3

THE STATIC MAGNETIC FIELD

In Chapter 2 we introduced the electric field as a force field associated with a region of space in which charges at rest experience forces. In this chapter we introduce a second kind of force field, known as the magnetic field and associated with a region in which charges in motion experience forces. These forces experienced by moving charges are in addition to any electric forces experienced by them by virtue of an electric field in the region. Just as we were concerned only with the static electric field in free space in Chapter 2, we are in this chapter concerned only with the static magnetic field in free space. We know that the motion of charges constitutes a current. Currents are, however, classified into different categories according to how they are produced. Currents arising from movement of charges such as space charges in vacuum tubes and electron beams in cathode-ray tubes are called convection currents. Two other types of current known as conduction and polarization currents result from different effects on charges in material media under the influence of electric fields, as we will learn in Chapter 5. Yet another type of current is the magnetization current which results from magnetic effects in materials, as we will learn also in Chapter 5. For the purposes of this chapter, it is not necessary to distinguish between them because they are all basically equivalent to rate of flow of charges with time in free space. Thus the laws which we will learn in this chapter can be applied equally well to all of these currents.

3.1 The Magnetic Field Concept

In Section 2.1 we learned that if, in a region of space, a fixed test charge q experiences a force $\mathbf{F}$, then the region is characterized by an electric field of intensity $\mathbf{E}$ given by

$$\mathbf{E} = \frac{\mathbf{F}}{q} \tag{2-3}$$

Here we introduce the concept of magnetic field by considering a test charge moving in a region of space. If the test charge q moving with a velocity $\mathbf{v}$ experiences a force $\mathbf{F}$, then the region is said to be characterized by a magnetic field, which we will represent by the symbol $\mathbf{B}$. This force $\mathbf{F}$ is related to q, $\mathbf{v}$, and $\mathbf{B}$ as given by

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} \tag{3-1}$$

According to (3-1), the force experienced by the moving charge due to the magnetic field is directed normal to both $\mathbf{v}$ and $\mathbf{B}$, as shown in Fig. 3.1, in

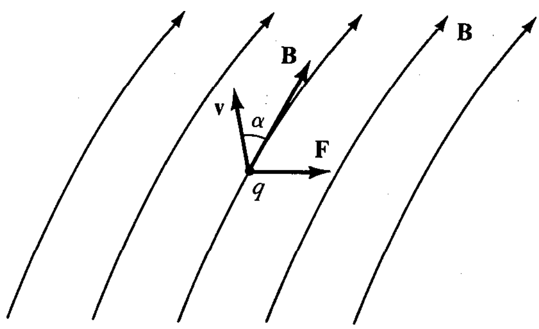

Fig. 3.1. Force experienced by a test charge moving with a velocity **v** in a magnetic field **B**.

contrast to the same directions of electric force and electric field intensity. The magnitude of the force is equal to $qvB \sin \alpha$, where α is the angle between $\mathbf{v}$ and $\mathbf{B}$. Since the force is always normal to $\mathbf{v}$, there is no acceleration along the direction of motion. Thus, the magnetic field changes only the direction of motion of the charge and does not alter the kinetic energy associated with it.

From Eq. (3-1), we note that if the test charge moves in, or opposite to, the direction of $\mathbf{B}$, it does not experience a force. Also, rewriting Eq. (3-1) as

$$\mathbf{F} = qvB\, \mathbf{i}_v \times \mathbf{i}_B = qvB \sin \alpha\, \mathbf{i}_F \tag{3-2}$$

where $\mathbf{i}_v$, $\mathbf{i}_B$, and $\mathbf{i}_F$ are unit vectors along $\mathbf{v}$, $\mathbf{B}$, and $\mathbf{F}$, respectively, we observe that it is only possible to deduce $B \sin \alpha$ by knowing the force for only one direction of motion of the test charge. On the other hand, if we know two nonzero forces $\mathbf{F}_1$ and $\mathbf{F}_2$ for two velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ in different directions, then we have

$$\begin{aligned} \mathbf{F}_1 \times \mathbf{F}_2 &= (q\mathbf{v}_1 \times \mathbf{B}) \times (q\mathbf{v}_2 \times \mathbf{B}) \\ &= q^2[(\mathbf{v}_1 \times \mathbf{B} \cdot \mathbf{B})\mathbf{v}_2 - (\mathbf{v}_1 \times \mathbf{B} \cdot \mathbf{v}_2)\mathbf{B}] \\ &= -q(\mathbf{F}_1 \cdot \mathbf{v}_2)\mathbf{B} \end{aligned} \tag{3-3}$$

or

$$\mathbf{B} = \frac{\mathbf{F}_2 \times \mathbf{F}_1}{q(\mathbf{F}_1 \cdot \mathbf{v}_2)} \tag{3-4}$$

Alternatively, we note from (3-1) or (3-2) that the force is maximum for $\mathbf{v}$ normal to $\mathbf{B}$ so that if we find a maximum force $\mathbf{F}_m$ by trying several directions of $\mathbf{v}$, keeping its magnitude constant, then

$$\mathbf{B} = \frac{\mathbf{F}_m \times \mathbf{i}_m}{qv} \tag{3-5}$$

where $\mathbf{i}_m$ is the direction of $\mathbf{v}$ for which the force is $\mathbf{F}_m$.

As in the case of defining the electric field, we assume that the movement of the test charge does not alter the magnetic field in which it is placed. From a practical point of view, the movement of the charge does influence the magnetic field irrespective of how small it is and how slowly it is moved. However, theoretically, we can define $\mathbf{B}$ as the right side of (3-5) in the limit that qv tends to zero; that is,

$$\mathbf{B} = \lim_{qv \to 0} \frac{\mathbf{F}_m \times \mathbf{i}_m}{qv} \tag{3-6}$$

From (3-5), we observe that the units of $\mathbf{B}$ are

$$\frac{\text{newtons per coulomb}}{\text{meters per second}} = \frac{\text{newton-seconds}}{\text{coulomb-meter}} = \frac{\text{newton-meter}}{\text{coulomb}} \times \frac{\text{seconds}}{(\text{meter})^2}$$

Recalling that newton-meter per coulomb is a volt, we can write these units as volt-seconds per square meter, commonly known as webers per square meter, and abbreviated Wb/m^2, giving the character of a flux density for $\mathbf{B}$. Accordingly, $\mathbf{B}$ is known as the magnetic flux density vector.

EXAMPLE 3-1. An electron moving with a velocity $\mathbf{v}_1 = \mathbf{i}_x$ m/sec at a point in a magnetic field experiences a force $\mathbf{F}_1 = e(-\mathbf{i}_y + \mathbf{i}_z)$ N, where e is the charge of the electron. If the electron is moving with a velocity $\mathbf{v}_2 = \mathbf{i}_y$ m/sec at the same point, it experiences a force $\mathbf{F}_2 = e(\mathbf{i}_x - \mathbf{i}_z)$ N. Find $\mathbf{B}$ at that point.

Using (3-4), we have

$$\begin{aligned} \mathbf{B} = \frac{\mathbf{F}_2 \times \mathbf{F}_1}{q(\mathbf{F}_1 \cdot \mathbf{v}_2)} &= \frac{e(\mathbf{i}_x - \mathbf{i}_z) \times e(-\mathbf{i}_y + \mathbf{i}_z)}{e[e(-\mathbf{i}_y + \mathbf{i}_z) \cdot \mathbf{i}_y]} \\ &= \frac{e^2(-\mathbf{i}_x - \mathbf{i}_y - \mathbf{i}_z)}{-e^2} \\ &= (\mathbf{i}_x + \mathbf{i}_y + \mathbf{i}_z) \text{ Wb/m}^2 \quad \blacksquare \end{aligned}$$

3.2 Force on a Current Element

In Section 3.1 we defined the magnetic field in terms of force experienced by a moving test charge, involving explicitly the charge and its velocity. This form of the definition, given by Eq. (3-1) is, however, not convenient for use with currents. Hence it is necessary to formulate Eq. (3-1) in terms of current. The current crossing a surface is defined as the rate at which charge flows across the surface; that is,

$$I = \frac{dQ}{dt} \tag{3-7}$$

Let us now consider a region in which charges distributed with a density ρ are moving with a velocity $\mathbf{v}$, where ρ and $\mathbf{v}$ can, in general, be nonuniform. At a point P in this region, let us consider an infinitesimal area dS normal to the direction of flow of charges as shown in Fig. 3.2. In a time dt, the

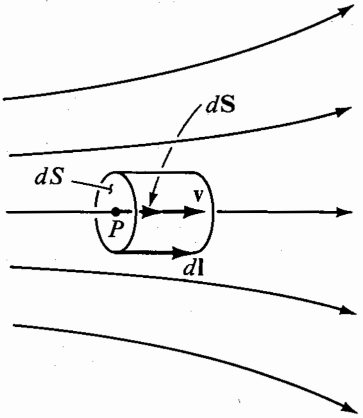

Fig. 3.2. Volume occupied by charge crossing a surface dS normal to it with a velocity $\mathbf{v}$, in time $dt = dl/v$.

distance traveled by the charge crossing this surface is equal to $\mathbf{v}\,dt$. Let $\mathbf{v}\,dt$ be equal to $d\mathbf{l}$, as shown in Fig. 3.2, so that the charge dQ crossing the surface dS in time dt is that contained in the infinitestimal volume $(d\mathbf{l} \cdot d\mathbf{S})$. The current crossing the surface dS is then given by

$$I = \frac{dQ}{dt} = \frac{dQ}{dl} v \tag{3-8}$$

But the current crossing the surface is also equal to $\mathbf{J} \cdot d\mathbf{S}$, where $\mathbf{J}$ is the current density at P. Since $\mathbf{J}$ and $d\mathbf{S}$ are in the same direction, $\mathbf{J} \cdot d\mathbf{S} = J\,dS$. Thus

$$J = \frac{I}{dS} = \frac{dQ}{(dl)(dS)} v = \rho v \tag{3-9}$$

and

$$\mathbf{J} = \rho \mathbf{v} \tag{3-10}$$

Now, the force experienced by the charge dQ moving with the velocity $\mathbf{v}$ is given by

$$\begin{aligned} d\mathbf{F} &= dQ\, \mathbf{v} \times \mathbf{B} \\ &= \rho(dl)(dS)\mathbf{v} \times \mathbf{B} \\ &= (d\mathbf{l} \cdot d\mathbf{S})\rho \mathbf{v} \times \mathbf{B} \\ &= \mathbf{J} \times \mathbf{B}\, d(\text{vol}) \end{aligned} \tag{3-11}$$

where $d(\text{vol})$ is the differential volume $(d\mathbf{l} \cdot d\mathbf{S})$. Thus the magnetic force experienced by the charges in a differential volume in a region of current is given by (3-11). To obtain the total force experienced in a large volume, we need to integrate the right side of (3-11) throughout the volume under consideration; that is,

$$\mathbf{F} = \int_{\text{vol}} \mathbf{J} \times \mathbf{B}\, d(\text{vol}) \tag{3-12}$$

For a filamentary wire carrying current I, the current density J is infinity since dS is zero but the product $\mathbf{J} \cdot d\mathbf{S}$ is equal to I so that (3-11) becomes

$$\begin{aligned} d\mathbf{F} &= (dl)\,(dS)\,\mathbf{J} \times \mathbf{B} \\ &= (\mathbf{J} \cdot d\mathbf{S})d\mathbf{l} \times \mathbf{B} \\ &= I\, d\mathbf{l} \times \mathbf{B} \end{aligned} \tag{3-13}$$

as illustrated in Fig. 3.3. The total force experienced by the filamentary wire is obtained by integrating the right side of (3-13) along the length of the wire. Thus

$$\mathbf{F} = \int_{\text{wire}} (I\, d\mathbf{l} \times \mathbf{B}) = I \int_{\text{wire}} (d\mathbf{l} \times \mathbf{B}) \tag{3-14}$$

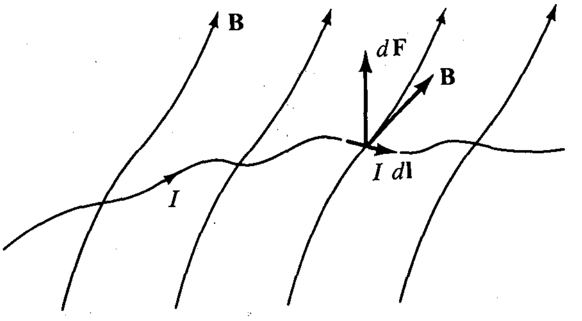

Fig. 3.3. Illustrating the force experienced by an infinitesimal segment of a filamentary wire carrying current I in a magnetic field $\mathbf{B}$.

EXAMPLE 3-2. Show that the total magnetic force experienced by a closed loop of wire carrying a current I in a uniform magnetic field $\mathbf{B}$ is equal to zero.

Applying (3-14) for the contour C of the wire, we have

$$\mathbf{F} = I\oint_C (d\mathbf{l} \times \mathbf{B}) = I\left(\oint_C d\mathbf{l}\right) \times \mathbf{B} \tag{3-15}$$

where, since $\mathbf{B}$ is uniform, we have taken it outside the integral on the right side of (3-15). Now,

$$\begin{aligned}\oint_C d\mathbf{l} &= \oint_C (dx\,\mathbf{i}_x + dy\,\mathbf{i}_y + dz\,\mathbf{i}_z) \\ &= \left(\oint_C dx\right)\mathbf{i}_x + \left(\oint_C dy\right)\mathbf{i}_y + \left(\oint_C dz\right)\mathbf{i}_z = 0\end{aligned} \tag{3-16}$$

Hence $\mathbf{F} = 0$. ▮

3.3 Ampere's Law of Force

In Chapter 2 the concept of electric field was introduced in terms of force experienced by a small test charge placed in the presence of a larger charge in analogy with the gravitational force associated with two masses. We then presented an experimental law known as Coulomb's law and obtained from it the expression for the electric field intensity of a point charge. Just as static charges which are influenced by electric fields are themselves sources of electric fields, moving charges or currents which are influenced by magnetic fields are themselves sources of magnetic fields. To demonstrate this, we will in this section present an experimental law known as Ampere's law of force, analogous to Coulomb's law, and use it in the next section to obtain the expression for the magnetic field due to a current element.

Ampere's law of force is concerned with the forces experienced by two loops of wire carrying currents I_1 and I_2, as shown in Fig. 3.4. As a result of

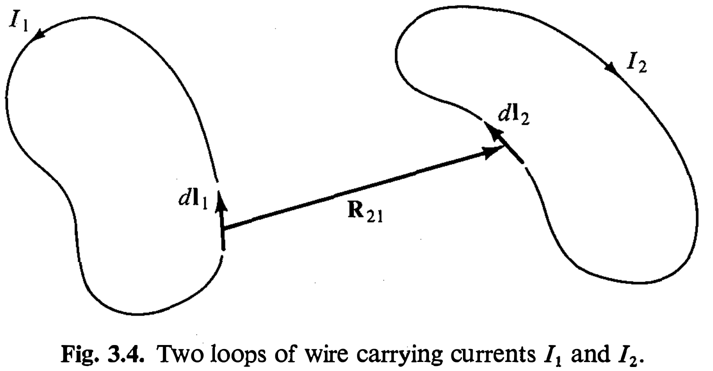

Fig. 3.4. Two loops of wire carrying currents I_1 and I_2.

experimental findings by Ampere, the force experienced by current loop 2 is given by

$$\mathbf{F}_{21} = k \oint_{C1} \oint_{C2} \frac{I_2\, d\mathbf{l}_2 \times (I_1\, d\mathbf{l}_1 \times \mathbf{R}_{21})}{R_{21}^3} \tag{3-17}$$

where C_1 and C_2 are the contours of loops 1 and 2, respectively, k is a constant of proportionality, and $\mathbf{R}_{21}$ is the vector drawn from a differential length element $d\mathbf{l}_1$ in loop 1 to a differential length element $d\mathbf{l}_2$ in loop 2. The constant of proportionality k is equal to $\mu_0/4\pi$ for free space and in the MKS system of units. The quanity μ_0 is known as the permeability of free space and is equal to $4\pi \times 10^{-7}$. Since (3-17) is valid for any orientation of the current loops, it follows that the differential force $d\mathbf{F}_{21}$ experienced by the differential current element $I_2\, d\mathbf{l}_2$ due to the differential current element $I_1\, d\mathbf{l}_1$ is

$$d\mathbf{F}_{21} = \frac{\mu_0}{4\pi} \frac{I_2\, d\mathbf{l}_2 \times (I_1\, d\mathbf{l}_1 \times \mathbf{R}_{21})}{R_{21}^3} \tag{3-18}$$

where we have substituted $\mu_0/4\pi$ for k. From (3-18), we note that μ_0 has the units newtons per ampere squared. These are commonly known as henrys per meter. Recalling that the permittivity of free space, ϵ_0, is equal to $10^{-9}/36\pi$ C^2/N-m^2, we note that

$$\begin{aligned} \frac{1}{\sqrt{\mu_0 \epsilon_0}} &= \frac{1}{\sqrt{4\pi \times 10^{-7} \times (10^{-9}/36\pi)}} \text{ amp-m/C} \\ &= 3 \times 10^8 \text{ m/sec} \end{aligned} \tag{3-19}$$

which is the velocity of light in free space.

Some of the features evident from Eq. (3-18) are as follows:

(a) The magnitude of the force is proportional to the product of the magnitudes of the currents.

(b) The magnitude of the force is inversely proportional to the square of the distance between the current elements.

(c) To determine the direction of the force, we first find the cross product $d\mathbf{l}_1 \times \mathbf{R}_{21}$ and then cross $d\mathbf{l}_2$ into the resulting vector. The parenthesis on the right side of (3-18) is very important since, for a triple cross product, $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$.

By interchanging $I_1\, d\mathbf{l}_1$ and $I_2\, d\mathbf{l}_2$ and replacing $\mathbf{R}_{21}$ by $\mathbf{R}_{12}$ in (3-18), we obtain the expression for the force experienced by $I_1\, d\mathbf{l}_1$ due to $I_2\, d\mathbf{l}_2$ as

$$d\mathbf{F}_{12} = \frac{\mu_0}{4\pi} \frac{I_1\, d\mathbf{l}_1 \times (I_2\, d\mathbf{l}_2 \times \mathbf{R}_{12})}{R_{12}^3} \tag{3-20}$$

It may be noted that $d\mathbf{F}_{12}$ is not necessarily equal to $-d\mathbf{F}_{21}$. This can be illustrated by considering a simple case in which $d\mathbf{l}_1$ and $d\mathbf{l}_2$ are normal, as shown in Fig. 3.5. The construction of Fig. 3.5(a) shows that $d\mathbf{F}_{21}$ is nonzero and directed parallel to $d\mathbf{l}_1$ whereas the construction of Fig. 3.5(b) shows that

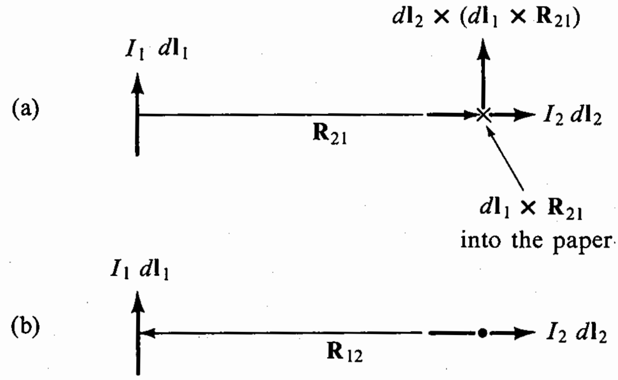

Fig. 3.5. For showing that the force experienced by $I_1\, d\mathbf{l}_1$ due to $I_2\, d\mathbf{l}_2$ is not necessarily equal and opposite to the force experienced by $I_2\, d\mathbf{l}_2$ due to $I_1\, d\mathbf{l}_1$.

$(d\mathbf{l}_2 \times \mathbf{R}_{12})$ is zero and hence $d\mathbf{F}_{12}$ is zero. The fact that $d\mathbf{F}_{21}$ and $d\mathbf{F}_{12}$ are not equal and opposite is not a violation of Newton's third law since isolated current elements do not exist without sources and sinks of charges at their ends. On the other hand, Newton's third law must and does hold for current loops. It is left as an exercise for the student to prove this (Problem 3.8).

3.4 The Magnetic Field of Filamentary Currents

In Section 3.2 we derived Eq. (3-13) for the differential force experienced by a filamentary current element located in a magnetic field. In Section 3.3 we introduced Ampere's law of force, which expresses the forces experienced by two current-carrying loops of wires, and from it obtained the expression (3-18) for the differential force experienced by a current element in the presence of another current element. Now, comparing the forms of the right sides of (3-13) and (3-18), we observe that the force experienced by $I_2\, d\mathbf{l}_2$ is due to the magnetic field of $I_1\, d\mathbf{l}_1$. If we denote this magnetic field by $d\mathbf{B}_1$, we can then write

$$I_2\, d\mathbf{l}_2 \times d\mathbf{B}_1 = \frac{\mu_0}{4\pi} \frac{I_2\, d\mathbf{l}_2 \times (I_1\, d\mathbf{l}_1 \times \mathbf{R}_{21})}{R_{21}^3} \tag{3-21}$$

where we have introduced the appropriate subscripts on the left side of (3-21). Similarly, by comparing the right sides of (3-13) and (3-20), we note that the force experienced by $I_1\, d\mathbf{l}_1$ is due to the magnetic field of $I_2\, d\mathbf{l}_2$. If we denote this magnetic field by $d\mathbf{B}_2$, we can then write

$$I_1\, d\mathbf{l}_1 \times d\mathbf{B}_2 = \frac{\mu_0}{4\pi} \frac{I_1\, d\mathbf{l}_1 \times (I_2\, d\mathbf{l}_2 \times \mathbf{R}_{12})}{R_{12}^3} \tag{3-22}$$

where we have introduced the appropriate subscripts on the left side of (3-22).

