2

THE STATIC ELECTRIC FIELD

In Chapter 1 we learned the mathematical language of vector analysis so that we are now ready to use it for the study of electromagnetic field theory. Electromagnetic field theory is built upon four equations known as Maxwell's equations and an associated set of relations known as the constitutive relations. It is our goal to learn how to interpret these equations and to use them for various applications, important among them being electromagnetic waves. Maxwell's equations, in their general form, relate the time-varying or dynamic electric and magnetic fields with one another and with the electric charges and currents present in the medium. It is possible to study electromagnetic theory by starting with Maxwell's equations and another equation known as the Lorentz force equation as postulates. The Lorentz force equation is the defining equation for the electric and magnetic fields in terms of the forces experienced by the charges. Alternatively, it is possible to develop Maxwell's equations gradually from the electric and magnetic field concepts based on forces experienced by charges and currents and from a few experimental facts. We will take this latter approach. The electromagnetic field is one in which the electric and magnetic effects are coupled. Before we venture to discuss the electromagnetic field, we will study the electric and magnetic fields separately. This is best done by considering static or time-independent fields in free space. With this approach in mind, the present chapter is devoted to the static electric field in free space.

2.1 The Electric Field Concept

In the study of mechanics, we are familiar with the gravitational field as a force field associated with the mutual attraction of material bodies in space. For example, a small test mass m placed in the gravitational field of the earth experiences a force equal to mMG/r^2 directed towards the center of mass of the earth, where M is the mass of the earth, G is the constant of universal gravitation, and r is the distance of the test mass from the center of mass of the earth. We associate with every point in the vicinity of the earth a vector quantity $\mathbf{g}$, known as the gravitational field intensity, having a magnitude MG/r^2 and directed towards the center of the earth as shown in Fig. 2.1. In terms of the value of the test mass and the force experienced by the test mass, the gravitational field intensity is given by

$$\mathbf{g} = \frac{\mathbf{F}}{m} \tag{2-1}$$

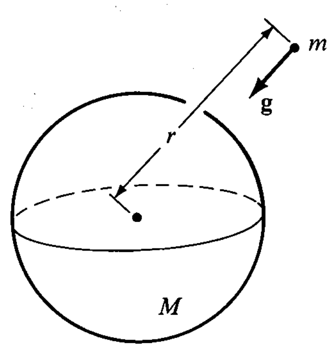

Fig. 2.1. Gravitational attraction of a test mass m towards the center of mass of the earth.

Just as the gravitational field is associated with the physical property known as "mass," a force field is associated with the physical property known as "charge" merely by virtue of its existence. This force field is known as the electric field. We will learn in the next chapter that a second kind of force field known as the magnetic field exists when charges are set in motion. A few words about charge are now in order. Matter can be regarded as composed of three types of elementary particles, known as protons, neutrons, and electrons. These particles are charged positive, zero, and negative, respectively. Table 2.1 gives the charge and mass for each of these particles.

TABLE 2.1. Charges and Masses of Elementary Particles

Particle	*Charge, C*	*Mass, kg*
Proton	1.6021×10^{-19}	1.6724×10^{-27}
Neutron	0	1.6747×10^{-27}
Electron	-1.6021×10^{-19}	9.1083×10^{-31}

Charges are conserved; that is, they can neither be created nor destroyed. They can only be transferred from one body to another. A material body is uncharged if it has no net charge. If the body acquires excess negative charge by some means, it is said to be negatively charged. On the other hand, if it loses some negative charge, it is said to be positively charged. The unit of charge is the coulomb (abbreviated C).

A small test charge q placed in the "electric field" of a larger charge Q experiences a force $\mathbf{F}$ given by

$$\mathbf{F} = q\mathbf{E} \tag{2-2}$$

as shown in Fig. 2.2, where $\mathbf{E}$ is the intensity of the electric field, analogous to the gravitational field intensity $\mathbf{g}$. Alternatively, we can say that if, in a

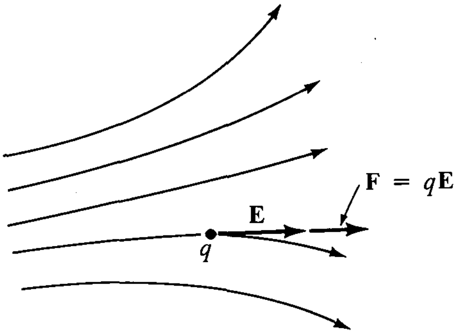

Fig. 2.2. Force experienced by a test charge in an electric field.

region of space, a test charge q experiences a force $\mathbf{F}$, then the region is characterized by an electric field of intensity $\mathbf{E}$ given by

$$\mathbf{E} = \frac{\mathbf{F}}{q} \tag{2-3}$$

Here we are assuming that the test charge q is so small that it does not alter the electric field in which it is placed. From a practical point of view, the test charge does influence the electric field irrespective of how small it is. However, theoretically, we can define $\mathbf{E}$ as the ratio of the force experienced by the test charge divided by the test charge in the limit that the test charge tends to zero; that is,

$$\mathbf{E} = \lim_{q \to 0} \frac{\mathbf{F}}{q} \tag{2-4}$$

The unit of electric field intensity is newton per coulomb (N/C).

Example 2-1. An electron placed at a point in an electric field experiences an acceleration of 10^5 m/sec^2 along the positive x axis. (a) What is the electric field intensity $\mathbf{E}$ at that point? (b) What acceleration does a proton placed at that point experience?

The force experienced by the electron is equal to $-1.6 \times 10^{-19}\,\mathbf{E}$. This is equal to the mass of the electron times the acceleration experienced by the electron. Hence

$$-1.6 \times 10^{-19}\,\mathbf{E} = 9.11 \times 10^{-31} \times 10^{5}\,\mathbf{i}_x$$

$$\mathbf{E} = \frac{9.11 \times 10^{-31} \times 10^{5}}{-1.6 \times 10^{-19}}\mathbf{i}_x = -5.7 \times 10^{-7}\,\mathbf{i}_x \text{ N/C}$$

Thus the electric field intensity has a magnitude of about 5.7×10^{-7} N/C and it is directed along the negative x axis.

Now, if a proton is placed at the same point, the acceleration **a** experienced by it is given by

$$\mathbf{a} = \frac{\text{charge of proton} \times \mathbf{E}}{\text{mass of proton}}$$

$$= \frac{1.6 \times 10^{-19} \times (-5.7 \times 10^{-7})\mathbf{i}_x}{1.67 \times 10^{-27}} = -54.6\,\mathbf{i}_x \text{ m/sec}^2$$

Thus the proton experiences an acceleration of about 54.6 m/sec² along the negative x axis. ∎

2.2 Coulomb's Law

In the previous section we introduced the concept of the electric field from an analogy with the gravitational field. It was mentioned that a small test charge placed in the electric field of a larger charge experiences a force. Actually, the larger charge also experiences a force just as two masses attract each other. This fact was proved experimentally by Coulomb. As a result of his experiments we have Coulomb's law, which relates the force between two charged bodies which are very small in size compared to their separation. Ideally, the charged bodies must be so small that they can be considered as "point charges." From Coulomb's experiments, the following conclusions were reached:

1. Like charges repel whereas unlike charges attract.
2. The magnitude of the force is proportional to the product of the magnitudes of the charges.
3. The magnitude of the force is inversely proportional to the square of the distance between the charges.
4. The direction of the force is along the line joining the charges.
5. The force depends upon the medium in which the charges are placed.

If we consider two point charges Q_1 and Q_2 C situated at points A and B separated by a distance R m, as shown in Fig. 2.3, we can express the foregoing five statements in equation form as

$$\mathbf{F}_A = k\frac{Q_1Q_2}{R^2}\mathbf{i}_{BA} \tag{2-5}$$

$$\mathbf{F}_B = k\frac{Q_1Q_2}{R^2}\mathbf{i}_{AB} \tag{2-6}$$

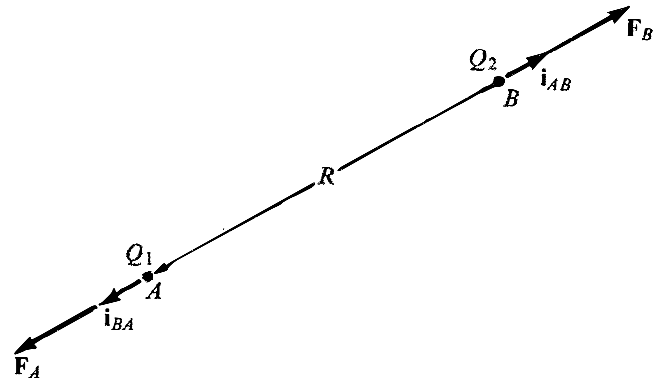

Fig. 2.3. Forces of repulsion between two charges Q_1 and Q_2 at points A and B.

where $\mathbf{F}_A$ and $\mathbf{F}_B$ are the forces experienced by Q_1 and Q_2, respectively, $\mathbf{i}_{BA}$ and $\mathbf{i}_{AB}$ are unit vectors along the line joining A and B (Fig. 2.3), and k is the constant of proportionality. Statement 1 is included in (2-5) and (2-6) since Q_1 and Q_2 represent the magnitudes as well as signs of the charges. If Q_1 and Q_2 are both positive charges or both negative charges, their product will be positive and hence positive forces act along $\mathbf{i}_{BA}$ and $\mathbf{i}_{AB}$. If one of the two charges is negative, then the product Q_1Q_2 will be negative; hence negative forces act along $\mathbf{i}_{BA}$ and $\mathbf{i}_{AB}$ or positive forces act along directions opposite to $\mathbf{i}_{BA}$ and $\mathbf{i}_{AB}$, respectively. The constant of proportionality k is equal to $1/4\pi\epsilon_0$ for free space and in MKS rationalized units. The quantity ϵ_0 is known as the permittivity of free space and its value is 8.854×10^{-12} or approximately equal to $10^{-9}/36\pi$. Substituting for k in (2-5) and (2-6), we have

$$\mathbf{F}_A = \frac{Q_1Q_2}{4\pi\epsilon_0R^2}\mathbf{i}_{BA} \tag{2-7}$$

$$\mathbf{F}_B = \frac{Q_1Q_2}{4\pi\epsilon_0R^2}\mathbf{i}_{AB} \tag{2-8}$$

Equations (2-7) and (2-8) represent Coulomb's law. From these equations, we note that ϵ_0 has the units (coulombs)^2 per [(newton)(meter)^2]. These are commonly known as farads per meter (F/m).

2.3 The Electric Field of Point Charges

Let one of the two charges considered in the preceding section, say Q_2, be a small test charge q. Then, from a knowledge of the force experienced by this test charge due to the presence of the charge Q_1, we can obtain the expression for the electric field intensity due to the charge Q_1 using (2-3).

According to Coulomb's law, the force experienced by the test charge is given by

$$\mathbf{F}_B = \frac{Q_1 q}{4\pi\epsilon_0 R^2}\mathbf{i}_{AB} \tag{2-9}$$

From (2-3) we then have the electric field intensity $\mathbf{E}_B$ at point B due to the charge Q_1 as

$$\mathbf{E}_B = \frac{\mathbf{F}_B}{q} = \frac{Q_1}{4\pi\epsilon_0 R^2}\mathbf{i}_{AB} \tag{2-10}$$

We can generalize this result by making R variable, that is, by moving the test charge around in the medium, writing the expression for the force experienced by it, and dividing the force by the test charge. The result is the same as (2-10) except that R is now a variable since point B is a variable. Thus, omitting the subscripts in (2-10), we write the electric field intensity $\mathbf{E}$ of a point charge Q as

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 R^2}\mathbf{i}_R \tag{2-11}$$

where R is the distance from the point charge to the point at which the field intensity is to be computed and $\mathbf{i}_R$ is the unit vector along the line joining the two points under consideration and directed away from the point charge. The electric field intensity of a point charge is thus directed everywhere radially away from the point charge, and on any spherical surface centered at the point charge its magnitude is constant. The situation is illustrated in Fig. 2.4. If the point charge is at the origin of a coordinate system, then we replace R by r and $\mathbf{i}_R$ by $\mathbf{i}_r$. The field represented by (2-11) is also known as the Coulomb field of a point charge.

If we now have several point charges $Q_1, Q_2, Q_3, \ldots, Q_n$ located at different points as shown in Fig. 2.5, we can invoke superposition and state that the force $\mathbf{F}$ experienced by a test charge situated at a point P is the vector

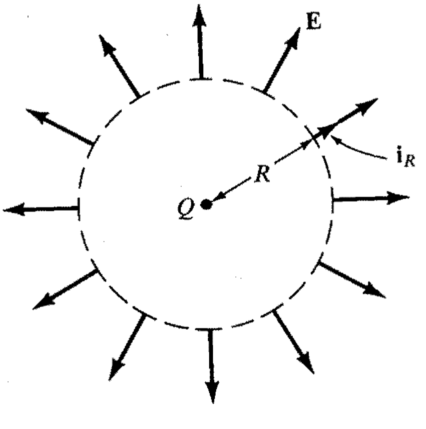

Fig. 2.4. The electric field of a point charge.

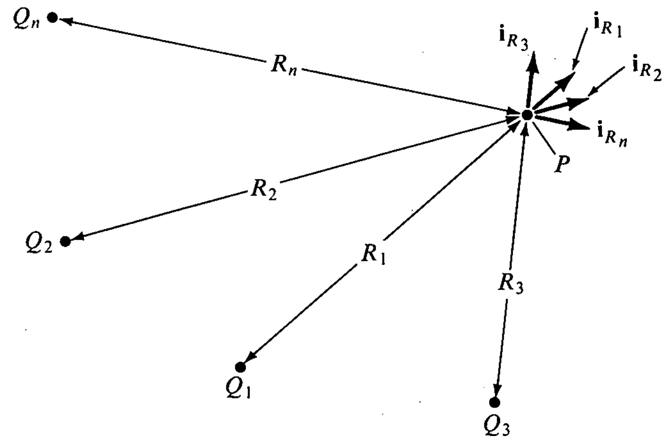

Fig. 2.5. Assembly of point charges and unit vectors along the direction of their electric field intensities at point P, due to the individual point charges.

sum of the forces experienced by the test charge due to the individual charges; that is,

$$\mathbf{F} = \frac{Q_1 q}{4\pi\epsilon_0 R_1^2}\mathbf{i}_{R_1} + \frac{Q_2 q}{4\pi\epsilon_0 R_2^2}\mathbf{i}_{R_2} + \frac{Q_3 q}{4\pi\epsilon_0 R_3^2}\mathbf{i}_{R_3} + \cdots + \frac{Q_n q}{4\pi\epsilon_0 R_n^2}\mathbf{i}_{R_n} \quad (2\text{-}12)$$

From (2-3) the electric field intensity $\mathbf{E}$ at the point P is

$$\begin{aligned}\mathbf{E} = \frac{\mathbf{F}}{q} &= \frac{Q_1}{4\pi\epsilon_0 R_1^2}\mathbf{i}_{R_1} + \frac{Q_2}{4\pi\epsilon_0 R_2^2}\mathbf{i}_{R_2} + \cdots + \frac{Q_n}{4\pi\epsilon_0 R_n^2}\mathbf{i}_{R_n} \\ &= \sum_{j=1}^{n} \frac{Q_j}{4\pi\epsilon_0 R_j^2}\mathbf{i}_{R_j}\end{aligned} \quad (2\text{-}13)$$

The electric field intensity due to the assembly of the point charges is thus the vector sum of the electric field intensities due to the individual point charges. Some examples are now in order.

