \sin \omega t \int_S dS = \pi r^2 B_0 \sin \omega t$$

For $r > a$,

$$\psi = \int_S \mathbf{B} \cdot d\mathbf{S} = \int_{S_1} \mathbf{B} \cdot d\mathbf{S} + \int_{S_2} \mathbf{B} \cdot d\mathbf{S} \tag{4-21}$$

where S_1 is the plane surface enclosed by the circular contour of radius a and S_2 is the remainder of the surface S. The magnetic field is zero, however, on the surface S_2 and hence the second integral on the right side of (4-21) is zero. Hence, for $r > a$,

$$\psi = \int_{S_1} \mathbf{B} \cdot d\mathbf{S} = \pi a^2 B_0 \sin \omega t$$

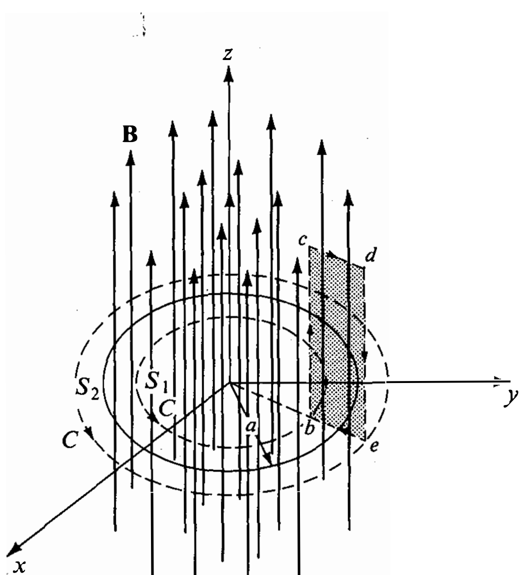

Fig. 4.5. For evaluating the induced electric field due a time-varying magnetic field possessing cylindrical symmetry.

Thus

$$\psi = \begin{cases} \pi r^2 B_0 \sin \omega t & \text{for } r < a \\ \pi a^2 B_0 \sin \omega t & \text{for } r > a \end{cases} \tag{4-22}$$

Now,

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = \int_{\phi=0}^{2\pi} E_\phi r \, d\phi = 2\pi r E_\phi \tag{4-23}$$

From Faraday's law, we then have

$$2\pi r E_\phi = -\frac{d\psi}{dt} = \begin{cases} -\pi r^2 B_0 \omega \cos \omega t & \text{for } r < a \\ -\pi a^2 B_0 \omega \cos \omega t & \text{for } r > a \end{cases}$$

or

$$E_\phi = \begin{cases} -\dfrac{B_0 r \omega}{2} \cos \omega t & \text{for } r < a \\ -\dfrac{B_0 a^2 \omega}{2r} \cos \omega t & \text{for } r > a \end{cases} \tag{4-24}$$

Any r component of $\mathbf{E}$ independent of ϕ and z will have nonzero curl and hence can be attributed to sources appropriate for a static electric field, that is, an electric field originating from charges at rest. Any z component will have to be independent of r since the magnetic field has no ϕ component. This is because if we consider a rectangular contour *bcdeb* in a plane containing the z axis as shown in Fig. 4.5, the magnetic flux enclosed by this contour is zero. Hence $\oint_{bcdeb} \mathbf{E} \cdot d\mathbf{l}$ is zero or $\int_b^c E_z \, dz + \int_d^e E_z \, dz$ is equal to zero, leading to the conclusion that E_z along *bc* is the same as E_z along *ed*. Since the curl of a field which has a z component independent of r and ϕ is zero, it can also be attributed to sources appropriate for a static field. Thus the induced electric field due to the time-varying magnetic field has a ϕ component only, thereby surrounding the magnetic field, and it is given by

$$\mathbf{E} = \begin{cases} -\dfrac{B_0 r \omega}{2} \cos \omega t \, \mathbf{i}_\phi & \text{for } r < a \\ -\dfrac{B_0 a^2 \omega}{2r} \cos \omega t \, \mathbf{i}_\phi & \text{for } r > a \end{cases} \tag{4-25}$$

The fact that the induced electric field surrounds the time-varying magnetic field can also be seen if we recognize that Faraday's law is similar in form to Ampere's circuital law

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 (\text{current } I \text{ enclosed by C})$$

The magnetic field due to the current I surrounds the current. Likewise, the electric field due to the changing magnetic flux should surround the flux. The induced electric field is thus solenoidal in character, as compared to the irrotational nature of the electric field due to charges at rest. ▮

One of the consequences of Faraday's law is that $\int \mathbf{E} \cdot d\mathbf{l}$ evaluated between two points a and b is, in general, dependent on the path followed from a to b to evaluate the integral, unlike in the case of the static electric field. To illustrate this, let us consider a region of uniform but time-varying magnetic field. Applying Faraday's law to two different closed paths $acbea$ and $adbea$ as shown in Fig. 4.6, we obtain two different results for $\oint \mathbf{E} \cdot d\mathbf{l}$

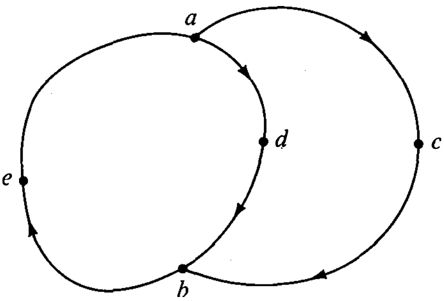

Fig. 4.6. Two different closed paths *acbea* and *adbea*.

since the paths enclose different areas. However, path *bea* is common to both the closed paths, and the contributions from the path *bea* to $\oint_{acbea} \mathbf{E} \cdot d\mathbf{l}$ and to $\oint_{adbea} \mathbf{E} \cdot d\mathbf{l}$ are the same. It then follows that $\int_{acb} \mathbf{E} \cdot d\mathbf{l}$ is not equal to $\int_{adb} \mathbf{E} \cdot d\mathbf{l}$. Thus the work done per unit charge in carrying a test charge from a to b in an electromagnetic field, that is, $\int_a^b \mathbf{E} \cdot d\mathbf{l}$ in an electromagnetic field, is not uniquely defined. It depends upon the path followed from a to b in evaluating $\int_a^b \mathbf{E} \cdot d\mathbf{l}$. The quantity $\int_a^b \mathbf{E} \cdot d\mathbf{l}$ is known as the voltage between the points a and b in the case of time-varying fields. The word "voltage" is interchangeable with "potential difference" for the case of static electric field only. For time-varying fields, the electric field cannot be expressed exclusively in terms of a time-varying electric scalar potential as we will learn in the following section. Hence, the two words are not interchangeable in the time-varying case.

Now, let us consider two different surfaces S_1 and S_2 bounded by a contour C with the normals defining the surfaces directed out of the volume bounded by $S_1 + S_2$ as shown in Fig. 4.7. Then, applying Faraday's law to C, we have

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_{S_1} \mathbf{B} \cdot d\mathbf{S} = \frac{d}{dt}\int_{S_2} \mathbf{B} \cdot d\mathbf{S} \tag{4-26}$$

It follows from (4-26) that

$$\frac{d}{dt}\left(\int_{S_1} \mathbf{B} \cdot d\mathbf{S} + \int_{S_2} \mathbf{B} \cdot d\mathbf{S}\right) = 0$$

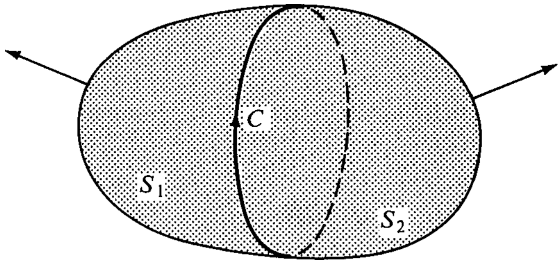

Fig. 4.7. Two surfaces S_1 and S_2 bounded by a contour C.

or

$$\oint_{S_1+S_2} \mathbf{B} \cdot d\mathbf{S} = \text{constant with time} \tag{4-27}$$

The constant on the right side of (4-27) must, however, be equal to zero since a nonzero value for any surface requires the existence forever of isolated magnetic charge within the volume bounded by that surface. There is no experimental evidence of the existence of such magnetic charge. Thus, it follows from Faraday's law in integral form that

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0 \tag{3-111}$$

where S is any closed surface.

4.3 Faraday's Law in Differential Form (Maxwell's First Curl Equation for the Electromagnetic Field)

In the previous section we introduced Faraday's law in integral form, given by

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S} \tag{4-16}$$

where S is any surface bounded by the contour C. According to Stokes' theorem, we have

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S}$$

where S is any surface bounded by the contour C. In particular, choosing the same surface as for the integral on the right side of (4-16), we obtain

$$\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S} \tag{4-28}$$

If the surface S is stationary, that is, independent of time, then

$$\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S} = \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} \tag{4-29}$$

and

$$\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = \int_S -\frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} \tag{4-30}$$

Comparing the integrands on the two sides of (4-30), we have

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{4-31}$$

This is the differential form of Faraday's law and Maxwell's first curl equation for the electromagnetic field.

If, in addition to the variation of the magnetic field with time, the surface S is also changing with time due to a displacement of the contour as shown in Fig. 4.8, then we evaluate $\frac{d}{dt}\int \mathbf{B} \cdot d\mathbf{S}$ by considering two times t_1 and t_2, where $t_2 = t_1 + \Delta t$. If S_1 and S_2 are the surfaces bounded

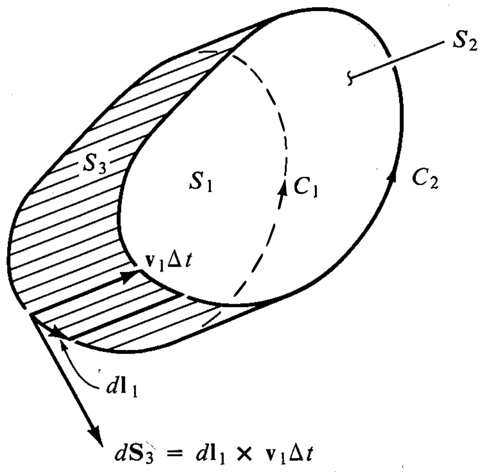

Fig. 4.8. Displacement of contour C_1 with time and the associated surfaces.

by C_1 and C_2 at t_1 and t_2, respectively, we have, from the definition of differentiation,

$$\left[\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S}\right]_{t_1} = \lim_{t_2 \to t_1} \frac{1}{t_2 - t_1}\left\{\left[\int_S \mathbf{B} \cdot d\mathbf{S}\right]_{t_2} - \left[\int_S \mathbf{B} \cdot d\mathbf{S}\right]_{t_1}\right\}$$
$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(\int_{S_2} \mathbf{B}_2 \cdot d\mathbf{S}_2 - \int_{S_1} \mathbf{B}_1 \cdot d\mathbf{S}_1\right) \tag{4-32}$$

where $\mathbf{B}_2$ and $\mathbf{B}_1$ are $\mathbf{B}(t_2)$ and $\mathbf{B}(t_1)$, respectively. Applying the divergence theorem at time t_2 to the volume V bounded by the two surfaces S_1 and S_2 and the surface S_3 formed by the movement of the contour C, we have

$$\int_V \nabla \cdot \mathbf{B}_2 \, dv = \oint_{S_1+S_2+S_3} \mathbf{B}_2 \cdot d\mathbf{S}$$
$$= -\int_{S_1} \mathbf{B}_2 \cdot d\mathbf{S}_1 + \int_{S_2} \mathbf{B}_2 \cdot d\mathbf{S}_2 + \int_{S_3} \mathbf{B}_2 \cdot d\mathbf{S}_3 \tag{4-33}$$

where the minus sign associated with the first of the three integrals on the right side of (4-33) is due to the direction of $d\mathbf{S}_1$ pointing into the volume V. Also, in the third integral, we choose the direction of $d\mathbf{S}_3$ as pointing out of the volume V.

Since $\nabla \cdot \mathbf{B} = 0$, we have, from (4-33),

$$\int_{S_2} \mathbf{B}_2 \cdot d\mathbf{S}_2 - \int_{S_1} \mathbf{B}_2 \cdot d\mathbf{S}_1 = -\int_{S_3} \mathbf{B}_2 \cdot d\mathbf{S}_3 \tag{4-34}$$

If the velocity with which an element $d\mathbf{l}_1$ in the contour C_1 is displaced is $\mathbf{v}_1$, the infinitesimal area $d\mathbf{S}_3$ swept by the element in the time Δt is $d\mathbf{l}_1 \times \mathbf{v}_1 \, \Delta t$ as shown in Fig. 4.8. Hence

$$\int_{S_3} \mathbf{B}_2 \cdot d\mathbf{S}_3 = \oint_{C_1} \mathbf{B}_2 \cdot d\mathbf{l}_1 \times \mathbf{v}_1 \, \Delta t \tag{4-35}$$

Substituting (4-35) into (4-34), we have

$$\int_{S_2} \mathbf{B}_2 \cdot d\mathbf{S}_2 - \int_{S_1} \mathbf{B}_2 \cdot d\mathbf{S}_1 = -\oint_{C_1} \mathbf{B}_2 \cdot d\mathbf{l}_1 \times \mathbf{v}_1 \, \Delta t \tag{4-36}$$

Now, expanding $\mathbf{B}(t)$ in a Taylor's series at time t_1, we have

$$\mathbf{B}_2 = \mathbf{B}_1 + \left[\frac{\partial \mathbf{B}}{\partial t}\right]_{t_1} \Delta t + \frac{1}{2}\left[\frac{\partial^2 \mathbf{B}}{\partial t^2}\right]_{t_1} (\Delta t)^2 + \cdots \tag{4-37}$$

and

$$\int_{S_1} \mathbf{B}_2 \cdot d\mathbf{S}_1 = \int_{S_1} \mathbf{B}_1 \cdot d\mathbf{S}_1 + \Delta t \int_{S_1} \left[\frac{\partial \mathbf{B}}{\partial t}\right]_{t_1} \cdot d\mathbf{S}_1 + \cdots \tag{4-38}$$

$$\begin{aligned}\oint_{C_1} \mathbf{B}_2 \cdot d\mathbf{l}_1 \times \mathbf{v}_1 \, \Delta t = \Delta t \oint_{C_1} \mathbf{B}_1 \cdot d\mathbf{l}_1 \times \mathbf{v}_1 \\ + (\Delta t)^2 \oint_{C_1} \left[\frac{\partial \mathbf{B}}{\partial t}\right]_{t_1} \cdot d\mathbf{l}_1 \times \mathbf{v}_1 + \cdots\end{aligned} \tag{4-39}$$

Substituting (4-38) and (4-39) into (4-36) and rearranging, we get

$$\begin{aligned}\int_{S_2} \mathbf{B}_2 \cdot d\mathbf{S}_2 - \int_{S_1} \mathbf{B}_1 \cdot d\mathbf{S}_1 = \Delta t \int_{S_1} \left[\frac{\partial \mathbf{B}}{\partial t}\right]_{t_1} \cdot d\mathbf{S}_1 - \Delta t \oint_{C_1} \mathbf{B}_1 \cdot d\mathbf{l}_1 \times \mathbf{v}_1 \\ + \text{higher-order terms in } \Delta t\end{aligned} \tag{4-40}$$

Substituting (4-40) into (4-32), we obtain

$$\begin{aligned}\left[\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S}\right]_{t_1} &= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left\{\Delta t \int_{S_1} \left[\frac{\partial \mathbf{B}}{\partial t}\right]_{t_1} \cdot d\mathbf{S}_1 - \Delta t \oint_{C_1} \mathbf{B}_1 \cdot d\mathbf{l}_1 \times \mathbf{v}_1 \right. \\ &\qquad \left. + \text{higher-order terms in } \Delta t\right\} \\ &= \int_{S_1} \left[\frac{\partial \mathbf{B}}{\partial t}\right]_{t_1} \cdot d\mathbf{S}_1 - \oint_{C_1} \mathbf{B}_1 \cdot d\mathbf{l}_1 \times \mathbf{v}_1 \\ &= \int_{S_1} \left[\frac{\partial \mathbf{B}}{\partial t}\right]_{t_1} \cdot d\mathbf{S}_1 - \oint_{C_1} [\mathbf{v} \times \mathbf{B}]_{t_1} \cdot d\mathbf{l}_1\end{aligned} \tag{4-41}$$

Since Eq. (4-41) must be true for any time t_1, we have, in general,

$$\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S} = \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} - \oint_C \mathbf{v} \times \mathbf{B} \cdot d\mathbf{l} \tag{4-42}$$

where C is the contour and S is the surface bounded by C at any arbitary time t.

To an observer moving with a point on the contour, the contour appars to be stationary and the observer will attribute the force experienced by a test charge at that point as due to an electric field alone. Denoting this electric field as $\mathbf{E}'$ and applying Faraday's law for the contour C and uing (4-42), we have

$$\begin{aligned}\oint_C \mathbf{E}' \cdot d\mathbf{l} &= -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S} \\ &= -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} + \oint_C \mathbf{v} \times \mathbf{B} \cdot d\mathbf{l}\end{aligned} \tag{4-43}$$

But, according to Stokes' theorem, we have

$$\oint_C \mathbf{E}' \cdot d\mathbf{l} = \int_S \nabla \times \mathbf{E}' \cdot d\mathbf{S} \tag{4-44a}$$

and

$$\oint_C \mathbf{v} \times \mathbf{B} \cdot d\mathbf{l} = \int_C \nabla \times (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{S} \tag{4-44b}$$

Substituting (4-44a) and (4-44b) into (4-43), we get

$$\int_S \nabla \times \mathbf{E}' \cdot d\mathbf{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} + \int_S \nabla \times (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{S} \tag{4-45}$$

or

$$\nabla \times \mathbf{E}' = -\frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{v} \times \mathbf{B}) \tag{4-46}$$

Equation (4-46) is Faraday's law in differential form, where $\mathbf{E}'$ is the electric field as measured by an observer moving with a velocity $\mathbf{v}$, relative to the magnetic field $\mathbf{B}$.

On the other hand, a stationary observer views the force experienced by the test charge moving with the point on the contour as being composed of two parts, electric-type and magnetic-type, that is, one due to an electric field acting on the charge and the other due to a magnetic field acting on the charge. Since the magnetic force acting on the test charge is $q\mathbf{v} \times \mathbf{B}$, the observer will attribute a force of $\mathbf{F} - q\mathbf{v} \times \mathbf{B}$ only to the electric field where $\mathbf{F}$ is the total force acting on the charge. The total force acting on the charge must of course be the same whether viewed by an observer moving with the contour or by a stationary observer. Hence it is equal to $q\mathbf{E}'$. Thus the force attributed to the electric field by the stationary observer is $q\mathbf{E}' - q\mathbf{v} \times \mathbf{B} = q(\mathbf{E}' - \mathbf{v} \times \mathbf{B})$ or the electric field as viewed by the stationary observer is given by

$$\mathbf{E} = \mathbf{E}' - \mathbf{v} \times \mathbf{B} \tag{4-47}$$

Rearranging (4-46), we have

$$\nabla \times (\mathbf{E}' - \mathbf{v} \times \mathbf{B}) = -\frac{\partial \mathbf{B}}{\partial t} \tag{4-48}$$

which, with the aid of (4-47), becomes

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{4-31}$$

which is the same as the result obtained in the case of the stationary contour. Thus Eq. (4-31) holds, in general, where **E** is the induced electric field as viewed by an observer stationary relative to the time-varying magnetic field **B**.

EXAMPLE 4-5. For the test charge of Example 4-2, find the electric field as viewed by an observer moving with the test charge.

From Example 4-2, the electric and magnetic fields as viewed by a stationary observer are

$$\mathbf{E} = E_0\mathbf{i}_y \qquad \text{and} \qquad \mathbf{B} = B_0\mathbf{i}_z$$

The velocity of motion of the test charge is given by

$$\begin{aligned}\mathbf{v} &= v_x\mathbf{i}_x + v_y\mathbf{i}_y \\ &= \left(\frac{E_0}{B_0} - \frac{E_0}{B_0}\cos\omega_c t\right)\mathbf{i}_x + \left(\frac{E_0}{B_0}\sin\omega_c t\right)\mathbf{i}_y\end{aligned} \tag{4-49}$$

where we have substituted for v_x and v_y from (4-8) and (4-9), respectively.

Rearranging (4-47), we note that the electric field **E**′ as viewed by an observer moving with a velocity **v** relative to the magnetic field is given by

$$\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B} \tag{4-50}$$

We can also obtain this result directly by noting that, for an observer moving with the test charge, the test charge appears to be stationary and hence the observer will attribute the force experienced by it to an electric field alone. Since the force experienced by the test charge is $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$, the observer views an electric field of $\mathbf{F}/q = \mathbf{E} + \mathbf{v} \times \mathbf{B}$. Substituting for **E**, **v**, and **B** in (4-50), we obtain

$$\begin{aligned}\mathbf{E}' &= E_0\mathbf{i}_y + \left[\left(\frac{E_0}{B_0} - \frac{E_0}{B_0}\cos\omega_c t\right)\mathbf{i}_x + \left(\frac{E_0}{B_0}\sin\omega_c t\right)\mathbf{i}_y\right] \times B_0\mathbf{i}_z \\ &= E_0 \sin\omega_c t\,\mathbf{i}_x + E_0\cos\omega_c t\,\mathbf{i}_y\end{aligned} \tag{4-51}$$

Thus the electric field as viewed by an observer moving with the test charge is $(E_0 \sin\omega_c t\,\mathbf{i}_x + E_0 \cos\omega_c t\,\mathbf{i}_y)$. ▮

EXAMPLE 4-6. In Example 4-3, we obtained the circulation of the induced electric field around a rectangular loop moving in a time varying magnetic field by the direct application of Faraday's law in integral form given by (4-16). It is here desired to verify the result of Example 4-3 by using (4-43).