Equations (3-21) and (3-22) yield a general expression for the magnetic flux density due to a current element $I\,d\mathbf{l}$ at any point located at a vector distance $\mathbf{R}$ from it as

$$d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{I\,d\mathbf{l} \times \mathbf{R}}{R^3} = \frac{\mu_0}{4\pi}\frac{I\,d\mathbf{l} \times \mathbf{i}_R}{R^2} \tag{3-23}$$

where $\mathbf{i}_R$ is the unit vector in the direction of $\mathbf{R}$. Equation (3-23) is known as the Biot–Savart law and is analogous to the expression for the electric field intensity of a point charge. The Biot–Savart law tells us that the magnetic flux density at a point P due to a current element is directed normal to the plane containing the current element and the line joining the current element to the point, as shown in Fig. 3.6. It is therefore directed circular

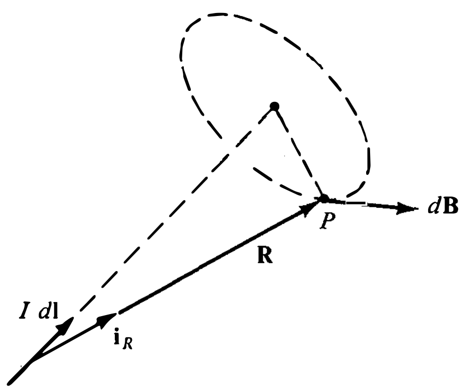

Fig. 3.6. The magnetic field $d\mathbf{B}$ due to a current element $I\,d\mathbf{l}$, at a distance $\mathbf{R}$ from the current element.

to the straight-line axis along the current element. In particular, the sense of the normal is that towards which the fingers are curled when the filamentary wire is grabbed with the right hand and with the thumb pointing in the direction of the current; it is the same as the sense of turning of a right-hand screw as it advances in the direction of $I\,d\mathbf{l}$. The magnitude of the magnetic flux density is proportional to the current I, the element length dl, and the sine of the angle between the current element and the line from it to the point P, and inversely proportional to the square of the distance from the current element to the point P. Hence the magnetic field is zero along the straight line in the direction of the current element. The magnetic flux density $\mathbf{B}$ due to a filamentary wire of any length can now be obtained by integrating the right side of (3-23) along the contour C of the wire. Thus

$$\mathbf{B} = \frac{\mu_0}{4\pi}\int_C \frac{I\,d\mathbf{l} \times \mathbf{i}_R}{R^2} \tag{3-24}$$

In evaluating the integral in (3-24), we note that $\mathbf{i}_R$ and $\mathbf{R}$ are functions of the location of $d\mathbf{l}$. In terms of source point-field point notation, (3-24) is written as

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{C'} \frac{I\, d\mathbf{l}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \tag{3-25}$$

where C' is the contour occupied by the wire.

EXAMPLE 3-3. A straight wire carrying current I amp lies along the z axis as shown in Fig. 3.7. Find the magnetic flux density vector due to the portion of the wire lying between $z = -a$ and $z = +a$ and then extend the result to that of an infinitely long wire.

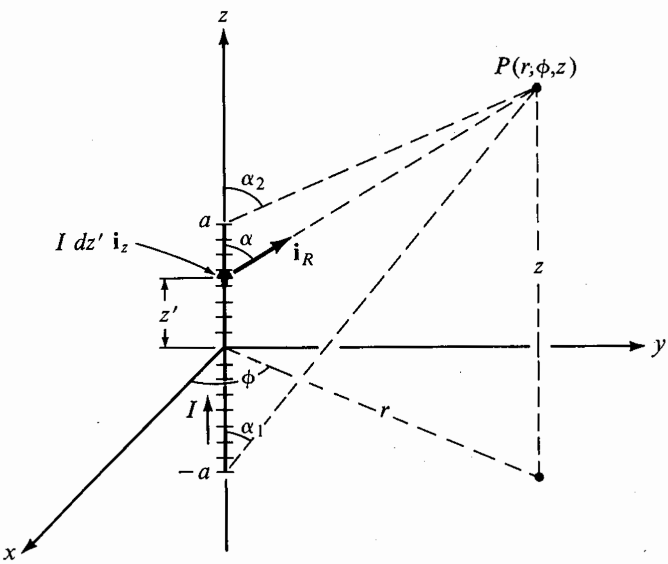

Fig. 3.7. For evaluating the magnetic flux density due to a straight wire carrying current I amp and lying along the z axis between $z = -a$ and $z = +a$.

First we divide the wire into a number of infinitesimal segments, each of which can be considered as a current element. The magnetic flux density due to a current element is given by (3-23). For a current element oriented along the z axis, $\mathbf{i}_z \times \mathbf{i}_R$ is in the $\mathbf{i}_\phi$ direction and hence the magnetic field is in the $\mathbf{i}_\phi$ direction. Also, its magnitude is independent of ϕ. Since all current elements making up the wire are along the z axis, the contributions due to them are all in the $\mathbf{i}_\phi$ direction and independent of ϕ. Thus the magnetic field has circular symmetry about the axis of the wire.

Let us now consider a point $P(r, \phi, z)$. The magnetic flux density at P due to a current element $I\, dz'\, \mathbf{i}_z$ at distance z' from the origin is given by

$$d\mathbf{B} = \frac{\mu_0}{4\pi} \frac{I\, dz'\, \mathbf{i}_z \times \mathbf{i}_R}{R^2} = \frac{\mu_0}{4\pi} \frac{I\, dz' \sin \alpha}{[r^2 + (z - z')^2]} \mathbf{i}_\phi$$

The magnetic flux density at P due to the segment of the wire lying between $z = -a$ and $z = +a$ is then given by

$$\mathbf{B} = \int_{z'=-a}^{a} d\mathbf{B} = \frac{\mu_0}{4\pi}\int_{z'=-a}^{a} \frac{I\,dz'\sin\alpha}{[r^2 + (z - z')^2]}\mathbf{i}_\phi \tag{3-26}$$

Introducing $(z - z') = r\cot\alpha$ in (3-26), we obtain

$$\mathbf{B} = \frac{\mu_0 I}{4\pi r}\int_{\alpha=\alpha_1}^{\alpha_2} \sin\alpha\,d\alpha\,\mathbf{i}_\phi = \frac{\mu_0 I}{4\pi r}(\cos\alpha_1 - \cos\alpha_2)\mathbf{i}_\phi \tag{3-27}$$

where α_1 and α_2 are the angles which the lines joining the ends of the wire segment to the point P make with the z axis. Now, for an infinitely long wire, $\alpha_1 = 0$ and $\alpha_2 = \pi$. Hence

$$\mathbf{B} = \frac{\mu_0 I}{4\pi r}(\cos 0 - \cos\pi)\mathbf{i}_\phi = \frac{\mu_0 I}{2\pi r}\mathbf{i}_\phi \tag{3-28}$$

Thus the magnetic flux density due to an infinitely long straight wire is dependent only on the distance away from the wire, analogous to the electric field intensity due to an infinitely long line charge of uniform density. The field is sketched in Fig. 3.8. ▮

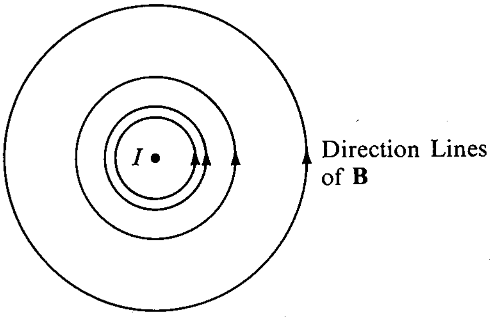

Fig. 3.8. The direction lines of magnetic field due to an infinitely long straight wire carrying current out of the plane of the paper.

EXAMPLE 3-4. A circular loop of wire of radius a m and carrying a current I amp lies in the xy plane with its center at the origin. Such an arrangement is known as a magnetic dipole. Obtain the expression for the magnetic flux density due to the magnetic dipole at distances very large from the origin compared to the radius a.

With reference to the geometry shown in Fig. 3.9, we note that, at any point P, the ϕ component of the magnetic field due to a current element in the ring is cancelled by the ϕ component of the magnetic field due to another current element situated symmetrically about P so that the ϕ component due to the entire ring is zero. Thus the magnetic field has only r and θ components. Furthermore, since the ring is circular about the origin and is in the xy plane, the magnetic field has circular symmetry about the z axis. Hence we consider, for simplicity, a point P having the spherical coordinates $(r, \theta, \pi/2)$. The magnetic field at P due to the current element 1 situated at

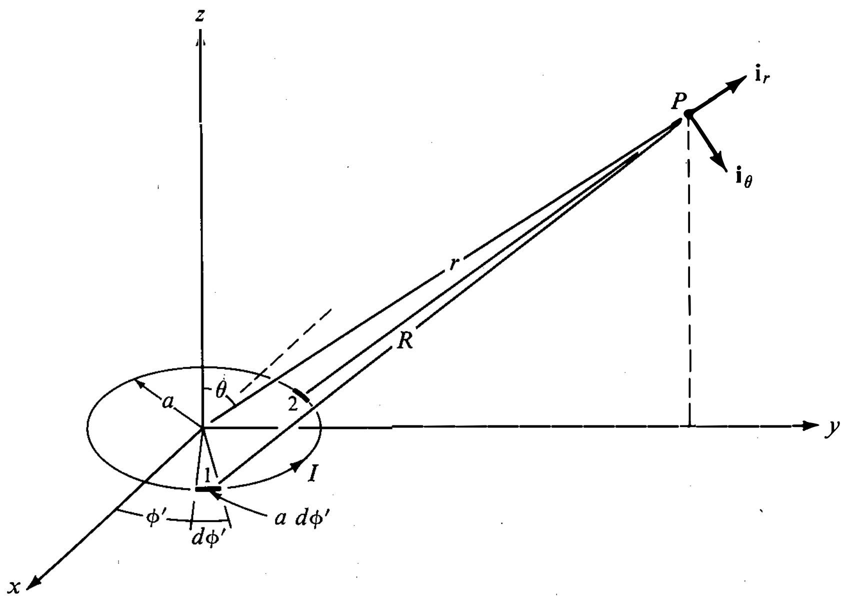

Fig. 3.9. For evaluating the magnetic field due to a magnetic dipole at distances very large from it compared to its radius.

$\phi = \phi'$ is then given by

$$d\mathbf{B}_1 = \frac{\mu_0}{4\pi}\frac{\left\{\begin{matrix} Ia\,d\phi'\,(-\sin\phi'\,\mathbf{i}_x + \cos\phi'\,\mathbf{i}_y) \\ \times [-a\cos\phi'\,\mathbf{i}_x + (r\sin\theta - a\sin\phi')\mathbf{i}_y + r\cos\theta\,\mathbf{i}_z]\end{matrix}\right\}}{(a^2 + r^2 - 2ar\sin\theta\sin\phi')^{3/2}}$$
$$= \frac{\mu_0 Ia\,d\phi'\,[r\cos\theta\cos\phi'\,\mathbf{i}_x + r\cos\theta\sin\phi'\,\mathbf{i}_y + (a - r\sin\theta\sin\phi')\mathbf{i}_z]}{4\pi(a^2 + r^2 - 2ar\sin\theta\sin\phi')^{3/2}}$$

The magnetic field at P due to the symmetrically situated current element 2 at $\phi = \pi - \phi'$ is given by

$$d\mathbf{B}_2 = \frac{\mu_0}{4\pi}\frac{\left\{\begin{matrix} Ia\,d\phi'\,(-\sin\phi'\,\mathbf{i}_x - \cos\phi'\,\mathbf{i}_y) \\ \times [a\cos\phi'\,\mathbf{i}_x + (r\sin\theta - a\sin\phi')\mathbf{i}_y + r\cos\theta\,\mathbf{i}_z]\end{matrix}\right\}}{(a^2 + r^2 - 2ar\sin\theta\sin\phi')^{3/2}}$$
$$= \frac{\mu_0 Ia\,d\phi'[-r\cos\theta\cos\phi'\,\mathbf{i}_x + r\cos\theta\sin\phi'\,\mathbf{i}_y + (a - r\sin\theta\sin\phi')\,\mathbf{i}_z]}{4\pi(a^2 + r^2 - 2ar\sin\theta\sin\phi')^{3/2}}$$

The contribution to the magnetic field at P due to the pair of current elements 1 and 2 is then given by

$$d\mathbf{B} = d\mathbf{B}_1 + d\mathbf{B}_2$$
$$= \frac{\mu_0 Ia\,d\phi'\,[r\cos\theta\sin\phi'\,\mathbf{i}_y + (a - r\sin\theta\sin\phi')\mathbf{i}_z]}{2\pi(a^2 + r^2 - 2ar\sin\theta\sin\phi')^{3/2}}$$

Denoting $d\mathbf{B} = dB_r\,\mathbf{i}_r + dB_\theta\,\mathbf{i}_\theta$, we have

$$\begin{aligned} dB_r &= d\mathbf{B}\cdot\mathbf{i}_r = d\mathbf{B}\cdot(\sin\theta\,\mathbf{i}_y + \cos\theta\,\mathbf{i}_z) \\ &= \frac{\mu_0 I a^2\cos\theta\,d\phi'}{2\pi(a^2 + r^2 - 2ar\sin\theta\sin\phi')^{3/2}} \end{aligned} \tag{3-29}$$

Proceeding further, we obtain

$$\begin{aligned} dB_r &= \frac{\mu_0 I a^2\cos\theta\,d\phi'}{2\pi r^3[(a/r)^2 + 1 - 2(a/r)\sin\theta\sin\phi']^{3/2}} \\ &\approx \frac{\mu_0 I a^2\cos\theta\,d\phi'}{2\pi r^3} \qquad \text{for } r \gg a \end{aligned} \tag{3-30}$$

Integrating the right side of Eq. (3-30) between the limits $\phi' = -\pi/2$ and $\phi' = \pi/2$, we obtain the r component of the magnetic flux density due to the entire ring as

$$B_r = \int_{\phi'=-\pi/2}^{\pi/2} \frac{\mu_0 I a^2\cos\theta\,d\phi'}{2\pi r^3} = \frac{\mu_0 I\pi a^2\cos\theta}{2\pi r^3} \tag{3-31}$$

Now, to find the θ component of $\mathbf{B}$, we note that

$$\begin{aligned} dB_\theta &= d\mathbf{B}\cdot\mathbf{i}_\theta = d\mathbf{B}\cdot(\cos\theta\,\mathbf{i}_y - \sin\theta\,\mathbf{i}_z) \\ &= \frac{\mu_0 I a\,d\phi'\,(-a\sin\theta + r\sin\phi')}{2\pi(a^2 + r^2 - 2ar\sin\theta\sin\phi')^{3/2}} \end{aligned} \tag{3-32}$$

Proceeding further, we obtain

$$\begin{aligned} dB_\theta &= \frac{\mu_0 I a\,d\phi'}{2\pi r^2}\left[-\left(\frac{a}{r}\right)\sin\theta + \sin\phi'\right]\left[1 - 2\left(\frac{a}{r}\right)\sin\theta\sin\phi' + \left(\frac{a}{r}\right)^2\right]^{-3/2} \\ &= \frac{\mu_0 I a\,d\phi'}{2\pi r^2}\left[-\left(\frac{a}{r}\right)\sin\theta + \sin\phi'\right]\left[1 + 3\left(\frac{a}{r}\right)\sin\theta\sin\phi' + \cdots\right] \\ &= \frac{\mu_0 I a\,d\phi'}{2\pi r^2}\left[-\left(\frac{a}{r}\right)\sin\theta + \sin\phi' + 3\left(\frac{a}{r}\right)\sin\theta\sin^2\phi'\right. \\ &\qquad\qquad \left. + \ldots \text{terms involving higher powers of } \left(\frac{a}{r}\right)\right] \\ &\approx \frac{\mu_0 I a}{2\pi r^2}\left[-\left(\frac{a}{r}\right)\sin\theta + \sin\phi' + 3\left(\frac{a}{r}\right)\sin\theta\sin^2\phi'\right]d\phi' \qquad \text{for } r \gg a \end{aligned} \tag{3-33}$$

where we have retained the (a/r) term since the $\sin\phi'$ term yields zero when integrated between $\phi' = -\pi/2$ and $\phi' = \pi/2$. Integrating the right side of Eq.(3-33) between these limits, we obtain the θ component of the magnetic flux density due to the entire ring as

$$\begin{aligned} B_\theta &= \int_{\phi'=-\pi/2}^{\pi/2} \frac{\mu_0 I a}{2\pi r^2}\left[-\left(\frac{a}{r}\right)\sin\theta + \sin\phi' + 3\left(\frac{a}{r}\right)\sin\theta\sin^2\phi'\right]d\phi' \\ &= \frac{\mu_0 I\pi a^2\sin\theta}{4\pi r^3} \end{aligned} \tag{3-34}$$

Thus

$$\mathbf{B} = \frac{\mu_0 I\pi a^2}{4\pi r^3}(2\cos\theta\,\mathbf{i}_r + \sin\theta\,\mathbf{i}_\theta) \tag{3-35}$$

We can consider Eq. (3-25) as the solution for the magnetic flux density at very large distances compared to the radius a or as the solution for the magnetic flux density at any point (r, θ, ϕ) in the limit that $a \longrightarrow 0$, keeping $I\pi a^2$ constant. It should be noted that to keep $I\pi a^2$ constant as $a \longrightarrow 0$ requires that $I \longrightarrow \infty$. The product $I\pi a^2$ is known as the magnetic dipole moment m. The magnetic dipole moment has also an orientation associated with it which is normal to the surface of the loop. In particular, the sense of the normal is that towards which the fingers pierce through the area of the ring when the loop is grabbed with the right hand and with the thumb pointing in the direction of the current. It is the same as the direction of advance of a right-hand screw as it is turned in the sense of the loop current. Substituting m for $I\pi a^2$ in (3-35), the magnetic flux density due to a magnetic dipole of moment m oriented along the positive z axis is given by

$$\mathbf{B} = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\,\mathbf{i}_r + \sin\theta\,\mathbf{i}_\theta) \tag{3-36}$$

The magnetic field given by (3-36) is analogous to the electric field due to an electric dipole of moment p oriented along the z-axis and given by (2-28). ▮

EXAMPLE 3-5. A solenoid consists of continuously wound, circular current loops. Let us consider an infinitely long, uniformly wound solenoid of radius a and n turns per unit length, each carrying the same current I and with the z axis as its axis. It is desired to find the magnetic flux density due to the infinitely long solenoid.

Since the solenoid is uniformly wound and infinitely long, and since it possesses cylindrical symmetry about the z axis, the magnetic flux density must be independent of z and must possess cylindrical symmetry about the z axis. Hence it is sufficient if we compute the magnetic flux density at a point P on the y axis. To do this, let us consider two sections of the solenoid symmetrically placed about the xy plane at distances z' from it and having infinitesimal lengths dz' as shown in Fig. 3.10. Since the lengths are infinitesimal, these sections can be considered as current loops carrying currents $nI\,dz'$.