Example 2-2. For a charge Q at an arbitrary point $A(x', y', z')$, obtain the x, y, and z components of the electric field intensity at an arbitrary point $B(x, y, z)$, as shown in Fig. 2.6.

From Coulomb's law, the electric field intensity at point B is given by

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0(AB)^2}\mathbf{i}_{AB} \quad (2\text{-}14)$$

where from Fig. 2.6,

$$AB = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2} \quad (2\text{-}15)$$

$$\mathbf{i}_{AB} = \frac{(x-x')\mathbf{i}_x + (y-y')\mathbf{i}_y + (z-z')\mathbf{i}_z}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} \quad (2\text{-}16)$$

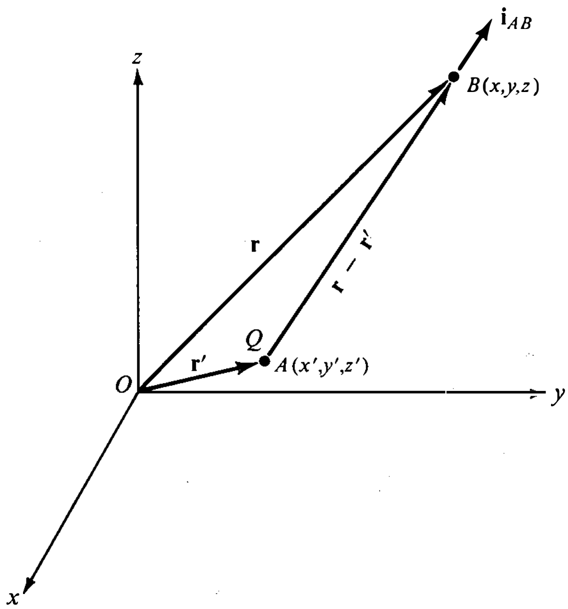

Fig. 2.6. Geometry pertinent to the computation of the electric field of a point charge located at an arbitrary point.

Substituting (2-15) and (2-16) into (2-14), we have

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0}\frac{(x-x')\mathbf{i}_x + (y-y')\mathbf{i}_y + (z-z')\mathbf{i}_z}{[(x-x')^2+(y-y')^2+(z-z')^2]^{3/2}} \tag{2-17}$$

The x, y, and z components of $\mathbf{E}$ are therefore given by

$$E_x = \mathbf{E}\cdot\mathbf{i}_x = \frac{Q}{4\pi\epsilon_0}\frac{(x-x')}{[(x-x')^2+(y-y')^2+(z-z')^2]^{3/2}} \tag{2-18a}$$

$$E_y = \mathbf{E}\cdot\mathbf{i}_y = \frac{Q}{4\pi\epsilon_0}\frac{(y-y')}{[(x-x')^2+(y-y')^2+(z-z')^2]^{3/2}} \tag{2-18b}$$

$$E_z = \mathbf{E}\cdot\mathbf{i}_z = \frac{Q}{4\pi\epsilon_0}\frac{(z-z')}{[(x-x')^2+(y-y')^2+(z-z')]^{3/2}} \tag{2-18c}$$

In vector notation, if we denote $\mathbf{r}'$ as the position vector for the source point A and $\mathbf{r}$ as the position vector for the point B at which the field is desired, then $AB = |\mathbf{r}-\mathbf{r}'|$ and $\mathbf{i}_{AB} = (\mathbf{r}-\mathbf{r}')/|\mathbf{r}-\mathbf{r}'|$ so that

$$\mathbf{E}(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^2}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} = \frac{Q}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^3}(\mathbf{r}-\mathbf{r}') \tag{2-19}$$

If a number of charges $Q_1, Q_2, Q_3, \ldots, Q_n$ are located at points defined by position vectors $\mathbf{r}'_1, \mathbf{r}'_2, \mathbf{r}'_3, \ldots, \mathbf{r}'_n$, respectively, then

$$\mathbf{E}(\mathbf{r}) = \sum_{j=1}^{n} \frac{Q_j}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'_j|^3}(\mathbf{r} - \mathbf{r}'_j) \tag{2-20}$$

where we have made use of superposition. ∎

EXAMPLE 2-3. Two equal and opposite point charges Q and $-Q$ are situated on the z axis at $d/2$ and $-d/2$, respectively, as shown in Fig. 2.7. Such an arrangement is known as an electric dipole. It is desired to obtain the expression for the electric field intensity due to the electric dipole at distances very large from the origin compared to the spacing d.

With reference to the geometry shown in Fig. 2.7, we note that the electric field intensity at any point P has only r and θ components if we use the spherical coordinate system, whereas it has all three components if we use the cartesian coordinate system. For fixed values of r and θ the field intensity is independent of ϕ; that is, it has circular symmetry about the z axis. Furthermore, we are interested only in the field at large distances from the dipole, that is, for $r \gg d$. Hence we use the spherical coordinate system. The electric field intensity $\mathbf{E}$ at P is the superposition of the electric field intensities due to the two charges. Thus, with reference to the notation in Fig. 2.7 we have

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 r_+^2}\mathbf{i}_{r+} - \frac{Q}{4\pi\epsilon_0 r_-^2}\mathbf{i}_{r-} \tag{2-21}$$

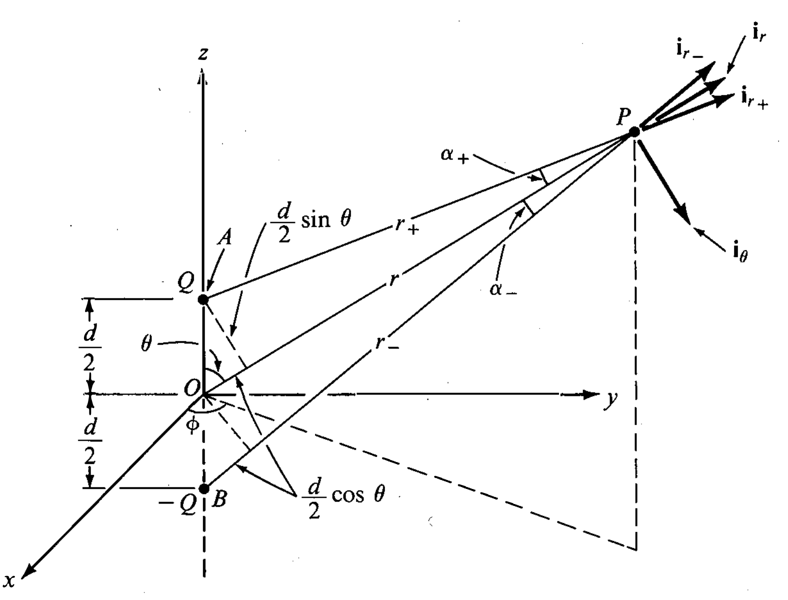

Fig. 2.7. Geometry pertinent to the computation of the electric field due to a dipole.

Now, the r component of $\mathbf{E}$ is given by

$$\begin{aligned} E_r &= \mathbf{E} \cdot \mathbf{i}_r \\ &= \frac{Q}{4\pi\epsilon_0 r_+^2}\mathbf{i}_{r_+} \cdot \mathbf{i}_r - \frac{Q}{4\pi\epsilon_0 r_-^2}\mathbf{i}_{r_-} \cdot \mathbf{i}_r \\ &= \frac{Q}{4\pi\epsilon_0 r_+^2}\cos\alpha_+ - \frac{Q}{4\pi\epsilon_0 r_-^2}\cos\alpha_- \end{aligned} \tag{2-22}$$

From the geometries of the triangles OAP and OBP, we have

$$\cos\alpha_+ = \frac{r_+^2 + r^2 - (d/2)^2}{2r_+ r} \tag{2-23}$$

$$\cos\alpha_- = \frac{r_-^2 + r^2 - (d/2)^2}{2r_- r} \tag{2-24}$$

Substituting (2-23) and (2-24) into (2-22), we obtain

$$\begin{aligned} E_r &= \frac{Q}{4\pi\epsilon_0}\left[\frac{r_+^2 + r^2 - (d/2)^2}{2r_+^3 r} - \frac{r_-^2 + r^2 - (d/2)^2}{2r_-^3 r}\right] \\ &= \frac{Q}{8\pi\epsilon_0 r_+^3 r_-^3 r}(r_- - r_+)\left\{r_+^2 r_-^2 + \left[r^2 - \left(\frac{d}{2}\right)^2\right](r_-^2 + r_- r_+ + r_+^2)\right\} \\ &\approx \frac{Q}{8\pi\epsilon_0 r^7}(r_- - r_+)(r_+^2 r_-^2 + r^2 r_-^2 + r^2 r_- r_+ + r^2 r_+^2) \\ &\approx \frac{Q}{2\pi\epsilon_0 r^3}(r_- - r_+) \approx \frac{Q}{2\pi\epsilon_0 r^3}d\cos\theta \end{aligned} \tag{2-25}$$

where we have used the approximations that, for $r \gg d$,

$$r_+ \approx r - \frac{d}{2}\cos\theta$$

$$r_- \approx r + \frac{d}{2}\cos\theta$$

The θ component of $\mathbf{E}$ is given by

$$\begin{aligned} E_\theta &= \mathbf{E} \cdot \mathbf{i}_\theta \\ &= \frac{Q}{4\pi\epsilon_0 r_+^2}\mathbf{i}_{r_+} \cdot \mathbf{i}_\theta - \frac{Q}{4\pi\epsilon_0 r_-^2}\mathbf{i}_{r_-} \cdot \mathbf{i}_\theta \\ &= \frac{Q}{4\pi\epsilon_0 r_+^2}\sin\alpha_+ + \frac{Q}{4\pi\epsilon_0 r_-^2}\sin\alpha_- \\ &\approx \frac{Q}{2\pi\epsilon_0 r^2}\sin\alpha_+ \\ &\approx \frac{Q}{4\pi\epsilon_0 r^3}d\sin\theta \end{aligned} \tag{2-26}$$

Thus

$$\mathbf{E} = \frac{Qd}{4\pi\epsilon_0 r^3}(2\cos\theta\,\mathbf{i}_r + \sin\theta\,\mathbf{i}_\theta) \tag{2-27}$$

Equation (2-27) can be considered as a solution for the electric field intensity at very large distances compared to a fixed spacing d between the two point charges, or it can be considered as the solution for the electric field intensity at any point (r, θ, ϕ) in the limit that $d \longrightarrow 0$, keeping Qd constant. It should be noted that to keep Qd constant as $d \longrightarrow 0$ requires that $Q \longrightarrow \infty$. The product Qd is known as the electric dipole moment p. The dipole moment also has an orientation associated with it which is from the negative charge to the positive charge. Substituting p for Qd in (2-27), we note that the electric field intensity due to an electric dipole moment p oriented along the positive z axis is given by

$$\mathbf{E} = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\mathbf{i}_r + \sin\theta\,\mathbf{i}_\theta) \tag{2-28}$$

We note that, as compared to the inverse square distance dependence of the electric field intensity of a point charge, the dipole field drops off inversely proportional to the cube of the distance. Likewise, by an arrangement of two dipoles, a "quadrupole" can be created for which the field varies as inversely proportional to r^4. The process can be extended to "multipoles" step by step, with the power of r increasing by one for each step. ▮

2.4 The Electric Field of Continuous Charge Distributions

In the previous section we considered collections of point charges at discrete points for which the field computation consists of finding the vector sums of the field intensities due to the individual point charges. In this section we will extend the computation to continuous charge distributions. Continuous charge distributions can be of three types:

(a) Line charge for which charge is distributed along a line (straight or curved).

(b) Surface charge for which charge is distributed on a surface (planar or nonplanar).

(c) Volume charge for which charge is distributed in a volume.

When a charge is distributed along a line or on a surface or in a volume, we have to deal with charge densities. The line charge density is the charge per unit length, the surface charge density is the charge per unit surface area, and the volume charge density is the charge per unit volume. We will use the symbols ρ_L, ρ_s, and ρ, respectively, for these charge densities. Obviously, the units of ρ_L, ρ_s, and ρ are coulombs per meter, coulombs per meter^2, coulombs per meter^3, respectively. In each case we can divide the total charge into several infinitesimal parts, each of which can be considered as a point charge. We thus represent the total charge as a continuous collection of point charges and obtain the field intensity at any point due to the total charge as the vector superposition of the field intensities due to the individual point charges. However, we now have to evaluate integrals instead of summations of a few terms since the distribution of charges is continuous instead of being discrete. We will illustrate this process by considering three examples: (a) infinitely long line charge, (b) infinite sheet charge, and (c) spherical volume charge.

EXAMPLE 2-4. An infinitely long line charge of uniform density ρ_{L0} C/m is situated along the z axis as shown in Fig. 2.8. We wish to obtain the electric field intensity due to this line charge.

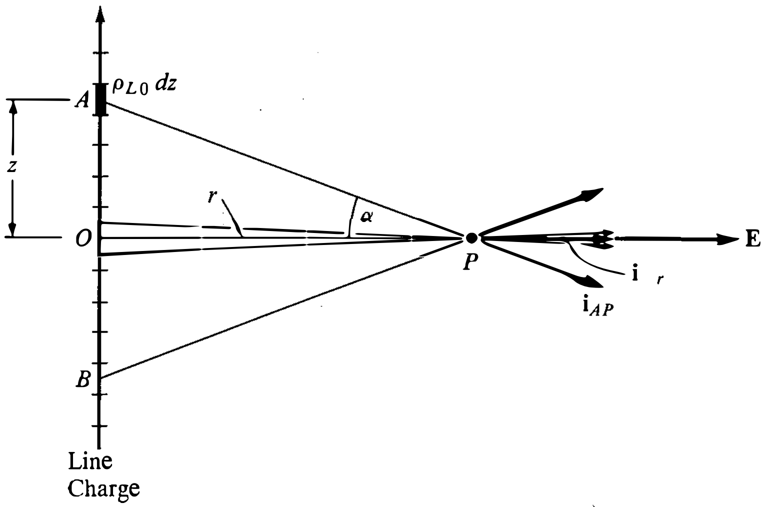

Fig. 2.8. Geometry for computing the electric field of an infinitely long line charge of uniform density ρ_{L0} C/m.