With reference to the notation of Fig. 4.4, the first integral on the right side of (4-43) is given by

$$-\int_{\substack{\text{plane surface } S\\ \text{bounded by } C}} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} = \int_S B_0\omega_1 \sin \omega_1 t\, \mathbf{i}_x \cdot (\cos \phi\, \mathbf{i}_x + \sin \phi\, \mathbf{i}_y)\, dS$$
$$= B_0 A\omega_1 \cos(\phi_0 + \omega_2 t) \sin \omega_1 t \tag{4-52}$$

To evaluate the second integral on the right side of (4-43), we note that, along side ef,

$$\mathbf{v} \times \mathbf{B} = \frac{(fg)}{2}\omega_2[\mathbf{i}_\phi]_{ef} \times B_0 \cos \omega_1 t\, \mathbf{i}_x$$
$$= -\frac{(fg)}{2}\omega_2 B_0 \cos \omega_1 t \sin \phi\, \mathbf{i}_z$$

so that

$$\int_e^f \mathbf{v} \times \mathbf{B} \cdot d\mathbf{l} = \frac{(ef)(fg)}{2}\omega_2 B_0 \cos \omega_1 t \sin \phi$$
$$= \frac{B_0 A\omega_2}{2} \cos \omega_1 t \sin \phi \tag{4-53a}$$

Along side fg, $\mathbf{v} \times \mathbf{B} \cdot d\mathbf{l} = 0$ so that

$$\int_f^g \mathbf{v} \times \mathbf{B} \cdot d\mathbf{l} = 0 \tag{4-53b}$$

Along side gh,

$$\mathbf{v} \times \mathbf{B} = \frac{(fg)}{2}\omega_2[\mathbf{i}_\phi]_{gh} \times B_0 \cos \omega_1 t\, \mathbf{i}_x$$
$$= \frac{(fg)}{2}\omega_2 B_0 \cos \omega_1 t \sin \phi\, \mathbf{i}_z$$

so that

$$\int_g^h \mathbf{v} \times \mathbf{B} \cdot d\mathbf{l} = \frac{(gh)(fg)}{2}\omega_2 B_0 \cos \omega_1 t \sin \phi$$
$$= \frac{B_0 A\omega_2}{2} \cos \omega_1 t \sin \phi \tag{4-53c}$$

Along side he, $\mathbf{v} \times \mathbf{B} \cdot d\mathbf{l} = 0$ so that

$$\int_h^e \mathbf{v} \times \mathbf{B} \cdot d\mathbf{l} = 0 \tag{4-53d}$$

From (4-53a)–(4-53d), we have

$$\oint_C \mathbf{v} \times \mathbf{B} \cdot d\mathbf{l} = \oint_{efghe} \mathbf{v} \times \mathbf{B} \cdot d\mathbf{l}$$
$$= B_0 A\omega_2 \cos \omega_1 t \sin \phi$$
$$= B_0 A\omega_2 \cos \omega_1 t \sin(\phi_0 + \omega_2 t) \tag{4-54}$$

Thus, from (4-52) and (4-54), we obtain

$$\oint_C \mathbf{E}' \cdot d\mathbf{l} = B_0 A\omega_1 \cos(\phi_0 + \omega_2 t)\sin\omega_1 t + B_0 A\omega_2 \cos\omega_1 t \sin(\phi_0 + \omega_2 t)$$

which agrees with (4-19). ▮

EXAMPLE 4-7. In Example 4-4 we obtained the expression for the induced electric field due to a time-varying magnetic field possessing cylindrical symmetry about the z axis, by using Faraday's law in integral form. It is desired to verify the result by using Faraday's law in differential form given by (4-31).

From Example 4-4, we have the induced electric field given by

$$\mathbf{E} = \begin{cases} -\dfrac{B_0 r\omega}{2}\cos\omega t\,\mathbf{i}_\phi & \text{for } r < a \\ -\dfrac{B_0 a^2\omega}{2r}\cos\omega t\,\mathbf{i}_\phi & \text{for } r > a \end{cases} \tag{4-25}$$

Hence

$$\nabla \times \mathbf{E} = \begin{vmatrix} \dfrac{\mathbf{i}_r}{r} & \mathbf{i}_\phi & \dfrac{\mathbf{i}_z}{r} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ 0 & rE_\phi & 0 \end{vmatrix} = \frac{\mathbf{i}_z}{r}\left[\frac{\partial}{\partial r}(rE_\phi)\right] = \begin{cases} -B_0\omega\cos\omega t\,\mathbf{i}_z & \text{for } r < a \\ 0 & \text{for } r > a \end{cases} \tag{4-55}$$

From Faraday's law in differential form, we then have

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} = \begin{cases} B_0\omega\cos\omega t\,\mathbf{i}_z & \text{for } r < a \\ 0 & \text{for } r > a \end{cases} \tag{4-56}$$

Equation (4-56) is consistent with

$$\mathbf{B} = \begin{cases} B_0 \sin\omega t\,\mathbf{i}_z & \text{for } r < a \\ 0 & \text{for } r > a \end{cases}$$

which is the magnetic field specified in Example 4-4. ▮

Returning to Eq. (4-31) and taking the divergence of both sides, we have

$$\nabla \cdot \nabla \times \mathbf{E} = -\nabla \cdot \frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial}{\partial t}(\nabla \cdot \mathbf{B}) \tag{4-57}$$

But, since $\nabla \cdot \nabla \times \mathbf{E} \equiv 0$, it follows from (4-57) that

$$\frac{\partial}{\partial t}(\nabla \cdot \mathbf{B}) = 0 \tag{4-58}$$

or

$$\nabla \cdot \mathbf{B} = \text{constant with time} \tag{4-59}$$

The constant on the right side of (4-59) must, however, be equal to zero since a nonzero value at any point in space requires the existence forever of isolated magnetic charge at that point. There is no experimental evidence of the existence of such magnetic charge. Thus, we note that Maxwell's equation for the divergence of the time-varying magnetic field given by

$$\nabla \cdot \mathbf{B} = 0 \tag{4-60}$$

follows from the Maxwell's equation for the curl of **E** given by (4-31). As a consequence of (4-60), we have

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{4-61}$$

where **A** is a time-varying vector potential. Substituting (4-61) into (4-31), we get

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{A}) = -\nabla \times \frac{\partial \mathbf{A}}{\partial t}$$

or

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0 \tag{4-62}$$

Thus $(\mathbf{E} + \partial \mathbf{A}/\partial t)$ can be expressed as the gradient of a time-varying scalar potential. In particular, we can write

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla V \tag{4-63}$$

where V is the time-varying scalar potential so that Eq. (4-63) reduces to $\mathbf{E} = -\nabla V$ for the static case. Rearranging (4-63), we obtain

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \tag{4-64}$$

We will have an opportunity to study the time-varying scalar and vector potentials in Section 6.16.

4.4 The Dilemma of Ampere's Circuital Law and the Displacement Current Concept; Modified Ampere's Circuital Law in Integral Form

In Section 3.6 we introduced Ampere's circuital law in integral form, given by

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0(\text{current enclosed by } C) \tag{3-57}$$

In that connection we discussed the uniqueness of a closed path enclosing a current by considering the case of a straight filamentary wire of finite length along which charge flows from one end to the other end (Fig. 3.16) and the case of an infinitely long filamentary wire. We found that the current enclosed by a closed path C is not uniquely defined in the case of the finitely long wire, whereas it is uniquely defined for the case of the infinitely long wire. On the other hand, the magnetic field due to a current-carrying wire is uniquely given at every point through the Biot–Savart law and hence $\oint_C \mathbf{B} \cdot d\mathbf{l}$ for a given closed path C has a unique value. Thus it seems to be meaningless to apply Ampere's circuital law as given by (3-57) for the case of the finitely long wire. What then is the fallacy of the situation? Is there any modification required for (3-57) so that the dilemma is resolved?

To answer these questions, let us consider a semiinfinitely long, straight filamentary wire occupying the upper half of the z axis. Let there be a point source of charge Q C at the origin and let the current flowing along the wire to infinity be I amp as shown in Fig. 4.9 so that the charge Q is decreas-

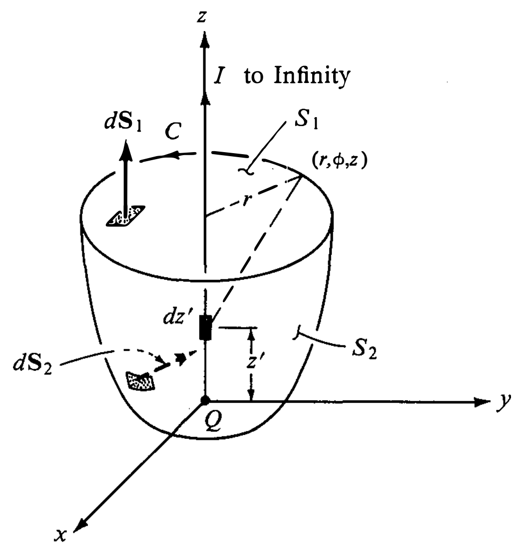

Fig. 4.9. For introducing the displacement current concept and deriving the modification to Ampere's circuital law.

ing at the rate of I C/sec. Let us consider a circular contour C of radius r in the plane normal to the wire and centered at a point on the wire a distance z from the origin, as shown in Fig. 4.9. The current enclosed by C is not uniquely defined since the current penetrating the plane surface S_1 bounded by the contour is I, whereas the current penetrating a bowl-shaped surface S_2 as shown in Fig. 4.9 is zero. On the other hand, $\oint_C \mathbf{B} \cdot d\mathbf{l}$ is unique since $\mathbf{B}$ along C is given by the application of the Biot–Savart law to the semiinfinitely long wire. According to the Biot–Savart law, the magnetic flux density at a point (r, ϕ, z) on the contour C due to an infinitesimal segment dz' of the wire at distance z' from the origin is given by

$$d\mathbf{B} = \frac{\mu_0 I r\, dz'}{4\pi[(z - z')^2 + r^2]^{3/2}} \mathbf{i}_\phi \tag{4-65}$$

The magnetic flux density at (r, ϕ, z) due to the entire semiinfinitely long wire is given by

$$\begin{aligned} \mathbf{B} &= \int_{z'=0}^{\infty} d\mathbf{B} = \frac{\mu_0 I r}{4\pi} \int_{z'=0}^{\infty} \frac{dz'}{[(z - z')^2 + r^2]^{3/2}} \mathbf{i}_\phi \\ &= \frac{\mu_0 I}{4\pi r}\left(1 + \frac{z}{\sqrt{z^2 + r^2}}\right)\mathbf{i}_\phi \end{aligned} \tag{4-66}$$

From (4-66), we have

$$\begin{aligned} \oint_C \mathbf{B} \cdot d\mathbf{l} &= \int_{\phi=0}^{2\pi} \frac{\mu_0 I}{4\pi r}\left(1 + \frac{z}{\sqrt{z^2 + r^2}}\right) \mathbf{i}_\phi \cdot r\, d\phi\, \mathbf{i}_\phi \\ &= \frac{\mu_0 I}{2}\left(1 + \frac{z}{\sqrt{z^2 + r^2}}\right) \end{aligned} \tag{4-67}$$

If we apply Ampere's circuital law (3-57) to the contour C in conjunction with the surface S_1 without regard to the uniqueness of the current enclosed, we obtain

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 I \tag{4-68}$$

Comparing (4-67) and (4-68), we note that the discrepancy between the right sides is by the amount

$$\frac{\mu_0 I}{2}\left(1 + \frac{z}{\sqrt{z^2 + r^2}}\right) - \mu_0 I = \frac{\mu_0 I}{2}\left(\frac{z}{\sqrt{z^2 + r^2}} - 1\right) \tag{4-69}$$

We have to resolve this discrepancy by some means. The only recourse seems to be the point charge at the origin whose value is decreasing at the rate of I C/sec. We have not as yet considered the electric field due to the point charge Q. As Q varies with time, the electric field flux due to it also varies with time. Let us consider the electric field flux through the surface S_1. Since the electric field intensity due to a point charge is spherically symmetric about the point charge, the electric field flux through any surface is equal to the solid angle subtended at the point charge by that surface times the point charge value divided by $4\pi\epsilon_0$.

To find the solid angle subtended by S_1 at Q, let us consider an infinitesimal area $dS_1 = r_1\, dr_1\, d\phi_1$ at the point (r_1, ϕ_1, z) on S_1. The projection of this area onto the plane normal to the line drawn from the origin to (r_1, ϕ_1, z) is $(r_1 z/\sqrt{r_1^2 + z^2})\, dr_1\, d\phi_1$. The projection of dS_1 onto the surface of a sphere of radius unity and centered at the origin or the infinitesimal solid angle subtended at the origin by dS_1 is given by

$$d\Omega_1 = \frac{r_1 z}{(r_1^2 + z^2)^{3/2}}\, dr_1\, d\phi_1$$

The solid angle subtended at the origin by the entire area S_1 is then given by

$$\Omega_1 = \int_{S_1} d\Omega_1 = \int_{r_1=0}^{r} \int_{\phi_1=0}^{2\pi} \frac{r_1 z}{(r_1^2 + z^2)^{3/2}}\, dr_1\, d\phi_1 = 2\pi\left(1 - \frac{z}{\sqrt{z^2 + r^2}}\right) \tag{4-70}$$

Since the normal to the surface S_1 drawn towards the direction of advance of a right-hand screw as it is turned in the sense of C is directed away from the point charge, the electric field flux passing through the surface S_1 towards the side of that normal is given by

$$\int_{S_1} \mathbf{E} \cdot d\mathbf{S}_1 = \frac{Q}{4\pi\epsilon_0}\Omega_1 = \frac{Q}{2\epsilon_0}\left(1 - \frac{z}{\sqrt{z^2 + r^2}}\right) \tag{4-71}$$

This electric field flux is changing with time. The rate at which it is changing with time is given by

$$\begin{aligned} \frac{d}{dt}\int_{S_1} \mathbf{E} \cdot d\mathbf{S}_1 &= \frac{d}{dt}\left[\frac{Q}{2\epsilon_0}\left(1 - \frac{z}{\sqrt{z^2 + r^2}}\right)\right] \\ &= \frac{1}{2\epsilon_0}\left(1 - \frac{z}{\sqrt{z^2 + r^2}}\right)\frac{dQ}{dt} \end{aligned} \tag{4-72}$$

But, since the charge Q is decreasing at the rate of I C/sec, we have

$$\frac{dQ}{dt} = -I \tag{4-73}$$

Substituting (4-73) into (4-72), we obtain

$$\frac{d}{dt}\int_{S_1} \mathbf{E} \cdot d\mathbf{S}_1 = \frac{I}{2\epsilon_0}\left(\frac{z}{\sqrt{z^2 + r^2}} - 1\right) \tag{4-74}$$

The right side of (4-74) is exactly the same as the right side of (4-69) divided by $\mu_0\epsilon_0$. Suppose we now modify (3-57) to read

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0\left(\text{current due to charges flowing through a surface } S \text{ bounded by } C + \frac{d}{dt}\int_S \epsilon_0 \mathbf{E} \cdot d\mathbf{S}\right) \tag{4-75}$$

and apply it to the surface S_1, we obtain

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0\left[I + \frac{I}{2}\left(\frac{z}{\sqrt{z^2 + r^2}} - 1\right)\right] = \frac{\mu_0 I}{2}\left(1 + \frac{z}{\sqrt{z^2 + r^2}}\right)$$

which agrees with (4-67), deduced by using the Biot–Savart law. Thus our dilemma seems to be resolved!

Before we discuss the meaning of $\frac{d}{dt}\int_S \epsilon_0 \mathbf{E} \cdot d\mathbf{S}$, let us apply (4-75) to the bowl-shaped surface S_2 bounded by C to see if it gives the correct result for $\oint_C \mathbf{B} \cdot d\mathbf{l}$. To do this, we note that the solid angle subtended by S_2 at Q is simply 4π minus the solid angle subtended by S_1 at Q. Thus the required solid angle Ω_2 is given by

$$\Omega_2 = (4\pi - \Omega_1) \tag{4-76}$$

where Ω_1 is given by (4-70). Substituting (4-70) into (4-76), we obtain

$$\Omega_2 = 2\pi\left(1 + \frac{z}{\sqrt{z^2 + r^2}}\right) \tag{4-77}$$

Now, noting that a right-hand screw advances into the bowl as it is turned in the sense of C from below the bowl whereas the electric field due to Q is directed away from Q, the electric field flux passing through the surface S_2 into the bowl is given by

$$\int_{S_2} \mathbf{E} \cdot d\mathbf{S}_2 = -\frac{Q}{4\pi\epsilon_0}\Omega_2 = -\frac{Q}{2\epsilon_0}\left(1 + \frac{z}{\sqrt{z^2 + r^2}}\right) \tag{4-78}$$

The rate at which this flux is changing with time is given by

$$\begin{aligned} \frac{d}{dt}\int_{S_2} \mathbf{E} \cdot d\mathbf{S}_2 &= \frac{d}{dt}\left[-\frac{Q}{2\epsilon_0}\left(1 + \frac{z}{\sqrt{z^2 + r^2}}\right)\right] \\ &= \frac{I}{2\epsilon_0}\left(1 + \frac{z}{\sqrt{z^2 + r^2}}\right) \end{aligned} \tag{4-79}$$

Substituting this result into (4-75) applied to S_2, we obtain

$$\begin{aligned} \oint_C \mathbf{B} \cdot d\mathbf{l} &= \mu_0 \biggl(\text{current due to charges flowing through } S_2 \\ &\qquad + \frac{d}{dt}\int_{S_2} \epsilon_0 \mathbf{E} \cdot d\mathbf{S}_2\biggr) \\ &= \mu_0\left[0 + \frac{I}{2}\left(1 + \frac{z}{\sqrt{z^2 + r^2}}\right)\right] \\ &= \frac{\mu_0 I}{2}\left(1 + \frac{z}{\sqrt{z^2 + r^2}}\right) \end{aligned}$$

which agrees with (4-67), deduced by using the Biot–Savart law. Thus the modified law (4-75) gives the correct result for $\oint_C \mathbf{B} \cdot d\mathbf{l}$ irrespective of the surface bounded by C chosen to apply it.

We note that the quantity $\frac{d}{dt}\int_S \epsilon_0 \mathbf{E} \cdot d\mathbf{S}$ has the units of current. This can be easily seen if we recognize from Gauss' law that $\int \mathbf{E} \cdot d\mathbf{S}$ has the units of Q/ϵ_0 and hence $\frac{d}{dt}\int_S \epsilon_0 \mathbf{E} \cdot d\mathbf{S}$ has the units of dQ/dt or current. Equation (4-75) therefore suggests that there are two kinds of current penetrating a surface S bounded by C. The first kind is due to the actual flow of charges across the surface S. The second kind is due to the flux of $\epsilon_0\mathbf{E}$ penetrating S changing with time; Maxwell attributed to it the name "displacement current." Physically, the displacement current is not a current in the sense that there is no flow of a physical quantity, like charge, across the surface. Although the term "time rate of change of the flux of $\epsilon_0\mathbf{E}$" is more apt, we shall follow Maxwell's terminology and use the term "displacement current." The reason behind this terminology will become evident in Chapter 5.

To summarize the discussion thus far in this section, we have found that the dilemma of Ampere's circuital law given by (3-57) is resolved by modifying it to read

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0\{[I_c]_S + [I_d]_S\} \tag{4-80}$$

where $[I_c]_S$ is the current due to the actual flow of charges across the surface S bounded by C in the direction of advance of a right-hand screw as it is turned in the sense of C, and $[I_d]_S = \frac{d}{dt}\int_S \epsilon_0\mathbf{E} \cdot d\mathbf{S}$ is the displacement current penetrating the surface S in the same direction. We shall refer to Eq. (4-80) as the modified Ampere's circuital law in integral form. While Faraday's law was a consequence of experimental observations by Faraday, the modified Ampere's circuital law was a result of theoretical investigations by Maxwell.

Although we have here derived the modified Ampere's circuital law by considering a particular case, Maxwell provided a general proof based on Gauss' law and the law of conservation of charge. Since charge is conserved, the current due to flow of charge out of a closed surface S bounding a volume V must be equal to the time rate of decrease of the charge enclosed by the surface. This is the law of conservation of charge. If the current flowing out of the surface is $[I_c]_S$ and the charge enclosed by S is Q, we then have

$$[I_c]_S = -\frac{dQ}{dt} \tag{4-81}$$

But, from Gauss' law, we have

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q}{\epsilon_0}$$

or

$$Q = \oint_S \epsilon_0\mathbf{E} \cdot d\mathbf{S} \tag{4-82}$$

Substituting (4-82) into (4-81) and rearranging, we obtain

$$[I_c]_S + \frac{d}{dt}\oint_S \epsilon_0\mathbf{E} \cdot d\mathbf{S} = 0 \tag{4-83}$$

or

$$[I_c]_S + [I_d]_S = 0 \tag{4-84}$$

Thus the law of conservation of charge states that the sum of the current due to the flow of charges and the displacement current across any closed surface must be equal to zero. We will now show that (4-80) is consistent but (3-57) is not consistent with (4-84). To do this, let us consider a closed path C in an electromagnetic field. Let S_1 and S_2 be two different surfaces bounded by C with their normals defined as shown in Fig. 4.7. The normal to S_1 is directed towards the side of advance of a right-hand screw as it is turned in the sense of C. Hence, from (4-80), we have

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0\{[I_c]_{S_1} + [I_d]_{S_1}\} \tag{4-85}$$

The normal to S_2 is directed opposite to the side of advance of a right-hand screw as it is turned in the sense of C. Hence, from (4-80), we have

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = -\mu_0\{[I_c]_{S_2} + [I_d]_{S_2}\} \tag{4-86}$$

Now, since $\oint_C \mathbf{B} \cdot d\mathbf{l}$ is unique, the right sides of (4-85) and (4-86) are equal, giving us

$$[I_c]_{S_1+S_2} + [I_d]_{S_1+S_2} = 0 \tag{4-87}$$

which is consistent with (4-84), since $(S_1 + S_2)$ is a closed surface. On the other hand, if we use (3-57) we obtain, for the surface S_1,

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0[I_c]_{S_1} \tag{4-88}$$

and for the surface S_2,

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = -\mu_0[I_c]_{S_2} \tag{4-89}$$

From (4-88) and (4-89), we have

$$[I_c]_{S_1+S_2} = 0 \tag{4-90}$$

which is inconsistent with (4-84) unless $[I_d]_{S_1+S_2}$ is equal to zero, which is true only in the static case. It is this inconsistency that prompted Maxwell to modify Ampere's circuital law by adding the displacement current term. A consequence of the displacement current term in the modified Ampere's circuital law is that the current enclosed by a closed path C in an electromagnetic field is generally not equal to $(1/\mu_0)\oint_C \mathbf{B} \cdot d\mathbf{l}$, unlike in the static magnetic field case.