In each of these current loops, let us consider two differential elements of lengths $a\,d\phi'$ symmetrically situated about the yz plane, as shown in Fig. 3.10. Applying the notation of Fig. 3.10 to (3-29) and (3-32), we obtain the magnetic field at P due to the pair of current elements 1 and 2 as

$$d\mathbf{B}_1 = \mu_0 nI\,dz'\,\frac{a^2\cos\alpha\,d\phi'\,\mathbf{i}_1 + a\,d\phi'\,[-a\sin\alpha + (y/\sin\alpha)\sin\phi']\mathbf{i}_2}{2\pi(a^2 + y^2 + z'^2 - 2ay\sin\phi')^{3/2}} \tag{3-37}$$

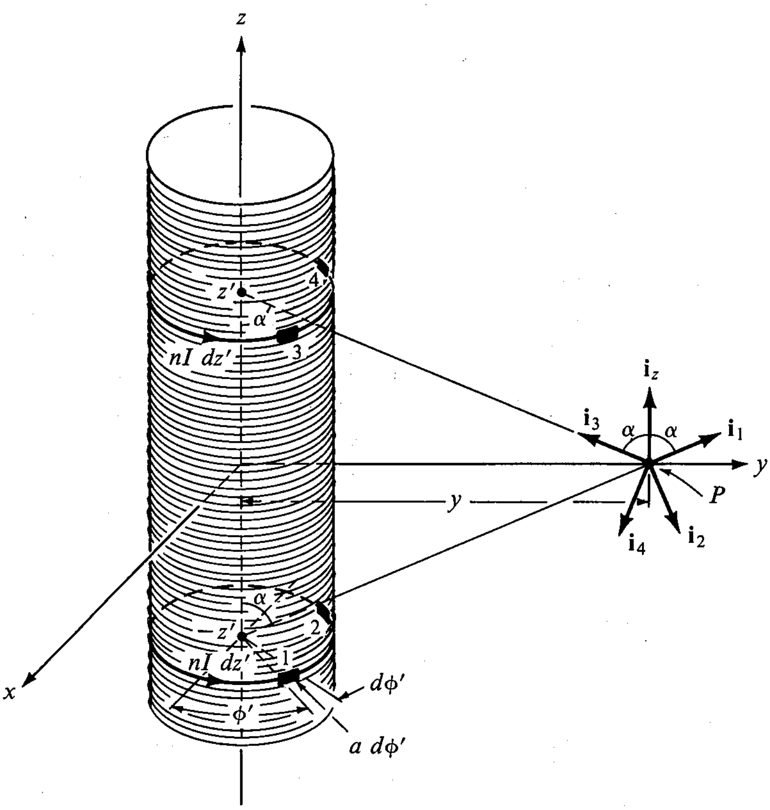

Fig. 3.10. For evaluating the magnetic field due to an infinitely long, uniformly wound solenoid of radius a and n turns per unit length.

and the magnetic field at P due to the pair of current elements 3 and 4 as

$$d\mathbf{B}_2 = \mu_0 nI\,dz' \frac{a^2 \cos\alpha\,d\phi'\,\mathbf{i}_3 + a\,d\phi'\,[-a\sin\alpha + (y/\sin\alpha)\sin\phi']\mathbf{i}_4}{2\pi(a^2 + y^2 + z'^2 - 2ay\sin\phi')^{3/2}} \tag{3-38}$$

where α, $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$, and $\mathbf{i}_4$ are defined in Fig. 3.10. Adding (3-37) and (3-38) and simplifying, we obtain the magnetic field at P due to the four current elements 1, 2, 3, and 4 as

$$d\mathbf{B} = d\mathbf{B}_1 + d\mathbf{B}_2 = \mu_0 nI\,dz' \frac{(a^2 - ay\sin\phi')\,d\phi'}{\pi(a^2 + y^2 + z'^2 - 2ay\sin\phi')^{3/2}}\mathbf{i}_z \tag{3-39}$$

Performing double integration of the right side of (3-39) between the appropriate limits, we obtain the magnetic flux density at P due to the entire solenoid as

$$\begin{aligned}\mathbf{B} &= \int_{\phi'=-\pi/2}^{\pi/2}\int_{z'=0}^{\infty} \frac{\mu_0 nI(a^2 - ay\sin\phi')\,d\phi'\,dz'}{\pi(a^2+y^2+z'^2-2ay\sin\phi')^{3/2}}\mathbf{i}_z \\ &= \frac{\mu_0 nIa}{\pi}\int_{\phi'=-\pi/2}^{\pi/2}\frac{(a-y\sin\phi')d\phi'}{(a^2+y^2-2ay\sin\phi')}\mathbf{i}_z \\ &= \begin{cases} 0 & \text{for } y > a \\ \mu_0 nI\mathbf{i}_z & \text{for } y < a \end{cases}\end{aligned} \qquad (3\text{-}40)$$

Thus the magnetic field due to the infinitely long solenoid is zero outside the solenoid and uniform inside the solenoid, having a value $\mu_0 nI$ and directed along the axis of the solenoid. ▮

3.5 The Magnetic Field of Current Distributions

In the previous section we considered the magnetic field computation for filamentary wires carrying current. In this section we will extend the computation to current distributions. Current distributions can be of two types:

(a) Surface current for which current is distributed on a surface (planar or nonplanar).

(b) Volume current for which current is distributed in a volume.

As in the case of continuous charge distributions, introduced in Section 2.4, we have to work with current densities when a current is distributed on a surface or in a volume. We have already introduced the current density for volume currents in Sections 1.7 and 3.2. The magnitude of the volume current density **J** at a point is the current per unit area crossing an infinitesimal area at that point with the orientation of the area adjusted so as to maximize the current, in the limit that the area tends to zero. The direction of **J** at that point is the direction to which the normal to the area approaches in the limit. Similarly, the magnitude of the surface current density at a point is the current per unit width crossing an infinitesimal line segment at that point with the orientation of the segment adjusted so as to maximize the current, in the limit that the width of the line segment tends to zero. The direction of the surface current density at that point is the direction to which the normal to the line and tangent to the surface approaches in the limit. We will use the symbol $\mathbf{J}_s$ for the surface current density, in contrast to **J** for the volume current density. In each case, we represent the total current as a continuous collection of appropriate filamentary currents and evaluate the magnetic field as the vector superposition of the contributions due to the individual filamentary currents.

EXAMPLE 3-6. A sheet of current with the surface current density given by

$$\mathbf{J}_s = J_{s0}\mathbf{i}_z \text{ amp/m}$$

where J_{s0} is a constant, occupies the entire xz plane. Find the magnetic flux density vector due to the portion of the current sheet lying between $x = -a$ and $x = +a$ as shown in Fig. 3.11(a) and then extend the result to that of the infinite sheet.

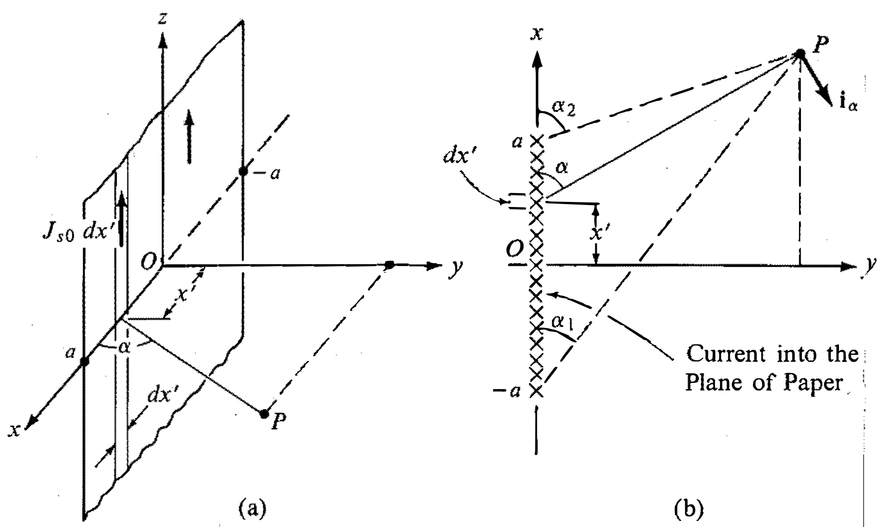

Fig. 3.11. For evaluating the magnetic field due to a sheet of current flowing in the z direction and lying in the xz plane between $x = -a$ and $x = +a$.

We divide the current sheet into a number of filaments of infinitesimal width in the x direction, each of which can be considered as an infinitely long wire parallel to the z axis. Let us consider a filament of width dx' located at $x = x'$ in the plane of the sheet, as shown in Fig. 3.11(a). From Example 3-3, we know that the magnetic flux density due to an infinitely long wire is dependent only on the distance away from the wire and is oriented circular to the wire. Hence the magnetic field due to the current sheet will not be dependent on the z coordinate and also will have only x and y components, so that it is sufficient if we consider the two-dimensional geometry shown in Fig. 3.11(b). Since the current density is $J_{s0}\mathbf{i}_z$, the current flowing in the filament of width dx' is $J_{s0}\,dx'$. Applying (3-28) to the geometry associated with this filament, we obtain the magnetic flux density due to it at any point $P(x, y, z)$ as

$$d\mathbf{B} = \frac{\mu_0 J_{s0}\,dx'}{2\pi\sqrt{(x - x')^2 + y^2}}\mathbf{i}_\alpha \tag{3-41}$$

where $\mathbf{i}_\alpha$ is the unit vector normal to the line drawn from the filament to the point P as shown in Fig. 3.11(b). Expressing $d\mathbf{B}$ in terms of its components along the coordinate axes, we have

$$d\mathbf{B} = \frac{\mu_0 J_{s0}\, dx'}{2\pi\sqrt{(x-x')^2+y^2}}(-\sin\alpha\,\mathbf{i}_x + \cos\alpha\,\mathbf{i}_y) \tag{3-42}$$

The magnetic flux density at P due to the portion of the infinite current sheet between $x = -a$ and $x = +a$ is then given by

$$\begin{aligned} \mathbf{B} &= \int_{x'=-a}^{a} d\mathbf{B} \\ &= \int_{x'=-a}^{a}\left[-\frac{\mu_0 J_{s0}\sin\alpha\, dx'}{2\pi\sqrt{(x-x')^2+y^2}}\mathbf{i}_x + \frac{\mu_0 J_{s0}\cos\alpha\, dx'}{2\pi\sqrt{(x-x')^2+y^2}}\mathbf{i}_y\right] \\ &= \frac{\mu_0 J_{s0}}{2\pi}\left[(\alpha_1-\alpha_2)\mathbf{i}_x + \ln\left(\frac{\sin\alpha_2}{\sin\alpha_1}\right)\mathbf{i}_y\right] \end{aligned} \tag{3-43}$$

where we have used the transformation $(x - x') = y\cot\alpha$ for evaluating the integrals in (3-43), and the angles α_1 and α_2 are as shown in Fig. 3.11(b). Now, for the infinite sheet of current, $\alpha_1 = 0$ and $\alpha_2 = \pi$ for $y > 0$, and $\alpha_1 = 2\pi$ and $\alpha_2 = \pi$ for $y < 0$. However, to evaluate $\ln(\sin\alpha_2/\sin\alpha_1)$, we note that

$$\lim_{a\to\infty}\frac{\sin\alpha_2}{\sin\alpha_1} = \lim_{a\to\infty}\left[\frac{(x+a)^2+y^2}{(x-a)^2+y^2}\right]^{1/2} = 1$$

and hence

$$\lim_{a\to\infty}\ln\frac{\sin\alpha_2}{\sin\alpha_1} = 0 \tag{3-44}$$

Substituting for α_1 and α_2 in (3-43), we then obtain the magnetic flux density due to the infinite sheet of current as

$$\begin{aligned} \mathbf{B} &= \begin{cases} -\dfrac{\mu_0 J_{s0}}{2}\mathbf{i}_x & \text{for } y > 0 \\ \dfrac{\mu_0 J_{s0}}{2}\mathbf{i}_x & \text{for } y < 0 \end{cases} \\ &= \frac{\mu_0}{2}\mathbf{J}_s \times \mathbf{i}_n \qquad \text{where } \mathbf{i}_n = \begin{cases} \mathbf{i}_y & \text{for } y > 0 \\ -\mathbf{i}_y & \text{for } y < 0 \end{cases} \end{aligned} \tag{3-45}$$

The field given by (3-45) is sketched in Fig. 3.12. If the sheet current occupies the $y = y_0$ plane, it follows from (3-45) that

$$\mathbf{B} = \begin{cases} -\dfrac{\mu_0 J_{s0}}{2}\mathbf{i}_x & \text{for } y > y_0 \\ \dfrac{\mu_0 J_{s0}}{2}\mathbf{i}_x & \text{for } y < y_0 \end{cases} \quad \blacksquare$$

EXAMPLE 3-7. Current flows in the axial direction in an infinitely long cylinder of radius a with uniform density J_0 amp/m². Find the magnetic flux density both inside and outside the cylinder.

Choosing the z axis as the axis of the infinitely long cylinder as shown

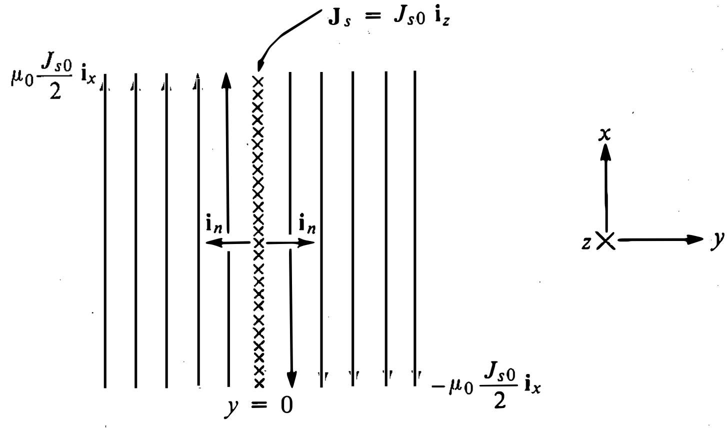

Fig. 3.12. The direction lines of magnetic field due to an infinite sheet of current flowing into the plane of the paper with uniform density.

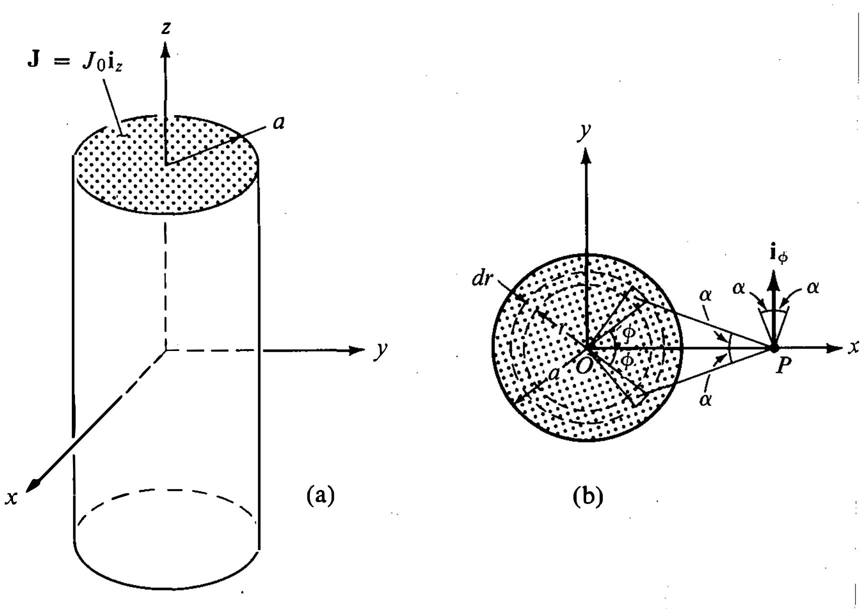

Fig. 3.13. For evaluating the magnetic field due to a volume current flowing along an infinitely long cylinder of radius a with uniform density.

in Fig. 3.13(a), we have the volume current density as

$$\mathbf{J} = J_0 \mathbf{i}_z$$

The cylindrical current distribution can be thought of as a superpositon of filamentary currents parallel to the z axis so that the magnetic field is independent of z. Hence it is sufficient if we consider the two-dimensional geometry shown in Fig. 3.13(b). Furthermore, for every filamentary current and for a given point P, there is another filamentary current so that the combined magnetic field due to these two filamentary currents is entirely in the ϕ direction. This is illustrated in Fig. 3.13(b) for a point P on the x axis. Thus the magnetic field due to the entire current distribution has only a ϕ component and possesses cylindrical symmetry about the z axis. Let us therefore consider two filamentary currents corresponding to the infinitesimal areas $r\,dr\,d\phi$ at (r, ϕ) and $(r, -\phi)$ as shown in Fig. 3.13(b). The magnetic field at P due to these two filamentary currents is given by

$$\begin{aligned} d\mathbf{B} &= \frac{\mu_0 J_0 r\,dr\,d\phi}{2\pi(r^2 + x^2 - 2rx\cos\phi)^{1/2}} 2\cos\alpha\,\mathbf{i}_\phi \\ &= \frac{\mu_0 J_0 r\,dr\,d\phi(x - r\cos\phi)}{\pi(r^2 + x^2 - 2rx\cos\phi)}\mathbf{i}_\phi \end{aligned} \tag{3-46}$$

The magnetic field at P due to the entire current distribution is then given by

$$\begin{aligned} \mathbf{B} &= \int_{r=0}^{a} \int_{\phi=0}^{\pi} d\mathbf{B} \\ &= \frac{\mu_0 J_0}{\pi} \int_{r=0}^{a} r\,dr \int_{\phi=0}^{\pi} \frac{(x - r\cos\phi)\,d\phi}{(r^2 + x^2 - 2rx\cos\phi)}\mathbf{i}_\phi \\ &= \frac{\mu_0 J_0}{\pi} \int_{r=0}^{a} r\,dr \begin{pmatrix} 0 & \text{for } x < r \\ \dfrac{\pi}{x} & \text{for } x > r \end{pmatrix} \mathbf{i}_\phi \\ &= \begin{cases} \dfrac{\mu_0 J_0}{\pi} \displaystyle\int_{r=0}^{a} \frac{\pi}{x} r\,dr\,\mathbf{i}_\phi & \text{for } x > a \\ \dfrac{\mu_0 J_0}{\pi} \displaystyle\int_{r=0}^{x} \frac{\pi}{x} r\,dr\,\mathbf{i}_\phi & \text{for } x < a \end{cases} \\ &= \begin{cases} \dfrac{\mu_0 J_0}{\pi x} \dfrac{\pi a^2}{2} \mathbf{i}_\phi & \text{for } x > a \\ \dfrac{\mu_0 J_0}{\pi x} \dfrac{\pi x^2}{2} \mathbf{i}_\phi & \text{for } x < a \end{cases} \end{aligned} \tag{3-47}$$

Recalling that $\mathbf{B}$ has cylindrical symmetry about the z axis, we substitute r for x in (3-47) and obtain

$$\mathbf{B} = \begin{cases} \mu_0 \dfrac{J_0(\pi a^2/2)}{2\pi r} \mathbf{i}_\phi & \text{for } r > a \\ \mu_0 \dfrac{J_0(\pi r^2/2)}{2\pi r} \mathbf{i}_\phi & \text{for } r < a \end{cases} \tag{3-48}$$

Noting that $\pi r^2/2$ is the area of cross section of a wire of radius r, and that there is no current for $r > a$, we can combine the two results on the right side of (3-48) as

$$\mathbf{B}(r) = \mu_0 \frac{\text{current enclosed by the circular path of radius } r}{2\pi r} \mathbf{i}_\phi \tag{3-49}$$

Viewed from any distance r from the axis of the infinitely long cylinder carrying current, the current distribution is equivalent to an infinitely long filamentary current of value equal to the current enclosed by the circular path of radius r. ▮

3.6 Ampere's Circuital Law in Integral Form

In Section 2.6 we started with the electric field intensity of a point charge and derived Gauss' law, which was later found to be very convenient for computing the electric field due to certain symmetrical charge distributions. Similarly, in this section we will start with the magnetic flux density due to an infinitely long wire carrying current and derive Ampere's circuital law. We will later find Ampere's circuital law to be very useful compared to the Biot–Savart law for computing the magnetic field due to certain symmetrical current distributions.

Let us consider an infinitely long filamentary wire along the z axis carrying current I amp. The magnetic flux density due to this wire is directed everywhere circular to the wire and its magnitude is dependent only on the distance from the wire. Let us consider a circular path C of radius r in the plane normal to the wire and centered at the wire as shown in Fig. 3.14. For an infinitesimal length $d\mathbf{l} = dl\,\mathbf{i}_\phi$ on this contour C, we have

$$\mathbf{B} \cdot d\mathbf{l} = \frac{\mu_0 I}{2\pi r}\mathbf{i}_\phi \cdot dl\,\mathbf{i}_\phi = \frac{\mu_0 I\,dl}{2\pi r} \tag{3-50}$$

The integral of $\mathbf{B} \cdot d\mathbf{l}$ along the entire path C is then given by

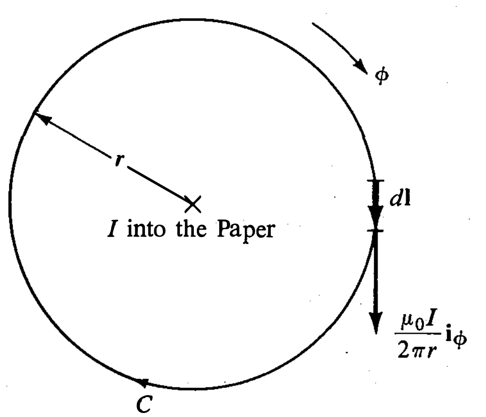

Fig. 3.14. For evaluating $\oint_C \mathbf{B} \cdot d\mathbf{l}$, where C is a circular path of radius r in the plane normal to a straight, infinitely long wire carrying current I and centered at the wire.

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \oint_C \frac{\mu_0 I\, dl}{2\pi r} = \frac{\mu_0 I}{2\pi r} \oint_C dl \tag{3-51}$$

where we have taken $\mu_0 I/2\pi r$ outside the integral since r is constant for the contour C. Proceeding further, we have

$$\begin{aligned}\oint_C \mathbf{B} \cdot d\mathbf{l} &= \frac{\mu_0 I}{2\pi r}(\text{circumference of } C) \\ &= \frac{\mu_0 I}{2\pi r}(2\pi r) = \mu_0 I\end{aligned} \tag{3-52}$$

Equation (3-52) states that the line integral of **B** around a circular path in the plane normal to an infinitely long wire carrying current I and centered at the wire is equal to $\mu_0 I$. It is independent of the radius r of the circular path. Whether $r = 1$ micron or 1000 km, the value of the line integral is the same (provided, of course, that there is no other magnetic field in the medium). It should be noted that the current I in (3-52) is the current which flows in the direction of advance of a right-hand screw as it is turned in the sense in which the line integral around C is evaluated.

Before we proceed further, a few words about the line integral of **B** are in order. In Chapter 2 we learned that $\int_a^b \mathbf{E} \cdot d\mathbf{l}$ has the meaning of work or change in potential energy per unit charge associated with the movement of a test charge from point a to point b in the electric field **E**. This is because the force experienced by a charge due to an electric field is in the same direction as the electric field. On the other hand, in a magnetic field **B**, the force experienced by a test charge moving in the direction of $d\mathbf{l}$ (or by a current element $I\,d\mathbf{l}$) is perpendicular to both **B** and $d\mathbf{l}$. Hence the work associated with the movement of the test charge is zero. Thus $\int \mathbf{B} \cdot d\mathbf{l}$ does not have the meaning of work. Just as $\oint_S \mathbf{E} \cdot d\mathbf{S}$ provides us information about charges enclosed by S, $\oint_C \mathbf{B} \cdot d\mathbf{l}$ tells us about the current enclosed by C. Therefore, in this respect $\oint_C \mathbf{B} \cdot d\mathbf{l}$ is analogous to $\oint_S \mathbf{E} \cdot d\mathbf{S}$. We will simply call it the circulation of **B**.