First, we divide the line into a number of infinitesimal segments each of length dz, as shown in Fig. 2.8, such that the charge $\rho_{L0}\,dz$ in each segment can be considered as a point charge. The electric field intensity due to each point charge is directed radially away from that point charge and varies inversely as the square of the distance from that charge. Now let us consider a point P at a distance r from the z axis, with the projection of the point P onto the z axis being the point O. The electric field intensity vectors at point P due to the infinitesimal segment immediately above O and the infinitesimal segment immediately below O have equal magnitudes and make equal angles with the line OP as shown in Fig. 2.8. The components of these two vectors perpendicular to OP (parallel to the z axis) therefore cancel, whereas the components along OP add to each other. Thus the resultant electric field intensity at P due to the two segments, one directly above O and another directly below O, is entirely directed along OP, that is, normal to the axis of the line charge. A similar argument can be made for the resultant electric field intensity vector at point P due to any other two segments which are equidistant from O with one above it and the other below it. Now, since there are as many (semiinfinite) segments above O as there are below it, the resultant field intensity at point P due to the entire line charge is directed radially away from it. The situation remains unchanged if we move P up or down, keeping r constant, since there are always a semiinfinite number of segments above the projection of P onto the line charge as well as below it. Thus the electric field intensity of an infinite line charge of uniform density at any arbitrary point is directed radially away from the line charge and is independent of the position of P parallel to the z axis. It is dependent only on the distance of P from the z axis. We have thus simplified the problem to one of finding the magnitude of the field intensity.

To determine the magnitude of $\mathbf{E}$, let us once again refer to Fig. 2.8, and consider the segment at the point A at a distance z above O. The electric field intensity at point P due to this segment is equal to

$$\frac{\rho_{L0}\,dz}{4\pi\epsilon_0(r^2+z^2)}\mathbf{i}_{AP}$$

The component of this electric field intensity along OP is

$$\frac{\rho_{L0}\,dz}{4\pi\epsilon_0(r^2+z^2)}\mathbf{i}_{AP}\cdot\mathbf{i}_r=\frac{\rho_{L0}\,dz}{4\pi\epsilon_0(r^2+z^2)}\cos\alpha=\frac{\rho_{L0}r\,dz}{4\pi\epsilon_0(r^2+z^2)^{3/2}}$$

We need not consider the component normal to OP since it gets cancelled from the contribution due to another segment at the point B at a distance z below O. The component along OP is, on the other hand, doubled from the contribution due to this second segment. Thus the magnitude of the resultant electric field intensity at P due to the two segments at A and B is given by

$$dE=\frac{2\rho_{L0}r\,dz}{4\pi\epsilon_0(r^2+z^2)^{3/2}} \tag{2-29}$$

The magnitude of the electric field intensity at P due to the entire line charge is now given by the integral of dE where the integration is to be performed between the limits $z=0$ and $z=\infty$. Thus

$$E=\int_{z=0}^{\infty}dE=\frac{2\rho_{L0}r}{4\pi\epsilon_0}\int_{z=0}^{\infty}\frac{dz}{(r^2+z^2)^{3/2}} \tag{2-30}$$

Introducing $z=r\tan\alpha$ in (2-30), we obtain

$$E=\frac{\rho_{L0}}{2\pi\epsilon_0 r}\int_{\alpha=0}^{\pi/2}\cos\alpha\,d\alpha=\frac{\rho_{L0}}{2\pi\epsilon_0 r} \tag{2-31}$$

Recalling that $\mathbf{E}$ is directed radially away from the line charge, we have

$$\mathbf{E}=\frac{\rho_{L0}}{2\pi\epsilon_0 r}\mathbf{i}_r \tag{2-32}$$

Equation (2-32) indicates that the electric field intensity of an infinite line charge of uniform density falls off only as the inverse of the distance from the line charge compared to the inverse square distance dependence in the case of the point charge. ▮

EXAMPLE 2-5. A sheet charge of uniform density ρ_{s0} C/m² extends over the entire xy plane as shown in Fig. 2.9. We wish to obtain the electric field intensity due to this infinite sheet charge.

Let us consider a point P at a distance z from the xy plane, with the projection of the point P on the xy plane being O, as shown in Fig. 2.9. The electric field intensities at point P due to two point charges situated at the diametrically opposite points A and B as shown in Fig. 2.9 have equal magnitudes but their directions are such that the resultant electric field intensity is directed along the line OP and away from the sheet charge. In fact, for any point charge on the ring of radius r, there is a diametrically opposite point charge which results in a resultant electric field intensity entirely along OP. Thus the field intensity at point P due to the charge on the entire ring of radius r and width dr is directed normally away from the sheet charge. This suggests that we divide the area of the xy plane into several

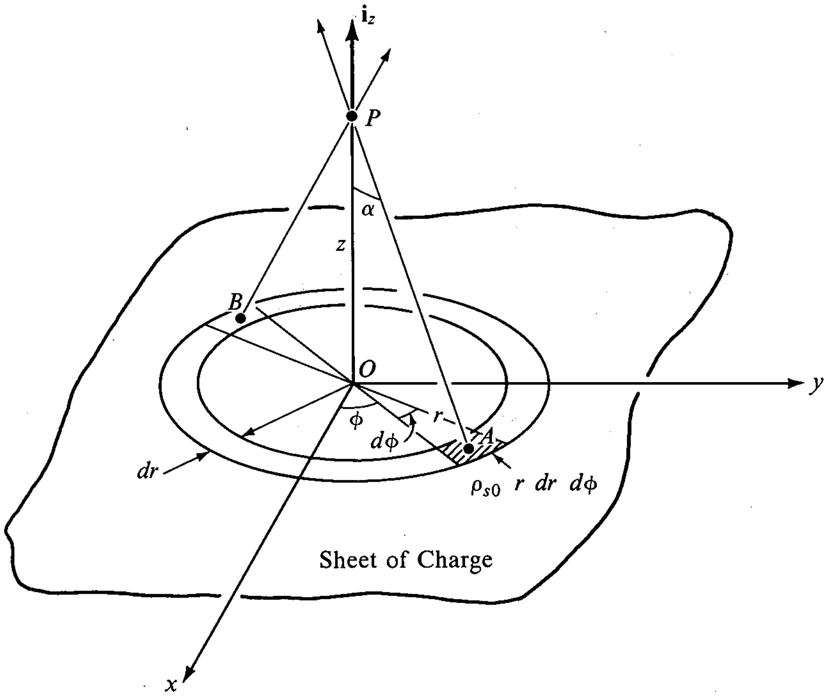

Fig. 2.9. Geometry for computing the electric field of an infinite sheet charge of uniform density ρ_{s0} C/m².

rings, each of width dr, and divide each ring into angular increments of $d\phi$, thus creating infinitesimal areas $r\,dr\,d\phi$ having charges $\rho_{s0}r\,dr\,d\phi$ as shown in Fig. 2.9.

Now, since each ring results in an electric field intensity at point P, only along OP, the field intensity due to the entire sheet charge will also be along the same direction. If we move P sideways while keeping z constant, the situation remains unchanged so that the field intensity is independent of the position of P in planes parallel to the sheet charge. Once again, we have reduced the problem to one of finding the magnitude of **E**.

To find the magnitude of **E**, we note that the component along OP of the field intensity at P, due to the infinitesimal charge $\rho_{s0}r\,dr\,d\phi$ at point A, is given by

$$dE = \frac{\rho_{s0}r\,dr\,d\phi}{4\pi\epsilon_0(r^2+z^2)}\cos\alpha = \frac{\rho_{s0}rz\,dr\,d\phi}{4\pi\epsilon_0(r^2+z^2)^{3/2}} \tag{2-33}$$

The resultant electric field intensity due to the ring of charge passing through A and B is obtained by adding up all the contributions due to the infinitesimal areas on the ring, that is, by integrating (2-33) with respect to ϕ between the limits 0 and 2π. We then add up all the contributions due to the several rings by integrating this result with respect to r between the limits 0 and ∞. We thus obtain a double integral for E as

$$\begin{aligned} E &= \int_{r=0}^{\infty}\int_{\phi=0}^{2\pi} dE = \int_{r=0}^{\infty}\int_{\phi=0}^{2\pi} \frac{\rho_{s0}rz\,dr\,d\phi}{4\pi\epsilon_0(r^2+z^2)^{3/2}} \\ &= \frac{\rho_{s0}z}{2\epsilon_0}\int_{r=0}^{\infty}\frac{r\,dr}{(r^2+z^2)^{3/2}} \end{aligned} \tag{2-34}$$

Introducing $r = z\tan\alpha$ in (2-34), we obtain

$$E = \frac{\rho_{s0}}{2\epsilon_0}\int_{\alpha=0}^{\pi/2}\sin\alpha\,d\alpha = \frac{\rho_{s0}}{2\epsilon_0} \tag{2-35}$$

Recalling that **E** is directed normally away from the line charge, we have

$$\mathbf{E} = \frac{\rho_{s0}}{2\epsilon_0}\mathbf{i}_n \tag{2-36}$$

where $\mathbf{i}_n = \mathbf{i}_z$ above the xy plane and $\mathbf{i}_n = -\mathbf{i}_z$ below the xy plane in Fig. 2.9. Equation (2-36) indicates that the electric field intensity due to an infinite sheet charge of uniform density is independent not only of the position of P in planes parallel to the sheet charge, but also of the distance away from the sheet charge. The field is thus uniform in magnitude and directed normally away from the sheet. If the sheet charge occupies the $z = z_0$ plane, it follows from (2-36) that

$$\mathbf{E} = \begin{cases} \dfrac{\rho_{s0}}{2\epsilon_0}\mathbf{i}_z & \text{for } z > z_0 \\[2ex] -\dfrac{\rho_{s0}}{2\epsilon_0}\mathbf{i}_z & \text{for } z < z_0 \end{cases}$$ ∎

EXAMPLE 2-6. A volume charge is distributed throughout a sphere of radius a, and centered at the origin, with uniform density ρ_0 C/m³. We wish to obtain the electric field intensity due to this volume charge.

With the experience gained in Examples 2-4 and 2-5, we will shorten the discussion concerning the direction of **E** by stating that, for every infinitesimal charge $\rho_0 r^2 \sin\theta\, dr\, d\theta\, d\phi$ in the infinitesimal volume $r^2 \sin\theta\, dr\, d\theta\, d\phi$ at point A inside the sphere as shown in Fig. 2.10, there is another infinitesimal charge such that the resultant electric field intensity at point P due to these two charges is directed entirely along OP, that is, radially away from the center of the sphere. Also, moving P on the surface of a sphere of radius z does not change the situation so that the field intensity is a function only of the distance from the center of the sphere. Thus it is sufficient if we evaluate the component of the electric field intensity at P along OP due to the infinitesimal charge $\rho_0 r^2 \sin\theta\, dr\, d\theta\, d\phi$ and perform a volume integration to obtain the electric

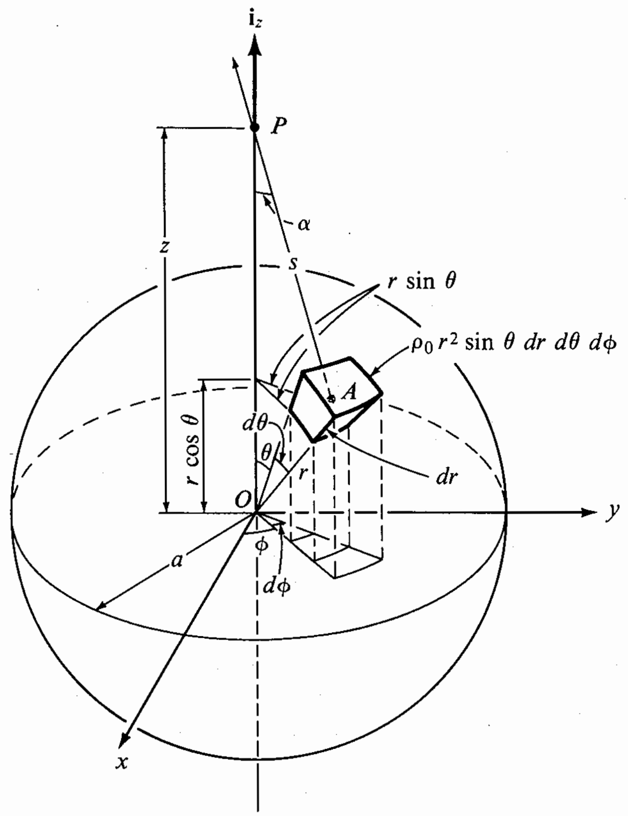

Fig. 2.10. Geometry for computing the electric field of a spherical volume charge of uniform density ρ_0 C/m³.

field intensity due to the entire spherical volume charge of radius a. The component, along OP, of the electric field intensity at P due to the infinitesimal charge at A is given by

$$dE = \frac{\rho_0 r^2 \sin\theta \, dr \, d\theta \, d\phi}{4\pi\epsilon_0 (r^2 + z^2 - 2\,rz\cos\theta)} \cos\alpha = \frac{\rho_0 (z - r\cos\theta) r^2 \sin\theta \, dr \, d\theta \, d\phi}{4\pi\epsilon_0 (r^2 + z^2 - 2\,rz\cos\theta)^{3/2}} \tag{2-37}$$

The electric field intensity due to the entire spherical charge is then given by

$$\begin{aligned} E &= \int_{r=0}^{a} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} dE = \int_{r=0}^{a} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{\rho_0 (z - r\cos\theta) r^2 \sin\theta \, dr \, d\theta \, d\phi}{4\pi\epsilon_0 (r^2 + z^2 - 2\,rz\cos\theta)^{3/2}} \\ &= \frac{\rho_0}{2\epsilon_0} \int_{r=0}^{a} \int_{\theta=0}^{\pi} \frac{(z - r\cos\theta) r^2 \sin\theta \, dr \, d\theta}{(r^2 + z^2 - 2\,rz\cos\theta)^{3/2}} \end{aligned} \tag{2-38}$$

Introducing $s^2 = r^2 + z^2 - 2\,z\cos\theta$, for integration with respect to θ, we have

$$\sin\theta \, d\theta = \frac{s\,ds}{rz} \tag{2-39a}$$

$$z - r\cos\theta = \frac{s^2 - r^2 + z^2}{2z} \tag{2-39b}$$

$$s = \begin{cases} z - r & \text{for} \quad \theta = 0, z > r \\ r - z & \text{for} \quad \theta = 0, 0 < z < r \\ z + r & \text{for} \quad \theta = \pi \end{cases} \tag{2-39c}$$