EXAMPLE 4-8. The arrangement shown in Fig. 4.10 is that of a V-shaped filamentary wire situated in the yz plane symmetrically about the z axis and with its vertex at the origin. Current flows along one leg from infinity to the origin at the rate of I_1 C/sec and leaves along another leg from the origin to infinity at the rate of I_2 C/sec. It is desired to find the values of $\oint \mathbf{B} \cdot d\mathbf{l}$ around two circular contours C_1 and C_2 of radii 1 m and centered at the origin, where (a) C_1 is in the xy plane and (b) C_2 is in the xz plane.

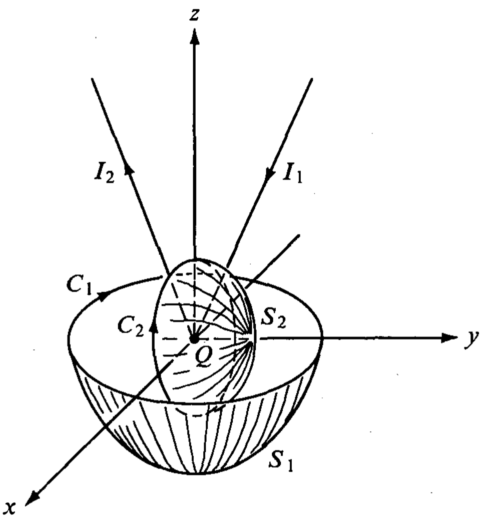

Fig. 4.10. For evaluating $\oint \mathbf{B} \cdot d\mathbf{l}$, around paths C_1 and C_2, due to a V-shaped filamentary wire with unequal currents in the two legs.

Since the current entering the origin is I_1 C/sec whereas the current leaving the origin is I_2 C/sec, there is a charge accumulation at the origin at the rate of $(I_1 - I_2)$ C/sec.

(a) To evaluate $\oint_{C_1} \mathbf{B} \cdot d\mathbf{l}$, let us choose the bowl-shaped surface S_1 bounded by C_1. $[I_c]_{S_1}$ is equal to zero since neither leg of the wire penetrates the surface. On the other hand, since half of the electric field flux emanating from the point charge penetrates the surfaces S_1 towards the side of advance of a right-hand screw as it is turned in the sense of C_1, $[I_d]_{S_1}$ is equal to $\frac{1}{2}(I_1 - I_2)$ C/sec. Thus, according to (4-80),

$$\oint_{C_1} \mathbf{B} \cdot d\mathbf{l} = \frac{\mu_0}{2}(I_1 - I_2)$$

(b) To evaluate $\oint_{C_2} \mathbf{B} \cdot d\mathbf{l}$, let us choose the bowl-shaped surface S_2 bounded by C_2. $[I_c]_{S_2}$ is equal to I_1 since that leg of the wire penetrates the surface with the current flowing towards the side of advance of a right-hand screw as it is turned in the sense of C_2. On the other hand, the electric field flux of the point charge penetrates S_2 in the opposite sense, and since half of the flux emanating from the point charge penetrates S_2, $[I_d]_{S_2}$ is equal to $-\frac{1}{2}(I_1 - I_2)$ C/sec. Thus, according to (4-80),

$$\oint_{C_2} \mathbf{B} \cdot d\mathbf{l} = \mu_0\left[I_1 - \frac{1}{2}(I_1 - I_2)\right] = \frac{\mu_0}{2}(I_1 + I_2)$$

Note that if $I_1 = I_2 = I$, $\oint_{C_1} \mathbf{B} \cdot d\mathbf{l} = 0$ and $\oint_{C_2} \mathbf{B} \cdot d\mathbf{l} = \mu_0 I$. ▮

4.5 Modified Ampere's Circuital Law in Differential Form (Maxwell's Second Curl Equation for the Electromagnetic Field) and the Continuity Equation

In the previous section we introduced the modified Ampere's circuital law in integral form, given by

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0\{[I_c]_S + [I_d]_S\} \tag{4-80}$$

where S is any surface bounded by C, $[I_c]_S$ is the current due to charges flowing across S, and $[I_d]_S$ is the displacement current through S. For a volume current of density $\mathbf{J}$, we have

$$[I_c]_S = \int_S \mathbf{J} \cdot d\mathbf{S} \tag{4-91}$$

Substituting for $[I_c]_S$ and $[I_d]_S$ in (4-80), we get

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0\left(\int_S \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt}\int_S \epsilon_0\mathbf{E} \cdot d\mathbf{S}\right) \tag{4-92}$$

According to Stokes' theorem, we have

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{S}$$

where S is any surface bounded by the contour C. In particular, choosing the same surface as for the integrals on the right side of (4-92), we obtain

$$\int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{S} = \mu_0\left(\int_S \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt}\int_S \epsilon_0\mathbf{E} \cdot d\mathbf{S}\right) \tag{4-93}$$

If the surface S is stationary, that is, independent of time,

$$\frac{d}{dt}\int_S \epsilon_0\mathbf{E} \cdot d\mathbf{S} = \int_S \frac{\partial}{\partial t}(\epsilon_0\mathbf{E}) \cdot d\mathbf{S} \tag{4-94}$$

and (4-93) becomes

$$\int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{S} = \int_S \mu_0\left[\mathbf{J} + \frac{\partial}{\partial t}(\epsilon_0\mathbf{E})\right] \cdot d\mathbf{S} \tag{4-95}$$

Comparing the integrands on both sides of (4-95), we have

$$\nabla \times \mathbf{B} = \mu_0\left[\mathbf{J} + \frac{\partial}{\partial t}(\epsilon_0\mathbf{E})\right] \tag{4-96}$$

Equation (4-96) is the differential form of the modified Ampere's circuital law and it is Maxwell's second curl equation for the electromagnetic field. While we have here derived (4-96) for a stationary S, it can be shown that it holds also for a time-varying surface S due to a moving C, where $\mathbf{E}$, $\mathbf{B}$, and $\mathbf{J}$ are the fields and the current density as viewed by a stationary observer. Following the terminology "displacement current" for the time rate of change of the flux of $\epsilon_0\mathbf{E}$, the time rate of change of $\epsilon_0\mathbf{E}$, that is $\frac{\partial}{\partial t}(\epsilon_0\mathbf{E})$ is known as the "displacement current density."

EXAMPLE 4-9. In the previous section we deduced the magnetic field [Eq. (4-66)] due to a semiinfinitely long filamentary wire along which current flows to infinity from a source of point charge at the origin (Fig. 4.7). It is here desired to verify the result by using (4-96).

From the previous section, the magnetic field due to the wire is given at a point (r, ϕ, z) by

$$\mathbf{B} = \frac{\mu_0 I}{4\pi r}\left(1 + \frac{z}{\sqrt{z^2 + r^2}}\right)\mathbf{i}_\phi$$

Hence

$$\begin{aligned}\nabla \times \mathbf{B} &= \frac{\mathbf{i}_r}{r}\left[-\frac{\partial}{\partial z}(rB_\phi)\right] + \frac{\mathbf{i}_z}{r}\left[\frac{\partial}{\partial r}(rB_\phi)\right] \\ &= \frac{\mu_0 I}{4\pi r}\left[-\mathbf{i}_r\frac{\partial}{\partial z}\left(1 + \frac{z}{\sqrt{z^2 + r^2}}\right) + \mathbf{i}_z\frac{\partial}{\partial r}\left(1 + \frac{z}{\sqrt{z^2 + r^2}}\right)\right] \\ &= -\frac{\mu_0 I}{4\pi(z^2 + r^2)^{3/2}}(r\mathbf{i}_r + z\mathbf{i}_z) \end{aligned} \tag{4-97}$$

Substituting $I = -dQ/dt$ in (4-97), we note that

$$\begin{aligned}\nabla \times \mathbf{B} &= \mu_0\epsilon_0\frac{d}{dt}\left[\frac{Q}{4\pi\epsilon_0(z^2 + r^2)^{3/2}}(r\mathbf{i}_r + z\mathbf{i}_z)\right] \\ &= \mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t}\end{aligned} \tag{4-98}$$

thereby satisfying (4-96) since $\mathbf{J}$ is zero at (r, ϕ, z). ▮

Returning to Eq. (4-96) and taking the divergence of both sides, we have

$$\begin{aligned}\nabla \cdot \nabla \times \mathbf{B} &= \nabla \cdot \mu_0\left[\mathbf{J} + \frac{\partial}{\partial t}(\epsilon_0\mathbf{E})\right] \\ &= \mu_0\left[\nabla \cdot \mathbf{J} + \frac{\partial}{\partial t}(\epsilon_0\nabla \cdot \mathbf{E})\right]\end{aligned} \tag{4-99}$$

Since $\nabla \cdot \nabla \times \mathbf{B} \equiv 0$, (4-99) gives us

$$\nabla \cdot \mathbf{J} + \frac{\partial}{\partial t}(\epsilon_0\nabla \cdot \mathbf{E}) = 0 \tag{4-100}$$

But, according to the law of conservation of charge,

$$[I_c]_S = -\frac{dQ}{dt} \tag{4-81}$$

where $[I_c]_S$ is the current due to the flow of charges out of a closed surface S and Q is the charge enclosed by S. In terms of current density $\mathbf{J}$ and charge density ρ, $[I_c]_S$ and Q are given by

$$[I_c]_S = \oint_S \mathbf{J} \cdot d\mathbf{S} \tag{4-101}$$

and

$$Q = \int_V \rho \, dv \tag{4-102}$$

where V is the volume bounded by S. Substituting (4-101) and (4-102) into (4-81), we obtain

$$\oint_S \mathbf{J} \cdot d\mathbf{S} = -\frac{d}{dt}\int_V \rho \, dv \tag{4-103}$$

Applying the divergence theorem to the left side of (4-103) and interchanging the differentiation and integration operations on the right side, we get

$$\int_V \nabla \cdot \mathbf{J} \, dv = -\int_V \frac{\partial \rho}{\partial t} \, dv \tag{4-104}$$

or

$$\int_V \left(\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t}\right) dv = 0 \tag{4-105}$$

Since (4-105) must be valid for any volume, it follows that

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0 \tag{4-106}$$

Equation (4-106) is the law of conservation of charge in differential form. It is also known as the continuity equation. For static fields, $\partial \rho / \partial t = 0$ and (4-106) reduces to $\nabla \cdot \mathbf{J} = 0$, which agrees with (3-113). Comparing (4-100) with (4-106), we have

$$\frac{\partial}{\partial t}(\epsilon_0 \nabla \cdot \mathbf{E}) = \frac{\partial \rho}{\partial t}$$

or

$$\frac{\partial}{\partial t}(\epsilon_0 \nabla \cdot \mathbf{E} - \rho) = 0 \tag{4-107}$$

or

$$(\epsilon_0 \nabla \cdot \mathbf{E} - \rho) = \text{constant with time} \tag{4-108}$$

The constant on the right side of (4-108) must, however, be equal to zero since a nonzero value at any point in space requires the existence forever of a source of nonsolenoidal electric field flux other than electric charge at that point. Thus we note that Maxwell's equation for the divergence of the time-varying electric field given by

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{4-109}$$

follows from the Maxwell's equation for the curl of **B** given by (4-96) with the aid of the continuity equation (4-106).

4.6 Energy Storage in an Electric Field

In Section 2.8 we introduced the concept of potential difference between two points in an electric field as equal to the work done per unit charge in moving a test charge from one point to the other. In Section 2.9 we extended this to the concept of potential, which is simply the potential difference between two points, one of which is a reference point having zero potential. If we transfer a test charge from a point of higher potential to a point of lower potential, the field does the work and hence there is loss in potential energy of the system, which is supplied to the test charge. Where in the system does this energy come from? Alternatively, if we transfer the test charge from a point of lower potential to a point of higher potential, an external agent moving the charge has to do work, thus increasing the potential energy of the system. Where in the system does this energy expended by the external agent reside? Wherever in the system the energy may reside, a convenient way is to think of the energy as being stored in the electric field. In the first case, part of the stored energy in the field is expended in moving the test charge, whereas in the second case the energy expended by the external agent increases the stored energy.

Let us then consider a system of two point charges Q_1 and Q_2 situated an infinite distance apart so that no forces are exerted on either charge and hence the charges are in equilibrium. According to the definition of potential difference, an amount of work equal to Q_2 times the potential of Q_1 at Q_2 must be expended by an external agent to bring Q_2 close to Q_1 as shown in Fig. 4.11(a). Thus the potential energy of the system is increased by the amount

$$W_2 = Q_2 V_2^1 \tag{4-110}$$

where V_2^1 is the potential of Q_1 at the location of Q_2. If we start with a system of three charges Q_1, Q_2, Q_3 situated an infinite distance apart from each other, then the amount of work required to bring Q_2 and Q_3 close to Q_1 can be determined in two steps. First we bring Q_2 close to Q_1, for which the work required is given by (4-110). Then we bring Q_3 close to Q_1 as shown in Fig. 4.11(b). But, this time, we have to overcome not only the force exerted on Q_3 by Q_1 but also the force exerted by Q_2. Hence the required work is given by

$$W_3 = Q_3 V_3^1 + Q_3 V_3^2 \tag{4-111}$$

Thus the total work required to bring Q_2 and Q_3 close to Q_1 is

$$W_e = W_2 + W_3 = Q_2 V_2^1 + (Q_3 V_3^1 + Q_3 V_3^2) \tag{4-112}$$

The potential energy of the system is increased by the amount given by (4-112).

We can proceed in this manner and consider a system of n point charges $Q_1, Q_2, Q_3, \ldots, Q_n$ initially located infinitely far apart from each other.

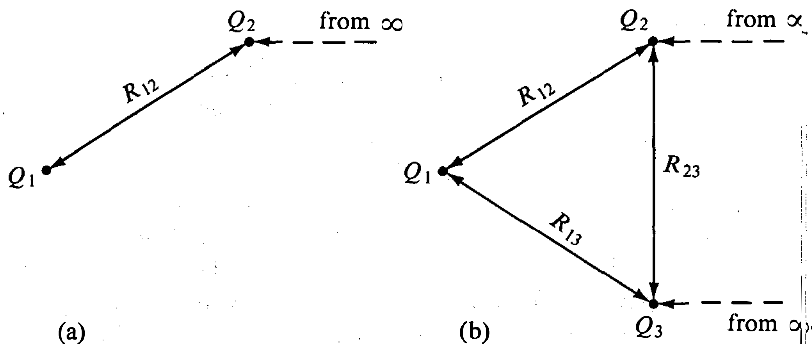

Fig. 4.11. Bringing point charges closer from infinity.

The total work required in bringing the charges close to each other is given by

$$\begin{aligned} W_e &= W_2 + W_3 + \cdots + W_n \\ &= Q_2V_2^1 + (Q_3V_3^1 + Q_3V_3^2) + (Q_4V_4^1 + Q_4V_4^2 + Q_4V_4^3) + \cdots \\ &= \sum_{i=2}^{n}\sum_{j=1}^{i-1} Q_iV_i^j \end{aligned} \tag{4-113}$$

where V_i^j is the potential of Q_j at the location of Q_i. However, we note that

$$Q_iV_i^j = Q_i\frac{Q_j}{4\pi\epsilon_0R_{ji}} = Q_j\frac{Q_i}{4\pi\epsilon_0R_{ij}} = Q_jV_j^i \tag{4-114}$$

Hence (4-113) may be written as

$$\begin{aligned} W_e &= Q_1V_1^2 + (Q_1V_1^3 + Q_2V_2^3) + (Q_1V_1^4 + Q_2V_2^4 + Q_3V_3^4) + \cdots \\ &= \sum_{i=2}^{n}\sum_{j=1}^{i-1} Q_jV_j^i \end{aligned} \tag{4-115}$$

Adding (4-113) and (4-115), we have

$$\begin{aligned} 2W_e &= Q_1(V_1^2 + V_1^3 + V_1^4 + \cdots) \\ &\quad + Q_2(V_2^1 + V_2^3 + V_2^4 + \cdots) \\ &\quad + Q_3(V_3^1 + V_3^2 + V_3^4 + \cdots) \\ &\quad + \cdots \\ &= Q_1(\text{potential at } Q_1 \text{ due to all other charges}) \\ &\quad + Q_2(\text{potential at } Q_2 \text{ due to all other charges}) \\ &\quad + Q_3(\text{potential at } Q_3 \text{ due to all other charges}) \\ &\quad + \cdots \\ &= Q_1V_1 + Q_2V_2 + Q_3V_3 + \cdots \\ &= \sum_{i=1}^{n} Q_iV_i \end{aligned} \tag{4-116}$$

where V_i is the potential at Q_i due to all other charges. Dividing both sides of (4-116) by 2, we have

$$W_e = \frac{1}{2}\sum_{i=1}^{n} Q_iV_i \tag{4-117}$$

Thus the potential energy stored in the system of n point charges is given by (4-117).

EXAMPLE 4-10. Three point charges of values 1, 2, and 3 C are situated at the corners of an equilateral triangle of sides 1 m. It is desired to find the work required to move these charges to the corners of an equilateral triangle of shorter sides $\frac{1}{2}$ m as shown in Fig. 4.12.

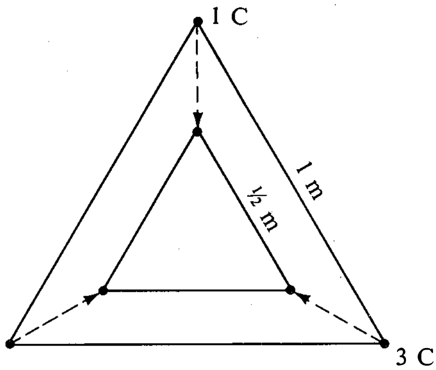

Fig. 4.12. Bringing three point charges from the corners of a larger equilateral triangle to the corners of a smaller equilateral triangle.

The potential energy stored in the system of three charges at the corners of the larger equilateral triangle is given by

$$\frac{1}{2}\sum_{i=1}^{3} Q_i V_i = \frac{1}{2}\left[1\left(\frac{2}{4\pi\epsilon_0} + \frac{3}{4\pi\epsilon_0}\right) + 2\left(\frac{1}{4\pi\epsilon_0} + \frac{3}{4\pi\epsilon_0}\right) + 3\left(\frac{1}{4\pi\epsilon_0} + \frac{2}{4\pi\epsilon_0}\right)\right]$$
$$= \frac{1}{2}\left[\frac{5+8+9}{4\pi\epsilon_0}\right] = \frac{11}{4\pi\epsilon_0} \text{ N-m}$$

The potential energy stored in the system of three charges at the corners of the smaller equilateral triangle is equal to twice the above value since all distances are halved. The increase in potential energy of the system in going from the larger to the smaller equilateral triangle is equal to $11/4\pi\epsilon_0$ N-m. Obviously, this increase in energy must be supplied by an external agent and hence the work required to move the charges to the corners of the equilateral triangle of sides $\frac{1}{2}$ m from the corners of the equilateral triangle of sides 1 m is equal to $11/4\pi\epsilon_0$ N-m. ∎

If we have a continuous distribution of charge with density $\rho(r, \theta, \phi)$ instead of an assembly of discrete charges, we can treat it as a continuous collection of infinitesimal charges of value $\rho(r, \theta, \phi)\,\Delta v$, each of which can be considered as a point charge, and obtain the potential energy of the system as

$$W_e = \frac{1}{2}\lim_{\Delta v \to 0} \sum [\rho(r, \theta, \phi)\,\Delta v]\, V(r, \theta, \phi)$$
$$= \frac{1}{2}\int_{\substack{\text{volume}\\ \text{containing } \rho}} \rho V\, dv \qquad (4\text{-}118a)$$

Similarly, for a surface charge distribution of density ρ_s on a surface S, we have

$$W_e = \frac{1}{2}\int_S \rho_s V\, dS \tag{4-118b}$$

Thus far, we have found the potential energy of the charge distribution by considering the work done in assembling the system. We stated at the beginning of this section that the potential energy can be thought of as being stored in the electric field set up by the system of charges. If so, we should be able to express the energy in terms of the electric field. To do this, we substitute for ρ in (4-118a) from (2-82) and obtain

$$W_e = \frac{1}{2}\int_{\substack{\text{volume}\\ \text{containing } \rho}} (\epsilon_0 \nabla \cdot \mathbf{E}) V\, dv \tag{4-119}$$

Since $\nabla \cdot \mathbf{E} = 0$ in the region not containing ρ, the value of the integral on the right side of (4-119) is not altered if we change the volume of integration from the volume containing ρ to the entire space. Thus

$$W_e = \frac{1}{2}\int_{\text{all space}} (\epsilon_0 \nabla \cdot \mathbf{E}) V\, dv \tag{4-120}$$

We now use the vector identity

$$\nabla \cdot V\mathbf{E} = V\nabla \cdot \mathbf{E} + \mathbf{E} \cdot \nabla V$$

to replace $V\nabla \cdot \mathbf{E}$ on the right side of (4-120) by $\nabla \cdot V\mathbf{E} - \mathbf{E} \cdot \nabla V$ and obtain

$$\begin{aligned} W_e &= \frac{1}{2}\epsilon_0 \int_{\text{all space}} (\nabla \cdot V\mathbf{E} - \mathbf{E} \cdot \nabla V)\, dv \\ &= \frac{1}{2}\epsilon_0 \int_{\text{all space}} \nabla \cdot V\mathbf{E}\, dv + \frac{1}{2}\epsilon_0 \int_{\text{all space}} \mathbf{E} \cdot \mathbf{E}\, dv \end{aligned} \tag{4-121}$$

where we have replaced ∇V by $-\mathbf{E}$ in accordance with (2-138). Using the divergence theorem, we equate the first integral on the right side of (4-121) to a surface integral thus:

$$\int_{\text{all space}} \nabla \cdot V\mathbf{E}\, dv = \int_{\substack{\text{surface}\\ \text{bounding}\\ \text{all space}}} V\mathbf{E} \cdot \mathbf{i}_n\, dS \tag{4-122}$$

However, as viewed from a surface bounding all space, a charge distribution of finite volume appears as a point charge, say Q. Hence, as $r \longrightarrow \infty$, we can write

$$\mathbf{E} \longrightarrow \frac{Q}{4\pi\epsilon_0 r^2}\mathbf{i}_r$$

$$V \longrightarrow \frac{Q}{4\pi\epsilon_0 r}$$

$$\int_{\substack{\text{surface}\\\text{bounding}\\\text{all space}}} V\mathbf{E}\cdot\mathbf{i}_n\,dS = \lim_{r\to\infty}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\frac{Q}{4\pi\epsilon_0 r}\frac{Q}{4\pi\epsilon_0 r^2}\mathbf{i}_r\cdot r^2\sin\theta\,d\theta\,d\phi\,\mathbf{i}_r$$

$$= \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\lim_{r\to\infty}\frac{Q^2}{(4\pi\epsilon_0)^2 r}\sin\theta\,d\theta\,d\phi = 0 \tag{4-123}$$

Equation (4-123) holds also for a charge distribution of infinite extent, provided the electric field due to the charge distribution falls off at least as $(1/r^2)\mathbf{i}_r$ and hence the potential falls off at least as $1/r$. Thus (4-121) reduces to

$$W_e = \frac{1}{2}\epsilon_0\int_{\text{all space}}\mathbf{E}\cdot\mathbf{E}\,dv = \int_{\text{all space}}\left(\frac{1}{2}\epsilon_0 E^2\right)dv \tag{4-124}$$

Equation (4-124) indicates clearly that the idea of energy residing in the electric field is a valid one provided we integrate $\frac{1}{2}\epsilon_0 E^2$ throughout the entire space. The quantity $\frac{1}{2}\epsilon_0 E^2$ is evidently the energy density in the electric field.