Let us now consider an arbitrary path C (not necessarily in a plane) enclosing the current as shown in Fig. 3.15. For an infinitesimal segment $d\mathbf{l}$ at P along this path,

$$\mathbf{B} \cdot d\mathbf{l} = \frac{\mu_0 I}{2\pi R}\mathbf{i}_\phi \cdot d\mathbf{l} = \frac{\mu_0 I\, dl \cos\alpha}{2\pi R} \tag{3-53}$$

where R is the distance of P from the wire, $\mathbf{i}_\phi$ is the unit vector at P directed circular to the wire, and α is the angle between $d\mathbf{l}$ and $\mathbf{i}_\phi$. The circulation of **B** around the arbitrary path C is

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \oint_C \frac{\mu_0 I\, dl \cos\alpha}{2\pi R} = \frac{\mu_0 I}{2\pi} \oint_C \frac{dl \cos\alpha}{R} \tag{3-54}$$

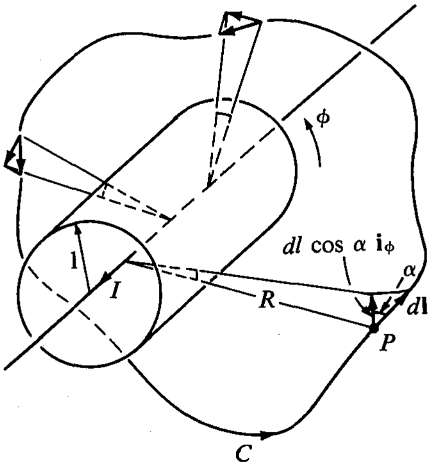

Fig. 3.15. For evaluating $\oint_C \mathbf{B} \cdot d\mathbf{l}$, where C is an arbitrary closed path enclosing a straight, infinitely long wire carrying current I.

In (3-54), $dl \cos \alpha$ is the projection of $d\mathbf{l}$ onto the circle of radius R centered at the wire and passing through P. Hence $(dl \cos \alpha)/R$ is the projection of $d\mathbf{l}$ on to the circle of radius unity in the plane normal to the wire and centered at the wire, and $\oint_C (dl \cos \alpha)/R$ is the sum of the projections of all infinitesimal segments comprising the contour C onto the circle of radius unity. Thus it is equal to the circumference of the circle of unit radius, that is, 2π. Substituting this result in (3-54), we have

$$\oint_{\substack{\text{contour}\\ \text{enclosing } I}} \mathbf{B} \cdot d\mathbf{l} = \frac{\mu_0 I}{2\pi}(2\pi) = \mu_0 I \tag{3-55}$$

If the arbitrary contour does not enclose the current, then, in evaluating $\oint_C (dl \cos \alpha)/R$, we start at one point on the circle of unit radius, traverse to another point on it and return to the starting point along the same path in the opposite direction, obtaining a result of zero in this process. Hence

$$\oint_{\substack{\text{contour not}\\ \text{enclosing } I}} \mathbf{B} \cdot d\mathbf{l} = 0 \tag{3-56}$$

Equations (3-55) and (3-56) may be combined into a single statement which reads as

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0(\text{current enclosed by the contour } C) \tag{3-57}$$

This is Ampere's circuital law. Although we have derived it here for an infinitely long straight wire, it can be proved for a current loop of arbitrary shape. Also, if we have a number of current loops or infinitely long wires carrying currents or continuous current distributions in the form of surface or volume current, we can invoke superposition and conclude that Ampere's circuital law as given by (3-57) holds for any closed path C provided the current enclosed by C is uniquely defined.

Let us now discuss the uniqueness of a closed path enclosing or not enclosing a current. To do this, let us consider the case of a straight filamentary wire of finite length in the plane of the paper carrying current I, as shown in Fig. 3.16. This can be achieved by having a source of point

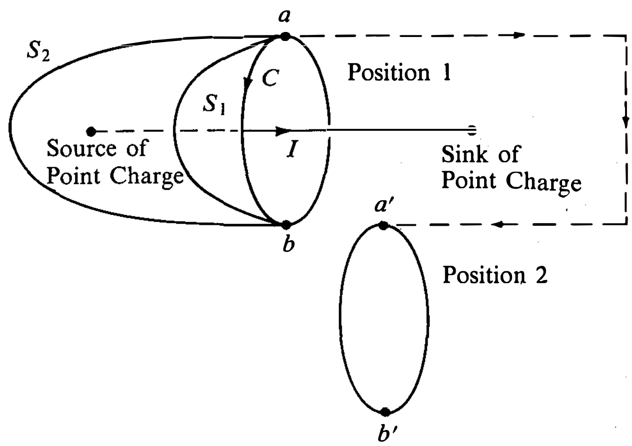

Fig. 3.16. For illustrating that the current enclosed by closed path C surrounding a finitely long filamentary wire is not uniquely defined.

charge at one end of the wire and a sink of point charge at the other end. Let a closed path C be in the plane normal to the paper, emerging out of the paper at a and going into it at b. Let us denote this position of the closed path as position 1. Imagining the closed path to be rigid, we can bring it to position 2 by sliding it parallel to the wire for some distance, pulling it down, and then sliding it back parallel to the wire as shown by the dashed lines. We are able to achieve this without cutting through the wire. We then say that the current enclosed by the closed path C is not uniquely defined. Alternatively, we can define the current enclosed by a path as that which pierces through (passes from one side to the other side of) a surface whose perimeter is the closed path. For the closed path C in Fig. 3.16, let us consider two bowl-shaped surfaces S_1 and S_2. It can be seen that the wire pierces through S_1 but not through S_2. This suggests that we cannot uniquely define the current enclosed by C in Fig. 3.16. It is clear that Ampere's circuital law (3-57) cannot be used for the case of Fig. 3.16. In fact, if we evaluate $\oint_C \mathbf{B} \cdot d\mathbf{l}$

around the contour C in Fig. 3.16, we will not obtain $\mu_0 I$ for the answer. On the other hand, if the wire is infinitely long, we cannot bring the closed path from position 1 to position 2 without cutting through the wire and there can be no surface whose perimeter is C and through which the wire does not pierce. The current enclosed by C is then uniquely defined. Similarly, for surfaces whose perimeter is position 2 of the closed path in Fig. 3.16, the infinitely long wire does not pierce at all or it pierces through an even number of times, entering from one side and emerging out on the same side so that the net current enclosed by the path is always zero. Thus we can summarize the discussion in this paragraph by stating that the current enclosed by a path is uniquely defined if the net current which passes through each possible surface whose perimeter is the closed path is the same.

EXAMPLE 3-8. An infinitely long filamentary wire along the z axis carries current I amp. Find $\int_P^Q \mathbf{B} \cdot d\mathbf{l}$ along the straight line joining P to Q, where P and Q are (1, −1, 0) and (1, 1, 0), respectively, in cartesian coordinates.

The geometry of the problem in the xy plane is shown in Fig. 3.17.

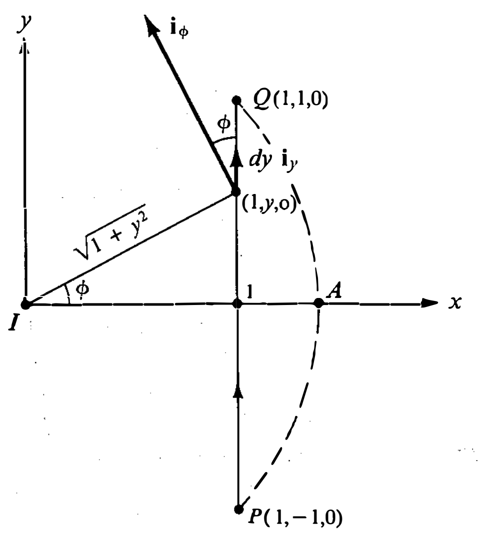

Fig. 3.17. For evaluating $\int_P^Q \mathbf{B} \cdot d\mathbf{l}$ along the straight line from P to Q in the field of an infinitely long wire carrying current I.

First we will solve this problem by actually evaluating $\int_P^Q \mathbf{B} \cdot d\mathbf{l}$ along the given path. To do this, let us consider an infinitesimal segment $d\mathbf{l} = dy\,\mathbf{i}_y$ at $(1, y, 0)$. Since $\mathbf{B}$ at this point due to the line current is $[\mu_0 I/(2\pi\sqrt{1+y^2})]\mathbf{i}_\phi$, we have

$$\mathbf{B} \cdot d\mathbf{l} = \frac{\mu_0 I}{2\pi\sqrt{1+y^2}}\mathbf{i}_\phi \cdot dy\,\mathbf{i}_y$$

$$= \frac{\mu_0 I\,dy}{2\pi\sqrt{1+y^2}} \cos\phi = \frac{\mu_0 I\,dy}{2\pi(1+y^2)}$$

Thus

$$\begin{aligned}\int_P^Q \mathbf{B} \cdot d\mathbf{l} &= \int_{y=-1}^{1} \frac{\mu_0 I\, dy}{2\pi(1+y^2)} \\ &= \frac{\mu_0 I}{2\pi} \int_{\phi=-\pi/4}^{\pi/4} d\phi = \frac{\mu_0 I}{4}\end{aligned} \tag{3-58}$$

This result can, however, be obtained without performing the integration if we note that, according to Ampere's circuital law,

$$\oint_{PQAP} \mathbf{B} \cdot d\mathbf{l} = 0 \tag{3-59}$$

where QAP is part of a circle centered at the line current. Equation (3-59) may be written as

$$\int_P^Q \mathbf{B} \cdot d\mathbf{l} + \int_{QAP} \mathbf{B} \cdot d\mathbf{l} = 0$$

which yields

$$\int_P^Q \mathbf{B} \cdot d\mathbf{l} = -\int_{QAP} \mathbf{B} \cdot d\mathbf{l} \tag{3-60}$$

However, from symmetry considerations, $\int_{QAP} \mathbf{B} \cdot d\mathbf{l}$ is equal to $-\mu_0 I(QAP)$ divided by the circumference of the circle, or $-\mu_0 I(\pi/2)/2\pi = -\mu_0 I/4$. From (3-60), we then obtain a value $\mu_0 I/4$ for $\int_P^Q \mathbf{B} \cdot d\mathbf{l}$, which agrees with (3-58). ▮

Given **B** and a closed path C, it is always possible to compute the current enclosed by the path by evaluating $\oint_C \mathbf{B} \cdot d\mathbf{l}$ analytically or numerically and then dividing the result by μ_0 in accordance with Ampere's circuital law given by (3-57). The inverse problem of finding **B** for a given current distribution by using (3-57) is possible only for certain simple cases involving a high degree of symmetry, just as in the case of the application of Gauss' law for finding **E** for a given charge distribution. First, the symmetry of the magnetic field must be determined from the Biot–Savart law and second, we should be able to choose a closed path C such that $\oint_C \mathbf{B} \cdot d\mathbf{l}$ can be reduced to an algebraic quantity involving the magnitude of **B**. Obviously, the closed path must be chosen such that the magnitude of **B** is uniform and the direction of **B** is tangential to the path along all or part of the path, while the magnitude of **B** is zero or the direction of **B** is normal to the path along the rest of the path in the latter case. We will illustrate this method of obtaining **B** by reconsidering Examples 3-6 and 3-7.

EXAMPLE 3-9. A sheet of current with the surface current density given by

$$\mathbf{J}_s = J_{s0}\mathbf{i}_z$$

where J_{s0} is a constant, occupies the entire xz plane as shown in Fig. 3.18.

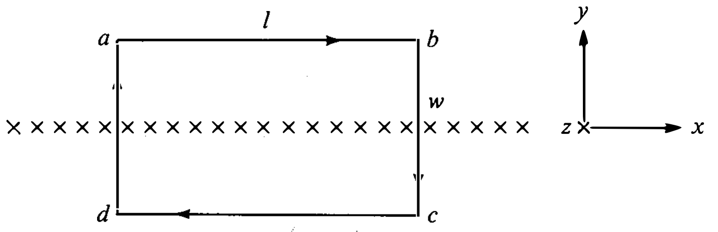

Fig. 3.18. For evaluating the magnetic flux density due to an infinite plane sheet of current.

The magnetic field due to such a current sheet was found in Example 3-6 by using the Biot–Savart law. It is here desired to find the magnetic flux density due to this infinite sheet of current using Ampere's circuital law.

From purely qualitative reasoning based upon the magnetic flux density due to an infinitely long, straight filamentary wire of current, we can conclude that the magnetic flux density due to the infinite sheet of current of uniform density is (a) entirely in the $+x$ direction for $y > 0$ and in the $-x$ direction for $y < 0$, (b) uniform in planes parallel to the current sheet, and (c) symmetrical about $y = 0$. Thus

$$\mathbf{B} = B_t \mathbf{i}_t \tag{3-61}$$

where $\mathbf{i}_t$ is the unit tangential vector to the current sheet given by

$$\mathbf{i}_t = \mathbf{i}_z \times \mathbf{i}_n \tag{3-62}$$

in which $\mathbf{i}_n$ is the unit normal vector to the current sheet. We can therefore choose a rectangular path *abcda* having length l parallel to the current sheet and width w normal to the current sheet and symmetrical about the current sheet as shown in Fig. 3.18. Then

$$\oint_{abcda} \mathbf{B} \cdot d\mathbf{l} = \int_a^b \mathbf{B} \cdot d\mathbf{l} + \int_b^c \mathbf{B} \cdot d\mathbf{l} + \int_c^d \mathbf{B} \cdot d\mathbf{l} + \int_d^a \mathbf{B} \cdot d\mathbf{l} \tag{3-63}$$

But $\int_b^c \mathbf{B} \cdot d\mathbf{l}$ and $\int_d^a \mathbf{B} \cdot d\mathbf{l}$ are equal to zero since $\mathbf{B}$ is normal to the paths *bc* and *da*. For paths *ab* and *cd*, $\mathbf{B}$ is parallel and directed along these paths. Furthermore, the magnitudes of $\mathbf{B}$ are the same for these paths since they are equidistant from the current sheet. Thus (3-63) reduces to

$$\begin{aligned}\oint_{abcda} \mathbf{B} \cdot d\mathbf{l} &= 2\int_a^b \mathbf{B} \cdot d\mathbf{l} = 2\int_a^b B_t \mathbf{i}_t \cdot dl\, \mathbf{i}_t \\ &= 2B_t \int_a^b dl = 2B_t l\end{aligned} \tag{3-64}$$

But, from Ampere's circuital law,

$$\oint_{abcda} \mathbf{B} \cdot d\mathbf{l} = \mu_0 \text{ (current enclosed by } abcda) = \mu_0 J_{s0} l \tag{3-65}$$

Comparing (3-64) and (3-65), we have

$$B_t = \frac{\mu_0 J_{s0}}{2} \tag{3-66}$$

$$\mathbf{B} = \frac{\mu_0 J_{s0}}{2}\mathbf{i}_z \times \mathbf{i}_n = \frac{\mu_0}{2}\mathbf{J}_s \times \mathbf{i}_n \tag{3-67}$$

which agrees with the result obtained in Example 3-6. ▮

EXAMPLE 3-10. Current flows in the axial (z) direction in an infinitely long cylinder of radius a with uniform density J_0 amp/m^2 as shown in Fig. 3.19. The magnetic field due to such a current distribution was found in Example 3-7 by using the Biot–Savart law. It is here desired to find the magnetic flux density both inside and outside the cylinder using Ampere's circuital law.

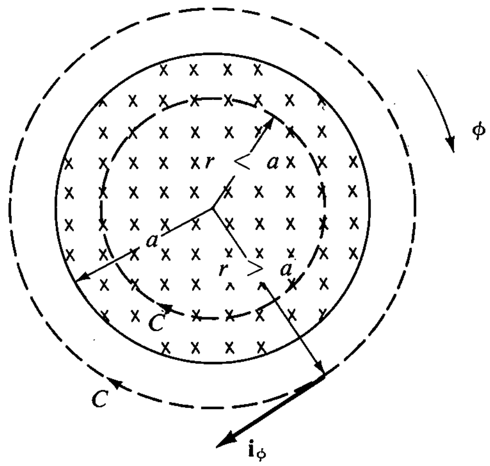

Fig. 3.19. For evaluating the magnetic flux density due to a volume current flowing with uniform density along an infinitely long cylinder.

In Example 3-7 we established from purely qualitative arguments that **B**, due to the given current distribution, has only a ϕ component and possesses cylindrical symmetry so that it is a function only of the distance from the axis of the cylinder. Thus

$$\mathbf{B} = B_\phi(r)\mathbf{i}_\phi \tag{3-68}$$

Choosing, therefore, a circular path C of radius $r \gtrless a$ centered at the axis of the cylinder and in the plane normal to the axis, as shown in Fig. 3.19, we have

$$\begin{aligned}\oint_C \mathbf{B} \cdot d\mathbf{l} &= \oint_C B_\phi \mathbf{i}_\phi \cdot dl\,\mathbf{i}_\phi = B_\phi \oint_C dl \\ &= B_\phi(\text{circumference of the circle of radius } r) \\ &= B_\phi(2\pi r)\end{aligned} \tag{3-69}$$

But, from Ampere's circuital law,

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0(\text{current enclosed by } C) = \mu_0(\text{current enclosed by circular path of radius } r) \tag{3-70}$$

Comparing (3-69) and (3-70), we have

$$B_\phi = \mu_0 \frac{\text{current enclosed by circular path of radius } r}{2\pi r}$$

$$\mathbf{B} = \mu_0 \frac{\text{current enclosed by circular path of radius } r}{2\pi r} \mathbf{i}_\phi \tag{3-71}$$

which agrees with the result of Example 3-7. ▮

3.7 Ampere's Circuital Law in Differential Form (Maxwell's Curl Equation for the Static Magnetic Field)

Let us consider a volume current distribution with the current density vector **J** as a given function of the coordinates. The current enclosed by an arbitrary closed path C is given by the surface integral of the current density over any surface S bounded by the closed path C; that is, $\int_S \mathbf{J} \cdot d\mathbf{S}$. According to Ampere's circuital law (3-57), we then have

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_S \mathbf{J} \cdot d\mathbf{S} \tag{3-72}$$

where C is traversed in the sense in which a right-hand screw needs to be turned if it is to advance to the side of S towards which the current on the right side of (3-72) is evaluated. If we now shrink the path C to a very small size ΔC so that the surface area bounded by it becomes very small, ΔS, we can write (3-72) as

$$\oint_{\Delta C} \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_{\Delta S} \mathbf{J} \cdot d\mathbf{S} \tag{3-73}$$

Since the surface area ΔS is very small, we can consider the current density to be uniform over the surface so that $\int_{\Delta S} \mathbf{J} \cdot d\mathbf{S} \approx \mathbf{J} \cdot \mathbf{i}_n \, \Delta S$, where $\mathbf{i}_n$ is the normal vector to ΔS pointed to the side towards which a right-hand screw advances as it is turned in the sense of the closed path. This relation becomes exact in the limit $\Delta S \longrightarrow 0$. Dividing both sides of (3-73) by ΔS and letting $\Delta S \longrightarrow 0$, we have

$$\lim_{\Delta S \to 0} \frac{\oint_{\Delta C} \mathbf{B} \cdot d\mathbf{l}}{\Delta S} = \lim_{\Delta S \to 0} \frac{\mu_0 \int_{\Delta S} \mathbf{J} \cdot d\mathbf{S}}{\Delta S} = \mu_0 \lim_{\Delta S \to 0} \frac{\mathbf{J} \cdot \mathbf{i}_n \, \Delta S}{\Delta S} = \mu_0 \mathbf{J} \cdot \mathbf{i}_n \tag{3-74}$$

Now, the curl of **B** is defined as the vector having the magnitude given by the maximum value of the quantity on the left side of (3-74) and the direction given by the normal to the ΔS for which the quantity is maximized. Looking at the right side of (3-74), we note that this maximum value occurs for an orientation of ΔS for which the direction of $\mathbf{i}_n$ coincides with the direction of **J** and it is equal to μ_0 times the magnitude of J. Thus

$$|\nabla \times \mathbf{B}| = \text{maximum value of} \left(\lim_{\Delta S \to 0} \frac{\oint_{\Delta C} \mathbf{B} \cdot d\mathbf{l}}{\Delta S} \right) = \mu_0 |\mathbf{J}| \qquad (3\text{-}75a)$$

$$\text{direction of } \nabla \times \mathbf{B} = \text{direction of } \mathbf{J} \qquad (3\text{-}75b)$$

so that

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \qquad (3\text{-}76)$$

Equation (3-76) is Ampere's circuital law in differential form. It states that the curl of the magnetic flux density at any point is equal to μ_0 times the volume current density at that point. This is Maxwell's curl equation for the static magnetic field.