Substituting these into (2-38), we obtain, for $z > a$,

$$\begin{aligned} E &= \frac{\rho_0}{2\epsilon_0} \int_{r=0}^{a} \frac{r\,dr}{2z^2} \int_{s=z-r}^{z+r} \frac{s^2 - r^2 + z^2}{s^2} ds \\ &= \frac{\rho_0}{2\epsilon_0} \int_{r=0}^{a} \frac{4r^2\,dr}{2z^2} = \frac{(4\pi a^3/3)\rho_0}{4\pi\epsilon_0 z^2} \end{aligned} \tag{2-40}$$

For $0 < z < a$, we have

$$\begin{aligned} E &= \frac{\rho_0}{2\epsilon_0} \int_{r=0}^{z} \frac{r\,dr}{2z^2} \int_{s=z-r}^{z+r} \frac{s^2 - r^2 + z^2}{s^2} ds + \frac{\rho_0}{2\epsilon_0} \int_{r=z}^{a} \frac{r\,dr}{2z^2} \int_{s=r-z}^{z+r} \frac{s^2 - r^2 + z^2}{s^2} ds \\ &= \frac{\rho_0}{2\epsilon_0} \int_{r=0}^{z} \frac{4r^2\,dr}{2z^2} + 0 = \frac{(4\pi z^3/3)\rho_0}{4\pi\epsilon_0 z^2} \end{aligned} \tag{2-41}$$

Equations (2-40) and (2-41) give the magnitude of $\mathbf{E}$ at any radial distance z greater than a and less than a, respectively, from the center of the charge. Recalling that the direction of $\mathbf{E}$ is radially away from the center of the charge distribution and substituting r for z, we have

$$\mathbf{E} = \begin{cases} \dfrac{(4\pi a^3/3)\rho_0}{4\pi\epsilon_0 r^2} \mathbf{i}_r & \text{for} \quad r > a \\ \dfrac{(4\pi r^3/3)\rho_0}{4\pi\epsilon_0 r^2} \mathbf{i}_r & \text{for} \quad r < a \end{cases} \tag{2-42}$$

Noting that $4\pi r^3/3$ is the volume of a sphere of radius r and that there is no charge in the region $r > a$, we can combine the two results on the right side of (2-42) as

$$\mathbf{E}(r) = \frac{\text{charge enclosed by the spherical surface of radius } r}{4\pi\epsilon_0 r^2}\mathbf{i}_r \tag{2-43}$$

Viewed from any distance r from the center of the volume charge, the volume charge is equivalent to a point charge of value equal to the charge enclosed by the spherical surface of radius r. ▮

In the examples we have considered in this section, it was possible to determine the electric field intensity by evaluating a single scalar integral in each case because of the symmetries involved. In the general case, it would be necessary to evaluate three scalar integrals. Furthermore, in order not to get confused between the field points (i.e., points at which the field is desired) and the source points (i.e., points in the volume, surface, or contour occupied by the charge distribution), we must use a notation which distinguishes the two sets of points. Usually, the coordinates of the source points are denoted by primes, whereas the coordinates of the field points are unprimed. The integration is then to be performed with respect to the primed coordinates. This notation is known as the source point-field point notation. Thus, in general, if a line charge of density $\rho_L(\mathbf{r}')$ occupies a contour C', where $\mathbf{r}'$ is the position vector in the source point coordinate system, then the electric field intensity $\mathbf{E}(\mathbf{r})$ at a field point defined by the position vector $\mathbf{r}$ is given by

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{C'} \frac{[\rho_L(\mathbf{r}')\,dl'](\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \tag{2-44a}$$

The right side of Eq. (2-44a) is a vector integral and, in general, it requires the evaluation of three separate scalar integrals. Expressions similar to (2-44a) can be written for surface and volume charge distributions. Thus, for a surface charge of density $\rho_s(\mathbf{r}')$ occupying a surface S', we have

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{S'} \frac{[\rho_s(\mathbf{r}')\,dS'](\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \tag{2-44b}$$

For a volume charge of density $\rho(\mathbf{r}')$ occupying a volume V', we have

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{V'} \frac{[\rho(\mathbf{r}')\,dv'](\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \tag{2-44c}$$

We will use the source point-field point notation only wherever the same coordinate or coordinates for the source and field points appear in the integral. For example, if we wish to evaluate the electric field intensity due to a finitely long line charge along the z axis at a point (r, ϕ, z), then we will have to define the points occupied by the line charge using a z' coordinate so that no confusion arises with the z coordinate of the field point.

2.5 Direction Lines

In the previous two sections we obtained the expressions for the electric field intensities due to certain charge distributions both discrete and continuous. In simple cases, such as for the point charge and for the three examples of the previous section, it is easy to visualize, from a glance at the field expression, the direction of the electric field intensity vector everywhere in space. However, in a case such as the electric dipole (Example 2-3), it is not easy to visualize the direction of the electric field intensity vector by a glance at the field expression [Eq. (2-28)]. If we want to attack the problem directly in such a case, we can assign numerical values for the coordinates in the field expression and compute the direction of the field intensity vector at several points in the medium and then draw arrows along the computed directions. Alternatively and more elegantly, we ask the question: Suppose we place a test charge at a point in the electric field, what is the direction along which it experiences acceleration? Obviously, the test charge experiences acceleration along the direction of the electric field intensity vector at that point. If we stop the test charge after each infinitesimal distance and trace its path in the limit that the infinitesimal distance tends to zero, we get a line along which the electric field is everywhere tangential to it. Such lines, called "direction lines," are of great help in understanding the behavior of a given field, as suggested in Chapter 1. They are also known as "stream lines" and "flux lines."

To develop the technique of sketching the direction lines for a given field, let us consider a small test charge placed at a point $P(x, y, z)$ in the field as shown in Fig. 2.11. At the point P the force on the test charge is

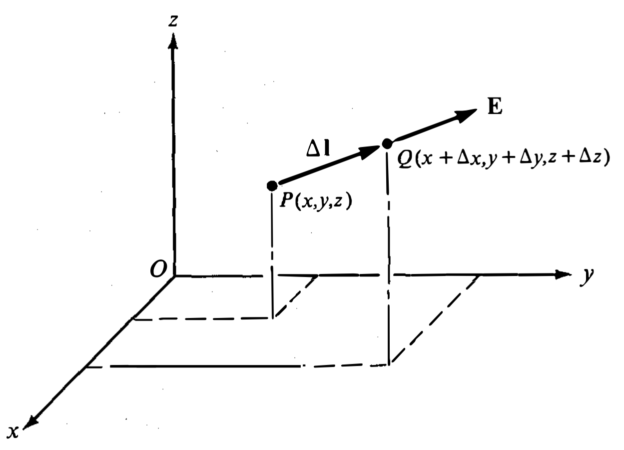

Fig. 2.11. Illustrating the proportionality of the electric field intensity vector **E** and the infinitesimal vector displacement **Δl** of a charge placed in the field.

directed along **E**. The test charge will travel for an infinitesimal distance Δl in the direction of **E** to point $Q(x + \Delta x, y + \Delta y, z + \Delta z)$. The vector displacement of the test charge is then equal to $\Delta x\, \mathbf{i}_x + \Delta y\, \mathbf{i}_y + \Delta z\, \mathbf{i}_z$. But this infinitesimal vector displacement is proportional to the force experienced by the charge which in turn is proportional to $\mathbf{E} = E_x\mathbf{i}_x + E_y\mathbf{i}_y + E_z\mathbf{i}_z$. Thus

$$\Delta x\, \mathbf{i}_x + \Delta y\, \mathbf{i}_y + \Delta z\, \mathbf{i}_z \propto E_x\mathbf{i}_x + E_y\mathbf{i}_y + E_z\mathbf{i}_z \tag{2-45}$$

Two vectors are proportional if and only if their respective components are proportional by the same amount. Hence we have, from (2-45),

$$\frac{\Delta x}{E_x} = \frac{\Delta y}{E_y} = \frac{\Delta z}{E_z} \tag{2-46}$$

But Eq. (2-46) is approximate since, in general, **E** varies continuously from point to point in magnitude and direction. However, it will be exact in the limit Δx, Δy, and Δz all tend to zero. It then reduces to

$$\frac{dx}{E_x} = \frac{dy}{E_y} = \frac{dz}{E_z} \tag{2-47a}$$

Knowing E_x, E_y, and E_z for a particular field, we can substitute in (2-47a) and solve the resulting differential equations to obtain the algebraic equations for the direction lines. We can obtain equations similar to (2-47a) for the cylindrical and spherical coordinate systems following similar arguments. These equations are

$$\frac{dr}{E_r} = \frac{r\,d\phi}{E_\phi} = \frac{dz}{E_z} \qquad \text{cylindrical} \tag{2-47b}$$

$$\frac{dr}{E_r} = \frac{r\,d\theta}{E_\theta} = \frac{r\sin\theta\,d\phi}{E_\phi} \qquad \text{spherical} \tag{2-47c}$$

We will now illustrate the use of these equations by considering an example.

EXAMPLE 2-7. In Example 2-3 we obtained the expression for **E** for an electric dipole of moment p oriented along the positive z axis as

$$\mathbf{E} = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\mathbf{i}_r + \sin\theta\,\mathbf{i}_\theta)$$

It is desired to obtain the equation for the direction lines for this field.

Noting that

$$E_r = \frac{2p\cos\theta}{4\pi\epsilon_0 r^3} \qquad E_\theta = \frac{p\sin\theta}{4\pi\epsilon_0 r^3} \qquad E_\phi = 0$$

we have, from (2-47c),

$$\frac{dr}{(2p\cos\theta)/4\pi\epsilon_0 r^3} = \frac{r\,d\theta}{(p\sin\theta)/4\pi\epsilon_0 r^3} = \frac{r\sin\theta\,d\phi}{0} \tag{2-48}$$

or

$$\frac{dr}{r} = 2 \cot \theta \, d\theta \qquad\qquad d\phi = 0$$
$$\ln r = -2 \ln \operatorname{cosec} \theta + \text{constant} \qquad \phi = \text{constant}$$
$$r \operatorname{cosec}^2 \theta = \text{constant} \qquad\qquad \phi = \text{constant} \tag{2-49}$$

The direction lines are thus intersections of the surfaces $r \operatorname{cosec}^2 \theta = \text{constant}$ and the planes $\phi = \text{constant}$. A few direction lines in constant ϕ plane are sketched in Fig. 2.12. The small arrow at the center indicates the dipole moment **p** with the direction of the arrow as the direction of orientation of the dipole. ▮

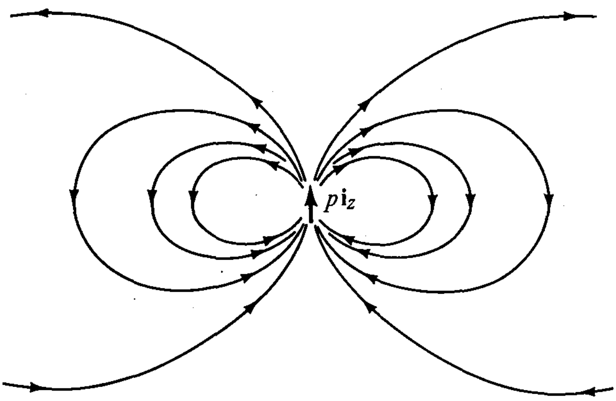

Fig. 2.12. Direction lines of **E** for electric dipole of moment $p\mathbf{i}_z$.

2.6 Gauss' Law in Integral Form

Let us consider the surface of a sphere of radius r and centered at a point charge Q at the origin. The electric field intensity due to the point charge is directed everywhere radially away from the point charge and hence is normal to the surface of the sphere as shown in Fig. 2.13. Its magnitude on the surface of the sphere is a constant equal to $Q/4\pi\epsilon_0 r^2$. If we now consider an infinitesimal area dS on the surface of the sphere, we have

$$\mathbf{E} \cdot d\mathbf{S} = \frac{Q}{4\pi\epsilon_0 r^2}\mathbf{i}_r \cdot dS\,\mathbf{i}_n = \frac{Q}{4\pi\epsilon_0 r^2}\mathbf{i}_r \cdot dS\,\mathbf{i}_r = \frac{Q\,dS}{4\pi\epsilon_0 r^2} \tag{2-50}$$

The integral of $\mathbf{E} \cdot d\mathbf{S}$ over the surface S of the sphere is given by

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \oint_S \frac{Q}{4\pi\epsilon_0 r^2}\,dS = \frac{Q}{4\pi\epsilon_0 r^2}\oint_S dS \tag{2-51}$$

since r is constant on the surface of the sphere. Proceeding further, we have

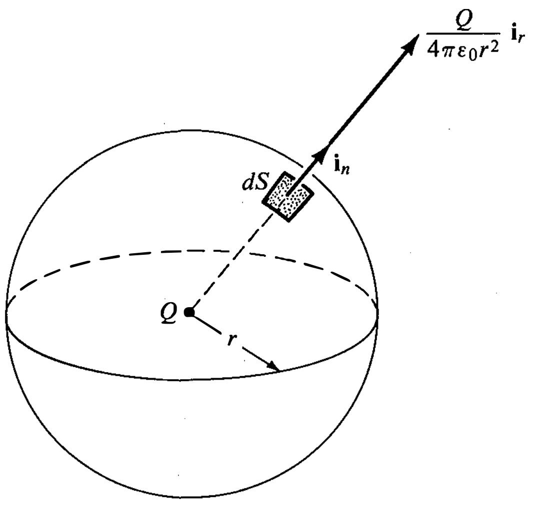

Fig. 2.13. For evaluating $\oint \mathbf{E} \cdot d\mathbf{S}$ on the surface of sphere centered at a point charge Q.

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q}{4\pi\epsilon_0 r^2} \text{ (surface area of the sphere)}$$

$$= \frac{Q}{4\pi\epsilon_0 r^2}(4\pi r^2) = \frac{Q}{\epsilon_0} \tag{2-52}$$

The physical significance of (2-52) is obvious if we compare the electric field lines emanating from the point charge with the flow of a fluid away from the location of the point charge. The surface integral of the fluid flow density vector is the net amount of fluid flowing out of the surface. Similarly, the surface integral of the electric field intensity vector can be interpreted as the net flux of electric field emanating from the surface, although the electric field is not a fluid in the sense that it does not flow like a fluid.