EXAMPLE 4-11. A volume charge is distributed throughout a sphere of radius a meters, and centered at the origin, with uniform density ρ_0 C/m³. We wish to find the energy stored in the electric field of this charge distribution.

From Example 2-6, the electric field of the uniformly distributed spherical charge, having its center at the origin, is given by

$$\mathbf{E} = \begin{cases}\dfrac{\rho_0 a^3}{3\epsilon_0 r^2}\mathbf{i}_r & \text{for } r > a\\[2ex] \dfrac{\rho_0 r}{3\epsilon_0}\mathbf{i}_r & \text{for } r < a\end{cases}$$

Hence the energy density in the electric field is given by

$$\frac{1}{2}\epsilon_0 E^2 = \begin{cases}\dfrac{\rho_0^2 a^6}{18\epsilon_0 r^4} & \text{for } r > a\\[2ex] \dfrac{\rho_0^2 r^2}{18\epsilon_0} & \text{for } r < a\end{cases}$$

The energy stored in the electric field is

$$\begin{aligned}W_e &= \int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\frac{\rho_0^2 r^2}{18\epsilon_0}r^2\sin\theta\,dr\,d\theta\,d\phi\\ &\quad+\int_{r=a}^{\infty}\int_{\theta=0}^{2\pi}\int_{\phi=0}^{2\pi}\frac{\rho_0^2 a^6}{18\epsilon_0 r^4}r^2\sin\theta\,dr\,d\theta\,d\phi\\ &= \frac{4\pi\rho_0^2 a^5}{15\epsilon_0}\quad\blacksquare\end{aligned}$$

4.7 Energy Storage in a Magnetic Field

In the previous section we derived an expression for the energy density in an electric field by first finding the work required to be done by an external agent in assembling a system of point charges and then extending the result to a continuous distribution of charge. Just as work is required for gathering point charges from infinity, it requires work to gather a set of current loops from infinity. Just as we can interpret the energy expended by an external agent in assembling the charges as being stored in the electric field of the charges, we can think of the energy expended by an external agent in assembling the current loops as being stored in the magnetic field of the current loops. It is possible to derive an expression for the energy density in a magnetic field by starting with a set of current loops at infinity and proceeding in a similar manner as in the previous section. To simplify the derivation, we will, however, consider directly the building up of a solenoidal volume current distribution.

Let us then consider a solenoidal volume current distribution of density $\mathbf{J}$ in a volume V where $\mathbf{J}$ increases linearly with time from zero to a value $\mathbf{J}_0$ in a time t_0, that is, $\mathbf{J} = \mathbf{J}_0 t/t_0$. The magnetic field $\mathbf{B}$ associated with the current distribution also increases linearly with time, that is, $\mathbf{B} = \mathbf{B}_0 t/t_0$. The time varying magnetic field induces an electric field in accordance with Faraday's law. The induced electric field exerts forces on charges constituting the current flow. The work done by these forces must be balanced by an external agent to maintain the current density at $\mathbf{J}_0 t/t_0$ and hence is stored in the magnetic field as the potential energy associated with the current distribution.

To find this energy, let us divide the cross-sectional area S of the current distribution into a number of infinitesimal areas ΔS_i. Through each infinitesimal area, a current loop C_i can be defined by the direction line of the current density vector $\mathbf{J}_i = \mathbf{J}_{i0} t/t_0$ corresponding to that area as shown in Fig. 4-13. The current I_i flowing around the loop C_i is equal to $\mathbf{J}_i \cdot \Delta\mathbf{S}_i$. The amount of charge dQ_i crossing ΔS_i in time dt is equal to $I_i\,dt$. Denoting the induced electric field at the point occupied by ΔS_i to be $\mathbf{E}_i$, we obtain the force exerted by this field on the charge dQ_i to be $dQ_i\,\mathbf{E}_i = I_i\,dt\,\mathbf{E}_i$. The work done by this force as the charge dQ_i is displaced by the infinitesimal distance $d\mathbf{l}_i$ along $\mathbf{J}_i$ is $I_i\,dt\,\mathbf{E}_i \cdot d\mathbf{l}_i$. Hence, the work required to be done against the induced electric field around the loop C_i in time dt is

$$dW_m = -\oint_{C_i} I_i\,dt\,\mathbf{E}_i \cdot d\mathbf{l}_i = -I_i\,dt \oint_{C_i} \mathbf{E}_i \cdot d\mathbf{l}_i \qquad (4\text{-}125)$$

Using Faraday's law and substituting $\mathbf{B} = \nabla \times \mathbf{A}$ and then using Stoke's theorem, we have

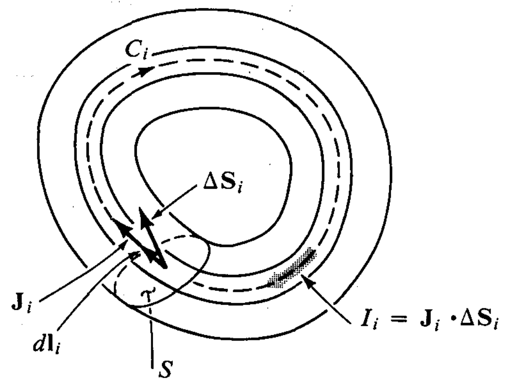

Fig. 4.13. Division of a solenoidal continuous distribution of current into a number of solenoidal current tubes having infinitesimal cross-sectional areas.

$$\begin{aligned}\oint_{C_i} \mathbf{E}_i \cdot d\mathbf{l}_i &= -\frac{d}{dt}\int_{S_i} \mathbf{B} \cdot d\mathbf{S} \\ &= -\frac{d}{dt}\int_{S_i} (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = -\frac{d}{dt}\oint_{C_i} \mathbf{A}_i \cdot d\mathbf{l}_i\end{aligned} \tag{4-126}$$

where $\mathbf{A}$ is the magnetic vector potential associated with $\mathbf{B}$ and S_i is any surface bounded by C_i. In view of the linear increase of $\mathbf{B}$ with time, $\mathbf{A}$ also increases linearly with time. Thus, denoting $\mathbf{A}_i = \mathbf{A}_{i0}t/t_0$, we have

$$\oint_{C_i} \mathbf{E}_i \cdot d\mathbf{l}_i = -\frac{d}{dt}\oint_{C_i} \mathbf{A}_{i0}\frac{t}{t_0} \cdot d\mathbf{l}_i = -\oint_{C_i} \frac{\mathbf{A}_{i0}}{t_0} \cdot d\mathbf{l}_i \tag{4-127}$$

Substituting (4-127) into (4-125), we obtain

$$dW_m = I_i\, dt \oint_{C_i} \frac{\mathbf{A}_{i0}}{t_0} \cdot d\mathbf{l}_i \tag{4-128}$$

The total work required to be done by an external agent from $t = 0$ to t and for the entire current distribution is then given by

$$\begin{aligned}W_m &= \sum_i \int_{t=0}^{t} I_i\, dt \oint_{C_i} \frac{\mathbf{A}_{i0}}{t_0} \cdot d\mathbf{l}_i \\ &= \sum_i \oint_{C_i} \int_{t=0}^{t} \left(\frac{\mathbf{J}_{i0}t}{t_0} \cdot \Delta\mathbf{S}_i\right)\left(\frac{\mathbf{A}_{i0}}{t_0} \cdot d\mathbf{l}_i\right) \\ &= \sum_i \frac{1}{2}\oint_{C_i} \left(\frac{\mathbf{J}_{i0}t}{t_0} \cdot \Delta\mathbf{S}_i\right)\left(\frac{\mathbf{A}_{i0}t}{t_0} \cdot d\mathbf{l}_i\right) \\ &= \sum_i \frac{1}{2}\oint_{C_i} (\mathbf{J}_i \cdot \Delta\mathbf{S}_i)(\mathbf{A}_i \cdot d\mathbf{l}_i) \\ &= \sum_i \frac{1}{2}\oint_{C_i} (\mathbf{J}_i \cdot \mathbf{A}_i)(\Delta\mathbf{S}_i \cdot d\mathbf{l}_i)\end{aligned} \tag{4-129}$$

since $\mathbf{J}_i$ and $d\mathbf{l}_i$ are parallel. Now, in the limit that all $\Delta\mathbf{S}_i \longrightarrow 0$, the summation on the right side of (4-129) becomes an integral to give us the potential energy associated with the volume current distribution as

$$W_m = \frac{1}{2}\int_S \oint_C (\mathbf{J}\cdot\mathbf{A})(d\mathbf{S}\cdot d\mathbf{l}) = \frac{1}{2}\int_{\substack{\text{volume}\\ \text{containing } \mathbf{J}}} \mathbf{J}\cdot\mathbf{A}\,dv \tag{4-130a}$$

Similarly, for a surface current distribution of density $\mathbf{J}_s$ on a surface S, we have

$$W_m = \frac{1}{2}\int_S \mathbf{J}_s\cdot\mathbf{A}\,dS \tag{4-130b}$$

To express the energy in terms of the magnetic field, we substitute for **J** in (4-130a) from (3-76) [instead of from (4-96)], in view of the solenoidal nature of **J**, and obtain

$$W_m = \frac{1}{2}\int_{\substack{\text{volume}\\ \text{containing } \mathbf{J}}} \frac{1}{\mu_0}\nabla\times\mathbf{B}\cdot\mathbf{A}\,dv \tag{4-131}$$

Since $\nabla\times\mathbf{B} = 0$ in the region not containing **J** [using again (3-76) instead of (4-96) for the same reason], the value of the integral on the right side of (4-131) is not altered if we change the volume of integration from the volume containing **J** to the entire space. Thus

$$W_m = \frac{1}{2}\int_{\text{all space}} \frac{1}{\mu_0}\nabla\times\mathbf{B}\cdot\mathbf{A}\,dv \tag{4-132}$$

We now use the vector identity

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot\nabla\times\mathbf{A} - \mathbf{A}\cdot\nabla\times\mathbf{B}$$

to replace $\nabla\times\mathbf{B}\cdot\mathbf{A}$ on the right side of (4-132) by $\mathbf{B}\cdot\nabla\times\mathbf{A} - \nabla\cdot(\mathbf{A}\times\mathbf{B})$ and obtain

$$W_m = \frac{1}{2\mu_0}\int_{\text{all space}} [\mathbf{B}\cdot\nabla\times\mathbf{A} - \nabla\cdot(\mathbf{A}\times\mathbf{B})]\,dv = \frac{1}{2\mu_0}\int_{\text{all space}} \mathbf{B}\cdot\mathbf{B}\,dv - \frac{1}{2\mu_0}\int_{\text{all space}} \nabla\cdot(\mathbf{A}\times\mathbf{B})\,dv \tag{4-133}$$

where we have replaced $\nabla\times\mathbf{A}$ by **B** in accordance with (3-82). Using the divergence theorem, we equate the second integral on the right side of (4-133) to a surface integral thus:

$$\int_{\text{all space}} \nabla\cdot(\mathbf{A}\times\mathbf{B})\,dv = \int_{\substack{\text{surface}\\ \text{bounding}\\ \text{all space}}} (\mathbf{A}\times\mathbf{B})\cdot\mathbf{i}_n\,dS \tag{4-134}$$

However, as viewed from a surface bounding all space, a solenoidal current distribution of finite volume appears as a dipole moment, say **m**. Hence, as $r \longrightarrow \infty$, we can write

$$\mathbf{B} \longrightarrow \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\,\mathbf{i}_r + \sin\theta\,\mathbf{i}_\theta)$$

$$\mathbf{A} \longrightarrow \frac{\mu_0 m}{4\pi r^2}\sin\theta\,\mathbf{i}_\phi$$

where the z axis is chosen to be along the direction of **m**. Thus

$$|\mathbf{A} \times \mathbf{B}| \sim \frac{1}{r^5}$$

whereas

$$dS \sim r^2$$

so that the integral on the right side of (4-134) is zero. This is true also for a current distribution of infinite extent, provided the magnetic flux density due to the current distribution falls off at least as $1/r^2$ and hence the magnetic vector potential falls off at least as $1/r$. Equation (4-133) then reduces to

$$W_m = \frac{1}{2\mu_0}\int_{\text{all space}} \mathbf{B} \cdot \mathbf{B}\,dv = \int_{\text{all space}} \left(\frac{1}{2}\frac{B^2}{\mu_0}\right) dv \qquad (4\text{-}135)$$

Equation (4-135) indicates clearly that the idea of energy residing in the magnetic field is a valid one provided we integrate $\frac{1}{2}B^2/\mu_0$ throughout the entire space. The quantity $\frac{1}{2}B^2/\mu_0$ is evidently the energy density in the magnetic field.

EXAMPLE 4-12. Current I flows in the $+z$ direction with uniform density on the cylindrical surface $r = a$ and returns in the $-z$ direction with uniform density on a second cylindrical surface $r = b$ so that the surface current distribution is given by

$$\mathbf{J}_s = \begin{cases} \dfrac{I}{2\pi a}\mathbf{i}_z & r = a \\ -\dfrac{I}{2\pi b}\mathbf{i}_z & r = b \end{cases}$$

We wish to find the energy stored in the magnetic field per unit length of the current distribution.

From application of Ampere's circuital law in integral form, we obtain the magnetic flux density due to the given current distribution as

$$\mathbf{B} = \begin{cases} 0 & r < a \\ \dfrac{\mu_0 I}{2\pi r}\mathbf{i}_\phi & a < r < b \\ 0 & r > b \end{cases} \qquad (4\text{-}136)$$

Since **B** is zero for $r > b$, the integral on the right side of (4-134) is zero so that we can use (4-135) for computing the energy. If we have a situation in which current flows on one surface in one direction and does not return on another surface, then the magnetic field will not be zero at $r = \infty$. In fact, it falls off as $1/r$ and the magnetic vector potential varies as $\ln r$ so that $|\mathbf{A} \times \mathbf{B}| \sim (1/r)\ln r$. But since $dS \sim r$, (4-134) does not reduce to zero. In such a case, we have to include the second term on the right side of (4-133) to compute W_m. However, in all physical situations, the current does return in the opposite direction on another surface and hence the magnetic field is zero at $r = \infty$. Now, returning to the solution of the example under consideration, we obtain, upon substitution of (4-136) into (4-135),

$$\begin{aligned} W_m &= \int_{z=-\infty}^{\infty}\int_{r=a}^{b}\int_{\phi=0}^{2\pi} \frac{1}{2\mu_0}\left(\frac{\mu_0 I}{2\pi r}\right)^2 r\,dr\,d\phi\,dz \\ &= \int_{z=-\infty}^{\infty}\left(\frac{\mu_0 I^2}{4\pi}\ln\frac{b}{a}\right)dz \end{aligned} \tag{4-137}$$

Thus the energy stored in the magnetic field per unit length of the current distribution is $(\mu_0 I^2/4\pi)\ln(b/a)$. ▮

4.8 Power Flow in an Electromagnetic Field; The Poynting Vector

In Section 4.6 we showed that the potential energy associated with a charge distribution can be thought of as residing in the electric field **E** set up by the charge distribution, with the energy density equal to $\frac{1}{2}\epsilon_0 E^2$. Similarly, in Section 4.7 we showed that the potential energy associated with a current distribution can be thought of as residing in the magnetic field **B** set up by the current distribution, with the energy density equal to $\frac{1}{2}B^2/\mu_0$. Let us now consider a point charge Q moving with a velocity **v** in a region of electromagnetic field characterized by electric and magnetic fields **E** and **B**. According to the Lorentz force equation, the force experienced by the point charge is given by

$$\mathbf{F} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

The work done by the force in displacing the charge by an infinitesimal distance $d\mathbf{l}$ is

$$\begin{aligned} dW &= \mathbf{F}\cdot d\mathbf{l} = Q(\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot d\mathbf{l} \\ &= Q\mathbf{E}\cdot d\mathbf{l} + Q\frac{d\mathbf{l}}{dt}\times\mathbf{B}\cdot d\mathbf{l} \\ &= Q\mathbf{E}\cdot d\mathbf{l} = Q\mathbf{E}\cdot\mathbf{v}\,dt \end{aligned} \tag{4-138}$$

This amount of work is done by the fields and the time rate at which it is done or the power supplied by the fields for the motion of the charge is

$$\frac{dW}{dt} = Q\mathbf{E}\cdot\mathbf{v} \tag{4-139}$$

If we have a volume charge distribution of density ρ instead of a point charge Q, we can divide the volume into a number of infinitesimal volumes dv and consider the charge $\rho\, dv$ in each infinitesimal volume as a point charge. Substituting $Q = \rho\, dv$ in (4-139), we then have the power supplied by the field for the motion of the charge $\rho\, dv$ as

$$\frac{dW}{dt} = \rho\, dv\, \mathbf{E} \cdot \mathbf{v} \tag{4-140}$$

The power supplied by the field to the entire volume charge distribution is given by the integral of (4-140) over the volume of the charge distribution. Thus, if a volume charge of density $\rho(r, \theta, \phi)$ is moving with a velocity $\mathbf{v}(r, \theta, \phi)$ in the region V of an electromagnetic field characterized by electric and magnetic fields $\mathbf{E}(r, \theta, \phi)$ and $\mathbf{B}(r, \theta, \phi)$, respectively, thereby constituting a current of density $\mathbf{J}(r, \theta, \phi)$, the power expended by the electromagnetic field is given by

$$P_d = \int_V \rho\, dv\, \mathbf{E} \cdot \mathbf{v} = \int_V \mathbf{E} \cdot \mathbf{J}\, dv \tag{4-141}$$

where we have substituted $\mathbf{J}$ for $\rho\mathbf{v}$ in accordance with (3-10).

We now make use of the vector identity

$$\nabla \cdot (\mathbf{E} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{B}$$

and Maxwell's curl equations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0\left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right)$$

to obtain

$$\nabla \cdot (\mathbf{E} \times \mathbf{B}) = -\mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} - \mu_0 \mathbf{E} \cdot \mathbf{J} - \mu_0 \epsilon_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} \tag{4-142}$$

Noting that

$$\mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{B} \cdot \mathbf{B}\right)$$

and

$$\mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{E} \cdot \mathbf{E}\right)$$

(4-142) can be written as

$$\mathbf{E} \cdot \mathbf{J} + \frac{\partial}{\partial t}\left(\frac{1}{2\mu_0}\mathbf{B} \cdot \mathbf{B}\right) + \frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon_0 \mathbf{E} \cdot \mathbf{E}\right) = -\nabla \cdot \left(\mathbf{E} \times \frac{\mathbf{B}}{\mu_0}\right) \tag{4-143}$$

Defining a vector $\mathbf{P}$ given by

$$\mathbf{P} = \mathbf{E} \times \frac{\mathbf{B}}{\mu_0} \tag{4-144}$$

and taking the volume integral on both sides of (4-143) over the volume V, we obtain

$$\int_V \mathbf{E} \cdot \mathbf{J}\, dv + \int_V \frac{\partial}{\partial t}\left(\frac{1}{2\mu_0}\mathbf{B} \cdot \mathbf{B}\right) dv + \int_V \frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon_0 \mathbf{E} \cdot \mathbf{E}\right) dv = -\int_V \nabla \cdot \mathbf{P}\, dv \tag{4-145}$$

Interchanging the differentiation operation with time and integration over volume in the second and third terms on the left side of (4-145) and replacing the volume integral on the right side of (4-145) by a closed surface integral in accordance with the divergence theorem, we get

$$\int_V \mathbf{E} \cdot \mathbf{J}\, dv + \frac{\partial}{\partial t}\int_V \left(\frac{1}{2\mu_0}\mathbf{B} \cdot \mathbf{B}\right) dv + \frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\epsilon_0 \mathbf{E} \cdot \mathbf{E}\right) dv = -\oint_S \mathbf{P} \cdot d\mathbf{S} \tag{4-146}$$

where S is the surface bounding the volume V.