The right side of (3-76) represents a volume current density. For problems involving line and surface currents, we make use of Dirac delta functions just as in the case of Gauss' law in differential form for point charges, line charges, and surface charges. For example, following the method employed in Example 2-12, we obtain for a surface current of density $\mathbf{J}_s$ occupying the $y = y_0$ plane,

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}_s \, \delta(y - y_0) \qquad (3\text{-}77)$$

3.8 Magnetic Vector Potential

Thus far we have discussed the determination of the magnetic field due to a current distribution directly from the current distribution using initially the Biot–Savart law and then Ampere's circuital law. In Chapter 2, we first discussed the determination of the electric field due to a charge distribution directly from the charge distribution using initially an integral formulation based on the electric field intensity due to a point charge and then Gauss' law. Later we introduced the electric potential field from energy considerations and discovered the relationship of the electric field intensity to the scalar potential through the gradient operation as an alternative approach to the determination of the electric field. In this section we introduce a similar alternative method for the computation of the magnetic field due to a given current distribution.

To do this, we note from (3-25) that, for a filamentary wire carrying current I, the magnetic flux density is given by

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{C'} \frac{I \, d\mathbf{l}' \times \mathbf{i}_R(\mathbf{r}, \mathbf{r}')}{R^2(\mathbf{r}, \mathbf{r}')} \qquad (3\text{-}78)$$

where C' is the contour of the wire, $\mathbf{r}'$ is the position vector defining the infinitesimal length element $d\mathbf{l}'$ on C', $\mathbf{r}$ is the position vector of the field point, $\mathbf{i}_R(\mathbf{r}, \mathbf{r}')$ is the unit vector along $\mathbf{r} - \mathbf{r}'$, and $R(\mathbf{r}, \mathbf{r}')$ is equal to $|\mathbf{r} - \mathbf{r}'|$. Substituting

$$\nabla\left(\frac{1}{R}\right) = -\frac{1}{R^2}\mathbf{i}_R$$

in (3-78), we have

$$\mathbf{B}(\mathbf{r}) = -\frac{\mu_0 I}{4\pi}\int_{C'} d\mathbf{l}' \times \nabla\left[\frac{1}{R(\mathbf{r}, \mathbf{r}')}\right] \tag{3-79}$$

Using the vector identity

$$\mathbf{A} \times \nabla V = V\nabla \times \mathbf{A} - \nabla \times (V\mathbf{A})$$

we can write (3-79) as

$$\mathbf{B}(\mathbf{r}) = -\frac{\mu_0 I}{4\pi}\int_{C'}\left[\frac{1}{R(\mathbf{r}, \mathbf{r}')}\nabla \times d\mathbf{l}' - \nabla \times \frac{d\mathbf{l}'}{R(\mathbf{r}, \mathbf{r}')}\right] \tag{3-80}$$

In (3-80), the integration is with respect to the points on the filamentary wire, whereas the curl operation has to do with differentiation with respect to the coordinates of the field point. Hence $\nabla \times d\mathbf{l}' = 0$ and also, the two operations can be interchanged to give us

$$\mathbf{B} = \frac{\mu_0 I}{4\pi}\int_{C'} \nabla \times \frac{d\mathbf{l}'}{R(\mathbf{r}, \mathbf{r}')} = \nabla \times \left(\frac{\mu_0}{4\pi}\int_{C'}\frac{I\,d\mathbf{l}'}{R}\right) \tag{3-81a}$$

If, instead of a filamentary wire, we have a surface current of density $\mathbf{J}_s$ on a surface S', or a volume current of density $\mathbf{J}$ in a volume V', we obtain similar relationships as follows, respectively:

$$\mathbf{B} = \nabla \times \left(\frac{\mu_0}{4\pi}\int_{S'}\frac{\mathbf{J}_s\,dS'}{R}\right) \tag{3-81b}$$

$$\mathbf{B} = \nabla \times \left(\frac{\mu_0}{4\pi}\int_{V'}\frac{\mathbf{J}\,dv'}{R}\right) \tag{3-81c}$$

In (3-81a)–(3-81c), we have expressions which permit us to compute $\mathbf{B}$ by finding the curl of a vector quantity. Denoting this vector quantity as $\mathbf{A}$, we have

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{3-82}$$

where

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int_{C'}\frac{I\,d\mathbf{l}'}{R} \quad \text{for line current} \tag{3-83a}$$

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int_{S'}\frac{\mathbf{J}_s\,dS'}{R} \quad \text{for surface current} \tag{3-83b}$$

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int_{V'}\frac{\mathbf{J}\,dv'}{R} \quad \text{for volume current} \tag{3-83c}$$

We note the similarity of the right sides of (3-83a)–(3-83c) with the expressions for the electrostatic potential V due to line, surface, and volume charges given, respectively, by

$$V = \frac{1}{4\pi\epsilon_0}\int_{C'} \frac{\rho_L\, dl'}{R} \qquad \text{for line charge}$$

$$V = \frac{1}{4\pi\epsilon_0}\int_{S'} \frac{\rho_S\, dS'}{R} \qquad \text{for surface charge}$$

$$V = \frac{1}{4\pi\epsilon_0}\int_{V'} \frac{\rho\, dv'}{R} \qquad \text{for volume charge}$$

In view of this similarity, and since **A** is a vector in contrast to the scalar nature of V, **A** is called the magnetic vector potential. Unlike V, **A** does not have a physical significance. It serves as a convenient intermediate step for the computation of **B**. This is especially so because of the similarity of the expressions for V and the expressions for **A**. The components of **A** due to a particular current distribution can be written without actually evaluating the integrals if the analogous integrals for the electrostatic potential have already been evaluated in the corresponding electrostatic problem.

EXAMPLE 3-11. An infinitely long straight wire carrying current I amp lies along the z axis. Obtain the magnetic vector potential due to this wire and then find the magnetic flux density by performing the curl operation on the vector potential.

Applying (3-83a) to the infinitely long wire, we have the vector potential given by

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int_{z'=-\infty}^{\infty}\left(\frac{I\,dz'\,\mathbf{i}_z}{R}\right)$$

or

$$\mathbf{A} = \left(\frac{\mu_0}{4\pi}\int_{z'=-\infty}^{\infty}\frac{I\,dz'}{R}\right)\mathbf{i}_z \tag{3-84}$$

where R is the distance of the point P at which **A** is to be computed from an infinitesimal current element $I\,dz'\,\mathbf{i}_z$, as shown in Fig. 3.20. Let us now consider the quantity

$$\left(\frac{1}{4\pi\epsilon_0}\int_{z'=-\infty}^{\infty}\frac{\rho_{L0}\,dz'}{R}\right)$$

This is the integral for computing the electrostatic potential due to an infinitely long line charge of uniform density ρ_{L0} lying along the z axis. This expression is analogous to the expression inside the parentheses on the right side of (3-84). Thus, finding the vector potential due to the infinitely long wire is analogous to determining the electrostatic potential due to the infinitely long line charge of uniform density. However, we already know the

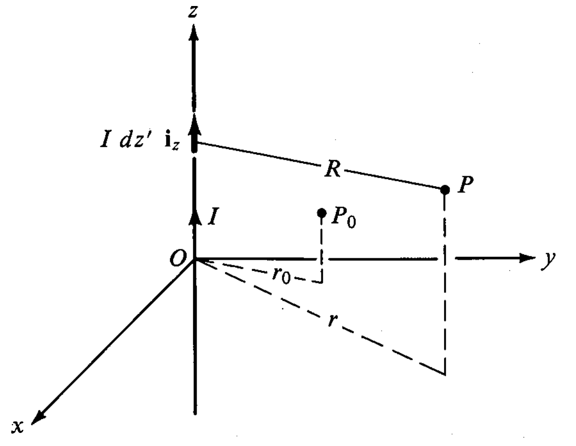

Fig. 3.20. For evaluating the magnetic vector potential due to an infinitely long, straight wire carrying current I.

solution for this electrostatic potential from Example 2-17. This is given by

$$V = -\frac{\rho_{L0}}{2\pi\epsilon_0}\ln\frac{r}{r_0} \tag{2-119}$$

where r is the distance of the point P, at which V is desired, from the line charge and r_0 is the distance from the line charge to the point at which the potential is zero, as explained in Example 2-17. Thus

$$\frac{1}{4\pi\epsilon_0}\int_{z'=-\infty}^{\infty}\frac{\rho_{L0}\,dz'}{R} = -\frac{\rho_{L0}}{2\pi\epsilon_0}\ln\frac{r}{r_0} \tag{3-85}$$

We can immediately write down by analogy that

$$\frac{\mu_0}{4\pi}\int_{z'=-\infty}^{\infty}\frac{I\,dz'}{R} = -\frac{\mu_0 I}{2\pi}\ln\frac{r}{r_0} \tag{3-86}$$

Substituting this result into (3-84), we obtain the vector potential due to the infinitely long wire as

$$\mathbf{A} = -\frac{\mu_0 I}{2\pi}\ln\frac{r}{r_0}\mathbf{i}_z \tag{3-87}$$

Using the expression for the curl in cylindrical coordinates, we then have

$$\mathbf{B} = \nabla\times\mathbf{A} = \begin{vmatrix} \frac{\mathbf{i}_r}{r} & \mathbf{i}_\phi & \frac{\mathbf{i}_z}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 0 & 0 & A_z \end{vmatrix}$$

$$= \frac{1}{r}\frac{\partial A_z}{\partial \phi}\mathbf{i}_r - \frac{\partial A_z}{\partial r}\mathbf{i}_\phi = \frac{\mu_0 I}{2\pi r}\mathbf{i}_\phi$$

which is the same as the result obtained in Example 3-3. ▮

EXAMPLE 3-12. A loop of wire carrying current I amp occupies an arbitrary contour C' as shown in Fig. 3.21. Find the vector potential due to this current loop at distances $\mathbf{r}$ from the origin large in magnitude compared to the distances of the points on the loop from the origin.

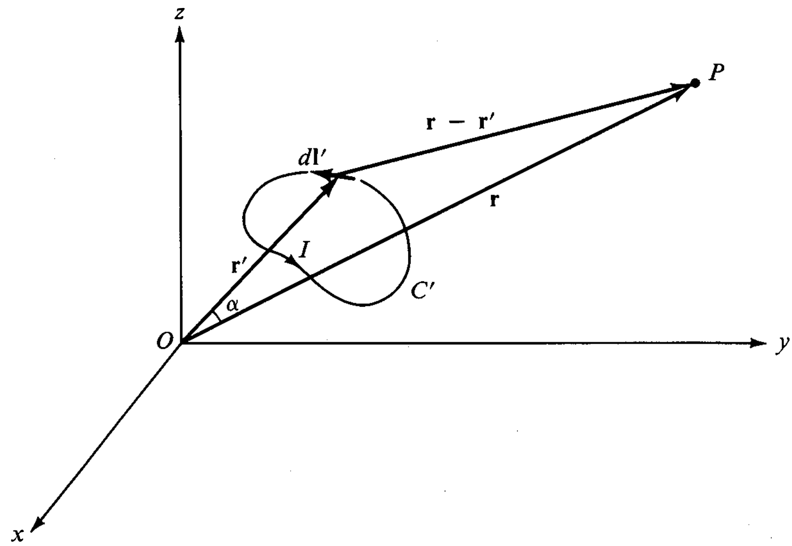

Fig. 3.21. For evaluating the vector potential due to an arbitrary loop of current I at large distances from the origin compared to the distances of the points on the loop from the origin.

Let P be the point at which the vector potential is desired. Then, from (3-83a), the vector potential at P due to the current loop is given by

$$\begin{aligned}\mathbf{A}(\mathbf{r}) &= \frac{\mu_0}{4\pi}\oint_{C'} \frac{I\,d\mathbf{l}'}{|\mathbf{r}-\mathbf{r}'|} \\ &= \frac{\mu_0 I}{4\pi}\oint_{C'} \frac{d\mathbf{l}'}{(r^2 + r'^2 - 2rr'\cos\alpha)^{1/2}} \\ &= \frac{\mu_0 I}{4\pi r}\oint_{C'} \left(1 + \frac{r'^2}{r^2} - \frac{2\mathbf{r}'\cdot\mathbf{r}}{r^2}\right)^{-1/2} d\mathbf{l}'\end{aligned} \tag{3-88}$$

Using the binomial expansion employed in Example 2-15, we have

$$\begin{aligned}\mathbf{A} &= \frac{\mu_0 I}{4\pi r}\oint_{C'} \left\{1 + \frac{\mathbf{r}'\cdot\mathbf{r}}{r^2} + \frac{1}{2r^4}[3(\mathbf{r}'\cdot\mathbf{r})^2 - r^2r'^2] \right. \\ &\quad \left. + \ldots \text{higher-order terms}\right\} d\mathbf{l}' \qquad (3\text{-}89) \\ &= \frac{\mu_0 I}{4\pi r}\left[\oint_{C'} d\mathbf{l}' + \oint_{C'} \frac{\mathbf{r}'\cdot\mathbf{r}}{r^2}\,d\mathbf{l}' + \oint_{C'} \frac{3(\mathbf{r}'\cdot\mathbf{r})^2 - r^2r'^2}{2r^4}\,d\mathbf{l}' + \ldots\right]\end{aligned}$$

In (3-89), $\oint_{C'} d\mathbf{l}' = 0$ for any C' so that the second term is the first significant term. Furthermore, for $r' \ll r$, it is sufficient if we consider the first significant term. Thus, for $r \gg r'$,

$$\mathbf{A} = \frac{\mu_0 I}{4\pi r^3} \oint_{C'} (\mathbf{r}' \cdot \mathbf{r})\, d\mathbf{l}' \tag{3-90}$$

Now, using the vector identity

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$$

we have

$$\mathbf{r} \times (d\mathbf{l}' \times \mathbf{r}') = (\mathbf{r} \cdot \mathbf{r}')\, d\mathbf{l}' - (\mathbf{r} \cdot d\mathbf{l}')\mathbf{r}'$$

or

$$\begin{aligned}(\mathbf{r}' \cdot \mathbf{r})\, d\mathbf{l}' &= \mathbf{r} \times (d\mathbf{l}' \times \mathbf{r}') + (\mathbf{r} \cdot d\mathbf{l}')\mathbf{r}' \\ &= \tfrac{1}{2}\mathbf{r} \times (d\mathbf{l}' \times \mathbf{r}') + \tfrac{1}{2}(\mathbf{r} \cdot d\mathbf{l}')\mathbf{r}' + \tfrac{1}{2}(\mathbf{r}' \cdot \mathbf{r})\, d\mathbf{l}'\end{aligned} \tag{3-91}$$

We further note that

$$\begin{aligned}&(\mathbf{r} \cdot d\mathbf{l}')\mathbf{r}' + (\mathbf{r}' \cdot \mathbf{r})\, d\mathbf{l}' \\ &\quad = (x\,dx' + y\,dy' + z\,dz')(x'\mathbf{i}_x + y'\mathbf{i}_y + z'\mathbf{i}_z) \\ &\qquad + (x'x + y'y + z'z)(dx'\,\mathbf{i}_x + dy'\,\mathbf{i}_y + dz'\,\mathbf{i}_z) \\ &\quad = (2xx'\,dx' + yy'\,dx' + zz'\,dx' + yx'\,dy' + zx'\,dz')\mathbf{i}_x \\ &\qquad + (xx'\,dy' + 2yy'\,dy' + zz'\,dy' + xy'\,dx' + zy'\,dz')\mathbf{i}_y \\ &\qquad + (xx'\,dz' + yy'\,dz' + 2zz'\,dz' + xz'\,dx' + yz'\,dy')\mathbf{i}_z \\ &\quad = d[(xx' + yy' + zz')(x'\mathbf{i}_x + y'\mathbf{i}_y + z'\mathbf{i}_z)] \\ &\quad = d[(\mathbf{r} \cdot \mathbf{r}')\mathbf{r}']\end{aligned} \tag{3-92}$$

Substituting (3-92) into (3-91), we obtain

$$(\mathbf{r}' \cdot \mathbf{r})\, d\mathbf{l}' = \tfrac{1}{2}\mathbf{r} \times (d\mathbf{l}' \times \mathbf{r}') + \tfrac{1}{2}\, d[(\mathbf{r} \cdot \mathbf{r}')\mathbf{r}'] \tag{3-93}$$

Substituting (3-93) into (3-90), we have

$$\begin{aligned}\mathbf{A} &= \frac{\mu_0 I}{4\pi r^3} \oint_{C'} \frac{1}{2}\mathbf{r} \times (d\mathbf{l}' \times \mathbf{r}') + \frac{\mu_0 I}{8\pi r^3} \oint_{C'} d[(\mathbf{r} \cdot \mathbf{r}')\mathbf{r}'] \\ &= \frac{\mu_0 I}{4\pi r^3} \oint_{C'} \frac{1}{2}\mathbf{r} \times (d\mathbf{l}' \times \mathbf{r}')\end{aligned} \tag{3-94}$$

since the second integral, being an integral of a total differential around a closed contour, is equal to zero. Finally, defining

$$\mathbf{m} = \frac{1}{2} \oint_{C'} \mathbf{r}' \times I\, d\mathbf{l}' \tag{3-95}$$

we obtain

$$\begin{aligned}\mathbf{A} &= \frac{\mu_0}{4\pi r^3} \left[\oint_{C'} \frac{1}{2}(\mathbf{r}' \times I\, d\mathbf{l}') \right] \times \mathbf{r} \\ &= \frac{\mu_0}{4\pi r^3} \mathbf{m} \times \mathbf{r}\end{aligned} \tag{3-96}$$

Thus, at large distances from the current loop, the vector potential falls off inversely as r^2 in contrast to the inverse distance dependence of the electrostatic potential at large distances from an arbitrary charge distribution, provided the total charge is not equal to zero. The quantity $\mathbf{m}$ is the dipole moment of the current loop about the origin. ▮

EXAMPLE 3-13. Show that, for a plane loop of wire carrying current I, the dipole moment $\mathbf{m}$ given by

$$\mathbf{m} = \frac{1}{2}\oint_{C'} \mathbf{r}' \times I\, d\mathbf{l}'$$

has a magnitude equal to the area of the loop and a direction normal to the plane of the loop drawn towards the direction of advance of a right-hand screw as it is turned in the sense of the contour C' of the loop.

First we show that the dipole moment about the origin is the same as the dipole moment about any other point. Letting the position vector of this arbitrary point be $\mathbf{r}_0$, we have

$$\begin{aligned}\mathbf{m} &= \frac{1}{2}\oint_{C'} \mathbf{r}' \times I\, d\mathbf{l}' \\ &= \frac{1}{2}\oint_{C'} (\mathbf{r}' - \mathbf{r}_0) \times I\, d\mathbf{l}' + \frac{1}{2}\oint_{C'} \mathbf{r}_0 \times I\, d\mathbf{l}' \\ &= \frac{1}{2}\oint_{C'} (\mathbf{r}' - \mathbf{r}_0) \times I\, d\mathbf{l}' + \frac{1}{2} I\mathbf{r}_0 \times \oint_{C'} d\mathbf{l}' \\ &= \frac{1}{2}\oint_{C'} (\mathbf{r}' - \mathbf{r}_0) \times I\, d\mathbf{l}'\end{aligned} \tag{3-97}$$

since $\oint_{C'} d\mathbf{l}' = 0$. Thus the dipole moment of the current loop is independent of the point about which it is computed. Let us therefore choose this point to be in the plane of the loop and inside the loop as shown in Fig. 3.22. Then

$$\tfrac{1}{2}(\mathbf{r}' - \mathbf{r}_0) \times d\mathbf{l}' = \tfrac{1}{2}|\mathbf{r}' - \mathbf{r}_0|\, dl' \sin\alpha\, \mathbf{i}_n \tag{3-98}$$

where α is the angle between $(\mathbf{r}' - \mathbf{r}_0)$ and $d\mathbf{l}'$ and $\mathbf{i}_n$ is the normal vector to the plane of the loop, drawn towards the direction of advance of a right-hand screw as it is turned in the sense of C' as shown in Fig. 3.22. But the magnitude on the right side of (3-98) is the area of the triangle formed by $(\mathbf{r}' - \mathbf{r}_0)$ and $d\mathbf{l}'$. Thus, since $(\mathbf{r}' - \mathbf{r}_0) \times d\mathbf{l}'$ is along $\mathbf{i}_n$ for all $d\mathbf{l}'$ on C', we have

$$\begin{aligned}\frac{1}{2}\oint_{C'} (\mathbf{r}' - \mathbf{r}_0) \times d\mathbf{l}' &= \begin{pmatrix}\text{sum of areas of triangles formed by all} \\ d\mathbf{l}' \text{ with the corresponding } \mathbf{r}' - \mathbf{r}_0\end{pmatrix} \mathbf{i}_n \\ &= (\text{area of the loop})\mathbf{i}_n\end{aligned} \tag{3-99}$$

This result is consistent with the dipole moment defined in Example 3-4 for the plane circular loop of radius a lying in the xy plane. ▮

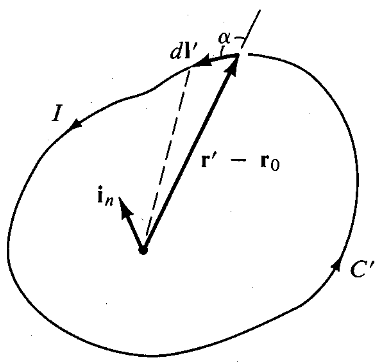

Fig. 3.22. For evaluating the dipole moment of a plane loop of wire carrying current *I*.