Thus Eq. (2-52) states that the net electric field flux emanating from the surface of a sphere of radius r centered at a point charge Q is equal to Q/ϵ_0. It is independent of the radius r of the spherical surface. Whether $r = 1$ micron or 1000 km, the electric field flux is the same (provided, of course, that there is no other electric field in the medium). This is not surprising if we once again compare the flux of the electric field with the flow of the fluid. If the fluid is flowing radially away from a point source of the fluid, then the amount of fluid crossing a spherical surface of one radius must be the same as the amount crossing a spherical surface of another radius or, for that matter, any arbitrary closed surface enclosing the point source (provided, of course, there is no other source or sink of the fluid). Likewise, if we choose an arbitrary surface enclosing the point charge, the net electric field flux emanating from this surface must be equal to Q/ϵ_0. To prove this

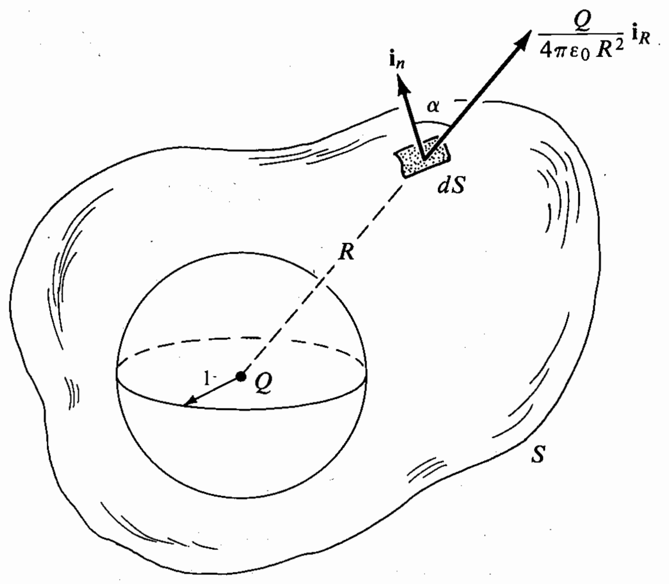

Fig. 2.14. For evaluating $\oint \mathbf{E} \cdot d\mathbf{S}$ over an arbitrary surface S enclosing a point charge Q.

mathematically, we refer to Fig. 2.14. Considering an infinitesimal area dS on the arbitrary surface, we find that the infinitesimal amount of electric field flux emanating from this area is given by

$$\mathbf{E} \cdot d\mathbf{S} = \frac{Q}{4\pi\epsilon_0 R^2}\mathbf{i}_R \cdot dS\,\mathbf{i}_n = \frac{Q\,dS}{4\pi\epsilon_0 R^2}\cos\alpha \tag{2-53}$$

where α is the angle between the radial vector away from the point charge and the normal vector to the area dS. The total flux emanating from the entire closed surface S is then given by

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \oint_S \frac{Q\,dS}{4\pi\epsilon_0 R^2}\cos\alpha = \frac{Q}{4\pi\epsilon_0}\oint_S \frac{dS\cos\alpha}{R^2} \tag{2-54}$$

In (2-54), $dS \cos \alpha$ is the projection of the area dS on the arbitrary surface S onto a spherical surface of radius R and centered at the point charge. Hence $(dS \cos \alpha)/R^2$ is the projection of dS onto a spherical surface of radius unity and centered at the point charge. It is known as the solid angle subtended at the point charge by the area dS. The unit of solid angle is steradian. The quantity $\oint (dS \cos \alpha)/R^2$ is the total solid angle subtended at the point charge by the closed surface S. It is the sum of the projections of all infinitesimal areas comprising the arbitrary surface S onto the spherical surface of radius unity and centered at the point charge. Thus it is equal to the surface area of the sphere of unit radius, that is, 4π. Substituting this result in (2-54), we have

$$\oint_{\substack{\text{surface}\\ \text{enclosing } Q}} \mathbf{E} \cdot d\mathbf{S} = \frac{Q}{4\pi\epsilon_0}(4\pi) = \frac{Q}{\epsilon_0} \tag{2-55}$$

If an arbitrary surface does not enclose a point source of fluid, then the net amount of fluid emanating from the surface must be zero since there are equal amounts of fluid flowing in and out of the surface. Likewise, if the arbitrary surface does not enclose the point charge, the net electric field flux emanating from the surface must be zero. Thus

$$\oint_{\substack{\text{surface not}\\ \text{enclosing } Q}} \mathbf{E} \cdot d\mathbf{S} = 0 \tag{2-56}$$

It will be left as an exercise for the student to provide a mathematical proof of (2-56).

If, instead of one point charge, we have five point charges Q_1, Q_2, Q_3, Q_4, Q_5 as shown in Fig. 2.15, then for an arbitrary surface S enclosing point

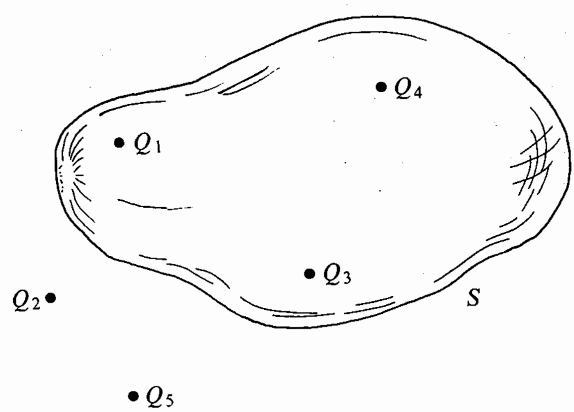

Fig. 2.15. An arbitrary surface enclosing three point charges.

charges Q_1, Q_3, and Q_4 but not Q_2 and Q_5, we can obtain the net electric field flux emanating from the surface using superposition. Thus, if $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$, $\mathbf{E}_4$, and $\mathbf{E}_5$ are the electric field intensity vectors due to Q_1, Q_2, Q_3, Q_4, and Q_5, respectively, we have

$$\begin{aligned}\oint_S \mathbf{E} \cdot d\mathbf{S} &= \oint_S \mathbf{E}_1 \cdot d\mathbf{S} + \oint_S \mathbf{E}_2 \cdot d\mathbf{S} + \oint_S \mathbf{E}_3 \cdot d\mathbf{S} + \oint_S \mathbf{E}_4 \cdot d\mathbf{S} \\ &\quad + \oint_S \mathbf{E}_5 \cdot d\mathbf{S} \\ &= \frac{1}{\epsilon_0}\left(Q_1 + 0 + Q_3 + Q_4 + 0\right) = \frac{1}{\epsilon_0}\left(Q_1 + Q_3 + Q_4\right) \\ &= \frac{1}{\epsilon_0}\left(\text{charge enclosed by the surface } S\right)\end{aligned} \tag{2-57}$$

The discussion can be extended to a continuous charge distribution if we note that a continuous charge distribution can be represented as a continuous collection of charges occupying infinitesimal volumes, each of which can be considered as a point charge. Those charges enclosed by the arbitrary surface result in a net electric field flux in accordance with (2-55), whereas those which are not enclosed by the surface result in zero flux in accordance with (2-56). We can summarize these conclusions in a single statement that "the net electric field flux emanating from a closed surface is equal to the net charge enclosed by the surface divided by ϵ_0." This statement is Gauss' law—one of the important laws in electromagnetic field theory. In equation form, Gauss' law is written as

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\epsilon_0}(\text{charge enclosed by the surface } S) \qquad (2\text{-}58)$$

EXAMPLE 2-8. An infinitely long line charge of uniform density ρ_{L0} C/m is situated along the z axis. It is desired to find the electric field flux cutting the portion of the plane $x = 1$ m lying between the planes $z = 0$ m and $z = 1$ m as shown in Fig. 2.16.

First we will solve this problem by actually evaluating $\int \mathbf{E} \cdot d\mathbf{S}$ over the given surface. To do this, we note that $\mathbf{E}$ due to the line charge is given by $(\rho_{L0}/2\pi\epsilon_0 r)\mathbf{i}_r$, where r is the radial distance from the line charge and $\mathbf{i}_r$ is the unit vector directed radially away from the line charge. Considering an infinitesimal area $dy\,dz$ at the location $(1, y, z)$ on the given plane, the infini-

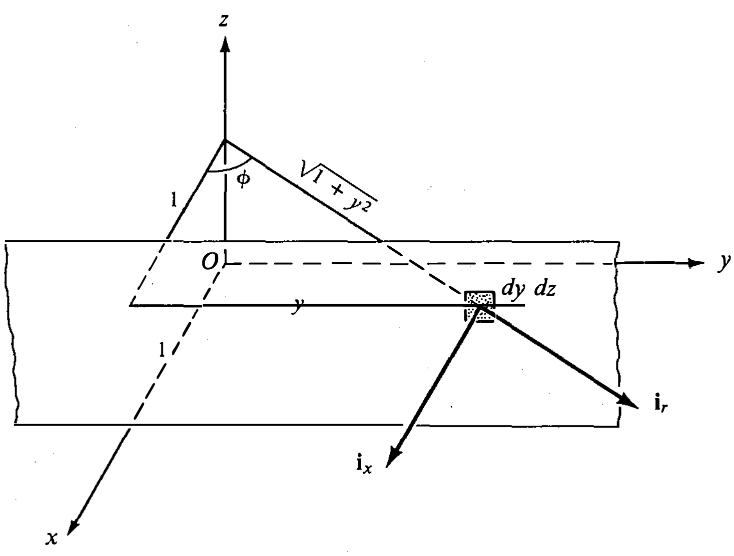

Fig. 2.16. For evaluation of electric field flux emanating from an infinite line charge and cutting a portion of the $x = 1$ plane.

tesimal amount of flux cutting this area is given by

$$\mathbf{E} \cdot d\mathbf{S} = \frac{\rho_{L0}}{2\pi\epsilon_0\sqrt{1+y^2}}\mathbf{i}_r \cdot dy\,dz\,\mathbf{i}_x = \frac{\rho_{L0}\,dy\,dz}{2\pi\epsilon_0(1+y^2)} \tag{2-59}$$

The total flux cutting the portion of the plane $x = 1$ m lying between the planes $z = 0$ m and $z = 1$ m is then given by

$$\begin{aligned}\int_{y=-\infty}^{\infty}\int_{z=0}^{1} \mathbf{E} \cdot d\mathbf{S} &= \int_{y=-\infty}^{\infty}\int_{z=0}^{1} \frac{\rho_{L0}\,dy\,dz}{2\pi\epsilon_0(1+y^2)} \\ &= \frac{\rho_{L0}}{2\pi\epsilon_0}\int_{\phi=-\pi/2}^{\pi/2} d\phi = \frac{\rho_{L0}}{2\epsilon_0}\end{aligned} \tag{2-60}$$

This result can, however, be obtained without performing the integration if we note that the electric field intensity due to the line charge is independent of ϕ and hence the electric field flux from the line charge emanates from it uniformly in ϕ. Thus half of the electric field flux emanating from that portion of the line charge lying between $z = 0$ m and $z = 1$ m cuts the given surface. Since the total flux emanating from this portion of the line charge is $\rho_{L0}(1)/\epsilon_0 = \rho_{L0}/\epsilon_0$, according to Gauss' law, the flux cutting the specified surface is $\rho_{L0}/2\epsilon_0$. ▮

Given $\mathbf{E}$ and a closed surface S, it is always possible to compute the charge enclosed by the surface by evaluating $\oint_S \mathbf{E} \cdot d\mathbf{S}$ analytically or numerically and then multiplying the result by ϵ_0 in accordance with Gauss' law as given by (2-58). The inverse problem of finding $\mathbf{E}$ for a given charge distribution by using (2-58) is possible only for certain simple cases involving a high degree of symmetry, since the unknown quantity $\mathbf{E}$ appears in the integrand. As a first step, the symmetry of the electric field must be determined by making use of the fact that the electric field due to a point charge is directed radially away from it. We have illustrated this in Examples 2-4, 2-5, and 2-6. Next, we should be able to choose a closed surface S such that $\oint_S \mathbf{E} \cdot d\mathbf{S}$ can be reduced to an algebraic quantity involving the magnitude of $\mathbf{E}$. Such a surface is known as a Gaussian surface. Obviously, the Gaussian surface must be such that the magnitude of $\mathbf{E}$ is uniform and the direction of $\mathbf{E}$ is normal to the surface over the whole or part of the surface, while the magnitude of $\mathbf{E}$ is zero or the direction of $\mathbf{E}$ is tangential to the surface over the rest of the surface in the latter case. We will illustrate this method of obtaining $\mathbf{E}$ by reconsidering Examples 2-4, 2-5, and 2-6.

EXAMPLE 2-9. An infinitely long line charge of uniform density ρ_{L0} C/m is situated along the z axis as shown in Fig 2.17. We wish to obtain the electric field intensity due to this line charge using Gauss' law.

In Example 2-4, we established from purely qualitative arguments that $\mathbf{E}$ due to the infinite line charge of uniform density is directed radially away from the line charge and its magnitude is dependent only on its distance

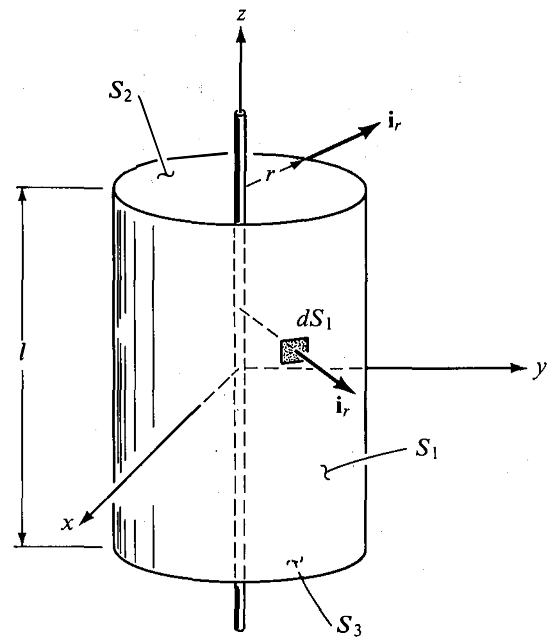

Fig. 2.17. Gaussian surface for computing the electric field of an infinitely long line charge of uniform density.

from the line charge. Thus

$$\mathbf{E} = E_r(r)\mathbf{i}_r \tag{2-61}$$

Choosing the Gaussian surface S as the surface of a cylinder of radius r with the line charge as its axis and of length l, as shown in Fig. 2.17, we have

$$\oint_{\substack{\text{surface of} \\ \text{cylinder, } S}} \mathbf{E} \cdot d\mathbf{S} = \int_{\substack{\text{curved} \\ \text{surface } S_1}} \mathbf{E} \cdot d\mathbf{S} + \int_{\substack{\text{plane sur-} \\ \text{faces } S_2, S_3}} \mathbf{E} \cdot d\mathbf{S} \tag{2-62}$$

The second integral on the right side of (2-62) is zero since $\mathbf{E}$ is tangential to the surfaces; that is, $\mathbf{E} \cdot d\mathbf{S}$ is zero throughout the surfaces. Noting that E_r is constant on the curved surface S_1, we find that the first integral can be written as

$$\begin{aligned}\int_{\substack{\text{curved} \\ \text{surface } S_1}} \mathbf{E} \cdot d\mathbf{S} &= \int_{S_1} E_r\mathbf{i}_r \cdot dS_1\,\mathbf{i}_r = E_r \int_{S_1} dS_1 \\ &= E_r \text{ (surface area of } S_1) = E_r(2\pi rl)\end{aligned} \tag{2-63}$$

Thus

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = 2\pi rlE_r \tag{2-64}$$

But, from Gauss' law,

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{\text{charge enclosed by } S}{\epsilon_0} = \frac{\rho_{L0} l}{\epsilon_0} \tag{2-65}$$

Comparing (2-64) and (2-65), we have

$$E_r = \frac{\rho_{L0}}{2\pi\epsilon_0 r} \tag{2-66}$$

$$\mathbf{E} = \frac{\rho_{L0}}{2\pi\epsilon_0 r}\mathbf{i}_r \tag{2-67}$$

which agrees with the result obtained in Example 2-4. ∎

EXAMPLE 2-10. A sheet charge of uniform density ρ_{s0} C/m^2 extends over the entire xy plane as shown in Fig. 2.18. We wish to obtain the electric field intensity due to this infinite sheet charge using Gauss' law.