On the left side of (4-146), the second and third terms represent the time rate of increase of energy stored in the magnetic and electric fields, respectively, in the volume V. Thus the left side is the sum of the power expended by the fields due to the motion of the charge and the time rate of increase of stored energy in the fields. Obviously then, the right side of (4-146) must represent the power flow into the volume V across the surface S, or

$$\text{the power flow out of volume } V \text{ across the surface } S = \oint_S \mathbf{P} \cdot d\mathbf{S} \tag{4-147}$$

It then follows that the vector $\mathbf{P}$ has the meaning of power density associated with the electromagnetic field at a point. The statement represented by (4-146) is known as Poynting's theorem after J. H. Poynting, who derived it in 1884, and the vector $\mathbf{P}$ is known as the Poynting vector. We note that the units of $\mathbf{P} = \mathbf{E} \times \mathbf{B}/\mu_0$ are

$$\begin{aligned}
&\frac{\text{newtons}}{\text{coulomb}} \times \frac{\text{newton-seconds}}{\text{coulomb-meter}} \div \frac{\text{newtons}}{(\text{ampere})^2} \\
&\qquad = \frac{\text{newton-amperes}}{\text{coulomb-meter}} = \frac{\text{newtons}}{\text{second-meter}} \\
&\qquad = \frac{\text{newton-meters}}{\text{second}} \times \frac{1}{(\text{meter})^2} = \frac{\text{watts}}{(\text{meter})^2}
\end{aligned}$$

and do indeed represent units of power density.

Caution must be exercised in the interpretation of the Poynting vector $\mathbf{P}$ as representing the power density at a point, since we can add to $\mathbf{P}$ on the right side of (4-146) any vector for which the surface integral over S vanishes, without affecting the equation. On the other hand, the interpretation of $\int_V (\nabla \cdot \mathbf{P})\, dv = \oint_S \mathbf{P} \cdot d\mathbf{S}$ as the power flow out of the volume V bounded by S should always give the correct answer. For example, let us consider a region free of charges and currents in which static electric and magnetic fields **E** and **B** exist. For such a situation, although $\mathbf{E} \times \mathbf{B}$ can be nonzero, $\nabla \cdot (\mathbf{E} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{B}) = 0$ since $\nabla \times \mathbf{E} = 0$ for a static electric field and $\nabla \times \mathbf{B} = 0$ for a static magnetic field in a current-free region. The fact that $\nabla \cdot (\mathbf{E} \times \mathbf{B}) = 0$ is consistent with the physical situation, since there is no change with time in the energy stored in the static electric and magnetic fields and hence there is no power flow associated with the fields. Thus the interpretation of the Poynting vector as the power density vector at a point in an electromagnetic field is strictly valid only in the sense that $\oint_S \mathbf{P} \cdot d\mathbf{S}$ gives the correct result for the power flow across the closed surface S.

EXAMPLE 4-13. The electric field intensity **E** in the radiation field of an antenna located at the origin of a spherical coordinate system is given by

$$\mathbf{E} = \frac{E_0 \sin \theta}{r} \cos (\omega t - \beta r)\, \mathbf{i}_\theta$$

where E_0, ω, and $\beta(=\omega\sqrt{\mu_0 \epsilon_0})$ are constants. It is desired to find the magnetic field **B** associated with this electric field and then find the power radiated by the antenna by integrating the Poynting vector over a spherical surface of radius r centered at the antenna.

From Maxwell's equation for the curl of **E**, we have

$$-\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{E}$$

$$= \begin{vmatrix} \dfrac{\mathbf{i}_r}{r^2 \sin \theta} & \dfrac{\mathbf{i}_\theta}{r \sin \theta} & \dfrac{\mathbf{i}_\phi}{r} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ 0 & E_0 \sin \theta \cos (\omega t - \beta r) & 0 \end{vmatrix}$$

$$= \frac{\beta}{r} E_0 \sin \theta \sin (\omega t - \beta r)\, \mathbf{i}_\phi$$

and

$$\mathbf{B} = \frac{\beta E_0}{\omega r} \sin \theta \cos (\omega t - \beta r)\, \mathbf{i}_\phi$$

The Poynting vector is then given by

$$\mathbf{P} = \mathbf{E} \times \frac{\mathbf{B}}{\mu_0}$$

$$= \frac{1}{\mu_0} \begin{vmatrix} \mathbf{i}_r & \mathbf{i}_\theta & \mathbf{i}_\phi \\ 0 & \dfrac{E_0 \sin \theta}{r} \cos (\omega t - \beta r) & 0 \\ 0 & 0 & \dfrac{\beta E_0}{\omega r} \sin \theta \cos (\omega t - \beta r) \end{vmatrix}$$

$$= \frac{\beta E_0^2 \sin^2 \theta}{\mu_0 \omega r^2} \cos^2 (\omega t - \beta r)\, \mathbf{i}_r$$

The power radiated by the antenna

$$= \oint_{\substack{\text{spherical surface} \\ \text{of radius } r}} \mathbf{P} \cdot d\mathbf{S}$$

$$= \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{\beta E_0^2 \sin^2 \theta}{\mu_0 \omega r^2} \cos^2 (\omega t - \beta r)\, \mathbf{i}_r \cdot r^2 \sin \theta \, d\theta \, d\phi \, \mathbf{i}_r$$

$$= \frac{2\pi \beta E_0^2 \cos^2 (\omega t - \beta r)}{\mu_0 \omega} \int_{\theta=0}^{\pi} \sin^3 \theta \, d\theta$$

$$= \frac{8\pi \beta E_0^2 \cos^2 (\omega t - \beta r)}{3\mu_0 \omega} \quad \blacksquare$$

4.9 The Phasor Concept and the Phasor Representation of Sinusoidally Time-Varying Fields and Maxwell's Equations for Sinusoidally Time-Varying Fields

In developing the electromagnetic field equations, we have thus far considered the time variation of the fields and the associated source quantities to be completely arbitrary. A very important special case of variation with time of the field and source quantities is the sinusoidal steady-state variation. Among the reasons for this importance are that, in practice, we do encounter such fields and that any function whose time variation is arbitrary can be expressed, in general, as an infinite sum of sinusoidal functions having a discrete or continuous spectrum of frequencies, depending upon whether the function is periodic or not. We therefore devote special attention to sinusoidally time-varying fields. In dealing with sinusoidally time-varying quantities, the phasor approach is convenient, as the student may have already learned in circuit analysis. However, we will here review the phasor concept and illustrate why it is convenient before applying it to electromagnetic fields.

A phasor is nothing but a complex number. It is represented graphically by the line drawn from the origin to the point, in the complex plane, corresponding to the complex number as shown in Fig. 4.14. The length of the line is equal to the magnitude of the complex number and the angle that the line makes with the positive real axis is the angle of the complex number. Sinusoidal functions of time are represented by phasors. In particular, when the sinusoidal function is expressed in cosinusoidal form, that is, in the form $A \cos (\omega t + \phi)$, the magnitude of the phasor is equal to the magnitude A of the cosinusoidal function and the angle of the phasor is equal to the phase angle ϕ of the cosinusoidal function for $t = 0$. The real part of the phasor is equal to $A \cos \phi$, which is the value of the function at $t = 0$. If we now imagine the phasor to be rotating about the origin in the counterclockwise direction at the rate of ω rad/sec as shown in Fig. 4.14, we can see that the instantaneous angle of the phasor is $(\omega t + \phi)$ and hence the

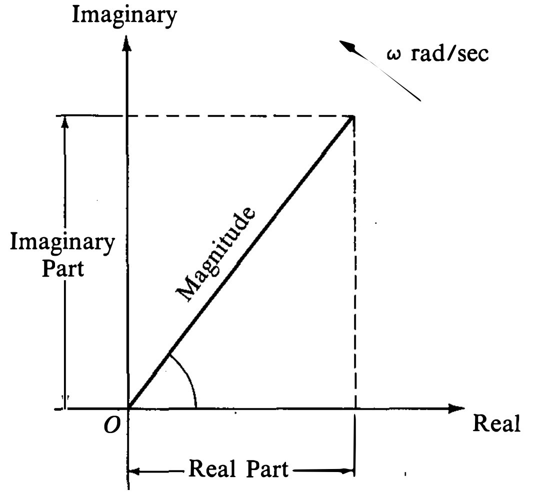

Fig. 4.14. Graphical representation of a phasor.

time variation of its projection on the real axis describes the time variation of the cosinusoidal function.

To illustrate why the phasor approach is convenient for solving sinusoidal steady-state problems, we consider the simple circuit shown in Fig. 4.15 in which a source of voltage $V(t) = V_m \cos(\omega t + \phi)$ drives a series combination of inductance L and resistance R. We will first find the solution for the current $I(t)$ in the steady state without using the phasor approach. Using Kirchhoff's voltage law, we have

$$L\frac{dI(t)}{dt} + RI(t) = V_m \cos(\omega t + \phi) \tag{4-148}$$

We know that the solution for the current in the steady state must also be a cosine function having the same frequency as that of the source voltage

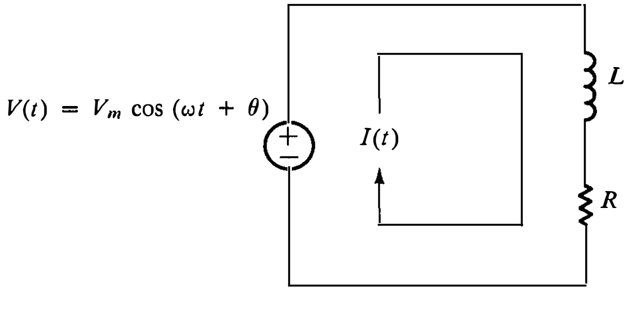

Fig. 4.15. A series *RL* circuit driven by a sinusoidally time-varying voltage source.

but having different magnitude and different phase angle in general. Thus let us assume the solution to be $I(t) = I_m \cos(\omega t + \theta)$. Substituting this solution in the differential equation, we have

$$L\frac{d}{dt}[I_m \cos(\omega t + \theta)] + RI_m \cos(\omega t + \theta) = V_m \cos(\omega t + \phi)$$

or

$$\begin{aligned}(&-\omega LI_m \sin \omega t \cos \theta - \omega LI_m \cos \omega t \sin \theta \\ &+ RI_m \cos \omega t \cos \theta - RI_m \sin \omega t \sin \theta) \\ &= V_m \cos \omega t \cos \phi - V_m \sin \omega t \sin \phi\end{aligned} \tag{4-149}$$

Since (4-149) must be true for all values of time, the coefficients of sin ωt on either side of it must be equal and, similarly, the coefficients of cos ωt on either side of it must also be equal. Thus we have

$$-\omega LI_m \cos \theta - RI_m \sin \theta = -V_m \sin \phi \tag{4-150a}$$

$$-\omega LI_m \sin \theta + RI_m \cos \theta = V_m \cos \phi \tag{4-150b}$$

Squaring (4-150a) and (4-150b) and adding, we obtain

$$V_m^2 = \omega^2 L^2 I_m^2 + R^2 I_m^2$$

or

$$I_m = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \tag{4-151}$$

Multiplying (4-150a) by $\cos\theta$ and (4-150b) by $\sin\theta$ and adding, we get

$$\omega LI_m = V_m \sin(\phi - \theta) \tag{4-152a}$$

Similarly, multiplying (4-150a) by $-\sin\theta$ and (4-150b) by $\cos\theta$ and adding, we get

$$RI_m = V_m \cos(\phi - \theta) \tag{4-152b}$$

From (4-152a) and (4-152b), we have

$$\tan(\phi - \theta) = \frac{\omega L}{R}$$

or

$$\theta = \phi - \tan^{-1}\frac{\omega L}{R} \tag{4-153}$$

Hence the solution for $I(t)$ in the steady state is given by

$$I(t) = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos\left(\omega t + \phi - \tan^{-1}\frac{\omega L}{R}\right) \tag{4-154}$$

Let us now use the phasor concept to solve the same simple problem. Noting that

$$V_m \cos(\omega t + \phi) = \mathcal{R}e[V_m e^{j(\omega t + \phi)}] \tag{4-155a}$$

and

$$I_m \cos(\omega t + \theta) = \mathcal{R}e[I_m e^{j(\omega t + \theta)}] \tag{4-155b}$$

where $\mathcal{R}e$ stands for "the real part of," we have from (4-148),

$$L\frac{d}{dt}\{\mathcal{R}e[I_m e^{j(\omega t+\theta)}]\} + R\{\mathcal{R}e[I_m e^{j(\omega t+\theta)}]\} = \mathcal{R}e[V_m e^{j(\omega t+\phi)}] \quad (4\text{-}156)$$

However, since L and R are constants and also since d/dt and $\mathcal{R}e$ can be interchanged, we can simplify (4-156) in accordance with the following steps:

$$\begin{aligned} \mathcal{R}e\{\frac{d}{dt}[LI_m e^{j(\omega t+\theta)}]\} + \mathcal{R}e[RI_m e^{j(\omega t+\theta)}] &= \mathcal{R}e[V_m e^{j(\omega t+\phi)}] \\ \mathcal{R}e[j\omega LI_m e^{j(\omega t+\theta)}] + \mathcal{R}e[RI_m e^{j(\omega t+\theta)}] &= \mathcal{R}e[V_m e^{j(\omega t+\phi)}] \\ \mathcal{R}e[(R + j\omega L)I_m e^{j(\omega t+\theta)}] &= \mathcal{R}e[V_m e^{j(\omega t+\phi)}] \end{aligned} \quad (4\text{-}157)$$

Equation (4-157) states that the real parts of two complex numbers are equal. Does this mean that the two complex numbers are equal? No, not in general! For example, consider $4 + j2$ and $4 + j5$. Their real parts are equal but the numbers themselves are not equal. However, (4-157) must hold for all values of time. Let us consider two times t_1 and t_2 corresponding to $(\omega t + \theta)$ equal to zero and $(\omega t + \theta)$ equal to $\pi/2$, respectively. Then, for time t_1, we have

$$\mathcal{R}e[(R + j\omega L)I_m] = \mathcal{R}e[V_m e^{j(\phi-\theta)}] \quad (4\text{-}158)$$

For time t_2, we have

$$\mathcal{R}e\{[R + j\omega L]I_m e^{j(\pi/2)}\} = \mathcal{R}e\{V_m e^{j[(\pi/2)-\theta+\phi]}\}$$

or

$$\mathcal{R}e\{j[(R + j\omega L)I_m]\} = \mathcal{R}e\{j[V_m e^{j(\phi-\theta)}]\}$$

or

$$\mathcal{I}m[(R + j\omega L)I_m] = \mathcal{I}m[V_m e^{j(\phi-\theta)}] \quad (4\text{-}159)$$

where $\mathcal{I}m$ stands for "the imaginary part of." Equations (4-158) and (4-159) state that the real parts as well as the imaginary parts of two complex numbers are equal. Hence the two complex numbers must be equal. Thus we obtain

$$(R + j\omega L)I_m = V_m e^{j(\phi-\theta)}$$

or

$$(R + j\omega L)I_m e^{j\theta} = V_m e^{j\phi} \quad (4\text{-}160)$$

Multiplying both sides of (4-160) by $e^{j\omega t}$, we note that the two complex numbers in (4-157) are equal. Now, defining phasors $\bar{I}$ and $\bar{V}$ as

$$\bar{I} = I_m e^{j\theta} \quad \text{so that} \quad I(t) = \mathcal{R}e(\bar{I}e^{j\omega t}) \quad (4\text{-}161\text{a})$$

$$\bar{V} = V_m e^{j\phi} \quad \text{so that} \quad V(t) = \mathcal{R}e(\bar{V}e^{j\omega t}) \quad (4\text{-}161\text{b})$$

Eq. (4-160) can be written as

$$(R + j\omega L)\bar{I} = \bar{V} \quad (4\text{-}162)$$

Note that an overscore associated with a symbol represents the phasor (or complex) character of the quantity represented by the symbol. We can easily show that (4-162) leads to the same result as (4-154), since

$$I_m e^{j\theta} = \bar{I} = \frac{\bar{V}}{R + j\omega L}$$
$$= \frac{V_m e^{j\phi}}{\sqrt{R^2 + \omega^2 L^2} e^{j\tan^{-1}\omega L/R}} = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} e^{j(\phi - \tan^{-1}\omega L/R)}$$

and

$$I(t) = I_m \cos(\omega t + \theta) = \mathcal{R}e(I_m e^{j\theta} e^{j\omega t})$$
$$= \mathcal{R}e\left[\frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} e^{j(\phi - \tan^{-1}\omega L/R)} e^{j\omega t}\right]$$
$$= \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos\left(\omega t + \phi - \tan^{-1}\frac{\omega L}{R}\right)$$

which is the same as (4-154).

In the foregoing illustration of the phasor technique, we have included several steps merely to understand the basis behind the phasor technique. It is clear that, hereafter, we can omit all steps up to (4-162) and write the phasor equation (4-162) directly from the differential equation (4-148) by simply replacing $I(t)$ and $V(t)$ by their phasors $\bar{I}$ and $\bar{V}$, respectively, and by replacing d/dt by $j\omega$. The phasor equation is then solved for the phasor $\bar{I}$ from which the time function $I(t)$ is obtained. Comparing with the trigonometric manipulations involved in the steps from (4-149) to (4-153) which have to be carried out for each different problem, we can now appreciate the simplicity of the phasor technique. As a numerical example, let us consider $V(t) = 10 \cos 1000t$, $L = 10^{-3}$ henry, and $R = 1$ ohm for the network of Fig. 4.15. The differential equation for $I(t)$ is given by

$$10^{-3}\frac{dI}{dt} + I = 10 \cos 1000t$$

Replacing the current and voltage by their phasors and d/dt by $j\omega$, we have

$$(j\omega 10^{-3} + 1)\bar{I} = 10e^{j0}$$

or, since $\omega = 1000$ rad/sec,

$$(j1 + 1)\bar{I} = 10e^{j0}$$

The phasor $\bar{I}$ is then given by

$$\bar{I} = \frac{10e^{j0}}{1 + j1} = \frac{10e^{j0}}{\sqrt{2}\, e^{j\pi/4}} = \frac{10}{\sqrt{2}} e^{-j\pi/4}$$

Finally,

$$I(t) = \mathcal{R}e[\bar{I} e^{j(1000t)}]$$
$$= \mathcal{R}e\left[\frac{10}{\sqrt{2}} e^{-j\pi/4} e^{j(1000t)}\right]$$
$$= 7.07 \cos(1000t - 45^\circ)$$

The voltage and current phasors and the corresponding time functions are shown in Figs. 4.16(a) and 4.16(b), respectively. Note that in Fig. 4.16(a) we have turned the complex plane around by 90° in the counterclockwise direction to illustrate that the time variations of the projections of the phasors as they rotate in the counterclockwise direction describe the curves shown in Fig. 4.16(b).

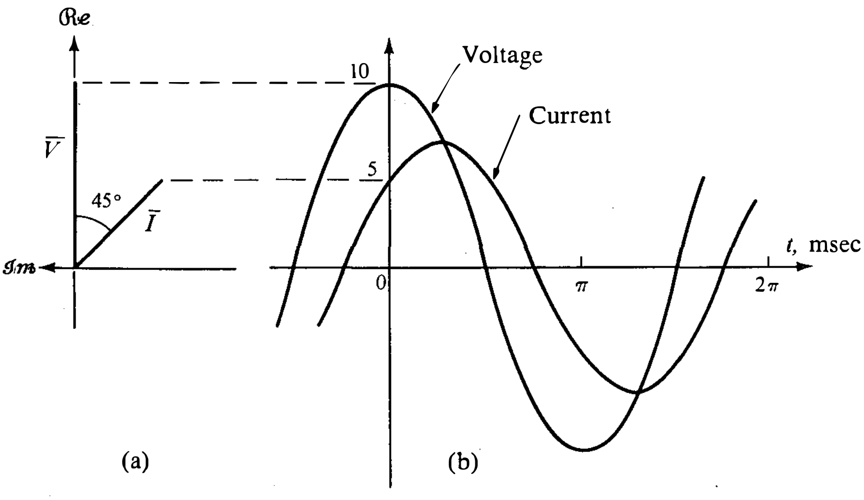

Fig. 4.16. (a) Voltage and current phasors for numerical values $V = 10$ volts, $\omega = 1000$ rad/sec, $L = 10^{-3}$ henry, and $R = 1$ ohm for the series *RL* circuit of Fig. 4.15. (b) Time functions corresponding to the voltage and current phasors of (a).

Extension of the phasor technique to vector quantities whose magnitudes vary sinusoidally with time follows from its application to the individual components of the vector along the coordinate axes. However, some confusion is bound to arise since both vectors and phasors are represented graphically in the same manner except that the vector has an arrowhead associated with it. A vector represents the magnitude and space direction of a quantity whereas a phasor represents the magnitude and phase angle of a sinusoidally varying function of time. Thus the angle which a phasor makes with the real axis of the complex plane has nothing to do with direction in space, and the angle which a vector makes with a reference axis in a spatial coordinate system has nothing to do with the phase angle which is associated with the time variation of the quantity. Nevertheless, there are certain similarities between vectors and phasors. These are pertinent to manipulations involving addition, subtraction, and multiplication by a constant. They both use the same graphical rules for carrying out these manipulations. Hence we must be careful, in performing these manipulations, not to get confused between the space angles associated with the vectors and the phase angles associated with the phasors. We will now consider an example to illustrate these differences and similarities between vectors and phasors.