Returning to Eq. (3-82) and taking the curl of both sides, we obtain

$$\nabla \times \mathbf{B} = \nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} \tag{3-100}$$

where we have used the vector identity for $\nabla \times \nabla \times \mathbf{A}$. But, from Ampere's circuital law in differential form, we have

$$\nabla \times \mathbf{B} = \mu_0\mathbf{J} \tag{3-76}$$

Thus, from (3-100) and (3-76), we get

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} = \mu_0\mathbf{J} \tag{3-101}$$

However, considering a current loop, we have

$$\begin{aligned}\nabla \cdot \mathbf{A} &= \nabla \cdot \oint_{C'} \frac{\mu_0 I\, d\mathbf{l}'}{4\pi R} \\ &= \frac{\mu_0 I}{4\pi}\oint_{C'} \nabla \cdot \frac{d\mathbf{l}'}{R}\end{aligned} \tag{3-102}$$

where C' is the contour of the current loop and $d\mathbf{l}'$ is an infinitesimal length element on C'. Using the vector identity

$$\nabla \cdot V\mathbf{A} = \mathbf{A} \cdot \nabla V + V\nabla \cdot \mathbf{A}$$

we write (3-102) as

$$\nabla \cdot \mathbf{A} = \frac{\mu_0 I}{4\pi}\left(\oint_{C'} d\mathbf{l}' \cdot \nabla\frac{1}{R} + \oint_{C'} \frac{1}{R}\nabla \cdot d\mathbf{l}'\right) \tag{3-103}$$

On the right side of (3-103), the second integral is zero since $\nabla \cdot d\mathbf{l}' = 0$. Using $\nabla(1/R) = -\nabla'(1/R)$ where the prime denotes differentiation with respect to the primed variables, and then using Stoke's theorem, the first integral can be written as

$$\oint_{C'} d\mathbf{l}' \cdot \nabla\frac{1}{R} = -\oint_{C'} \nabla'\frac{1}{R} \cdot d\mathbf{l}' = -\int_{S'} \nabla' \times \nabla'\frac{1}{R} \cdot d\mathbf{S}' \tag{3-104}$$

where S' is any surface whose perimeter is C'. But the curl of the gradient of a scalar is identically equal to zero. Hence, the right side of (3-104) is zero. Thus, for a current loop, $\nabla \cdot \mathbf{A} = 0$. If we now consider a region of volume current in which there is no accumulation of charge, we can represent the volume current as a superposition of a number of current loops for each of which $\nabla \cdot \mathbf{A} = 0$ so that, for the entire volume current, $\nabla \cdot \mathbf{A} = 0$. Substituting this result in (3-101), we obtain

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} \tag{3-105}$$

In analogy with

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \tag{2-140}$$

Equation (3-105) is known as the Poisson's equation for the vector potential. It is a differential equation which relates the magnetic vector potential at a point to the volume current density at that point, just as (2-140) is a differential equation which relates the electrostatic potential at a point to the volume charge density at that point. Equation (3-105) is a vector equation and hence it is equivalent to three scalar equations. For example, in rectangular coordinates,

$$\nabla^2 \mathbf{A} = (\nabla^2 A_x)\mathbf{i}_x + (\nabla^2 A_y)\mathbf{i}_y + (\nabla^2 A_z)\mathbf{i}_z$$

so that we have

$$\nabla^2 A_x = -\mu_0 J_x \tag{3-106a}$$

$$\nabla^2 A_y = -\mu_0 J_y \tag{3-106b}$$

$$\nabla^2 A_z = -\mu_0 J_z \tag{3-106c}$$

If the volume current density is zero in a region, then the right side of (3-105) is zero for that region so that (3-105) reduces to

$$\nabla^2 \mathbf{A} = 0 \qquad \text{for } \mathbf{J} = 0 \tag{3-107}$$

which is Laplace's equation for the magnetic vector potential, in analogy with Laplace's equation for the electrostatic potential given by

$$\nabla^2 V = 0 \qquad \text{for } \rho = 0 \tag{2-141}$$

It states that the Laplacian of the magnetic vector potential in a region devoid of current is zero, just as (2-141) states that the Laplacian of the electrostatic potential in a region devoid of charges is zero. Again, using the expansion for $\nabla^2 \mathbf{A}$ in rectangular coordinates, we obtain the three component equations for (3-107) as

$$\nabla^2 A_x = 0 \tag{3-108a}$$

$$\nabla^2 A_y = 0 \tag{3-108b}$$

$$\nabla^2 A_z = 0 \tag{3-108c}$$

For a given current distribution, the solution to Poisson's equation (3-105) is obtained by solving the three component equations (3-106a)–(3-106c). Again, we can take advantage of the similarity of (3-106a)–(3-106c) with (2-140) and in many cases simply write down the solution from previous knowledge of electrostatics, without the necessity of solving the differential equations.

3.9 Maxwell's Divergence Equation for the Magnetic Field

The divergence of the curl of a vector is identically zero. Since

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{3-82}$$

it then follows that

$$\nabla \cdot \mathbf{B} = 0 \tag{3-109}$$

Equation (3-109) is Maxwell's divergence equation for the magnetic field. Together with Maxwell's curl equation for the static magnetic field given by (3-76), (3-109) completely defines the properties of the static magnetic field. Equation (3-109) determines whether or not a given vector field is realizable as a magnetic field, whereas Eq. (3-76) relates the field to the current distribution responsible for producing the field. When compared with Maxwell's divergence equation for the electric field intensity,

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{2-82}$$

Eq. (3-109) reveals the fact that isolated magnetic charges do not exist.

Taking the volume integral of both sides of (3-109) in a volume V, we have

$$\int_V (\nabla \cdot \mathbf{B})\, dv = 0 \tag{3-110}$$

But, according to the divergence theorem,

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = \int_V (\nabla \cdot \mathbf{B})\, dv$$

where S is the surface bounding the volume V. Since (3-110) is true for any volume, we obtain the result that

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0 \tag{3-111}$$

for any closed surface S. Equation (3-111) is the integral form of the divergence equation (3-109). Since $\mathbf{B}$ is the magnetic flux density, $\oint_S \mathbf{B} \cdot d\mathbf{S}$ is the total magnetic flux emanating from the surface S. Thus Eq. (3-111) states that the total magnetic flux emanating from any closed surface is equal to zero. Whatever flux goes into the volume bounded by the surface must come out of it. The magnetic field lines form closed paths, unlike electric field lines which begin from positive charges and terminate on negative charges. Since

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 (\text{current enclosed by } C) \tag{3-57}$$

the closed paths must form around the current producing the magnetic field. Vectors which, in this manner, are characterized by zero net flux over all possible closed surfaces are said to be solenoidal. The current density vector **J** for static fields is another example of a solenoidal vector since, from

$$\nabla \cdot \nabla \times \mathbf{B} = 0 \tag{3-112}$$

we have

$$\nabla \cdot \mu_0 \mathbf{J} = 0$$

or

$$\nabla \cdot \mathbf{J} = 0 \tag{3-113}$$

The solenoidal nature of **J** follows from the fact that, in the absence of accumulation of charge at a point with time, current must flow in closed paths. Since we are here considering static phenomena, there cannot be any accumulation of charge and hence $\nabla \cdot \mathbf{J} = 0$. On the other hand, when we consider time-varying or dynamic fields, we can allow for the accumulation of charge, in which case we will find that (3-113) does not necessarily hold everywhere.

EXAMPLE 3-14. Determine if the following vector fields are realizable as magnetic fields:

(a) $\mathbf{F}_a = (-y\mathbf{i}_x + x\mathbf{i}_y)$ cartesian coordinates

(b) $\mathbf{F}_b = \dfrac{\mu_0 m_L}{2\pi r^2}(-\sin\phi\, \mathbf{i}_r + \cos\phi\, \mathbf{i}_\phi)$ cylindrical coordinates

(c) $\mathbf{F}_c = (\sin\theta\, \mathbf{i}_r + \cos\theta\, \mathbf{i}_\theta)$ spherical coordinates

(a) $\nabla \cdot \mathbf{F}_a = \dfrac{\partial}{\partial x}(-y) + \dfrac{\partial}{\partial y}(x) = 0$

Hence $\mathbf{F}_a$ can be realized as a magnetic field. In fact, if we note that, in cylindrical coordinates, $\mathbf{F}_a = r\mathbf{i}_\phi$, the solenoidal nature of $\mathbf{F}_a$ becomes obvious.

(b) $\nabla \cdot \mathbf{F}_b = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(-\dfrac{\mu_0 m_L}{2\pi r}\sin\phi\right) + \dfrac{1}{r}\dfrac{\partial}{\partial\phi}\left(\dfrac{\mu_0 m_L}{2\pi r^2}\cos\phi\right) = 0$

Hence $\mathbf{F}_b$ can be realized as a magnetic field. It is left as an exercise (Problem 3.21) for the student to show that $\mathbf{F}_b$ is the magnetic field due to a two-dimensional magnetic dipole of moment m_L.

(c) $\nabla \cdot \mathbf{F}_c = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}(r^2 \sin\theta) + \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial\theta}(\sin\theta\cos\theta) \neq 0$

Hence $\mathbf{F}_c$ cannot be realized as a magnetic field. ▌

EXAMPLE 3-15. In Example 3-5, the magnetic field due to an infinitely long, uniformly wound solenold of radius a and n turns per unit length carrying current I was found by using the Biot–Savart law. It is here desired to find the magnetic field due to the solenoid from Ampere's circuital law and the solenoidal character of the magnetic field.

Employing a cylindrical coordinate system with the z axis as the axis of the solenoid, let us assume that the magnetic field due to the solenoid has all three components B_r, B_ϕ, and B_z. Because of the cylindrical symmetry and infinite length of the solenoid, all three components must be independent of ϕ and z. Thus B_r, B_ϕ, and B_z can be functions of r only. Now, applying (3-111) to a cylindrical box of radius b, length l and coaxial with the solenoid, as shown in Fig. 3.23(a), we have

$$\oint_{\substack{\text{surface of the}\\\text{cylindrical box}}} \mathbf{B} \cdot d\mathbf{S} = 0 \tag{3-114}$$

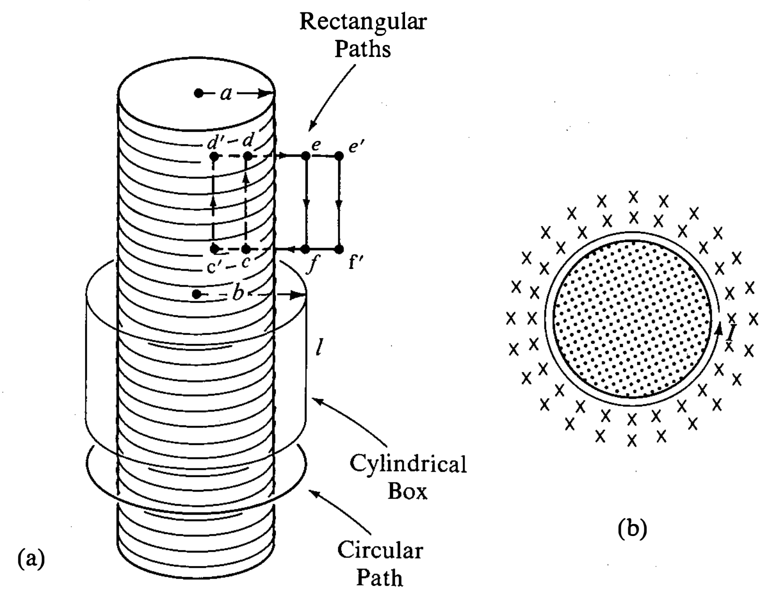

Fig. 3.23. For evaluating the magnetic field due to an infinitely long, uniformly wound solenoid using Ampere's circuital law and the solenoidal character of the magnetic field.

But

$$\oint_{\substack{\text{surface of the}\\\text{cylindrical box}}} \mathbf{B} \cdot d\mathbf{S} = \int_{\substack{\text{cylindrical}\\\text{surface}}} \mathbf{B} \cdot d\mathbf{S} + \int_{\substack{\text{upper plane}\\\text{surface}}} \mathbf{B} \cdot d\mathbf{S} + \int_{\substack{\text{lower plane}\\\text{surface}}} \mathbf{B} \cdot d\mathbf{S} \tag{3-115}$$

On the cylindrical surface,

$$\mathbf{B} \cdot d\mathbf{S} = [B_r\mathbf{i}_r + B_\phi\mathbf{i}_\phi + B_z\mathbf{i}_z]_{r=b} \cdot b\, d\phi\, dz\, \mathbf{i}_r = [B_r]_{r=b} b\, d\phi\, dz$$

$$\int \mathbf{B} \cdot d\mathbf{S} = \int_{z=z}^{z+l} \int_{\phi=0}^{2\pi} [B_r]_{r=b} b\, d\phi\, dz = 2\pi bl[B_r]_{r=b} \tag{3-116}$$

since $[B_r]_{r=b}$ is a constant.

On the upper plane surface,

$$\mathbf{B} \cdot d\mathbf{S} = (B_r\mathbf{i}_r + B_\phi\mathbf{i}_\phi + \mathbf{B}_z\mathbf{i}_z) \cdot r\,dr\,d\phi\,\mathbf{i}_z = B_z(r)\,r\,dr\,d\phi \tag{3-117a}$$

On the lower plane surface

$$\mathbf{B} \cdot d\mathbf{S} = (B_r\mathbf{i}_r + B_\phi\mathbf{i}_\phi + B_z\mathbf{i}_z) \cdot (-r\,dr\,d\phi\,\mathbf{i}_z) = -B_z(r)\,r\,dr\,d\phi \tag{3-117b}$$

We see from (3-117a) and (3-117b) that $\int \mathbf{B} \cdot d\mathbf{S}$ on the upper plane surface cancels exactly with $\int \mathbf{B} \cdot d\mathbf{S}$ on the lower plane surface since the integrands are equal and opposite and the limits of integration are the same. Thus

$$\oint_{\substack{\text{surface of the} \\ \text{cylindrical box}}} \mathbf{B} \cdot d\mathbf{S} = 2\pi bl[B_r]_{r=b} \tag{3-118}$$

Comparing (3-118) and (3-114), we obtain the result that $[B_r]_{r=b} = 0$. Since the radius b can be chosen to be any value, it follows that

$$B_r = 0 \qquad \text{for all } r$$

Applying Ampere's circuital law to a circular path of radius b, as shown in Fig. 3.23(a) in the plane normal to the axis of the solenoid and centered at the axis of the solenoid, we have

$$\oint_{\substack{\text{circular} \\ \text{path}}} \mathbf{B} \cdot d\mathbf{l} = 0 \tag{3-119}$$

since the path does not enclose any current. But, along the circular path,

$$\mathbf{B} \cdot d\mathbf{l} = [B_r\mathbf{i}_r + B_\phi\mathbf{i}_\phi + B_z\mathbf{i}_z]_{r=b} \cdot b\,d\phi\,\mathbf{i}_\phi = [B_\phi]_{r=b}b\,d\phi$$
$$\oint \mathbf{B} \cdot d\mathbf{l} = \int_{\phi=0}^{2\pi} [B_\phi]_{r=b}b\,d\phi = 2\pi b[B_\phi]_{r=b} \tag{3-120}$$

since $[B_\phi]_{r=b}$ is a constant. Comparing (3-120) with (3-119), we obtain the result that $[B_\phi]_{r=b} = 0$. Since the radius b can be chosen to be any value, it follows that

$$B_\phi = 0 \qquad \text{for all } r$$

Thus the magnetic field due to the solenoid has only a z component and we are now left with the task of finding this component.

Applying Ampere's circuital law for two rectangular paths *cdefc* and *cde′f′c* in the plane containing the solenoid axis, as shown in Fig. 3.23(a), we have

$$\oint_{\text{cdefc}} \mathbf{B} \cdot d\mathbf{l} = \oint_{\text{cde}'\text{f}'\text{c}} \mathbf{B} \cdot d\mathbf{l} = \mu_0 nI(cd) \tag{3-121}$$

Since the three sides *cd*, *de*, and *fc* are common to the two rectangular paths, (3-121) gives us

$$\int_e^f \mathbf{B} \cdot d\mathbf{l} = \int_{e'}^{f'} \mathbf{B} \cdot d\mathbf{l} \tag{3-122}$$

Along paths ef and $e'f'$,

$$\mathbf{B} \cdot d\mathbf{l} = [B_r\mathbf{i}_r + B_\phi\mathbf{i}_\phi + B_z\mathbf{i}_z] \cdot dz\,\mathbf{i}_z = B_z\,dz$$

and since B_z is independent of z, (3-122) yields

$$[B_z]_{ef}(ef) = [B_z]_{e'f'}(e'f')$$

or

$$[B_z]_{ef} = [B_z]_{e'f'} \tag{3-123}$$

Thus B_z is independent of r (in addition to ϕ and z) outside the solenoid. Similarly, by applying Ampere's circuital law to the two rectangular paths $cdefc$ and $c'd'efc'$ in the plane containing the solenoid axis, we can show that B_z is independent of r (in addition to ϕ and z) inside the solenoid. Thus the values of B_z both inside and outside the solenoid are constants. This requires that B_z outside the solenoid be equal to zero since, if it is nonzero, the amount of magnetic flux outside the solenoid will be infinity and for this flux to return in the opposite direction inside the solenoid as shown in Fig. 3.23(b), the flux density inside the solenoid must be infinity. But then, if the flux density inside the solenoid is infinity and that outside the solenoid is finite, (3-121) cannot be satisfied. On the other hand, for a finite amount of flux inside the solenoid in one direction to return in the opposite direction outside the solenoid, it requires zero flux density outside the solenoid since the area of cross section outside the solenoid is infinity ($\infty \times 0 =$ nonzero). Thus we conclude that B_z is zero outside the solenoid. It remains to evaluate B_z inside the solenoid. To do this, we write (3-121) as

$$\int_c^d \mathbf{B} \cdot d\mathbf{l} + \int_d^e \mathbf{B} \cdot d\mathbf{l} + \int_e^f \mathbf{B} \cdot d\mathbf{l} + \int_f^c \mathbf{B} \cdot d\mathbf{l} = \mu_0 nI(cd) \tag{3-124}$$

In (3-124),

$$\int_c^d \mathbf{B} \cdot d\mathbf{l} = [B_z]_{cd}(cd) \tag{3-125a}$$

$$\int_d^e \mathbf{B} \cdot d\mathbf{l} = 0 \qquad \text{since } \mathbf{B} \text{ is normal to the path} \tag{2-125b}$$

$$\int_e^f \mathbf{B} \cdot d\mathbf{l} = 0 \qquad \text{since } \mathbf{B} \text{ is zero outside the solenoid} \tag{3-125c}$$

$$\int_f^c \mathbf{B} \cdot d\mathbf{l} = 0 \qquad \text{since } \mathbf{B} \text{ is normal to the path} \tag{3-125d}$$

Substituting (3-125a)–(3-125d) into (3-124), we obtain

$$[B_z]_{cd}(cd) = \mu_0 nI(cd)$$

or

$$[B_z]_{cd} = \mu_0 nI \tag{3-126}$$

The constant value of B_z inside the solenoid is equal to $\mu_0 nI$. Thus

$$\mathbf{B} = \begin{cases} \mu_0 nI\,\mathbf{i}_z & \text{inside the solenoid} \\ 0 & \text{outside the solenoid} \end{cases}$$

which agrees with the result obtained in Example 3-5 by using the Biot–Savart law. However, compared with Example 3-5, we have here obtained the solution in a conceptual manner, gaining in this process considerable insight into the properties of the magnetic field. ▌

3.10 Summary and Further Discussion of Static Electric and Magnetic Field Laws and Formulas

Now that we have gained familiarity with the static magnetic field as well as the static electric field, it is worthwhile to list the basic laws governing the two fields and important formulas derived from them and make a few further comments. Accordingly, these laws and formulas are summarized in Table 3.1. Note that we have repeated Maxwell's equations at the end of the table. These equations pertain to the divergence and curl of the static electric and magnetic fields. We note from these equations that static vector fields, that is, vector fields independent of time, may be classified into four groups, depending on the values of their divergence and curl in the region of interest. These groups are as follows:

(a) Divergence of the field is not zero but its curl is zero. This represents a static electric field.

(b) Divergence of the field is zero but its curl is not zero. This represents a static magnetic field.

(c) Both divergence and curl of the field are zero. This represents either a static electric field in a charge-free region or a static magnetic field in a current-free region.

(d) Both divergence and curl of the field are not zero. Obviously, this represents a combination of the fields belonging to groups (a) and (b) and hence cannot be realized solely as a static electric field or solely as a static magnetic field.