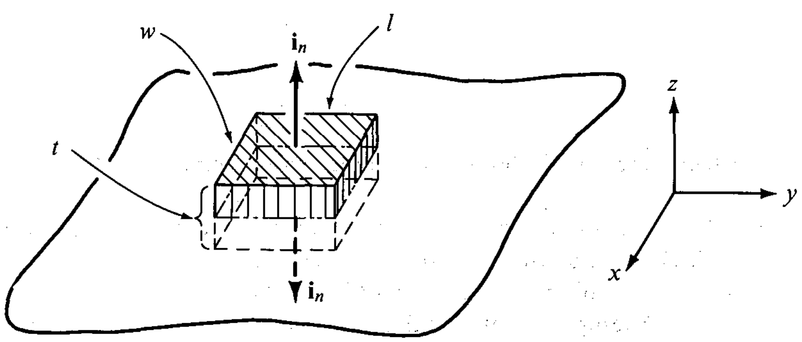

Fig. 2.18. Gaussian surface for computing the electric field of an infinite sheet charge of uniform density.

In Example 2-5 we established from purely qualitative arguments that $\mathbf{E}$ due to the infinite sheet charge of uniform density is directed normally away from the sheet charge and that it is uniform in planes parallel to the sheet charge. Thus

$$\mathbf{E} = E_n \mathbf{i}_n \tag{2-68}$$

Choosing the Gaussian surface S as the surface of a rectangular pill box of sides l, w, and t as shown in Fig. 2.18, such that half of the box is above the sheet charge and the other half below it, we have

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \int_{\substack{\text{top}\\\text{surface}}} \mathbf{E} \cdot d\mathbf{S} + \int_{\substack{\text{bottom}\\\text{surface}}} \mathbf{E} \cdot d\mathbf{S} + \int_{\substack{\text{side}\\\text{surfaces}}} \mathbf{E} \cdot d\mathbf{S} \tag{2-69}$$

But the last integral on the right side of (2-69) is equal to zero since $\mathbf{E}$ is parallel to the side surfaces and hence $\mathbf{E} \cdot d\mathbf{S}$ is zero throughout these surfaces. Because E_n is constant on both the top and bottom surfaces and E_n is the same on both these surfaces, since they are equidistant from the sheet charge, Eq. (2-69) then reduces to

$$\begin{aligned}\oint_S \mathbf{E} \cdot d\mathbf{S} &= 2 \int_{\substack{\text{top}\\\text{surface}}} \mathbf{E} \cdot d\mathbf{S} = 2 \int_{\substack{\text{top}\\\text{surface}}} E_n \mathbf{i}_n \cdot dS\,\mathbf{i}_n \\ &= 2E_n \int_{\substack{\text{top}\\\text{surface}}} dS = 2E_n \text{ (surface area of top surface)} \\ &= 2E_n lw\end{aligned} \tag{2-70}$$

But, from Gauss' law,

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{\text{charge enclosed by } S}{\epsilon_0} = \frac{\rho_{s0} lw}{\epsilon_0} \tag{2-71}$$

Comparing (2-70) and (2-71), we have

$$E_n = \frac{\rho_{s0}}{2\epsilon_0} \tag{2-72}$$

$$\mathbf{E} = \frac{\rho_{s0}}{2\epsilon_0}\mathbf{i}_n \tag{2-73}$$

which agrees with the result obtained in Example 2-5. ▮

EXAMPLE 2-11. A volume charge is distributed throughout a sphere of radius a with uniform density ρ_0 C/m³. We wish to obtain the electric field intensity due to this volume charge using Gauss' law.

In Example 2-6 we established from purely qualitative arguments that **E** due to the spherical volume charge of uniform density is directed radially away from the center of the charge and is a function only of the distance from the center of the sphere. Thus

$$\mathbf{E} = E_r(r)\mathbf{i}_r \tag{2-74}$$

Choosing the Gaussian surface S as the surface of a sphere of radius $r \gtrless a$, concentric with the spherical charge, as shown in Fig. 2.19, we have

$$\begin{aligned}\oint_S \mathbf{E} \cdot d\mathbf{S} &= \oint_S E_r \mathbf{i}_r \cdot dS\,\mathbf{i}_r = E_r \oint_S dS \\ &= E_r \text{ (surface area of the sphere of radius } r) \\ &= E_r(4\pi r^2)\end{aligned} \tag{2-75}$$

But, from Gauss' law,

$$\begin{aligned}\oint_S \mathbf{E} \cdot d\mathbf{S} &= \frac{\text{charge enclosed by } S}{\epsilon_0} \\ &= \frac{\text{charge enclosed by spherical surface of radius } r}{\epsilon_0}\end{aligned} \tag{2-76}$$

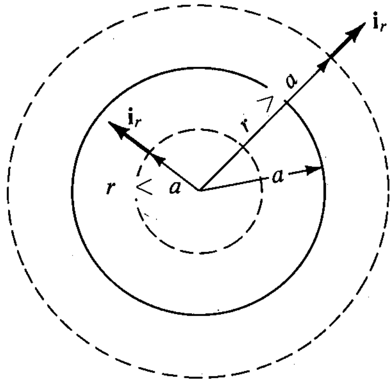

Fig. 2.19. Gaussian surfaces for computing the electric field of a spherical volume charge of uniform density.

Comparing (2-75) and (2-76), we have

$$E_r = \frac{\text{charge enclosed by spherical surface of radius } r}{4\pi\epsilon_0 r^2} \tag{2-77}$$

$$\mathbf{E} = \frac{\text{charge enclosed by spherical surface of radius } r}{4\pi\epsilon_0 r^2}\,\mathbf{i}_r \tag{2-78}$$

which agrees with the result of Example 2-6. ∎

2.7 Gauss' Law in Differential Form (Maxwell's Divergence Equation for the Electric Field)

Let us consider a volume charge distribution with the charge density ρ as a given function of the coordinate system. The charge enclosed by an arbitrary closed surface S is given by the volume integral of the charge density throughout the volume V enclosed by the surface S; that is, $\int_V \rho\, dv$. According to Gauss' law (2-58), we have

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\epsilon_0}\int_V \rho\, dv \tag{2-79}$$

If we now shrink the volume to a very small value ΔV, so that the surface area becomes very small ΔS, we can write (2-79) for this infinitesimal surface as

$$\oint_{\Delta S} \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\epsilon_0}\int_{\Delta v} \rho\, dv \tag{2-80}$$

Since the volume is very small, we can consider the charge density ρ to be uniform inside that volume so that $\int_{\Delta v} \rho\, dv \approx \rho\, \Delta v$. This is exact in the limit that $\Delta v \longrightarrow 0$. Dividing both sides of (2-80) by Δv and letting $\Delta v \longrightarrow 0$, we have

$$\lim_{\Delta v \to 0} \frac{\oint_{\Delta S} \mathbf{E} \cdot d\mathbf{S}}{\Delta v} = \lim_{\Delta v \to 0} \frac{(1/\epsilon_0)\int_{\Delta v} \rho \, dv}{\Delta v} = \frac{1}{\epsilon_0} \lim_{\Delta v \to 0} \frac{\rho \, \Delta v}{\Delta v} = \frac{1}{\epsilon_0} \rho \tag{2-81}$$

The left side of (2-81) is the divergence of $\mathbf{E}$ so that we have

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho \tag{2-82}$$

Equation (2-82) is Gauss' law in differential form, which states that the divergence of the electric field intensity at any point is equal to $1/\epsilon_0$ times the volume charge density at that point. This is Maxwell's divergence equation for the electric field.

The right side of (2-82) represents a volume charge density. Suppose we are considering problems involving point charges, line charges, and surface charges. The question then arises as to how we should represent the right side of (2-82) since, for such charges, the volume charge density is infinity. We can resolve this problem by resorting to the Dirac delta function or the impulse function. We will illustrate this for the case of a surface charge in the following example.

EXAMPLE 2-12. A sheet charge of uniform density ρ_{s0} C/m² extends over the entire xy plane. It is desired to write Gauss' law in differential form for this sheet charge.

Let us consider a slab of charge lying between the planes $z = -a$ and $z = +a$ and of uniform density ρ_0 C/m³ as shown in Fig. 2.20(a). The volume charge density as a function of z for such a charge distribution is sketched in

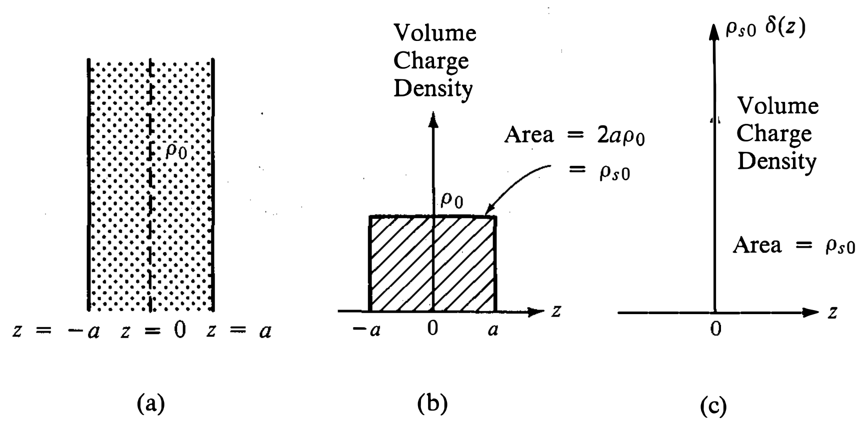

Fig. 2.20. For deriving the volume charge density corresponding to a surface charge.

Fig. 2.20(b). The charge per unit surface area of the slab charge is given by $\int_{z=-a}^{a} \rho_0\, dz = \rho_0 2a =$ area under the curve of Fig. 2.20(b). Let this quantity be ρ_{s0}. Suppose we now shrink a to zero, increasing ρ_0 such that ρ_{s0} remains constant. We then obtain a sheet charge of density ρ_{s0} C/m^2. What happens to the sketch of Fig. 2.20(b)? The width of the pulse-shaped sketch decreases to zero and the height increases to infinity but maintaining the area under it equal to ρ_{s0}. The resulting function is sketched in Fig. 2.20(c). This function is known as the Dirac delta function of strength ρ_{s0} and is represented as $\rho_{s0}\,\delta(z)$, where $\delta(z)$ satisfies the properties

$$\delta(z) = \begin{cases} 0 & \text{for } z \neq 0 \\ \infty & \text{for } z = 0 \end{cases} \tag{2-83}$$

$$\int_{z=-\infty}^{\infty} \delta(z)\, dz = \int_{z=0-}^{0+} \delta(z)\, dz = \lim_{a\to 0} \int_{z=-a}^{a} \frac{1}{2a}\, dz = 1 \tag{2-84}$$

$$\int_{z=-\infty}^{\infty} f(z)\,\delta(z)\, dz = f(0) \tag{2-85}$$

Thus the volume charge density corresponding to the sheet charge of density ρ_{s0} lying in the $z = 0$ plane is $\rho_{s0}\,\delta(z)$. Gauss' law in differential form for the sheet charge is then given by

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho_{s0}\,\delta(z) \tag{2-86}$$

If the sheet charge lies in the $z = z_0$ plane, then the Dirac delta function is shifted to $z = z_0$ and is written as $\delta(z - z_0)$, having the properties

$$\delta(z - z_0) = \begin{cases} 0 & \text{for } z \neq z_0 \\ \infty & \text{for } z = z_0 \end{cases} \tag{2-87}$$

$$\int_{z=-\infty}^{\infty} \delta(z - z_0)\, dz = 1 \tag{2-88}$$

$$\int_{z=-\infty}^{\infty} f(z)\,\delta(z - z_0)\, dz = f(z_0) \tag{2-89}$$

Gauss' law in differential form is modified to read

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho_{s0}\,\delta(z - z_0) \tag{2-90}$$

It is left to the student to derive equations similar to (2-90) for line and point charges, involving two-dimensional and three-dimensional Dirac delta functions, respectively (See Problems 2.33 and 2.34). ▮

2.8 Potential Difference

In the study of mechanics, we are familiar with potential energy associated with the movement of a mass in the gravitational field of the earth. If the movement of the mass is along the direction of the gravitational field, that is, from a higher elevation to a lower elevation, the gravitational field does the work. If the movement is opposite to the direction of the gravitational field, that is, from a lower elevation to a higher elevation, certain work has to be performed by an external source to overcome the gravitational force. Likewise, since the electric field is a force field in so far as charges are concerned, there is work associated with the movement of charges in an electric field. If a test charge is moved along the direction of the field, work is done by the field since the force exerted by the field on the charge is in the direction of its movement and hence it accelerates the test charge. If the charge is moved against the direction of the field, an external agent has to supply the energy to overcome the force exerted on the charge by the field, since this force is opposite to the direction of movement of the charge.