EXAMPLE 4-14. In the arrangement shown in Fig. 4.17(a), three line charges, infinitely long in the direction normal to the plane of the paper and having uniform densities varying sinusoidally with time, are situated at the corners of an equilateral triangle. The amplitudes of the sinusoidally time-varying charge

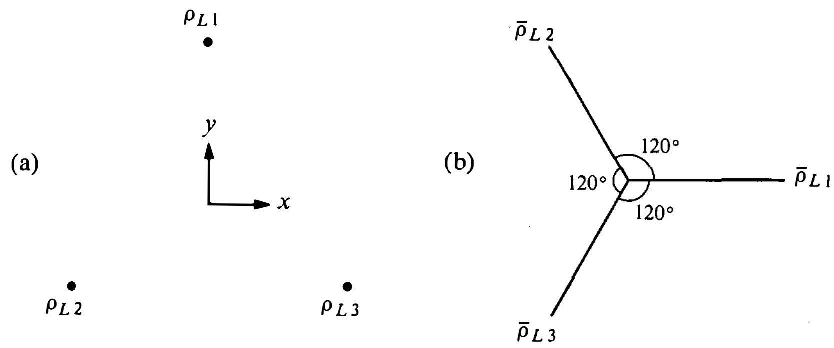

Fig. 4.17. (a) Geometrical arrangement of infinitely long uniform and sinusoidally time-varying line charges. (b) Phasor diagram of the sinusoidally time-varying line charge densities.

densities are such that, considered alone, each line charge produces unit peak electric field intensity at the geometric center of the triangle. The phasor diagram of the charge densities is shown in Fig. 4.17(b).

(a) Find the phasors representing the x and y components of the electric field intensity vector at the geometric center of the triangle.

(b) Determine how the magnitude and direction of the electric field intensity vector at the geometric center of the triangle vary with time.

The phasor diagram indicates that the line charge densities are given by

$$\rho_{L1} = \rho_{Lm} \cos \omega t$$
$$\rho_{L2} = \rho_{Lm} \cos (\omega t + 120°)$$
$$\rho_{L3} = \rho_{Lm} \cos (\omega t + 240°)$$

where ρ_{Lm} is the peak value of the charge densities.

(a) The electric field intensity vector due to an infinitely long line charge of uniform density is directed radially away from the line charge. Hence the field intensities due to the different line charges are directed as shown in Fig. 4. 18(a), with the complex numbers beside the vectors representing their phasors. For example, the phasor $1\underline{/0°}$ associated with the field intensity

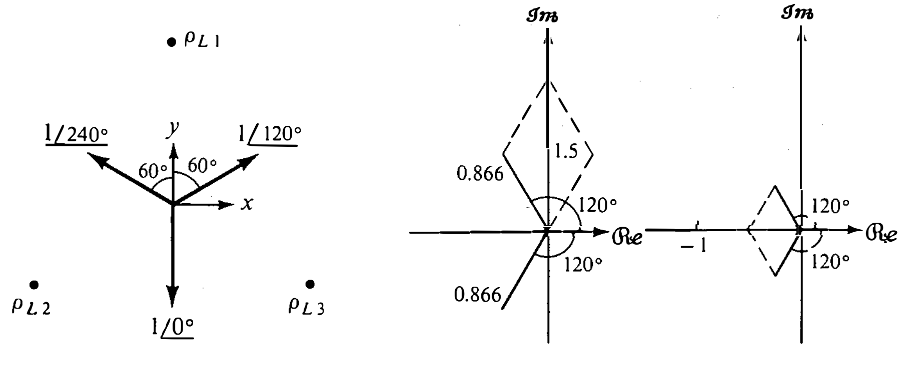

Fig. 4.18. For evaluating the phasors representing the x and y components of the electric field intensity vector at the geometric center of the line charge arrangement of Fig. 4.17.

vector due to the line charge of density ρ_{L1} indicates that the time variation of the magnitude of the vector is given by $1 \cos \omega t$. Thus, timewise, the vector oscillates back and forth along the y axis starting with a magnitude of 1 in the negative y direction, shrinking gradually to zero in a sinusoidal manner, then reversing its direction and growing in magnitude in the positive y direction until it reaches a maximum of unity, then shrinking back to zero, and so on.

Now, the x component of the phasor electric field intensity vector at the geometric center of the triangle is given by

$$\begin{aligned}\bar{E}_x &= (1 \cos 30°)\underline{/120°} - (1 \cos 30°)\underline{/240°}\\ &= 0.866\underline{/120°} - 0.866\underline{/240°} = 1.5\underline{/90°}\end{aligned}$$

where we have used the construction shown in Fig. 4.18(b). We note that, in the above steps, certain manipulations are vector manipulations whereas certain other manipulations have to do with phasors. For example, in finding the x component of the phasor vector $1\underline{/120°}$ pointing away from the line charge of density ρ_{L2}, the phase angle 120° is preserved and the magnitude 1 is multiplied by the cosine of the angle which the vector makes with the x axis, giving us $(1 \cos 30°)\underline{/120°}$ or $0.866\underline{/120°}$. Similarly, the y component of the phasor electric field intensity vector at the geometric center of the triangle is given by

$$\begin{aligned}\bar{E}_y &= -1\underline{/0°} + (1 \cos 60°)\underline{/120°} + (1 \cos 60°)\underline{/240°}\\ &= -1\underline{/0°} + 0.5\underline{/120°} + 0.5\underline{/240°}\\ &= 1\underline{/180°} + 0.5\underline{/180°} = 1.5\underline{/180°}\end{aligned}$$

where we have used the construction shown in Fig. 4.18(c). The phasor diagram of the x and y components of the electric field intensity vector at the geometric center of the triangle relative to the phasor diagram of the line charge densities is shown in Fig. 4.19(a).

(b) From the results of part (a), we have

$$E_x(t) = \mathcal{R}e(\bar{E}_x e^{j\omega t}) = 1.5\cos(\omega t + 90°) = -1.5\sin\omega t$$
$$E_y(t) = \mathcal{R}e(\bar{E}_y e^{j\omega t}) = 1.5\cos(\omega t + 180°) = -1.5\cos\omega t$$

Now, since $\mathbf{E}(t) = E_x(t)\mathbf{i}_x + E_y(t)\mathbf{i}_y$, the magnitude of $\mathbf{E}(t)$ is given by

$$|\mathbf{E}(t)| = \sqrt{E_x^2(t) + E_y^2(t)}$$
$$= \sqrt{(-1.5\sin\omega t)^2 + (-1.5\cos\omega t)^2} = 1.5$$

The angle which the vector $\mathbf{E}(t)$ makes with the x axis is given by

$$\tan^{-1}\frac{E_y(t)}{E_x(t)} = \tan^{-1}\frac{-1.5\cos\omega t}{-1.5\sin\omega t} = \tan^{-1}\frac{-1.5\sin(\omega t + \pi/2)}{1.5\cos(\omega t + \pi/2)}$$
$$= \tan^{-1}[-\tan(\omega t + \pi/2)] = -(\omega t + \pi/2)$$

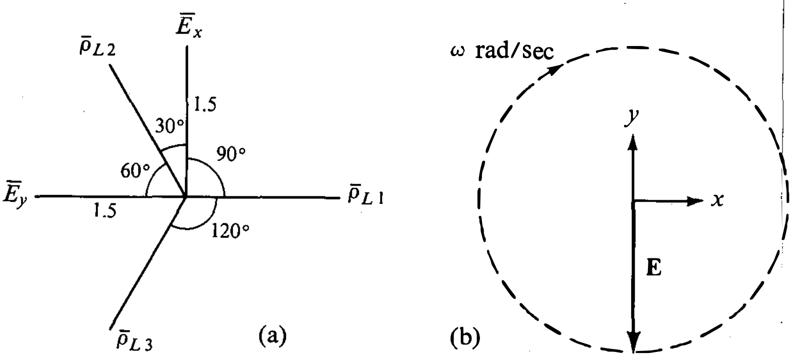

Fig. 4.19. (a) Phasor diagram of the x and y components of the electric field intensity vector at the geometric center of the line charge arrangement of Fig. 4.17, relative to the phasor diagram of the line charge densities. (b) For describing the time variation of the electric field intensity vector corresponding to the phasor diagram of (a).

Thus the magnitude of the electric field intensity vector at the geometric center of the triangle remains constant at 1.5 units and the angle which the vector makes with the x axis varies as $-(\omega t + \pi/2)$ with time; that is, the vector rotates with a constant magnitude and at a rate of ω rad/sec, with the direction at $t = 0$ along the negative y axis and in the sense shown in Fig. 4.19(b). The field is then said to be circularly polarized. ▮

We will now discuss briefly polarization of vector fields. Polarization is the characteristic by means of which we describe how the magnitude and the direction of the field vary with time. For an arbitrarily time-varying field characterized by random time-variations of its components along the coordinate axes at a point in space, the magnitude and direction of the field vary randomly with time. The field is then said to be unpolarized or randomly polarized. For a sinusoidally time-varying field at a particular frequency ω, the field vector is characterized by a well-defined polarization. In the most general case, the magnitude and direction of such a field vector at a point change with time in such a manner that the tip of the vector drawn at that point describes an ellipse as time progresses, as shown in Fig. 4.20(a). The field is then said to be elliptically polarized. There are two special cases of elliptical polarization. These are linear polarization and circular polarization.

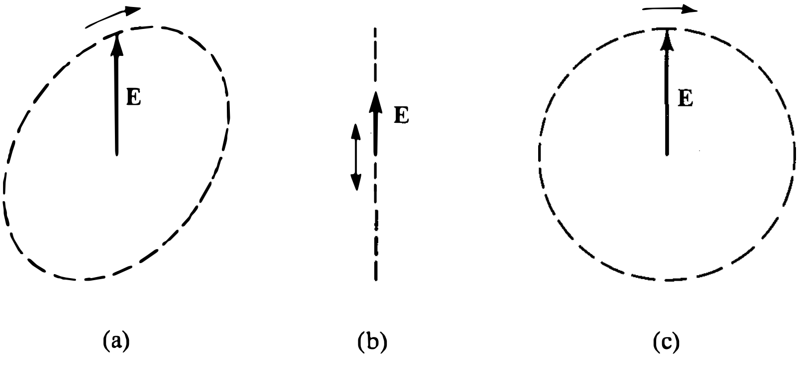

Fig. 4.20. For illustrating (a) elliptical polarization, (b) linear polarization, and (c) circular polarization of a field vector.

If the field vector at a point in space lies along the same straight line through that point as time progresses, as shown in Fig. 4.20(b), the field is said to be linearly polarized. Obviously, the components of a field vector along the coordinate axes are linearly polarized. If all the components of the field vector along the coordinate axes have the same phase, although possessing different magnitudes, then the field vector itself is linearly polarized. If the tip of a field vector drawn at a point in space describes a circle as time progresses, as shown in Fig. 4.20(c) the field is said to be circularly polarized. Circular polarization is realized by the superposition of two field components oriented perpendicular to each other and having the same magnitude but differing in phase by $\pi/2$ or 90° as in the case of the two components in Example 4-14. Elliptical polarization is realized by the superposition of two field components having in general different magnitudes as well as different phase angles. Since a circle and ellipse can be traversed in one of two senses, we have to distinguish between the opposite senses of rotation in the cases of circular and elliptical polarizations. The distinction is made as follows. Considering the vector to be the electric field intensity vector **E**, the field is said to be clockwise or right circularly (or elliptically) polarized if the vector rotates in the clockwise sense as seen looking along the direction of the Poynting vector $\mathbf{P} = \mathbf{E} \times \mathbf{B}/\mu_0$ where $\mathbf{B}$ is the magnetic field associated with $\mathbf{E}$. The field is said to be counterclockwise or left circularly (or elliptically) polarized if the vector rotates in the counterclockwise sense as seen looking along the direction of the Poynting vector.

Having illustrated the application of the phasor technique in dealing with sinusoidally time-varying vector fields, we now turn to the phasor representation of Maxwell's equations for sinusoidally time-varying fields. Maxwell's equations for time-varying fields are given by

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{4-163}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{4-164}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{4-165}$$

$$\nabla \times \mathbf{B} = \mu_0\left[\mathbf{J} + \frac{\partial}{\partial t}(\epsilon_0 \mathbf{E})\right] \tag{4-166}$$

whereas the continuity equation is given by

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0 \tag{4-167}$$

In (4-163)–(4-167), the quantities $\mathbf{E}$, $\mathbf{B}$, ρ, and $\mathbf{J}$ are also functions of all three space coordinates in general. Thus we have

$$\mathbf{E} = \mathbf{E}(x, y, z, t) = E_x(x, y, z, t)\mathbf{i}_x + E_y(x, y, z, t)\mathbf{i}_y + E_z(x, y, z, t)\mathbf{i}_z$$
$$\mathbf{B} = \mathbf{B}(x, y, z, t) = B_x(x, y, z, t)\mathbf{i}_x + B_y(x, y, z, t)\mathbf{i}_y + B_z(x, y, z, t)\mathbf{i}_z$$
$$\rho = \rho(x, y, z, t)$$
$$\mathbf{J} = \mathbf{J}(x, y, z, t) = J_x(x, y, z, t)\mathbf{i}_x + J_y(x, y, z, t)\mathbf{i}_y + J_z(x, y, z, t)\mathbf{i}_z$$

For the particular case of sinusoidal variation with time, we have

$$\begin{aligned}
\mathbf{E} &= \mathbf{E}(x, y, z, t) \\
&= E_{x0}(x, y, z)\cos[\omega t + \phi_x(x, y, z)]\,\mathbf{i}_x \\
&\quad + E_{y0}(x, y, z)\cos[\omega t + \phi_y(x, y, z)]\,\mathbf{i}_y \\
&\quad + E_{z0}(x, y, z)\cos[\omega t + \phi_z(x, y, z)]\,\mathbf{i}_z \\
&= \mathcal{R}e[E_{x0}(x, y, z)e^{j\phi_x(x, y, z)}e^{j\omega t}\mathbf{i}_x \\
&\qquad + E_{y0}(x, y, z)e^{j\phi_y(x, y, z)}e^{j\omega t}\mathbf{i}_y \\
&\qquad + E_{z0}(x, y, z)e^{j\phi_z(x, y, z)}e^{j\omega t}\mathbf{i}_z] \\
&= \mathcal{R}e\{[\bar{E}_x(x, y, z)\mathbf{i}_x + \bar{E}_y(x, y, z)\mathbf{i}_y + \bar{E}_z(x, y, z)\mathbf{i}_z]e^{j\omega t}\} \\
&= \mathcal{R}e[\bar{\mathbf{E}}(x, y, z)e^{j\omega t}]
\end{aligned} \tag{4-168}$$

Similarly, we have

$$\mathbf{B} = \mathbf{B}(x, y, z, t) = \Re e[\bar{\mathbf{B}}(x, y, z)e^{j\omega t}] \tag{4-169}$$

$$\rho = \rho(x, y, z, t) = \Re e[\bar{\rho}(x, y, z)e^{j\omega t}] \tag{4-170}$$

$$\mathbf{J} = \mathbf{J}(x, y, z, t) = \Re e[\bar{\mathbf{J}}(x, y, z)e^{j\omega t}] \tag{4-171}$$

In (4-168)–(4-171), the complex quantities $\bar{\mathbf{E}}(x, y, z)$, $\bar{\mathbf{B}}(x, y, z)$, $\bar{\rho}(x, y, z)$, and $\bar{\mathbf{J}}(x, y, z)$ are the phasor representations for the sinusoidally time-varying quantities $\mathbf{E}(x, y, z, t)$, $\mathbf{B}(x, y, z, t)$, $\rho(x, y, z, t)$, and $\mathbf{J}(x, y, z, t)$, respectively.

Substituting the respective phasors for the quantities $\mathbf{E}, \mathbf{B}, \rho$, and $\mathbf{J}$ and replacing $\partial/\partial t$ by $j\omega$ in (4-163)–(4-167), we obtain the phasor representations of Maxwell's equations as

$$\nabla \cdot \bar{\mathbf{E}} = \frac{\bar{\rho}}{\epsilon_0} \tag{4-172}$$

$$\nabla \cdot \bar{\mathbf{B}} = 0 \tag{4-173}$$

$$\nabla \times \bar{\mathbf{E}} = -j\omega\bar{\mathbf{B}} \tag{4-174}$$

$$\nabla \times \bar{\mathbf{B}} = \mu_0(\bar{\mathbf{J}} + j\omega\epsilon_0\bar{\mathbf{E}}) \tag{4-175}$$

whereas the corresponding continuity equation is given by

$$\nabla \cdot \bar{\mathbf{J}} + j\omega\bar{\rho} = 0 \tag{4-176}$$

In (4-172)–(4-176), we understand that $\bar{\mathbf{E}}, \bar{\mathbf{B}}, \bar{\rho}$, and $\bar{\mathbf{J}}$ are functions of x, y, z (but not t). Note that (4-173) follows from (4-174) whereas (4-172) follows from (4-175) with the aid of (4-176).

EXAMPLE 4-15. A sinusoidally time-varying electric field intensity vector is characterized by its phasor $\bar{\mathbf{E}}$, given by

$$\bar{\mathbf{E}} = (3e^{j\pi/2}\mathbf{i}_x + 5\mathbf{i}_y - 4e^{j\pi/2}\mathbf{i}_z)e^{-j0.02\pi(4x+3z)} \tag{4-177}$$

(a) Show that the surfaces of constant phase of $\bar{\mathbf{E}}$ are planes. Find the equation of the planes.

(b) Show that the electric field is circularly polarized in the planes of constant phase.

(c) Obtain the magnetic flux density phasor $\bar{\mathbf{B}}$ associated with the given $\bar{\mathbf{E}}$ and determine if the field is right circularly polarized or left circularly polarized.

(a) The phase angle associated with $\bar{\mathbf{E}}$ is equal to $-0.02\pi(4x + 3z)$. Hence the surfaces of constant phase of $\bar{\mathbf{E}}$ are given by

$$-0.02\pi(4x + 3z) = \text{constant}$$

or

$$(4x + 3z) = \text{constant} \tag{4-178}$$

Equation (4-178) represents planes and hence the surfaces of constant phase are planes.

(b) Combining the x and z components of $\bar{\mathbf{E}}$, we obtain

$$\bar{\mathbf{E}} = (5\mathbf{i}_y + 5\mathbf{i}_{xz}e^{j\pi/2})e^{-j0.02\pi(4x+3z)} \tag{4-179}$$

where $\mathbf{i}_{xz} = (3\mathbf{i}_x - 4\mathbf{i}_z)/5$ is the unit vector in the xz plane and making an angle of $-\tan^{-1}\frac{4}{3}$ or $-53.1°$ with the positive x axis. Thus the electric field is made up of two components perpendicular to each other and having equal magnitudes but differing in phase by $\pi/2$. Hence the field is circularly polarized. From (4-179), we observe that the field vector lies in planes defined by $\mathbf{i}_y$ and $\mathbf{i}_{xz}$. The equation of these planes is given by

$$\mathbf{i}_y \cdot \mathbf{i}_{xz} \times (\mathbf{r} - \mathbf{r}_0) = 0 \tag{4-180}$$

where $\mathbf{r}$ is the position vector of an arbitrary point (x, y, z) and $\mathbf{r}_0$ is the position vector of a reference point (x_0, y_0, z_0), both points lying in a particular plane. Simplifying (4-180), we obtain

$$4x + 3z = 4x_0 + 3z_0 = \text{constant}$$

which is the same as Eq. (4-178). Thus the field is circularly polarized in the planes of constant phase.

(c) The magnetic flux density phasor $\bar{\mathbf{B}}$ associated with the given $\bar{\mathbf{E}}$ can be obtained by using

$$\nabla \times \bar{\mathbf{E}} = -j\omega\bar{\mathbf{B}} \tag{4-174}$$

Substituting for $\bar{\mathbf{E}}$ in (4-174) from (4-177) and simplifying, we obtain

$$\bar{\mathbf{B}} = \frac{0.1\pi}{\omega}(-3\mathbf{i}_x + 5e^{j\pi/2}\mathbf{i}_y + 4\mathbf{i}_z)e^{-j0.02\pi(4x+3z)} \tag{4-181}$$

Let us now consider, for simplicity, the field vectors in the plane $4x + 3z = 0$. The phasor vectors $\bar{\mathbf{E}}_0$ and $\bar{\mathbf{B}}_0$ in this plane are given by

$$\bar{\mathbf{E}}_0 = 3e^{j\pi/2}\mathbf{i}_x + 5\mathbf{i}_y - 4e^{j\pi/2}\mathbf{i}_z \tag{4-182}$$

$$\bar{\mathbf{B}}_0 = \frac{0.1\pi}{\omega}(-3\mathbf{i}_x + 5e^{j\pi/2}\mathbf{i}_y + 4\mathbf{i}_z) \tag{4-183}$$

The corresponding real field vectors are given by

$$\begin{aligned}\mathbf{E}_0 &= \mathcal{R}e(\bar{\mathbf{E}}_0e^{j\omega t}) \\ &= -3\sin\omega t\,\mathbf{i}_x + 5\cos\omega t\,\mathbf{i}_y + 4\sin\omega t\,\mathbf{i}_z\end{aligned} \tag{4-184}$$

$$\begin{aligned}\mathbf{B}_0 &= \mathcal{R}e(\bar{\mathbf{B}}_0e^{j\omega t}) \\ &= \frac{0.1\pi}{\omega}(-3\cos\omega t\,\mathbf{i}_x - 5\sin\omega t\,\mathbf{i}_y + 4\cos\omega t\,\mathbf{i}_z)\end{aligned} \tag{4-185}$$

Substituting (4-184) and (4-185) into

$$\mathbf{P} = \mathbf{E}_0 \times \frac{\mathbf{B}_0}{\mu_0}$$

and simplifying, we obtain the Poynting vector $\mathbf{P}$ as

$$\mathbf{P} = \frac{0.1\pi}{\omega\mu_0}(20\mathbf{i}_x + 15\mathbf{i}_z) \tag{4-186}$$

Now, we note from (4-184) that the direction of $\mathbf{E}_0$ is along $5\mathbf{i}_y$ for $\omega t = 0$ and along $(-3\mathbf{i}_x + 4\mathbf{i}_z)$ for $\omega t = \pi/2$. These two directions and the direction of the Poynting vector are shown in Fig. 4.21. It can be seen that the electric field vector rotates in the clockwise sense as seen looking along the direction of the Poynting vector. Hence the field is right circularly polarized. ▮

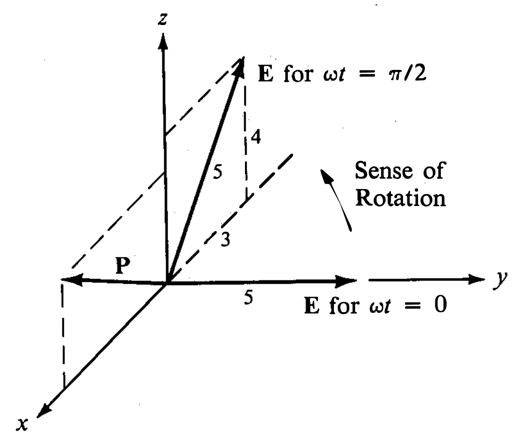

Fig. 4.21. For the determination of the sense of rotation of the circularly polarized vector of Example 4-15.