In Table 3.2 we list the expressions for the electric and magnetic fields for two simple analogous pairs of source distributions: infinitely long line charge of uniform density versus infinitely long filamentary wire of current along the z axis, and infinite sheet charge of uniform density versus infinite sheet current of uniform density. For each pair, the analogy between the two fields is obvious from the expressions: The magnitudes of the fields are proportional to each other whereas their directions are orthogonal. This analogy is actually more general than is indicated by these two cases. To illustrate this, let us consider a charge distribution of density $\rho(x, y)$ and a current distribution of density $\mathbf{J} = J_z(x, y)\mathbf{i}_z$ such that

$$J_z(x, y) = k\rho(x, y) \tag{3-127}$$

where k is a proportionality constant. The electrostatic potential $V(x, y)$ corresponding to $\rho(x, y)$ and the magnetic vector potential $\mathbf{A} = A_z(x, y)\mathbf{i}_z$

TABLE 3.1. Summary of Basic Laws and Important Formulas Associated with the Static Electric and Magnetic Fields

	Static Electric Field	Static Magnetic Field
Definition	$\mathbf{F} = q\mathbf{E}$	$\mathbf{F} = q\mathbf{v} \times \mathbf{B} = I\,d\mathbf{l} \times \mathbf{B}$
Experimental laws	Coulomb's law: $\mathbf{F}_{21} = \dfrac{Q_1 Q_2}{4\pi\epsilon_0 R_{21}^3}\mathbf{R}_{21}$	Ampere's law of force: $\mathbf{F}_{21} = \dfrac{\mu_0}{4\pi}\oint_{C_1}\oint_{C_2} \dfrac{I_2\,d\mathbf{l}_2 \times (I_1\,d\mathbf{l}_1 \times \mathbf{R}_{21})}{R_{21}^3}$
Fields due to point sources	$\mathbf{E} = \dfrac{Q}{4\pi\epsilon_0 R^3}\mathbf{R}$	$\mathbf{B} = \mu_0 \dfrac{I\,d\mathbf{l} \times \mathbf{R}}{4\pi R^3}$
Fields due to continuous source distributions:		
Line	$\mathbf{E} = \displaystyle\int_{C'} \frac{[\rho_L(\mathbf{r}')](\mathbf{r} - \mathbf{r}')\,dl'}{4\pi\epsilon_0 \lvert\mathbf{r} - \mathbf{r}'\rvert^3}$	$\mathbf{B} = \displaystyle\int_{C'} \frac{\mu_0 I\,d\mathbf{l}' \times (\mathbf{r} - \mathbf{r}')}{4\pi \lvert\mathbf{r} - \mathbf{r}'\rvert^3}$
Surface	$\mathbf{E} = \displaystyle\int_{S'} \frac{[\rho_s(\mathbf{r}')](\mathbf{r} - \mathbf{r}')\,dS'}{4\pi\epsilon_0 \lvert\mathbf{r} - \mathbf{r}'\rvert^3}$	$\mathbf{B} = \displaystyle\int_{S'} \frac{\mu_0 \mathbf{J}_s(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')\,dS'}{4\pi \lvert\mathbf{r} - \mathbf{r}'\rvert^3}$
Volume	$\mathbf{E} = \displaystyle\int_{V'} \frac{[\rho(\mathbf{r}')](\mathbf{r} - \mathbf{r}')\,dv'}{4\pi\epsilon_0 \lvert\mathbf{r} - \mathbf{r}'\rvert^3}$	$\mathbf{B} = \displaystyle\int_{V'} \frac{\mu_0 \mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')\,dv'}{4\pi \lvert\mathbf{r} - \mathbf{r}'\rvert^3}$
Integral laws involving sources	Gauss' law: $\oint_S \mathbf{E} \cdot d\mathbf{S} = \dfrac{1}{\epsilon_0}\begin{pmatrix}\text{charge} \\ \text{enclosed by } S\end{pmatrix}$	Ampere's circuital law: $\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 \begin{pmatrix}\text{current enclosed} \\ \text{by } C\end{pmatrix}$
Differential laws involving sources	$\nabla \cdot \mathbf{E} = \dfrac{\rho}{\epsilon_0}$	$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$
Integral laws independent of sources	Conservative property: $\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$	Solenoidal property: $\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$
Differential laws independent of sources	$\nabla \times \mathbf{E} = 0$	$\nabla \cdot \mathbf{B} = 0$
Potentials	Scalar potential: $\mathbf{E} = -\nabla V$	Vector potential: $\mathbf{B} = \nabla \times \mathbf{A}$
Potentials due to point sources	$V = \dfrac{Q}{4\pi\epsilon_0 R}$	$\mathbf{A} = \dfrac{\mu_0 I\,d\mathbf{l}}{4\pi R}$
Potentials due to continuous source distributions:		
Line	$V = \displaystyle\int_{C'} \frac{\rho_L(\mathbf{r}')\,dl'}{4\pi\epsilon_0 \lvert\mathbf{r} - \mathbf{r}'\rvert}$	$\mathbf{A} = \displaystyle\int_{C'} \frac{\mu_0 I\,d\mathbf{l}'}{4\pi \lvert\mathbf{r} - \mathbf{r}'\rvert}$
Surface	$V = \displaystyle\int_{S'} \frac{\rho_s(\mathbf{r}')\,dS'}{4\pi\epsilon_0 \lvert\mathbf{r} - \mathbf{r}'\rvert}$	$\mathbf{A} = \displaystyle\int_{S'} \frac{\mu_0 \mathbf{J}_s(\mathbf{r}')\,dS'}{4\pi \lvert\mathbf{r} - \mathbf{r}'\rvert}$
Volume	$V = \displaystyle\int_{V'} \frac{\rho(\mathbf{r}')\,dv'}{4\pi\epsilon_0 \lvert\mathbf{r} - \mathbf{r}'\rvert}$	$\mathbf{A} = \displaystyle\int_{V'} \frac{\mu_0 \mathbf{J}(\mathbf{r}')\,dv'}{4\pi \lvert\mathbf{r} - \mathbf{r}'\rvert}$
Differential equations for potentials	$\nabla^2 V = -\dfrac{\rho}{\epsilon_0}$	$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$
Maxwell's equations:		
Divergence equation	$\nabla \cdot \mathbf{E} = \dfrac{\rho}{\epsilon_0}$	$\nabla \cdot \mathbf{B} = 0$
Curl equation	$\nabla \times \mathbf{E} = 0$	$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$

TABLE 3.2. Electric and Magnetic Fields for Two Pairs of Analogous Source Distributions

Electric Field	*Magnetic Field*
Infinitely long, straight line charge of uniform density ρ_{L0}:	Infinitely long, straight wire of current I:
$\mathbf{E} = \frac{\rho_{L0}}{2\pi\epsilon_0 r}\mathbf{i}_r$	$\mathbf{B} = \frac{\mu_0 I}{2\pi r}\mathbf{i}_\phi$
Infinite sheet charge of uniform density ρ_{s0}:	Infinite sheet current of uniform density $\mathbf{J}_{s0}$:
$\mathbf{E} = \frac{\rho_{s0}}{2\epsilon_0}\mathbf{i}_n$	$\mathbf{B} = \frac{\mu_0}{2}\mathbf{J}_{s0} \times \mathbf{i}_n$

corresponding to $J_z(x, y)\mathbf{i}_z$ satisfy the equations

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \tag{3-128}$$

and

$$(\nabla^2 A_z)\mathbf{i}_z = -\mu_0 J_z \mathbf{i}_z = -\mu_0 k\rho\mathbf{i}_z \tag{3-129}$$

respectively. Comparing (3-128) and (3-129), we have

$$A_z = k\mu_0\epsilon_0 V \tag{3-130}$$

We then obtain

$$\begin{aligned}\frac{E}{B} &= \frac{|\nabla V|}{|\nabla \times A_z\mathbf{i}_z|} = \frac{[(\partial V/\partial x)^2 + (\partial V/\partial y)^2]^{1/2}}{[(\partial A_z/\partial x)^2 + (\partial A_z/\partial y)^2]^{1/2}} \\ &= \frac{1}{k\mu_0\epsilon_0}\frac{[(\partial V/\partial x)^2 + (\partial V/\partial y)^2]^{1/2}}{[(\partial V/\partial x)^2 + (\partial V/\partial y)^2]^{1/2}} = \frac{1}{k\mu_0\epsilon_0}\end{aligned} \tag{3-131}$$

and

$$\begin{aligned}\mathbf{E} \cdot \mathbf{B} &= -\nabla V \cdot (\nabla \times A_z\mathbf{i}_z) \\ &= -\nabla V \cdot (\nabla \times k\mu_0\epsilon_0 V\mathbf{i}_z) \\ &= -k\mu_0\epsilon_0 \nabla V \cdot (\nabla \times V\mathbf{i}_z) \\ &= -k\mu_0\epsilon_0 \nabla V \cdot (\nabla V \times \mathbf{i}_z + V\nabla \times \mathbf{i}_z) \\ &= -k\mu_0\epsilon_0[\nabla V \cdot \nabla V \times \mathbf{i}_z] = 0\end{aligned} \tag{3-132}$$

Thus, for analogous charge and current distributions which vary only in two dimensions x and y (or r and ϕ in cylindrical coordinates) and with the current flow along the z direction, the electric and magnetic fields are proportional in magnitude and orthogonal in direction. We will use this important result in chapter 6.

PROBLEMS

3.1. An electron moving with a velocity $\mathbf{v}_1 = \mathbf{i}_x$ m/sec at a point in a magnetic field experiences a force $\mathbf{F}_1 = -e\mathbf{i}_y$ N, where e is the charge of the electron. If the electron is moving with a velocity $\mathbf{v}_2 = (\mathbf{i}_y + \mathbf{i}_z)$ m/sec at the same point, it experiences a force $\mathbf{F}_2 = e\mathbf{i}_x$ N. Find the force the electron would experience if it were moving with a velocity $\mathbf{v}_3 = \mathbf{v}_1 \times \mathbf{v}_2$ at the same point.

3.2. A mass spectrograph is a device for separating charged particles having different masses. Consider two particles of the same charge q but different masses m_1 and m_2 injected into the region of a uniform magnetic field $\mathbf{B}$ with a known velocity $\mathbf{v}$ normal to the magnetic field as shown in Fig. 3.24. Show that the particles are separated by a distance $d = |2(m_2 - m_1)\mathbf{v}|/|q\mathbf{B}|$ in the plane normal to the incident velocity.

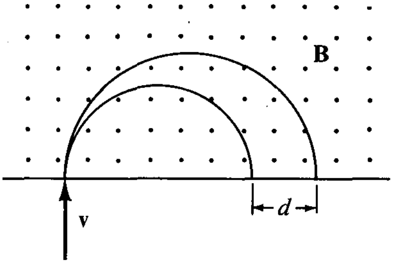

Fig. 3.24. For Problem 3.2.

3.3. A magnetic field given by

$$\mathbf{B} = B_0\mathbf{i}_z$$

where B_0 is a constant exists in the space between two parallel metallic plates of length L as shown in Fig. 3.25. A small test charge q having a mass m enters the

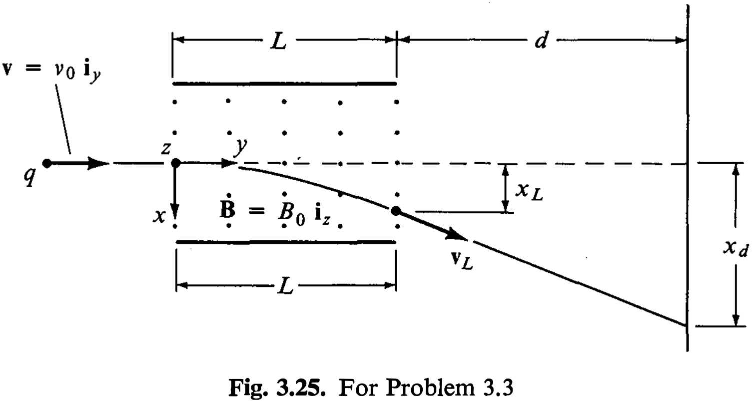

Fig. 3.25. For Problem 3.3

region between the plates at $t = 0$ with a velocity $\mathbf{v} = v_0\mathbf{i}_y$ as shown in the figure.
(a) Show that the path of the test charge between the plates is circular.
(b) Find the position x_L along the x direction and the velocity $\mathbf{v}_L$ of the test charge just after it emerges from the field region.
(c) Find the deflection x_d undergone by the test charge along the x direction at a distance d from the plates in the y direction.

3.4. In a region of magnetic field $\mathbf{B} = B_0\mathbf{i}_z$, where B_0 is a constant, an electron starts out at the origin with an initial velocity $\mathbf{v}_0 = v_{x0}\mathbf{i}_x + v_{y0}\mathbf{i}_y + v_{z0}\mathbf{i}_z$. Obtain the equations of motion of the electron and show that the path of the electron is a helix of radius $m\sqrt{v_{x0}^2 + v_{y0}^2}/|eB_0|$ and pitch $2\pi m|v_{z0}|/|eB_0|$, where e and m are the charge and mass of the electron.

3.5. Find the current required to counteract the earth's gravitational force on a horizontal filamentary wire of length l and mass m and oriented in the east-west direction in a uniform magnetic field B_0 directed northward. Compute the value of this current for a wire of length 1 meter and mass 30 grams situated in the earth's magnetic field at the magnetic equator assuming a value of 0.3×10^{-4} Wb/m² for B_0.

3.6. A rigid loop of wire in the form of a square of sides a m is hung by pivoting one side along the x axis as shown in Fig. 3.26. The loop is free to swing about the pivoted side without friction. The mass of the wire is m kg/m. If the wire is situated in a uniform magnetic field $\mathbf{B} = B_0\mathbf{i}_z$ and carries a current I amp, find the angle by which the loop swings from the vertical.

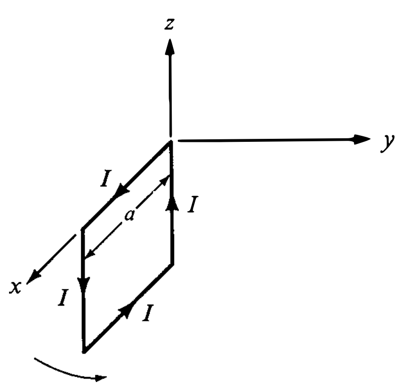

Fig. 3.26. For Example 3.6

3.7. A rigid rectangular loop of wire carrying current I amp and symmetrically situated about the z axis is in the yz plane as shown in Fig. 3.27. If the loop is situated in a uniform magnetic field $\mathbf{B}$ and is free to swing about the z axis, show that the torque acting on the loop is $IA(\mathbf{i}_y \cdot \mathbf{B})\mathbf{i}_z$ where A is the area of the loop.

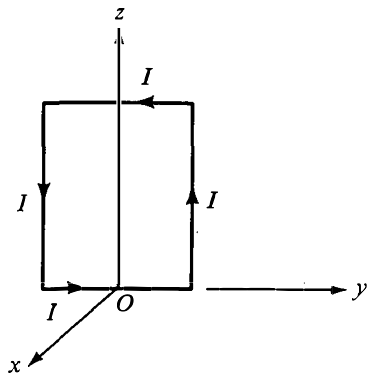

Fig. 3.27. For Problem 3.7.

3.8. Show that the total force experienced by a current loop C_1 carrying current I_1 due to another current loop C_2 carrying current I_2 is equal and opposite to the total force experienced by the current loop C_2 due to the current loop C_1; that is, show that Newton's third law holds for current loops.

3.9. Two circular loops of radii 1 m carrying currents I_1 and I_2 amp are situated in the $z = 0$ m and $z = 1$ m planes, respectively, and with their centers on the z axis, as shown in Fig. 3.28. Find the forces experienced by the current elements $I_1\,d\mathbf{l}_1$, $I_2\,d\mathbf{l}_2$ and $I_2\,d\mathbf{l}_3$ due to each other.

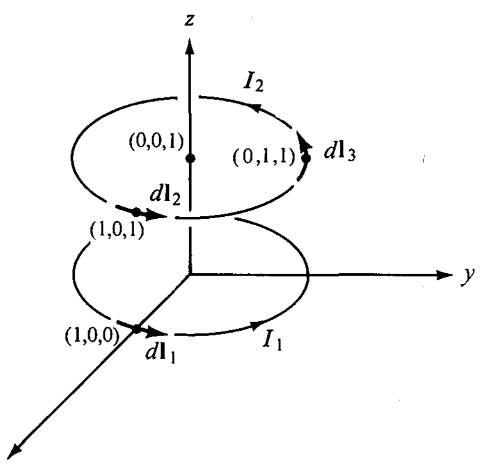

Fig. 3.28. For Problem 3.9.

3.10. Two square loops of sides a m are placed parallel to each other and separated by a distance d m as shown in Fig. 3.29. If the currents carried by the loops are I_1 and I_2 amp, respectively, as shown in Fig. 3.29, find the force acting on one loop due to the second loop.

3.14. A circular loop of wire of radius a lying in the xy plane with its center at the origin carries a current I in the ϕ direction. Find $\mathbf{B}$ at the point $(0, 0, z)$. Verify your answer by letting $z \longrightarrow 0$.

3.15. A loop of wire lying in the xy plane and carrying a current I is in the shape of a regular polygon of n sides inscribed in a circle of radius a with its center at the origin. If the current flow is in the sense of the ϕ direction, find $\mathbf{B}$ at the point $(0, 0, z)$. Verify your answer by letting $n \longrightarrow \infty$ and comparing the result with the answer to Problem 3.14.

3.16. A V-shaped filamentary wire with semi-infinitely long legs making an angle α at its vertex P and lying in the plane of the paper carries a current I amp as shown in Fig. 3.31. Find $\mathbf{B}$ at a point distance d directly above the vertex P. Verify your result by letting $\alpha \longrightarrow \pi$.

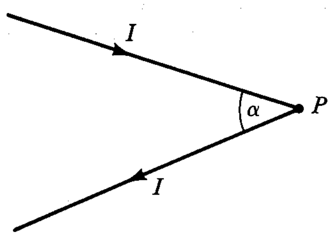

Fig. 3.31. For Problem 3.16.

3.17. Two circular loops of filamentary wire each of radius a and with their centers on the z axis are situated parallel to and symmetrically about the xy plane with the separation equal to $2b$ as shown in Fig. 3.32. The loops carry a current of I amp each in the ϕ direction. (a) Obtain the expression for $\mathbf{B}$ at a point on the z axis. (b) Show that if $b = a/2$, the first three derivatives of $\mathbf{B}$ evaluated at the origin are equal to zero.

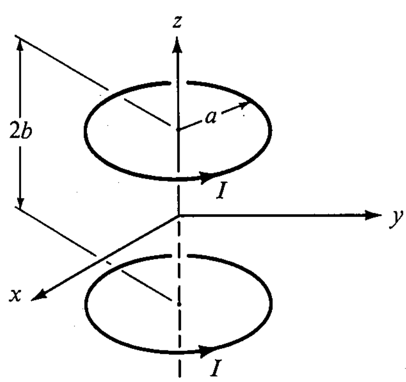

Fig. 3.32. For Problem 3.17.

3.18. A finitely long, uniformly wound solenoid of radius a, consisting of n turns per unit length and carrying a current I in the ϕ direction, lies between $z = -L_1$ and $z = L_2$ with the z axis as its axis. Find $\mathbf{B}$ at a point $(0, 0, z)$. Verify your answer by letting L_1 and $L_2 \longrightarrow \infty$.

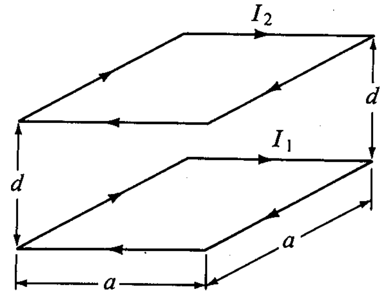

Fig. 3.29. For Problem 3.10.

3.11. An infinitely long straight wire carrying current I_1 amp is situated in the plane of and parallel to one side of a rectangular loop of wire carrying current I_2 amp as shown in Fig. 3.30. Evaluate independently the force experienced by the infinitely long wire due to the magnetic field of the rectangular loop of wire and the force experienced by the rectangular loop of wire due to the magnetic field of the infinitely long wire.

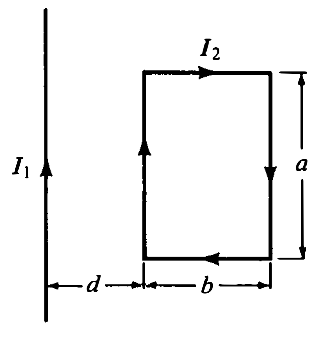

Fig. 3.30. For Problem 3.11.

3.12. Four infinitely long, straight filamentary wires occupy the lines $x = 0$, $y = 0$; $x = 1$, $y = 0$; $x = 1$, $y = 1$ and $x = 0$, $y = 1$. Each wire carries a current of value 1 amp in the z direction.

(a) Find the force experienced per unit length of each wire.

(b) Find the magnetic flux density at the point (2, 2, 0).

(c) Find the magnetic flux density at the point (0, 2, 0).

3.13. Two identical rigid filamentary wires, each of length l and weight W, are suspended horizontally from the ceiling by long weightless threads, each of length L. The wires are arranged to be parallel and separated by a distance d, where d is very small compared to l and L. A current I amp is passed through both wires through flexible connections so as to cause the wires to be attracted towards each other. If the current is gradually increased from zero, the wires will gradually approach each other. A condition may be reached at which any further increase of current will cause the wires to swing and touch each other. Determine the critical current at which this would happen.

3.19. A filamentary wire closely wound in the form of a spiral in the xy plane, starting at the origin and ending at radius a, carries a current I in the ϕ direction. Consider the turn density n to be an arbitrary function of r and show that the magnetic flux density at a point $(0, 0, z)$ is given by

$$\mathbf{B} = \frac{\mu_0 I}{2} \int_{r=0}^{a} \frac{nr^2\, dr}{(r^2 + z^2)^{3/2}} \mathbf{i}_z$$

Evaluate $\mathbf{B}$ for the following turn density distributions:

(a) $n = n_0$

(b) $n = \dfrac{n_0}{r}$

(c) $n = \dfrac{n_0}{r^2}$

where n_0 is a constant.