Let us consider the displacement of a test charge q by an infinitesimal distance $d\mathbf{l}$ from A to B at an angle α with the electric field $\mathbf{E}$ at the point A as shown in Fig. 2.21(a). The force exerted on the test charge by the field

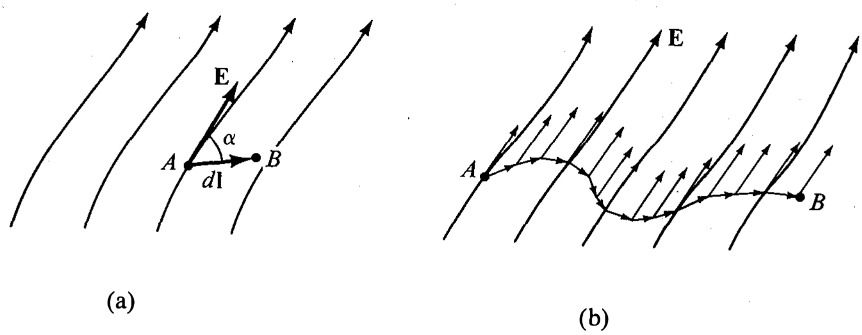

Fig. 2.21. Movement of a test charge in an electric field.

has magnitude qE and is directed along $\mathbf{E}$. Its component along the line from A to B is $qE\cos\alpha$. If the charge is moved from A to B, the amount of work dW done by the field is the product of the force and the displacement; that is,

$$dW = qE\cos\alpha\, dl = q\mathbf{E}\cdot d\mathbf{l} \tag{2-91}$$

where $d\mathbf{l}$ is the vector from A to B. Note that dW is positive if $0 < \alpha < 90°$ so that work is done by the field; dW is negative if $90° < \alpha < 180°$ so that negative work is done by the field, which amounts to stating that work is done against the field by an external agent. For $\alpha = 90°$, dW is zero, which is analogous to the movement of a mass on a frictionless surface at right angles to the gravitational field. Now let us consider two points A and B which are widely separated as shown in Fig. 2.21(b). The work W_{AB} done by the field in moving a test charge q from A to B along a given path can be obtained by dividing the path into several segments of infinitesimal length dl, then applying (2-91) to each segment, and adding up all the contributions.

The result is a line integral expression given by

$$W_{AB} = q \int_A^B \mathbf{E} \cdot d\mathbf{l} \tag{2-92}$$

where the integration is performed along the given path from A to B. The evaluation of line integrals was discussed in Section 1.7.

In the gravitational field, when a mass moves from a higher elevation to a lower elevation, it loses some potential energy and vice versa. Likewise, in the electric field, we can state that the test charge has certain potential energy associated with it by virtue of its location in the electric field. W_{AB} as given by (2-92) is then the loss of potential energy associated with the movement of the charge from A to B. If we divide W_{AB} by q, we obtain the loss of potential energy per unit charge. This quantity denoted by V_{AB} is known as the potential difference between the points A and B. Thus

$$V_{AB} = \frac{W_{AB}}{q} = \int_A^B \mathbf{E} \cdot d\mathbf{l} \tag{2-93}$$

If V_{AB} is positive, there is a loss in potential energy associated with the movement of the charge from A to B; that is, the field does the work. If V_{AB} is negative, there is a gain in potential energy associated with the movement of the charge from A to B; that is, an external agent has to do the work. The units of potential difference are newton-meters per coulomb or joules per coulomb, commonly known as volts. This gives the units of volts per meter to the electric field intensity.

EXAMPLE 2-13. In cartesian coordinates, the electric field intensity is given by

$$\mathbf{E} = yz\mathbf{i}_x + zx\mathbf{i}_y + xy\mathbf{i}_z$$

Find the potential difference between the points $A(0, 22.7, 99)$ and $B(1, 1, 1)$. Is it necessary to specify a path for line integration between the two points?

In cartesian coordinates, $d\mathbf{l} = dx\mathbf{i}_x + dy\mathbf{i}_y + dz\mathbf{i}_z$ so that

$$\begin{aligned} V_{AB} &= \int_A^B \mathbf{E} \cdot d\mathbf{l} = \int_A^B (yz\mathbf{i}_x + zx\mathbf{i}_y + xy\mathbf{i}_z) \cdot (dx\,\mathbf{i}_x + dy\,\mathbf{i}_y + dz\,\mathbf{i}_z) \\ &= \int_A^B (yz\,dx + zx\,dy + xy\,dz) \\ &= \int_A^B d(xyz) = [xyz]_A^B \end{aligned}$$

Since $\mathbf{E} \cdot d\mathbf{l}$ is the total derivative of a function of x, y, z, it is not necessary to specify a path for the line integration between the two points. V_{AB} is dependent only on the coordinates of the end points A and B. We will find in Section 2.11 that this is a general characteristic of the static electric field. Here, we have

$$V_{AB} = [xyz]_A^B = [xyz]_{0,\,22.7,\,99}^{1,\,1,\,1} = 1. \quad \blacksquare$$

2.9 The Potential Field of Point Charges

Let us now consider two points A and B in the electric field of a point charge Q situated at distances r_A and r_B, respectively, from the point charge as shown in Fig. 2.22. Using (2-93), the potential difference between A and B can be computed for any specified path from A to B. Noting that $\mathbf{E} = (Q/4\pi\epsilon_0 r^2)\mathbf{i}_r$ for a point charge and that the differential length vector $d\mathbf{l}$ is given in spherical coordinates as

$$d\mathbf{l} = dr\,\mathbf{i}_r + r\,d\theta\,\mathbf{i}_\theta + r\sin\theta\,d\phi\,\mathbf{i}_\phi \tag{2-94}$$

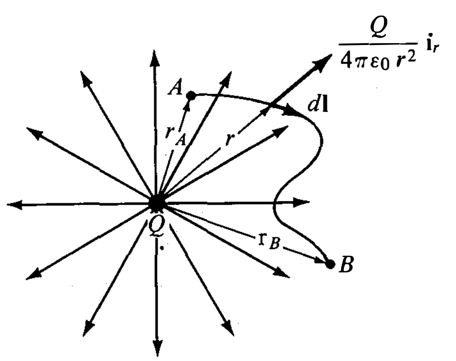

Fig. 2.22. Computation of the potential difference between two points in the electric field of a point charge.

we have, from (2-93),

$$\begin{aligned} V_{AB} &= \int_A^B \mathbf{E}\cdot d\mathbf{l} = \int_A^B \left(\frac{Q}{4\pi\epsilon_0 r^2}\mathbf{i}_r\right)\cdot(dr\,\mathbf{i}_r + r\,d\theta\,\mathbf{i}_\theta + r\sin\theta\,d\phi\,\mathbf{i}_\phi) \\ &= \int_{r=r_A}^{r_B} \frac{Q}{4\pi\epsilon_0 r^2}\,dr = \frac{Q}{4\pi\epsilon_0 r_A} - \frac{Q}{4\pi\epsilon_0 r_B} \end{aligned} \tag{2-95}$$

Equation (2-95) indicates that, for a given charge Q, the potential difference between the two points is dependent only upon their distances from the point charge and not on the path from A to B chosen for its evaluation. Furthermore, the potential difference is the difference between two terms, one of which is dependent on r_A only and the other dependent on r_B only. We can call these terms the potentials at r_A and r_B, respectively. If we denote these potentials as V_A and V_B, respectively, we have, from (2-95),

$$V_A = \frac{Q}{4\pi\epsilon_0 r_A} \tag{2-96}$$

$$V_B = \frac{Q}{4\pi\epsilon_0 r_B} \tag{2-97}$$

The right sides of Eqs. (2-96) and (2-97) are, however, not unique expressions for V_A and V_B since, on the right side of (2-95), we can add and subtract any arbitrary constant C without altering its value; that is,

$$V_{AB} = \left(\frac{Q}{4\pi\epsilon_0 r_A} + C\right) - \left(\frac{Q}{4\pi\epsilon_0 r_B} + C\right) \tag{2-98}$$

which then leads to

$$V_A = \frac{Q}{4\pi\epsilon_0 r_A} + C \tag{2-99}$$

$$V_B = \frac{Q}{4\pi\epsilon_0 r_B} + C \tag{2-100}$$

If we let $C = Q/4\pi\epsilon_0 r_0$, where r_0 is a constant, we have

$$V_A = \frac{Q}{4\pi\epsilon_0 r_A} - \frac{Q}{4\pi\epsilon_0 r_0} \tag{2-101a}$$

$$V_B = \frac{Q}{4\pi\epsilon_0 r_B} - \frac{Q}{4\pi\epsilon_0 r_0} \tag{2-101b}$$

Comparing (2-101a) with (2-95), we note that V_A is the potential difference between point A and another point situated at a distance r_0 from the point charge, which we will call the reference point. Similarly, V_B is the potential difference between the point B and the same reference point. Thus the potential at any point is simply the potential difference between that point and an arbitrary reference point. But then, what is the potential at the reference point? The answer to this question is obtained by substituting $r_A = r_0$ in (2-101a) or $r_B = r_0$ in (2-101b), both of which result in zero. The potential at the reference point is therefore zero. To complete the definition, we state that the potential at any point is the potential difference between that point and an arbitrary reference point at which the potential is zero. In the case of a point charge, a convenient reference point is $r_0 = \infty$. We then have

$$V(r) = \frac{Q}{4\pi\epsilon_0 r} \tag{2-102}$$

The potential at a distance r from the point charge is thus the work done per unit charge by the field in the movement of a test charge from that point to infinity or, it is the work done per unit charge by an external agent in bringing a test charge from infinity to that point; that is,

$$V(r) = \int_r^\infty \mathbf{E} \cdot d\mathbf{l} = -\int_\infty^r \mathbf{E} \cdot d\mathbf{l} \tag{2-103}$$

The right side of (2-102) represents the potential field of a point charge. It is also known as the Coulomb potential of a point charge. In contrast to the vector nature of the electric field intensity, the potential field is a scalar field.

Surfaces on which potential is a constant are known as equipotential surfaces. If a test charge is moved on such a surface from one point to another, no work is involved since the potential difference between any two points is zero. For the point charge, the equipotential surfaces are, according to (2-102), $r =$ constant, that is, surfaces of spheres centered at the point charge. The equipotential surfaces are thus orthogonal to the direction lines of **E** which are radial, as shown in Fig. 2.23. This result is to be expected not only for a point charge but for any charge distribution, since if we move a test charge along a path everywhere normal to the direction lines, there is no component of force acting on the charge along the direction of the path and hence the work involved is zero.

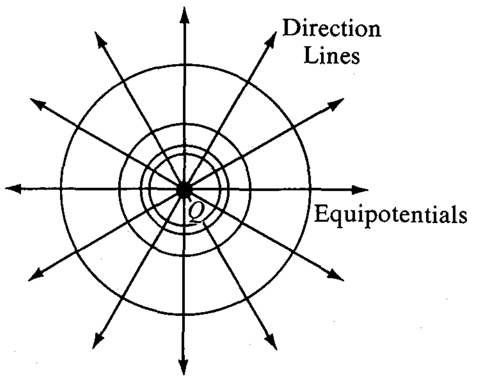

Fig. 2.23. Cross sections of equipotential surfaces and direction lines of **E** for a point charge.

For several point charges located at different points as shown in Fig. 2.5, the potential at any point P is the work done per unit charge by an external agent in bringing a test charge from infinity to that point in the combined electric field **E** of all the charges; that is,

$$\begin{aligned} V(P) &= -\int_{\infty}^{P} \mathbf{E} \cdot d\mathbf{l} \\ &= -\int_{\infty}^{P} (\mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3 + \cdots + \mathbf{E}_n) \cdot d\mathbf{l} \\ &= -\int_{\infty}^{P} \mathbf{E}_1 \cdot d\mathbf{l} - \int_{\infty}^{P} \mathbf{E}_2 \cdot d\mathbf{l} - \cdots - \int_{\infty}^{P} \mathbf{E}_n \cdot d\mathbf{l} \end{aligned} \tag{2-104}$$

where $\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3, \ldots, \mathbf{E}_n$ are the electric field intensities due to the individual point charges $Q_1, Q_2, Q_3, \ldots, Q_n$, respectively. But each term on the right side of (2-104) is equal to the potential at the point P due to the corresponding charge. Thus

$$\begin{aligned} V(P) &= \frac{Q_1}{4\pi\epsilon_0 R_1} + \frac{Q_2}{4\pi\epsilon_0 R_2} + \cdots + \frac{Q_n}{4\pi\epsilon_0 R_n} \\ &= \sum_{j=1}^{n} \frac{Q_j}{4\pi\epsilon_0 R_j} \end{aligned} \tag{2-105}$$

The potential at P due to the collection of point charges is the sum of the potentials at P due to the individual charges. In the vector notation defined in connection with Eq. (2-20), we write

$$V(\mathbf{r}) = \sum_{j=1}^{n} \frac{Q_j}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'_j|} \tag{2-106}$$

EXAMPLE 2-14. For the electric dipole arrangement of Fig. 2.7, it is desired to find the potential at distances very far from the dipole compared to the spacing d.

With reference to the notation of Fig. 2.7, the electric potential at point P is given by

$$V(r) = \frac{Q}{4\pi\epsilon_0 r_+} - \frac{Q}{4\pi\epsilon_0 r_-} \tag{2-107}$$

For $r \gg d$, (2-107) can be approximated as

$$\begin{aligned} V(r) &\approx \frac{Q}{4\pi\epsilon_0[r - (d/2)\cos\theta]} - \frac{Q}{4\pi\epsilon_0[r + (d/2)\cos\theta]} \\ &= \frac{Qd\cos\theta}{4\pi\epsilon_0[r^2 - (d^2/4)\cos^2\theta]} \approx \frac{Qd\cos\theta}{4\pi\epsilon_0 r^2} \end{aligned} \tag{2-108}$$

Equation (2-108) becomes exact in the limit $d \longrightarrow 0$, keeping the dipole moment $p = Qd$ constant. We then have the potential field of dipole moment $\mathbf{p} = p\mathbf{i}_z$ given by

$$V(r) = \frac{p\cos\theta}{4\pi\epsilon_0 r^2} = \frac{\mathbf{p}\cdot\mathbf{i}_r}{4\pi\epsilon_0 r^2} = \frac{\mathbf{p}\cdot\mathbf{r}}{4\pi\epsilon_0 r^3} \tag{2-109}$$

The potential field of a dipole drops off inversely as the square of the distance, as compared to the inverse distance dependence of the potential field of a point charge. Likewise, the potential field of a quadrupole can be shown to vary inversely as r^3. The potential fields of successive higher-order multipoles vary inversely as $r^4, r^5, \ldots$. From (2-109), we note that the equipotential surfaces for the dipole field are $(\cos\theta)/r^2 =$ constant, or

$$r^2 \sec\theta = \text{constant} \tag{2-110}$$

Cross sections of these surfaces are sketched in Fig. 2.24, in which the direction lines of **E** taken from Fig. 2.12 are also shown. It is left as an exercise for the student to show that the equipotential surfaces given by (2-110) and the direction lines given by (2-49) are orthogonal. ∎

EXAMPLE 2-15. A point charge Q is situated at a vector distance $\mathbf{r}'$ from the origin of a coordinate system as shown in Fig. 2.25. It is desired to find the potential due to this point charge at distances $\mathbf{r}$ from the origin large in magnitude compared to $\mathbf{r}'$ in the form of a power series in r.