4.10 Power and Energy Considerations for Sinusoidally Time-Varying Electromagnetic Fields

In Section 4.8 we introduced the Poynting vector **P** given by

$$\mathbf{P} = \mathbf{E} \times \frac{\mathbf{B}}{\mu_0} \tag{4-144}$$

as the power density associated with the electromagnetic field at a point. The surface integral of the Poynting vector evaluated over a closed surface S always gives the correct result for the power flow across the surface out of the volume bounded by it. For a sinusoidally time-varying electromagnetic field characterized by complex field vectors,

$$\bar{\mathbf{E}} = \mathbf{E}_0 e^{j\phi}$$

$$\bar{\mathbf{B}} = \mathbf{B}_0 e^{j\theta}$$

the instantaneous Poynting vector **P** is given by

$$\begin{aligned}
\mathbf{P} &= \mathbf{E} \times \frac{\mathbf{B}}{\mu_0} \\
&= (\mathcal{R}e\, \bar{\mathbf{E}} e^{j\omega t}) \times \left(\frac{1}{\mu_0}\, \mathcal{R}e\, \bar{\mathbf{B}} e^{j\omega t}\right) \\
&= \mathbf{E}_0 \cos(\omega t + \phi) \times \frac{\mathbf{B}_0}{\mu_0} \cos(\omega t + \theta) \\
&= \mathbf{E}_0 \times \frac{\mathbf{B}_0}{\mu_0}[\cos(\omega t + \phi) \cos(\omega t + \theta)] \\
&= \mathbf{E}_0 \times \frac{\mathbf{B}_0}{\mu_0}\left[\frac{1}{2} \cos(2\omega t + \phi + \theta) + \frac{1}{2} \cos(\phi - \theta)\right] \\
&= \frac{1}{2}\mathbf{E}_0 \times \frac{\mathbf{B}_0}{\mu_0} \cos(\phi - \theta) + \frac{1}{2}\mathbf{E}_0 \times \frac{\mathbf{B}_0}{\mu_0} \cos(2\omega t + \phi - \theta)
\end{aligned} \tag{4-187}$$

The first term on the right side of (4-187) is independent of time whereas the second term varies sinusoidally with time. The time-average value of the second term obtained by integrating it through one period T and dividing by the period is equal to zero since the integral of a cosine or sine function over one period is equal to zero. Thus the time-average value of the Poynting vector $\mathbf{P}$, denoted as $\langle \mathbf{P} \rangle$ is given by

$$\begin{aligned}
\langle \mathbf{P} \rangle &= \frac{1}{T}\int_0^T \mathbf{P}\, dt \\
&= \left\langle \frac{1}{2}\mathbf{E}_0 \times \frac{\mathbf{B}_0}{\mu_0} \cos(\phi - \theta) \right\rangle + \left\langle \frac{1}{2}\mathbf{E}_0 \times \frac{\mathbf{B}_0}{\mu_0} \cos(2\omega t + \phi - \theta) \right\rangle \\
&= \frac{1}{2}\mathbf{E}_0 \times \frac{\mathbf{B}_0}{\mu_0} \cos(\phi - \theta) \\
&= \mathcal{R}e\left[\frac{1}{2}\mathbf{E}_0 \times \frac{\mathbf{B}_0}{\mu_0} e^{j(\phi-\theta)}\right] \\
&= \mathcal{R}e\left(\frac{1}{2}\mathbf{E}_0 e^{j\phi} \times \frac{\mathbf{B}_0 e^{-j\theta}}{\mu_0}\right) \\
&= \mathcal{R}e\left(\frac{1}{2}\bar{\mathbf{E}} \times \frac{\bar{\mathbf{B}}^*}{\mu_0}\right)
\end{aligned} \tag{4-188}$$

where $\bar{\mathbf{B}}^*$ denotes the complex conjugate of $\bar{\mathbf{B}}$.

We now define the complex Poynting vector $\bar{\mathbf{P}}$ as

$$\bar{\mathbf{P}} = \frac{1}{2}\bar{\mathbf{E}} \times \frac{\bar{\mathbf{B}}^*}{\mu_0} \tag{4-189}$$

so that the time-average Poynting vector $\langle \mathbf{P} \rangle$ can be written as

$$\langle \mathbf{P} \rangle = \mathcal{R}e(\bar{\mathbf{P}}) \tag{4-190}$$

We note that Eq. (4-189) is analogous to the expression for the complex power in sinusoidal steady-state circuit theory given by

$$\bar{P} = \frac{1}{2}\bar{V}\bar{I}^* \tag{4-191}$$

where $\bar{V}$ and $\bar{I}$ are the complex voltage and complex current, respectively. By integrating the complex Poynting vector over a closed surface S, we obtain the complex power flowing across S out of the volume V bounded by it. Thus

$$\begin{aligned}\oint_S \bar{\mathbf{P}} \cdot d\mathbf{S} &= \oint_S \frac{1}{2}\bar{\mathbf{E}} \times \frac{\bar{\mathbf{B}}^*}{\mu_0} \cdot d\mathbf{S} \\ &= \frac{1}{2\mu_0}\int_V \nabla \cdot (\bar{\mathbf{E}} \times \bar{\mathbf{B}}^*)\, dv\end{aligned} \tag{4-192}$$

where we have used the divergence theorem to replace the surface integral by a volume integral. Now, using the vector identity

$$\nabla \cdot (\bar{\mathbf{E}} \times \bar{\mathbf{B}}^*) = \bar{\mathbf{B}}^* \cdot \nabla \times \bar{\mathbf{E}} - \bar{\mathbf{E}} \cdot \nabla \times \bar{\mathbf{B}}^* \tag{4-193}$$

and Maxwell's curl equations for complex fields given by

$$\nabla \times \bar{\mathbf{E}} = -j\omega\bar{\mathbf{B}} \tag{4-174}$$

$$\nabla \times \bar{\mathbf{B}} = \mu_0(\bar{\mathbf{J}} + j\omega\epsilon_0\bar{\mathbf{E}}) \tag{4-175}$$

we have

$$\begin{aligned}\nabla \cdot (\bar{\mathbf{E}} \times \bar{\mathbf{B}}^*) &= \bar{\mathbf{B}}^* \cdot (-j\omega\bar{\mathbf{B}}) - \bar{\mathbf{E}} \cdot \mu_0(\bar{\mathbf{J}} + j\omega\epsilon_0\bar{\mathbf{E}})^* \\ &= -j\omega\bar{\mathbf{B}}^* \cdot \bar{\mathbf{B}} - \mu_0(\bar{\mathbf{E}} \cdot \bar{\mathbf{J}}^* - j\omega\epsilon_0\bar{\mathbf{E}} \cdot \bar{\mathbf{E}}^*)\end{aligned} \tag{4-194}$$

However, the time-average stored energy density in the electric field is given by

$$\begin{aligned}\langle w_e\rangle &= \left\langle\frac{1}{2}\epsilon_0 E^2\right\rangle \\ &= \left\langle\frac{1}{2}\epsilon_0|\mathbf{E}_0|^2\cos^2(\omega t + \phi)\right\rangle \\ &= \left\langle\frac{1}{4}\epsilon_0|\mathbf{E}_0|^2 + \frac{1}{4}\epsilon_0|\mathbf{E}_0|^2\cos 2(\omega t + \phi)\right\rangle \\ &= \frac{1}{4}\epsilon_0|\mathbf{E}_0|^2 = \frac{1}{4}\epsilon_0\mathbf{E}_0 e^{j\phi} \cdot \mathbf{E}_0 e^{-j\phi} \\ &= \frac{1}{4}\epsilon_0\bar{\mathbf{E}} \cdot \bar{\mathbf{E}}^*\end{aligned} \tag{4-195}$$

Similarly, the time-average stored energy density in the magnetic field is given by

$$\langle w_m\rangle = \left\langle\frac{1}{2}\frac{B^2}{\mu_0}\right\rangle = \frac{1}{4}\bar{\mathbf{B}} \cdot \frac{\bar{\mathbf{B}}^*}{\mu_0} \tag{4-196}$$

The time-average power density expended by the field due to the current flow is given by

$$\langle p_d\rangle = \langle\mathbf{E} \cdot \mathbf{J}\rangle = \mathcal{R}e\left(\frac{1}{2}\bar{\mathbf{E}} \cdot \bar{\mathbf{J}}^*\right) \tag{4-197}$$

so that the complex power density associated with the current flow is given by

$$\bar{p}_d = \frac{1}{2}\bar{\mathbf{E}} \cdot \bar{\mathbf{J}}^* \tag{4-198}$$

Substituting (4-195), (4-196), and (4-198) into (4-194), we get

$$\nabla \cdot (\bar{\mathbf{E}} \times \bar{\mathbf{B}}^*) = -2\mu_0 \bar{p}_d - j4\omega\mu_0(\langle w_m \rangle - \langle w_e \rangle) \tag{4-199}$$

Finally, substituting (4-199) into (4-192), we obtain

$$\oint_S \bar{\mathbf{P}} \cdot d\mathbf{S} = -\int_V \bar{p}_d \, dv - j2\omega \int_V (\langle w_m \rangle - \langle w_e \rangle) \, dv \tag{4-200}$$

Equation (4-200) is known as the complex Poynting's theorem. Equating the real and imaginary parts on both sides of (4-200), we have

$$\mathcal{Re}\left(\int_V \bar{p}_d \, dv\right) = -\mathcal{Re}\left(\oint_S \bar{\mathbf{P}} \cdot d\mathbf{S}\right) \tag{4-201}$$

or

$$\int_V \mathcal{Re}(\bar{p}_d) \, dv = -\oint_S [\mathcal{Re}(\bar{\mathbf{P}})] \cdot d\mathbf{S}$$

or

$$\int_V \langle p_d \rangle \, dv = -\oint_S \langle \mathbf{P} \rangle \cdot d\mathbf{S} \tag{4-202}$$

and

$$\mathcal{Im}\left(\int_V \bar{p}_d \, dv\right) + 2\omega \int_V (\langle w_m \rangle - \langle w_e \rangle) \, dv = -\mathcal{Im}\left(\oint_S \bar{\mathbf{P}} \cdot d\mathbf{S}\right)$$

or

$$2\omega \int_V (\langle w_m \rangle - \langle w_e \rangle) \, dv = -\mathcal{Im}\left(\oint_S \bar{\mathbf{P}} \cdot d\mathbf{S}\right) - \mathcal{Im}\left(\int_V \bar{p}_d \, dv\right) \tag{4-203}$$

Equation (4-202) states that the time-average power expended by the field due to the current flow in the volume V is equal to the time-average power flowing into the volume V as given by the surface integral of the time-average Poynting vector over the surface S bounding V. If $\oint_S \langle \mathbf{P} \rangle \cdot d\mathbf{S}$ is zero, it means that there is no time-average power expended by the field in the volume V; whatever time-average power enters the volume V through part of the surface S leaves through the rest of that surface. Equation (4-203) provides a physical interpretation for the imaginary part of the complex Poynting vector. It relates the difference between the time-average magnetic and electric stored energies in the volume V to the reactive power flowing into the volume V as given by the imaginary part of the surface integral of the complex Poynting vector over the surface S and to the reactive power associated with the current flow in the volume V. We note that the complex Poynting theorem is analogous to a similar relationship in sinusoidal steady-state circuit theory given by

$$\frac{1}{2} \bar{V}\bar{I}^* = \langle P_d \rangle + j2\omega(\langle W_m \rangle - \langle W_e \rangle)$$

where $\langle P_d \rangle$ is the average power dissipated in the resistors, and $\langle W_m \rangle$ and $\langle W_e \rangle$ are the time-average stored energies in the inductors and capacitors, respectively.

TABLE 4.1. Summary of Electromagnetic Field Laws and Formulas

Description	*Arbitrarily Time-Varying Fields*	*Sinusoidally Time-Varying Fields*
Definition	$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$	
Maxwell's equations and the continuity equation in differential form	$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$	$\nabla \cdot \bar{\mathbf{E}} = \frac{\bar{\rho}}{\epsilon_0}$
	$\nabla \cdot \mathbf{B} = 0$	$\nabla \cdot \bar{\mathbf{B}} = 0$
	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\nabla \times \bar{\mathbf{E}} = -j\omega\bar{\mathbf{B}}$
	$\nabla \times \mathbf{B} = \mu_0\left[\mathbf{J} + \frac{\partial}{\partial t}(\epsilon_0\mathbf{E})\right]$	$\nabla \times \bar{\mathbf{B}} = \mu_0(\bar{\mathbf{J}} + j\omega\epsilon_0\bar{\mathbf{E}})$
	$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0$	$\nabla \cdot \bar{\mathbf{J}} + j\omega\bar{\rho} = 0$
Maxwell's equations and the continuity equation in integral form	$\oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\epsilon_0}\int_V \rho\, dv$	$\oint_S \bar{\mathbf{E}} \cdot d\mathbf{S} = \frac{1}{\epsilon_0}\int \bar{\rho}\, dv$
	$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$	$\oint_S \bar{\mathbf{B}} \cdot d\mathbf{S} = 0$
	$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S}$	$\oint_C \bar{\mathbf{E}} \cdot d\mathbf{l} = -j\omega\int_S \bar{\mathbf{B}} \cdot d\mathbf{S}$
	$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0\left(\int_S \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt}\int_S \epsilon_0\mathbf{E} \cdot d\mathbf{S}\right)$	$\oint_C \bar{\mathbf{B}} \cdot d\mathbf{l} = \mu_0\left(\int_S \bar{\mathbf{J}} \cdot d\mathbf{S} + j\omega\int_S \epsilon_0\bar{\mathbf{E}} \cdot d\mathbf{S}\right)$
	$\oint_S \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt}\int_V \rho\, dv = 0$	$\oint_S \bar{\mathbf{J}} \cdot d\mathbf{S} + j\omega\int_V \bar{\rho}\, dv = 0$
Energy density in the electric field	$w_e = \frac{1}{2}\epsilon_0\mathbf{E} \cdot \mathbf{E}$	$\langle w_e \rangle = \frac{1}{4}\epsilon_0\bar{\mathbf{E}} \cdot \bar{\mathbf{E}}^*$
Energy density in the magnetic field	$w_m = \frac{1}{2}\mathbf{B} \cdot \frac{\mathbf{B}}{\mu_0}$	$\langle w_m \rangle = \frac{1}{4}\bar{\mathbf{B}} \cdot \frac{\bar{\mathbf{B}}^*}{\mu_0}$
Power density expended by the field due to current flow	$p_d = \mathbf{E} \cdot \mathbf{J}$	$\langle p_d \rangle = \frac{1}{2}\bar{\mathbf{E}} \cdot \bar{\mathbf{J}}^*$
Poynting vector	$\mathbf{P} = \mathbf{E} \times \frac{\mathbf{B}}{\mu_0}$	$\bar{\mathbf{P}} = \frac{1}{2}\bar{\mathbf{E}} \times \frac{\bar{\mathbf{B}}^*}{\mu_0}$

4.11 Summary of Electromagnetic Field Laws and Formulas

We now summarize in Table 4.1 the basic laws governing the electromagnetic field and the power and energy relations for the electromagnetic field. We recall that all four Maxwell's equations for time-varying fields are not independent. The divergence equation for the magnetic field follows from the curl equation for the electric field as shown in Section 4.3, whereas the divergence equation for the electric field follows from the curl equation for the magnetic field and the continuity equation as shown in Section 4.5.

Comparing Maxwell's equations for time-varying fields with those for the static fields discussed in Chapters 2 and 3, we observe a coupling between the time-varying electric field and the time-varying magnetic field. This is because the curl of the electric field is dependent on the time derivative of the magnetic field and the curl of the magnetic field is dependent on the time derivative of the electric field. Thus the solution for the electric field requires a knowledge of the magnetic field whereas the solution for the magnetic field requires a knowledge of the electric field. The two curl equations must therefore be solved simultaneously to obtain the solution for the electromagnetic field. It is precisely this two-way coupling between the time-varying electric and magnetic fields that gives rise to the phenomenon of electromagnetic wave propagation, as we will learn in Chapter 6.

PROBLEMS

4.1. The forces experienced by a test charge q C at a point in a region of electric and magnetic fields **E** and **B**, respectively, are given as follows for three different velocities:

Velocity, m/sec	*Force, N*
$\mathbf{i}_x$	$q\mathbf{i}_x$
$\mathbf{i}_y$	$q(2\mathbf{i}_x + \mathbf{i}_y)$
$\mathbf{i}_z$	$q(\mathbf{i}_x + \mathbf{i}_y)$

Find **E** and **B** at that point.

4.2. The forces experienced by a test charge q C at a point in a region of electric and magnetic fields **E** and **B**, respectively, are given as follows for three different velocities:

Velocity, m/sec	*Force, N*
$\mathbf{i}_x - \mathbf{i}_y$	0
$\mathbf{i}_x - \mathbf{i}_y + \mathbf{i}_z$	0
$\mathbf{i}_z$	$q(\mathbf{i}_x + \mathbf{i}_y)$

Find **E** and **B** at that point.

4.3. A region is characterized by crossed electric and magnetic fields $\mathbf{E} = E_0\mathbf{i}_y$ and $\mathbf{B} = B_0\mathbf{i}_z$, where E_0 and B_0 are constants. A test charge q having a mass m starts from the origin at $t = 0$ with an initial velocity v_0 in the y direction. Obtain the parametric equations of motion of the test charge. Sketch the path of the test charge.

4.4. A region is characterized by crossed electric and magnetic fields $\mathbf{E} = E_0\mathbf{i}_y$ and $\mathbf{B} = B_0\mathbf{i}_z$, where E_0 and B_0 are constants. A test charge q having a mass m starts from the origin at $t = 0$ with an initial velocity v_0 in the x direction. Obtain the parametric equations of motion of the test charge. Sketch the paths of the test charge for the following cases: (a) $v_0 = 0$, (b) $v_0 = E_0/2B_0$, (c) $v_0 = E_0/B_0$, (d) $v_0 = 2E_0/B_0$, and (e) $v_0 = 3E_0/B_0$.

4.5. A region is characterized by crossed electric and magnetic fields $\mathbf{E} = E_0 \cos \omega t\, \mathbf{i}_y$ and $\mathbf{B} = B_0\mathbf{i}_z$, where E_0 and B_0 are constants. A test charge q having a mass m starts from the origin at $t = 0$ with zero initial velocity. Obtain the parametric equations of motion of the test charge. Check your result with that of Example 4-2 by letting $\omega \longrightarrow 0$. Investigate the limiting case of $\omega \longrightarrow \omega_c$, where ω_c is equal to qB_0/m.

4.6. A region is characterized by crossed electric and magnetic fields given by

$$\mathbf{E} = E_0(-\sin \omega t\, \mathbf{i}_x + \cos \omega t\, \mathbf{i}_y) \qquad \mathbf{B} = B_0\mathbf{i}_z$$

where E_0 and B_0 are constants. A test charge q having a mass m starts from the origin at $t = 0$ with zero initial velocity. Obtain the parametric equations of motion of the test charge. Check your result with that of Example 4-2 by letting $\omega \longrightarrow 0$. Investigate the limiting case of $\omega \longrightarrow \omega_c$, where ω_c is equal to qB_0/m.

4.7. A magnetic field is given, in cylindrical coordinates, by

$$\mathbf{B} = \frac{B_0}{r}\mathbf{i}_\phi$$

where B_0 is a constant. A rectangular loop is situated in the yz plane and parallel to the z axis as shown in Fig. 4.22. If the loop is moving in that plane with a velocity $\mathbf{v} = v_0\mathbf{i}_y$, where v_0 is a constant, find the circulation of the induced electric field around the loop.

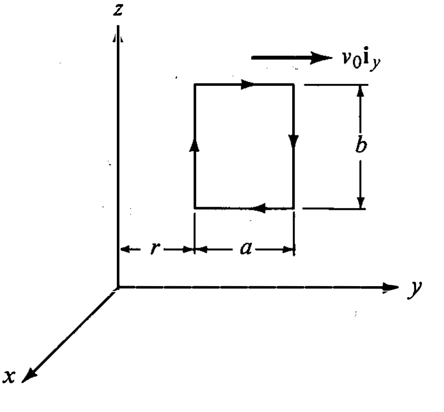

Fig. 4.22. For Problem 4.7.