3.20. A filamentary wire carrying a current I is closely wound on the surface of a sphere of radius a and centered at the origin. The winding starts at $(0, 0, a)$ and ends at $(0, 0, -a)$ with the turns in the planes normal to the z axis and carrying current in the ϕ direction. Consider the turn density to be an arbitrary function of θ and show that the magnetic flux density at a point $(0, 0, z)$ is given by

$$\mathbf{B} = \frac{\mu_0 I a^3}{2} \int_{\theta=0}^{\pi} \frac{n \sin^2 \theta\, d\theta}{[a^2 + z^2 - 2az \cos \theta]^{3/2}} \mathbf{i}_z$$

Evaluate $\mathbf{B}$ both for $|z| < a$ and for $|z| > a$ for the following turn density distributions:

(a) $n = n_0 \sin \theta$

(b) $n = n_0/\sin \theta$

where n_0 is a constant.

3.21. Two infinitely long, straight filamentary wires situated parallel to the z axis and passing through $(d/2, 0, 0)$ and $(-d/2, 0, 0)$, respectively, carry currents I in the $+z$ and $-z$ directions, respectively. The arrangement is known as a two-dimensional magnetic dipole in contrast to the three-dimensional magnetic dipole consisting of a circular loop of current. (a) Obtain the magnetic flux density due to the two-dimensional magnetic dipole in the limit that $d \to 0$, keeping the dipole moment Id constant. (b) Find and sketch the direction lines of the magnetic flux density.

3.22. Two infinitely long, straight filamentary wires situated parallel to the z axis and passing through $(d/2, 0, 0)$ and $(-d/2, 0, 0)$ carry currents I_1 and I_2, respectively, in the z direction. Show that the equation for the direction lines of the magnetic flux density is

$$I_1 \ln\left[\left(x + \frac{d}{2}\right)^2 + y^2\right] + I_2 \ln\left[\left(x - \frac{d}{2}\right)^2 + y^2\right] = \text{constant}$$

Obtain and sketch the direction lines for the following cases:

(a) $I_1 = I_2 = I_0$

(b) $I_1 = I_0, I_2 = -I_0$

3.23. Two circular loops of filamentary wire, each of radius a and with their centers on the z axis, are situated parallel to and symmetrically about the xy plane with the separation equal to $2d$. The loops carry currents of I amp each in opposite directions. Such an arrangement is known as the magnetic quadrupole. Obtain the magnetic flux density due to the magnetic quadrupole at distances from the origin large compared to a and d, at points along (a) the z-axis and (b) in the xy plane.

3.24. A sheet of surface current flowing in the z direction occupies the portion of the $y = 0$ plane lying between $x = -a$ and $x = +a$. Consider the z-directed surface current density $\mathbf{J}_s$ to be an arbitrary function of x and show that the components of the magnetic flux density at a point $(0, 0, y)$ are given in cartesian coordinates by

$$B_x = -\frac{\mu_0 y}{2\pi}\int_{x=-a}^{a}\frac{J_s\,dx}{(x^2+y^2)}$$

$$B_y = -\frac{\mu_0}{2\pi}\int_{x=-a}^{a}\frac{J_s x\,dx}{(x^2+y^2)}$$

$$B_z = 0$$

Evaluate the field components for the following surface current density distributions:

(a) $\mathbf{J}_s = J_{s0}\mathbf{i}_z$

(b) $\mathbf{J}_s = J_{s0}\left(1 - \frac{|x|}{a}\right)\mathbf{i}_z$

(c) $\mathbf{J}_s = J_{s0}\frac{x}{a}\mathbf{i}_z$

where J_{s0} is a constant.

3.25. Current flows on the xy plane radially away from the origin with density given by

$$\mathbf{J}_s = \frac{I}{2\pi r}\mathbf{i}_r \text{ amps/m}$$

Show that the magnetic flux density at any point above the xy plane is the same as that which would be produced by a filamentary wire along the negative z axis carrying current I from the origin to $z = -\infty$. Show also that the magnetic flux density at any point below the xy plane is the same as that which would be produced by a filamentary wire along the positive z axis carrying current I from the origin to $z = \infty$.

3.26. Current flows in the z direction in an infinite slab of thickness $2a$ symmetrically placed about the xz plane. Consider the z-directed current density $\mathbf{J}$ to be uniform in x but not necessarily in y and show that the magnetic flux density at any point (x, y, z) has only an x component given by

$$B_x = \begin{cases} -\dfrac{\mu_0}{2}\displaystyle\int_{y=-a}^{a} J\,dy & y > a \\ \dfrac{\mu_0}{2}\left(\displaystyle\int_{y=y}^{a} J\,dy - \int_{y=-a}^{y} J\,dy\right) & -a < y < a \\ \dfrac{\mu_0}{2}\displaystyle\int_{y=-a}^{a} J\,dy & y < -a \end{cases}$$

Evaluate B_x as a function of y for $-\infty < y < \infty$ for the following current distributions:

(a) $\mathbf{J} = J_0\mathbf{i}_z \qquad -a < y < a$

(b) $\mathbf{J} = \begin{cases} J_0\mathbf{i}_z & -a < y < 0 \\ -J_0\mathbf{i}_z & 0 < y < a \end{cases}$

(c) $\mathbf{J} = |y|\mathbf{i}_z \qquad -a < y < a$

(d) $\mathbf{J} = y\mathbf{i}_z \qquad -a < y < a$

where J_0 is a constant. Discuss your results from considerations of symmetry.

3.27. Current flows in the axial direction in an infinitely long cylinder of radius a having the z axis as its axis. Consider the z-directed current density $\mathbf{J}$ to be uniform in ϕ but an arbitrary function of r and show that the magnetic flux density is given by

$$\mathbf{B} = \frac{\mu_0}{r}\int_{r=0}^{r} Jr\,dr\,\mathbf{i}_\phi$$

Evaluate $\mathbf{B}$ for the following current density distributions:

(a) $\mathbf{J} = J_0\mathbf{i}_z, \qquad 0 < r < a$

(b) $\mathbf{J} = \begin{cases} 0 & 0 < r < a \\ J_0\mathbf{i}_z & a < r < b \\ 0 & b < r < \infty \end{cases}$

(c) $\mathbf{J} = J_0\left(\frac{r}{a}\right)^n\mathbf{i}_z, \; n \geq 1 \qquad 0 < r < a$

where J_0 is a constant.

3.28. An infinitely long straight filamentary wire occupying the z axis carries current I amp in the z direction. Evaluate $\int \mathbf{B} \cdot d\mathbf{l}$ for the following paths:

(a) From (1, 0, 0) to (0, $\frac{1}{2}$, 0) along the path $x + 2y = 1$, $z = 0$.

(b) From (2, 0, 0) to (1, 1, 1) along a straight line path.

Check your answers from considerations of symmetry and Ampere's circuital law in integral form.

3.29. Using Ampere's circuital law in integral form, obtain the magnetic flux densities due to the following volume current distributions in cartesian coordinates:

(a) $\mathbf{J} = \begin{cases} J_0\mathbf{i}_z & |y| < a \\ 0 & |y| > a \end{cases}$

(b) $\mathbf{J} = \begin{cases} J_0\mathbf{i}_z & -a < y < 0 \\ -J_0\mathbf{i}_z & 0 < y < a \end{cases}$

(c) $\mathbf{J} = \begin{cases} |y|\mathbf{i}_z & |y| < a \\ 0 & |y| > a \end{cases}$

(d) $\mathbf{J} = \begin{cases} y\mathbf{i}_z & |y| < a \\ 0 & |y| > a \end{cases}$

(e) $\mathbf{J} = \begin{cases} (a - |y|)\mathbf{i}_z & |y| < a \\ 0 & |y| > a \end{cases}$

where J_0 is a constant.

3.30. Using Ampere's circuital law in integral form, obtain the magnetic flux densities due to the following volume current distributions in cylindrical coordinates:

(a) $\mathbf{J} = \begin{cases} 0 & 0 < r < a \\ J_0\mathbf{i}_z & a < r < b \\ 0 & b < r < \infty \end{cases}$

(b) $\mathbf{J} = \begin{cases} J_0\left(\dfrac{r}{a}\right)^n \mathbf{i}_z, \ n \geq 1 & 0 < r < a \\ 0 & a < r < \infty \end{cases}$

(c) $\mathbf{J} = \begin{cases} \dfrac{I}{\pi a^2}\mathbf{i}_z & 0 < r < a \\ 0 & a < r < b \\ -\dfrac{I}{\pi(c^2 - b^2)}\mathbf{i}_z & b < r < c \\ 0 & c < r < \infty \end{cases}$

where J_0 and I are constants.

3.31. Using Ampere's circuital law in integral form, obtain the magnetic flux densities due to the following surface current distributions:

(a) $\mathbf{J}_s = \begin{cases} J_{s0}\mathbf{i}_z & y = a \\ -J_{s0}\mathbf{i}_z & y = -a \end{cases}$ cartesian coordinates

(b) $\mathbf{J}_s = J_{s0}\mathbf{i}_z \quad r = a$ cylindrical coordinates

(c) $\mathbf{J}_s = \begin{cases} J_{s0}\mathbf{i}_z & r = a \\ -J_{s0}\dfrac{a}{b}\mathbf{i}_z & r = b \end{cases}$ cylindrical coordinates

where J_{s0} is a constant.

3.32. A toroid with a circular cross section is formed by rotating about the z axis the circle of radius a ($< b$) in the xz plane and centered at $(b, 0, 0)$ as shown in Fig. 3.33. A filamentary wire carrying current I is closely wound around the toroid uniformly with n turns per unit length along the mean circumference. Using Ampere's circuital law in integral form, find the magnetic field both inside and outside the toroid.

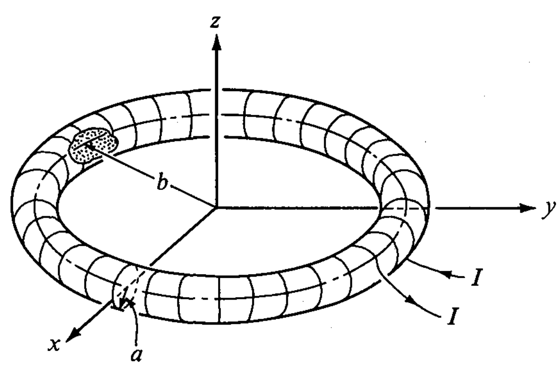

Fig. 3.33. For Problem 3.32.

3.33. Current I amp flows in a filamentary wire along the z axis from $z = \infty$ to $z = a$ and then to the point $z = -a$ via a spherical surface of radius a centered at the origin, continuing on to $z = -\infty$ along a filamentary wire from $z = -a$ to $z = -\infty$. The surface current density on the spherical surface is given by

$$\mathbf{J}_s = \frac{I}{2\pi a \sin\theta}\mathbf{i}_\theta \text{ amp/m}$$

Using Ampere's circuital law in integral form, find **B** both inside and outside the sphere of radius a.

3.34. Current flows axially with uniform density J_0 amp/m^2 in the region between two infinitely long, parallel cylindrical surfaces of radii a and b (< a) and with their axes separated by a distance c ($< a -$ b) as shown in Fig. 3.34. Find the magnetic flux density in the current-free region inside the cylindrical surface of radius b.

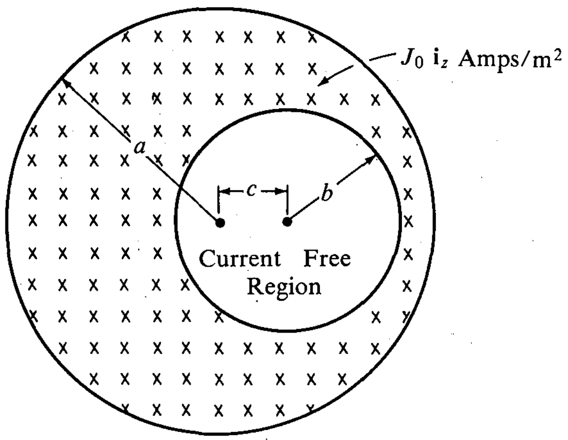

Fig. 3.34. For Problem 3.34.

3.35. Verify your answers to Problem 3.29 by using Ampere's circuital law in differential form.

3.36. Verify your answers to Problem 3.30 by using Ampere's circuital law in differential form.

3.37. For each of the following magnetic fields, find the current distribution which produces the field, using Ampere's circuital law in differential form:

(a) $$\mathbf{B} = \begin{cases} \mu_0 J_{s0}\mathbf{i}_x & -\infty < y < 0 \\ \dfrac{\mu_0 J_{s0}}{3}\mathbf{i}_x & 0 < y < a \\ -\mu_0 J_{s0}\mathbf{i}_x & a < y < \infty \end{cases} \quad \text{cartesian coordinates}$$

(b) $$\mathbf{B} = \begin{cases} \mu_0 J_0 r^2 \mathbf{i}_\phi & 0 < r < a \\ \mu_0 J_0 \dfrac{a^3}{r}\mathbf{i}_\phi & a < r < b \\ 0 & b < r < \infty \end{cases} \quad \text{cylindrical coordinates}$$

(c) $$\mathbf{B} = \begin{cases} \mu_0 J_{s0}(\cos\theta\,\mathbf{i}_r - \sin\theta\,\mathbf{i}_\theta) & 0 < r < a \\ \dfrac{\mu_0 J_{s0}}{2}\left(\dfrac{a}{r}\right)^3 (2\cos\theta\,\mathbf{i}_r + \sin\theta\,\mathbf{i}_\theta) & a < r < \infty \end{cases} \quad \text{spherical coordinates}$$

where J_{s0} and J_0 are constants.

3.38. A surface current of density $\mathbf{J}_s$ amp/m occupies the plane surface $y = y_0$. Show that

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}_s\, \delta(y - y_0)$$

3.39. A surface current of density $\mathbf{J}_s$ amp/m occupies the cylindrical surface $r = r_0$. Show that

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}_s \, \delta(r - r_0)$$

3.40. An infinitely long filamentary wire carrying current I amp in the z direction is situated parallel to the z axis and passes through the point (r_0, ϕ_0) in the $z = 0$ plane. Show that

$$\nabla \times \mathbf{B} = \mu_0 I \frac{\delta(r - r_0)\,\delta(\phi - \phi_0)}{r_0} \mathbf{i}_z$$

3.41. Obtain the magnetic vector potential at an arbitrary point due to a finitely long, straight filamentary wire lying along the z axis between $z = -a$ and $z = +a$ and carrying a current I amp in the $+z$ direction. Then evaluate $\mathbf{B}$ by performing the curl operation on the magnetic vector potential and compare the result with (3-27).

3.42. Two infinitely long, straight filamentary wires situated parallel to the z axis and passing through $(d/2, 0, 0)$ and $(-d/2, 0, 0)$, respectively, carry currents I in the $+z$ and $-z$ directions, respectively. (a) Obtain the magnetic vector potential $\mathbf{A}$. (b) Find $\mathbf{A}$ in the limit that $d \longrightarrow 0$, keeping Id constant. (c) Evaluate the curl of $\mathbf{A}$ found in part (b) and compare with the result of Problem 3.21.

3.43. For the magnetic dipole of Fig. 3-9, obtain the vector potential at distances very large from the dipole compared to the radius a. Find the magnetic flux density by performing the curl operation on the vector potential.

3.44. For the magnetic quadrupole arrangement of Problem 3.23, obtain the magnetic vector potential at distances from it large compared to the dimensions of the quadrupole. Then find $\mathbf{B}$ by evaluating the curl of the magnetic vector potential and verify the results for the special cases of Problem 3.23.

3.45. For the volume current distributions specified in Problem 3.29, obtain the magnetic vector potentials.

3.46. For the volume current distributions specified in Problem 3.30, obtain the magnetic vector potentials.

3.47. For the following surface current distributions, obtain the magnetic vector potentials:

(a) $$\mathbf{J}_s = \begin{cases} J_{s0}\mathbf{i}_z & y = a \\ -J_{s0}\mathbf{i}_z & y = -a \end{cases} \quad \text{cartesian coordinates}$$

(b) $$\mathbf{J}_s = \begin{cases} J_{s0}\mathbf{i}_z & r = a \\ -J_{s0}\dfrac{a}{b}\mathbf{i}_z & r = b \end{cases} \quad \text{cylindrical coordinates}$$

where J_{s0} is a constant.

3.48. For each of the arrangements of current loops shown in Fig. 3.35, find the magnetic vector potential at distances very far from the loop.

3.49. For the spirally wound filamentary wire of Problem 3.19, show that the magnetic dipole moment $\mathbf{m}$ is given by

$$\mathbf{m} = \pi I \left(\int_{r=0}^{a} n r^2 \, dr \right) \mathbf{i}_z$$

Evaluate $\mathbf{m}$ and hence $\mathbf{A}$ at large distances from the spiral for each of the three cases specified in Problem 3.19.

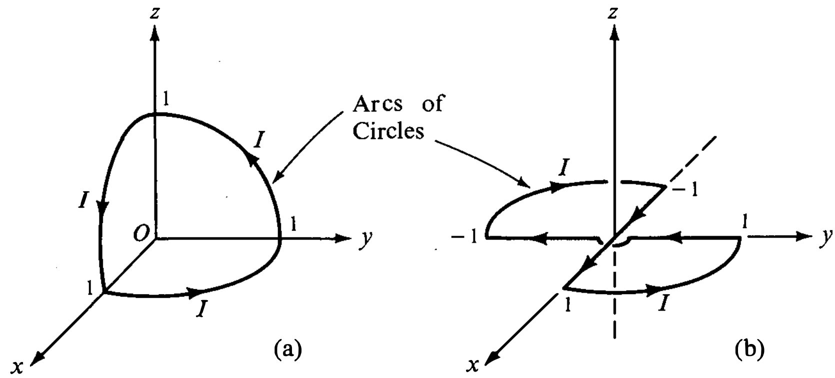

Fig. 3.35. For Problem 3.48.

3.50. For the filamentary wire wound on the surface of a sphere as specified in Problem 3.20, show that the magnetic dipole moment **m** is given by

$$\mathbf{m} = \pi a^3 I\left(\int_{\theta=0}^{\pi} n \sin^2\theta\, d\theta\right)\mathbf{i}_z$$

Evaluate **m** and hence **A** at large distances from the sphere for each of the two cases listed in Problem 3.20.

3.51. A spherical volume charge of radius a m and having uniform density ρ_0 C/m^3 and centered at the origin spins about the z axis with constant angular velocity ω_0 in the ϕ direction. Obtain the magnetic vector potential due to the spinning sphere of charge at distances from the origin large compared to a.

3.52. Show that the magnetic flux enclosed by a closed path C in a magnetic field **B** is equal to $\oint_C \mathbf{A} \cdot d\mathbf{l}$, where **A** is the magnetic vector potential corresponding to **B**. Use this result to find the magnetic flux enclosed by the rectangular loop of Fig. 3.30 due to the current flowing in the infinitely long wire. Check your answer by evaluating $\int_S \mathbf{B} \cdot d\mathbf{S}$, where S is the surface bounded by the rectangular loop.

3.53. Show that, if $\mathbf{A} = A_z\mathbf{i}_z$, where A_z is independent of z, the direction lines of $\mathbf{B} = \nabla \times \mathbf{A}$ are the cross sections of the constant $|\mathbf{A}|$ surfaces in the $z =$ constant plane. Use this result to find and sketch the direction lines of the magnetic flux density due to the infinitely long, filamentary wire-pair arrangement of Problem 3.42.

3.54. Determine if the following fields are realizable as magnetic fields:

(a) $\mathbf{A} = \frac{1}{y^2}(y\mathbf{i}_x - x\mathbf{i}_y)$ cartesian coordinates

(b) $\mathbf{B} = \frac{1}{r^n}\mathbf{i}_\phi$ cylindrical coordinates

(c) $\mathbf{C} = \left(1 + \frac{1}{r^2}\right)\cos\phi\,\mathbf{i}_r - \left(1 - \frac{1}{r^2}\right)\sin\phi\,\mathbf{i}_\phi$ cylindrical coordinates

(d) $\mathbf{D} = \left(1 + \frac{2}{r^3}\right)\cos\theta\,\mathbf{i}_r - \left(1 - \frac{1}{r^3}\right)\sin\theta\,\mathbf{i}_\theta$ spherical coordinates

3.55. For the following current distributions, start with the assumption that all three components of **B** exist and use Ampere's circuital law and the solenoidal nature of the magnetic field to eliminate some components and evaluate the remaining components:
(a) Infinite sheet of current with uniform density.
(b) Surface current flowing axially with uniform density along an infinitely long cylinder.

3.56. Make use of the solenoidal character of the magnetic field to find the radial derivative of the magnetic flux density due to a circular loop of current I at a point on its axis.

3.57. In Sec. 3-10, we classified static vector fields into four groups. Determine to which of the four groups does each of the following fields belong:
(a) $\mathbf{A} = x\mathbf{i}_x + y\mathbf{i}_y$
(b) $\mathbf{B} = xy\mathbf{i}_x + yz\mathbf{i}_y + zx\mathbf{i}_z$
(c) $\mathbf{C} = (x^2 - y^2)\mathbf{i}_x - 2xy\mathbf{i}_y + 4\mathbf{i}_z$
(d) $\mathbf{D} = \frac{e^{-r}}{r}\mathbf{i}_\phi$, cylindrical coordinates
(e) $\mathbf{E} = \frac{\cos\phi}{r^2}\mathbf{i}_r + \frac{\sin\phi}{r^2}\mathbf{i}_\phi$, cylindrical coordinates

3.58. From the examples and problems of Chapters 2 and 3, identify and prepare a table of analogous pairs of charge and current distributions which vary only in two dimensions x and y (or r and ϕ) and with the current flow in the z direction. List the expressions for the corresponding electric and magnetic fields and demonstrate that the fields are proportional in magnitude and orthogonal in direction.