4.8. For the rectangular loop arrangement of Fig. 4.22, find the circulation of the induced electric field around the loop if the loop is stationary but the magnetic field is varying with time in the manner

$$\mathbf{B} = \frac{B_0}{r} \cos \omega t \, \mathbf{i}_\phi$$

where B_0 is a constant.

4.9. For the rectangular loop arrangement of Fig. 4.22, find the circulation of the induced electric field around the loop if the loop is moving with a velocity $\mathbf{v} = v_0 \mathbf{i}_y$ and if the magnetic field is varying with time in the manner

$$\mathbf{B} = \frac{B_0}{r} \cos \omega t \, \mathbf{i}_\phi$$

where v_0 and B_0 are constants.

4.10. For each of the following magnetic fields, find the induced electric field everywhere, by using Faraday's law in integral form:

(a) $$\mathbf{B} = \begin{cases} B_0 \sin \omega t \, \mathbf{i}_z & |x| < a \\ 0 & |x| > a \end{cases}$$

(b) $$\mathbf{B} = \begin{cases} 0 & r < a \\ B_0 \sin \omega t \, \mathbf{i}_z & a < r < b \\ 0 & r > b \end{cases}$$

(c) $$\mathbf{B} = \begin{cases} B_0\left(1 - \frac{r^2}{a^2}\right) \sin \omega t \, \mathbf{i}_z & r < a \\ 0 & r > a \end{cases}$$

where B_0 is a constant.

4.11. In a region characterized by a magnetic field $\mathbf{B} = B_0 \mathbf{i}_z$, where B_0 is a constant, a test charge q having a mass m is moving along a circular path of radius a and in the xy plane. Find the electric field as viewed by an observer moving with the test charge.

4.12. A region is characterized by crossed electric and magnetic fields $\mathbf{E} = E_0 \mathbf{i}_y$ and $\mathbf{B} = B_0 \mathbf{i}_z$, where E_0 and B_0 are constants. A test charge q having a mass m starts from the origin at $t = 0$ with an initial velocity $\mathbf{v} = (E_0/B_0)\mathbf{i}_x$. Find the electric field as viewed by an observer moving with the test charge.

4.13. Verify your answer to Problem 4.9 by using (4-43).

4.14. Verify your answers to Problem 4.10 by using Faraday's law in differential form.

4.15. A current I C/sec flows from a point charge Q_1 C situated at $(0, 0, -d)$ to a point charge Q_2 C situated at $(0, 0, d)$ along a straight filamentary wire as shown in Fig. 4.23. Find $\oint_C \mathbf{B} \cdot d\mathbf{l}$, where C is a circular path centered at $(0, 0, z)$ and lying in the plane normal to the z axis, in two ways: (a) by applying the Biot–Savart law to find the magnetic field due to the current-carrying wire and (b) by applying the modified Ampere's circuital law in integral form to the path C.

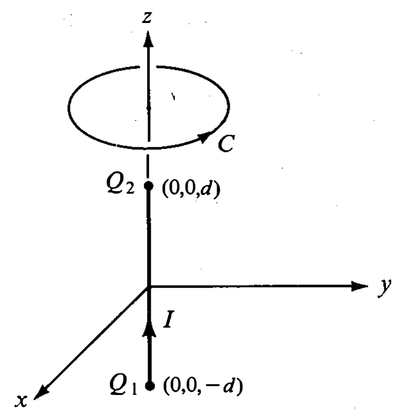

Fig. 4.23. For Problem 4.15.

4.16. Current flows away from a point charge Q C at the origin radially on the xy plane with density given by

$$\mathbf{J}_s = \frac{I}{2\pi r}\mathbf{i}_r \text{ amps/m}$$

Find $\oint_C \mathbf{B} \cdot d\mathbf{l}$ where C is a circular path centered at $(0, 0, z)$ and lying in the plane normal to the z axis in two ways: (a) by applying the Biot–Savart law to find the magnetic field due to the surface current and (b) by applying the modified Ampere's circuital law in integral form to the path C.

4.17. Current flows from a point charge Q_1 C at $(0, 0, a)$ to a point charge Q_2 coulombs at $(0, 0, -a)$ along a spherical surface of radius a and centered at the origin with density given by

$$\mathbf{J}_s = \frac{I}{2\pi a \sin\theta}\mathbf{i}_\theta \text{ amp/m}$$

Find $\oint_C \mathbf{B} \cdot d\mathbf{l}$, where C is a circular path centered at $(0, 0, z)$ and lying in the plane normal to the z axis. Consider both cases: path C outside the sphere and path C inside the sphere.

4.18. A point charge Q C moves along the z axis with a constant velocity v_0 m/sec. Assuming that the point charge crosses the origin at $t = 0$, find and sketch the variation with time of $\oint_C \mathbf{B} \cdot d\mathbf{l}$ where C is a circular path of radius a in the xy plane having its center at the origin, and traversed in the ϕ direction. From symmetry considerations, find $\mathbf{B}$ at points on C.

4.19. A point charge Q_1 C is situated at the origin. Current flows away from the point charge at the rate of I C/sec along a straight wire from the origin to infinity and passing through the point (1, 1, 1). Find $\oint \mathbf{B} \cdot d\mathbf{l}$ around the closed path formed by the triangle having the vertices (1, 0, 0), (0, 1, 0), and (0, 0, 1). Assume that the closed path is traversed in the clockwise direction as seen from the origin.

4.20. Repeat Prob. 4-19 if the straight wire, instead of extending to infinity, terminates on another point charge Q_2 C situated on the plane surface bounded by the triangular path and inside the closed path.

4.21. In the arrangement shown in Fig. 4.24, three point charges Q_1, Q_2, and Q_3 are situated along a straight line. A current of 2 amp flows from Q_1 to Q_2 whereas a current of 1 amp flows from Q_2 to Q_3. Find $\oint_C \mathbf{B} \cdot d\mathbf{l}$, where C is a circular path centered at Q_2 and in the plane normal to the line joining Q_1 to Q_3.

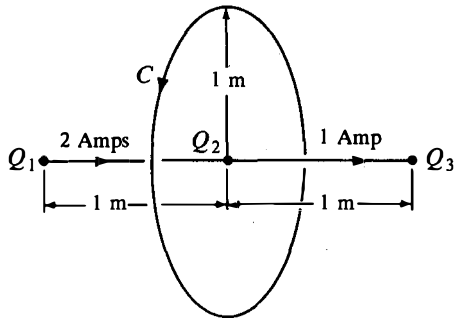

Fig. 4.24. For Problem 4.21.

4.22. Verify your result for the magnetic field due to the current-carrying wire of Problem 4.15, by using (4-96).

4.23. Verify your result for the magnetic field due to the moving charge of Problem 4.18, by using (4-96).

4.24. In a region containing no charges and currents, the magnetic field is given by

$$\mathbf{B} = B_0 \sin \beta z \sin \omega t \, \mathbf{i}_x$$

where B_0, β, and ω are constants. Using one of Maxwell's curl equations at a time, find two expressions for the associated electric field $\mathbf{E}$ and then find the relationship between β, ω, μ_0, and ϵ_0.

4.25. Four point charges having values 1, −2, 3, and 4 C are situated at the corners of a square of sides 1 m as shown in Fig. 4.25. Find the work required to move the point charges to the corners of a smaller square of sides $1/\sqrt{2}$ m.

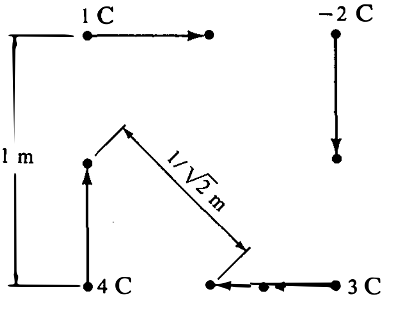

Fig. 4.25. For Problem 4.25.

4.26. Find the potential energy associated with the following volume charge distributions of density ρ in spherical coordinates using $W_e = \frac{1}{2}\int_{\text{vol}} \rho V \, dv$:

(a) $\rho = \begin{cases} 0 & 0 < r < a \\ \rho_0 & a < r < b \\ 0 & b < r < \infty \end{cases}$

(b) $\rho = \begin{cases} \rho_0 \dfrac{r}{a} & 0 < r < a \\ 0 & a < r < \infty \end{cases}$

where ρ_0 is a constant.

4.27. Verify your results for Problem 4.26 by performing volume integration of the electric energy densities associated with the charge distributions.

4.28. Two spherical charges, each of the same radius a m and the same uniform density ρ_0 C/m^3 are situated infinitely apart.

(a) The two spherical charges are now brought together and made into a single spherical charge having the same uniform density ρ_0 C/m^3 as those of the original charges. Find the work required.

(b) Instead of as in part (a), the two spherical charges are brought together and made into a single spherical charge of uniform density and of the same radius a as those of the original charges. Find the work required.

4.29. Show that the total energy stored in an electric field made up of two fields $\mathbf{E}_1$ and $\mathbf{E}_2$ is equal to the sum of the energies stored in the individual fields plus a coupling term, $\epsilon_0 \int_{\text{vol}} (\mathbf{E}_1 \cdot \mathbf{E}_2)\, dv$, that is,

$$W_e = \int_{\text{vol}} \left(\frac{1}{2}\epsilon_0 E_1^2 + \frac{1}{2}\epsilon_0 E_2^2 + \epsilon_0 \mathbf{E}_1 \cdot \mathbf{E}_2\right) dv$$

4.30. Find the energy stored in the electric field set up by charges Q and $-Q$ uniformly distributed on concentric spherical surfaces of radii a and b, respectively, in three ways:

(a) By using $W_e = \frac{1}{2}\int_{\text{vol}} \rho V\, dv$.

(b) By performing volume integration of the energy density in the electric field set up by the charge distribution.

(c) By considering the electric field as the superposition of the fields set up independently by the two spherical surface charges and using the result of Problem 4.29.

4.31. Find the energy associated with the following current distributions, in cylindrical coordinates, per unit length along the z axis, by using $W_m = \frac{1}{2}\int_{\text{vol}} \mathbf{J} \cdot \mathbf{A}\, dv$.

(a) $\mathbf{J} = \begin{cases} \dfrac{I_0}{\pi a^2}\mathbf{i}_z & 0 < r < a \\ 0 & a < r < b \\ -\dfrac{I_0}{\pi(c^2 - b^2)}\mathbf{i}_z & b < r < c \\ 0 & c < r < \infty \end{cases}$

(b) $\mathbf{J} = \begin{cases} J_0 \dfrac{r}{a}\mathbf{i}_z & 0 < r < a \\ -\dfrac{J_0 a^2}{3b}\delta(r - b)\,\mathbf{i}_z & a < r < \infty \end{cases}$

where I_0 and J_0 are constants.

4.32. Find the energy associated with the following current distributions, per unit area in the $y = 0$ plane, by using $W_m = \frac{1}{2}\int_{\text{vol}} \mathbf{J} \cdot \mathbf{A}\, dv$.

(a) $\mathbf{J} = J_{s0}[\delta(y + a) - \delta(y - a)]\, \mathbf{i}_z$, where J_{s0} is a constant

(b) $\mathbf{J} = \begin{cases} y\mathbf{i}_z & |y| < a \\ 0 & |y| > a \end{cases}$

4.33. Verify your results for Problems 4.31 and 4.32 by performing volume integration of the magnetic energy densities associated with the current distributions.

4.34. Show that the total energy stored in a magnetic field made up of two fields $\mathbf{B}_1$ and $\mathbf{B}_2$ is equal to the sum of the energies stored in the individual fields plus a coupling term, $(1/\mu_0)\int_{\text{vol}} (\mathbf{B}_1 \cdot \mathbf{B}_2)\, dv$, that is,

$$W_m = \int_{\text{vol}} \left(\frac{B_1^2}{2\mu_0} + \frac{B_2^2}{2\mu_0} + \frac{\mathbf{B}_1 \cdot \mathbf{B}_2}{\mu_0}\right) dv$$

4.35. A surface current distribution is given, in cylindrical coordinates, by

$$\mathbf{J}_s = \begin{cases} \dfrac{I_1}{a}\mathbf{i}_z & r = a \\[2ex] \dfrac{I_2}{b}\mathbf{i}_z & r = b \\[2ex] -\dfrac{I_1 + I_2}{c}\mathbf{i}_z & r = c \end{cases}$$

Find the energy stored in the magnetic field, set up by the current distribution, per unit length along the z axis in three ways:

(a) By using $W_m = \frac{1}{2}\int_S \mathbf{J}_s \cdot \mathbf{A}\, dS$.

(b) By performing volume integration of the energy density in the magnetic field set up by the current distribution.

(c) By considering the magnetic field as the superposition of the fields set up independently by two surface current distributions given by

$$\mathbf{J}_s = \begin{cases} \dfrac{I_1}{a}\,\mathbf{i}_z & r = a \\[2ex] -\dfrac{I_1}{c}\,\mathbf{i}_z & r = c \end{cases}$$

$$\mathbf{J}_s = \begin{cases} \dfrac{I_2}{b}\,\mathbf{i}_z & r = b \\[2ex] -\dfrac{I_2}{c}\,\mathbf{i}_z & r = c \end{cases}$$

and using the result of Problem 4.34.

4.36. An electric field intensity vector is given by

$$\mathbf{E} = 100 \cos(\omega t - \beta z)\, \mathbf{i}_x + 50 \sin(\omega t + \beta z)\, \mathbf{i}_y$$

where ω and β $(= \omega\sqrt{\mu_0\epsilon_0})$ are constants. Find the associated magnetic flux density vector $\mathbf{B}$. Find the Poynting vector $\mathbf{E} \times \mathbf{B}/\mu_0$.

4.37. Electric and magnetic fields are given in cylindrical coordinates by

$$\mathbf{E} = \begin{cases} \dfrac{V_0}{r \ln b/a} \cos \beta z \cos \omega t \, \mathbf{i}_r & a < r < b \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbf{B} = \begin{cases} \dfrac{\mu_0 I_0}{2\pi r} \sin \beta z \sin \omega t \, \mathbf{i}_\phi & a < r < b \\ 0 & \text{otherwise} \end{cases}$$

where V_0, I_0, ω, and β $(= \omega\sqrt{\mu_0 \epsilon_0})$ are constants. Find the expression for the power leaving the volume bounded by two constant z planes, one of which is the $z = 0$ plane. Draw a graph of the power versus z for $\omega t = \pi/4$.

4.38. In the region $r < a$ in cylindrical coordinates, charges are in motion under the combined influence of an electric field $\mathbf{E} = E_0 \mathbf{i}_z$ and a frictional mechanism, thereby constituting a current of density $\mathbf{J} = J_0 \mathbf{i}_z$, where E_0 and J_0 are constants. Obtain the magnetic field due to the current and show that $\mathbf{E} \times \mathbf{B}$ points everywhere towards the z axis, that is, in the $-\mathbf{i}_r$ direction. Show that $\oint_S (\mathbf{E} \times \mathbf{B}/\mu_0) \cdot d\mathbf{S}$, where S is the surface of a cylindrical volume of any radius r and length l, and with the z axis as its axis, gives the correct result for the power expended by the electric field in that volume.

4.39. The electric field intensity in the radiation field of an antenna located at the origin of a spherical coordinate system is given by

$$\mathbf{E} = E_0 \frac{\sin \theta \cos \theta}{r} \cos (\omega t - \beta r) \, \mathbf{i}_\theta$$

where E_0, ω, and β $(= \omega\sqrt{\mu_0 \epsilon_0})$ are constants. Find the magnetic field associated with this electric field and then find the power radiated by the antenna by integrating the Poynting vector over a spherical surface of radius r centered at the origin.

4.40. Obtain the steady-state solution for the following differential equation in two ways: (a) without using the phasor technique, and (b) by using the phasor technique:

$$2 \times 10^{-3} \frac{dV}{dt} + V = 10 \sin \left(500t + \frac{\pi}{6}\right)$$

4.41. Repeat Problem 4.40 for the following integrodifferential equation:

$$\frac{dI}{dt} + 2I + \int I \, dt = 10 \cos \left(2t - \frac{\pi}{3}\right)$$

4.42. Two infinitely long, straight parallel wires carry currents $I_1 = I_0 \cos \omega t$ and $I_2 = I_0 \cos (\omega t + 90°)$ amp, respectively, as shown in Fig. 4.26. Determine the x and y components of the magnetic flux density vector at each of the three points A, B, and C. Describe how the magnitude and direction of the magnetic flux density vector varies with time at each of the three points A, B, and C.

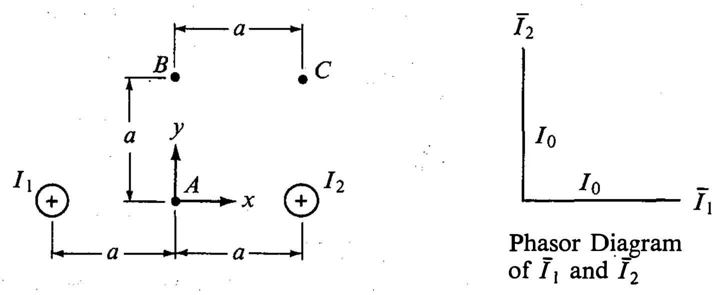

Fig. 4.26. For Problem 4.42.

4.43. In the arrangement shown in Fig. 4.27(a), four line charges, infinitely long in the direction normal to the plane of the paper and having uniform charge densities varying sinusoidally with time are situated at the corners of a square. The amplitudes of the sinusoidally time-varying charge densities are such that, considered alone, each line charge produces unit peak electric field intensity at the center of the square. The phasor diagram of the charge densities is shown in Fig. 4.27(b).

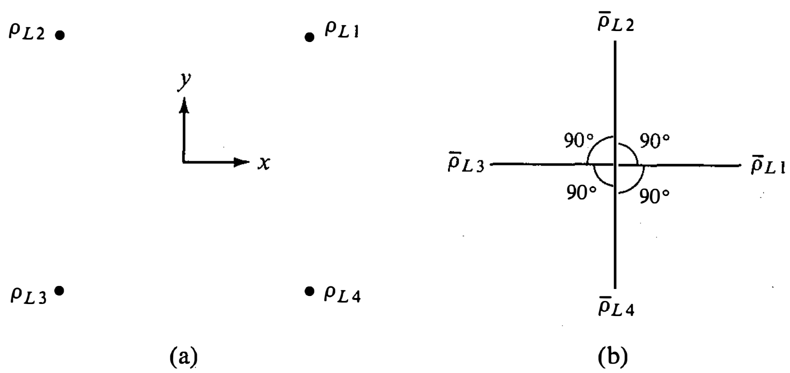

Fig. 4.27. For Problem 4.43.

(a) Find and sketch the phasor representing the x and y components of the electric field intensity vector at the center of the square.
(b) Determine how the magnitude and direction of the electric field intensity vector at the center of the square vary with time.

4.44. Repeat Problem 4.43 for the rectangular arrangement of line charges shown in Fig. 4.28.

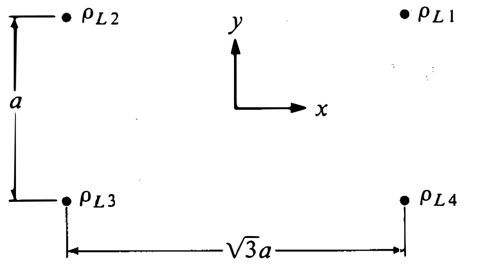

Fig. 4.28. For Problem 4.44.

4.45. A sinusoidally time-varying electric field intensity vector is characterized by its phasor $\bar{\mathbf{E}}$, given by

$$\bar{\mathbf{E}} = (-\mathbf{i}_x - 2\sqrt{3}\,\mathbf{i}_y + \sqrt{3}\,\mathbf{i}_z)e^{-j0.04\pi(\sqrt{3}x-2y-3z)}$$

(a) Show that the surfaces of constant phase of $\bar{\mathbf{E}}$ are planes. Find the equation of the planes.
(b) Show that the electric field is linearly polarized in the planes of constant phase.
(c) Find the direction of polarization.

4.46. A sinusoidally time-varying electric field intensity vector is characterized by its phasor $\bar{\mathbf{E}}$, given by

$$\bar{\mathbf{E}} = (-j1\mathbf{i}_x - 2\mathbf{i}_y + j\sqrt{3}\,\mathbf{i}_z)e^{-j0.05\pi(\sqrt{3}x+z)}$$

(a) Show that the surfaces of constant phase of $\bar{\mathbf{E}}$ are planes. Find the equation of the planes.
(b) Show that the electric field is circularly polarized in the planes of constant phase.
(c) Obtain the magnetic flux density phasor $\bar{\mathbf{B}}$ associated with the given $\bar{\mathbf{E}}$ and determine if the field is right circularly polarized or left circularly polarized.

4.47. Repeat Problem 4.46 for the following phasor electric field intensity vector:

$$\bar{\mathbf{E}} = \left[\left(-\sqrt{3} - j\frac{1}{2}\right)\mathbf{i}_x + \left(1 - j\frac{\sqrt{3}}{2}\right)\mathbf{i}_y + j\sqrt{3}\,\mathbf{i}_z\right]e^{-j0.02\pi(\sqrt{3}x+3y+2z)}$$

4.48. Show that a linearly polarized field vector can be expressed as the sum of left and right circularly polarized field vectors having equal magnitudes, and that an elliptically polarized field vector can be expressed as the sum of left and right circularly polarized field vectors having unequal magnitudes.

4.49. Find the time-average stored energy density in the electric field characterized by the phasor specified in Problem 4.47.

4.50. The electric field associated with a sinusoidally time-varying electromagnetic field is given by

$$\mathbf{E}(x, y, z, t) = 10 \sin \pi x \sin (6\pi \times 10^8 t - \sqrt{3}\pi z)\,\mathbf{i}_y \text{ volts/m}$$

Find (a) the time-average stored energy density in the electric field, (b) the time-average stored energy density in the magnetic field, (c) the time-average Poynting vector associated with the electromagnetic field, and (d) the imaginary part of the complex Poynting vector.