Let P be the point at which the potential is desired. Then, from (2-106), the potential at P due to Q is given by

$$\begin{aligned} V(\mathbf{r}) &= \frac{Q}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}'|} \\ &= \frac{Q}{4\pi\epsilon_0(r^2 + r'^2 - 2rr'\cos\alpha)^{1/2}} \\ &= \frac{Q}{4\pi\epsilon_0 r}\left(1 + \frac{r'^2}{r^2} - \frac{2\mathbf{r}'\cdot\mathbf{r}}{r^2}\right)^{-1/2} \end{aligned} \tag{2-111}$$

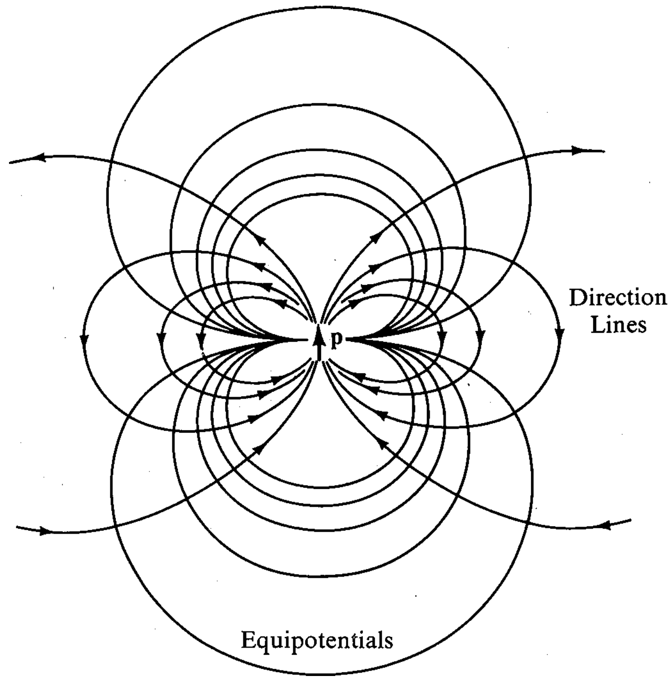

Fig. 2.24. Cross sections of equipotential surfaces and direction lines of **E** for an electric dipole.

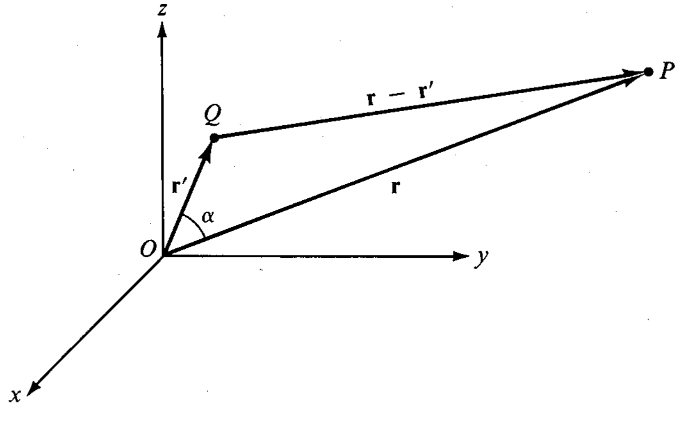

Fig. 2.25. For the computation of potential due to a point charge at distances large compared to its distance from the origin.

Using the binomial theorem,

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots$$

we have

$$V(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0 r}\left[1 + \left(-\frac{1}{2}\right)\left(\frac{r'^2}{r^2} - \frac{2\mathbf{r}' \cdot \mathbf{r}}{r^2}\right) + \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{r'^2}{r^2} - \frac{2\mathbf{r}' \cdot \mathbf{r}}{r^2}\right)^2 + \cdots \text{higher-order terms}\right]$$
$$= \frac{Q}{4\pi\epsilon_0 r}\left\{1 + \frac{\mathbf{r}' \cdot \mathbf{r}}{r^2} + \frac{1}{2r^4}[3(\mathbf{r}' \cdot \mathbf{r})^2 - r^2 r'^2] + \cdots \text{higher-order terms}\right\} \tag{2-112}$$

For $r'/r \ll 1$, the magnitudes of the successive terms on the right side of (2-112) decrease rapidly as can be seen by writing (2-112) as

$$V(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0 r}\left[1 + \left(\frac{r'}{r}\right)\cos\alpha + \left(\frac{r'}{r}\right)^2\left(\frac{3\cos^2\alpha - 1}{2}\right) + \cdots \text{higher-order terms}\right] \tag{2-113}$$

Hence, for $r' \ll r$, only the first few terms are significant. Furthermore, writing

$$V(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0 r} + \frac{Q\mathbf{r}' \cdot \mathbf{r}}{4\pi\epsilon_0 r^3} + \frac{Q}{8\pi\epsilon_0 r^5}[3(\mathbf{r}' \cdot \mathbf{r})^2 - r^2 r'^2] + \cdots$$
$$= \frac{Q}{4\pi\epsilon_0 r} + \frac{Qr'\cos\alpha}{4\pi\epsilon_0 r^2} + \frac{Qr'^2}{4\pi\epsilon_0 r^3}\left(\frac{3\cos^2\alpha - 1}{2}\right) + \cdots \tag{2-114}$$

we observe that, on the right side of (2-114), the first term is the potential at P due to a point charge Q at the origin; the second term is the potential at P due a dipole moment $\mathbf{p} = Q\mathbf{r}'$ at the origin; the third term seems like the potential at P due to a quadrupole at the origin since it varies as $1/r^3$, and so on.

If we have several point charges $Q_1, Q_2, Q_3, \ldots, Q_n$ situated at $\mathbf{r}'_1, \mathbf{r}'_2, \mathbf{r}'_3, \ldots, \mathbf{r}'_n$, the potential at $\mathbf{r}$ due to this collection of point charges can be written by applying superposition to (2-114) as

$$V(\mathbf{r}) = \sum_{j=1}^{n}\left\{\frac{Q_j}{4\pi\epsilon_0 r} + \frac{Q_j\mathbf{r}'_j \cdot \mathbf{r}}{4\pi\epsilon_0 r^3} + \frac{Q_j}{8\pi\epsilon_0 r^5}[3(\mathbf{r}'_j \cdot \mathbf{r})^2 - r^2 r'^2_j] + \cdots\right\}$$
$$= \frac{\sum_{j=1}^{n} Q_j}{4\pi\epsilon_0 r} + \frac{\sum_{j=1}^{n} Q_j\mathbf{r}'_j \cdot \mathbf{r}}{4\pi\epsilon_0 r^3} + \cdots \tag{2-115}$$

The potential due to the collection of point charges at large distances from the collection is thus a superposition of the potentials due to a point charge of value $\sum_{j=1}^{n} Q_j$, a dipole moment $\sum_{j=1}^{n} Q_j \mathbf{r}'_j$, and so on, all situated at the origin. We note that if the sum of the charges is zero, the first significant term is that of the dipole moment. Likewise, if the sum of the charges as well as the dipole moment are zero, the first significant term is the quadrupole term, and so on. Usually, two significant terms will suffice. ▮

EXAMPLE 2-16. Point charges are located at the corners of a cube of sides 1 m², with one corner placed at the origin and three edges coinciding with the coordinate axes as shown in Fig. 2.26. Values of the point charges in coulombs are indicated at the respective corners. Find the first two significant terms in the potential of this collection of charges at large distances from it.

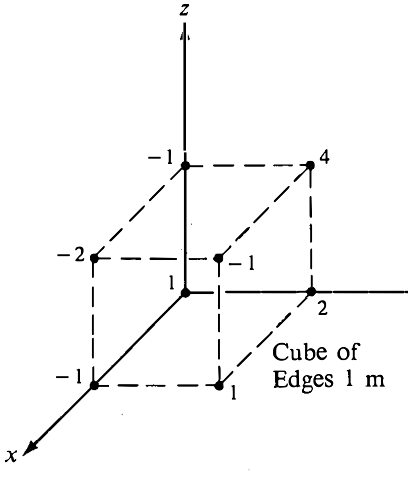

Fig. 2.26. Point charges located at the corners of a cube. Values of the point charges indicated at the respective corners are in coulombs.

The solution to this problem consists of evaluating $\sum Q$ and $\sum Q\mathbf{r}'$ for the collection of point charges and substituting the results in (2-115). These quantities are evaluated with the aid of Table 2.2.

The potential for large r correct to the first two significant terms is then given by

$$\begin{aligned} V &= \frac{\sum Q}{4\pi\epsilon_0 r} + \frac{\sum Q\mathbf{r}' \cdot \mathbf{r}}{4\pi\epsilon_0 r^3} \\ &= \frac{3}{4\pi\epsilon_0 r} + \frac{(-3\mathbf{i}_x + 6\mathbf{i}_y) \cdot \mathbf{i}_r}{4\pi\epsilon_0 r^2} \qquad (2\text{-}116) \\ &= \frac{3}{4\pi\epsilon_0 r} + \frac{-3 \sin\theta \cos\phi + 6 \sin\theta \sin\phi}{4\pi\epsilon_0 r^2} \end{aligned}$$

If, in Table 2.2, $\sum Q$ is zero, then we have to evaluate the third term if the result is to be correct to the first two significant terms, and so on. ▮

TABLE 2.2. Computation of $\sum Q$ and $\sum Q\mathbf{r}'$ for the Arrangement of Point Charges in Fig. 2.26

Location (x, y, z)	*Charge*, Q	$\mathbf{r}'$	$Q\mathbf{r}'$
0, 0, 0	1	0	0
1, 0, 0	-1	$\mathbf{i}_x$	$-\mathbf{i}_x$
0, 1, 0	2	$\mathbf{i}_y$	$2\mathbf{i}_y$
0, 0, 1	-1	$\mathbf{i}_z$	$-\mathbf{i}_z$
1, 1, 0	1	$\mathbf{i}_x + \mathbf{i}_y$	$\mathbf{i}_x + \mathbf{i}_y$
0, 1, 1	4	$\mathbf{i}_y + \mathbf{i}_z$	$4\mathbf{i}_y + 4\mathbf{i}_z$
1, 0, 1	-2	$\mathbf{i}_x + \mathbf{i}_z$	$-2\mathbf{i}_x - 2\mathbf{i}_z$
1, 1, 1	-1	$\mathbf{i}_x + \mathbf{i}_y + \mathbf{i}_z$	$-\mathbf{i}_x - \mathbf{i}_y - \mathbf{i}_z$
	$\sum Q = 3$		$\sum Q\mathbf{r}' = -3\mathbf{i}_x + 6\mathbf{i}_y$

2.10 The Potential Field of Continuous Charge Distributions

In the previous section we considered the potential field of collections of point charges at discrete points. In this section we will extend the discussion to continuous charge distributions. As in Section 2.4, we divide the continuous charge distribution into several infinitesimal parts, each of which can be considered as a point charge, and obtain the potential at any point due to the total charge as the superposition of the potentials due to the individual point charges. To do this, we again have to evaluate integrals as in Section 2.4. However, the integrals involve the scalar quantity potential instead of the vector quantity electric field intensity. Hence, for a particular charge distribution, the potential at any point is given by a single integral, whereas for the determination of the electric field intensity as in Section 2.4, it is necessary to evaluate three integrals for the three components in the general case. We will illustrate the determination of the potential for continuous charge distributions through some examples.

EXAMPLE 2-17. An infinitely long line charge of uniform density ρ_{L0} C/m is situated along the z axis. It is desired to obtain the potential field due to this charge.

First we divide the line into a number of infinitesimal segments each of length dz as shown in Fig. 2.27, such that the charge $\rho_{L0}\,dz$ in each segment can be considered as a point charge. Let us consider a point P at a distance r from the z axis, with the projection of P onto the z axis being O. For the sake of generality, we consider the point P_0 at a distance r_0 from O along OP as the reference point for zero potential and write the potential dV at P due to the infinitesimal charge $\rho_{L0}\,dz$ at A as

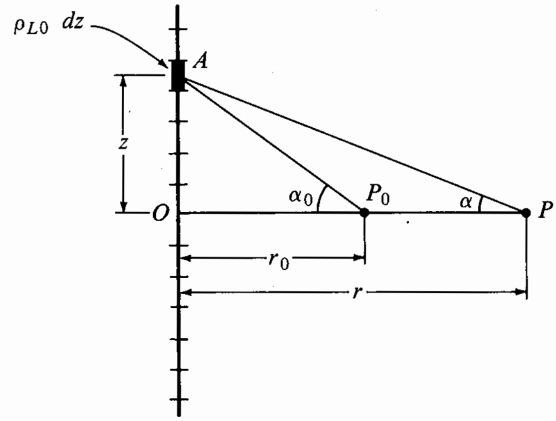

Fig. 2.27. Geometry for the computation of the potential field of an infinitely long line charge of uniform density ρ_{L0} C/m.

$$\begin{aligned} dV &= \frac{\rho_{L0}\,dz}{4\pi\epsilon_0(AP)} - \frac{\rho_{L0}\,dz}{4\pi\epsilon_0(AP_0)} \\ &= \frac{\rho_{L0}\,dz}{4\pi\epsilon_0\sqrt{r^2+z^2}} - \frac{\rho_{L0}\,dz}{4\pi\epsilon_0\sqrt{r_0^2+z^2}} \end{aligned} \tag{2-117}$$

We will, however, find later that we have to choose the reference point for zero potential at a finite value of r, in contrast to the case of the point charge for which the reference point can be chosen to be infinity. The potential V at P due to the entire line charge is now given by the integral of (2-117), where the integration is to be performed between the limits $z = -\infty$ and $z = \infty$. Thus

$$\begin{aligned} V &= \int_{z=-\infty}^{\infty} dV = \int_{z=-\infty}^{\infty}\left(\frac{\rho_{L0}\,dz}{4\pi\epsilon_0\sqrt{r^2+z^2}} - \frac{\rho_{L0}\,dz}{4\pi\epsilon_0\sqrt{r_0^2+z^2}}\right) \\ &= \frac{\rho_{L0}}{2\pi\epsilon_0}\int_{z=0}^{\infty}\left(\frac{dz}{\sqrt{r^2+z^2}} - \frac{dz}{\sqrt{r_0^2+z^2}}\right) \end{aligned} \tag{2-118}$$

Introducing $z = r\tan\alpha$ and $z = r_0\tan\alpha_0$ in the first and second terms, respectively, in the integrand on the right side of (2-118), we have

$$\begin{aligned} V &= \frac{\rho_{L0}}{2\pi\epsilon_0}\left(\int_{\alpha=0}^{\pi/2}\sec\alpha\,d\alpha - \int_{\alpha_0=0}^{\pi/2}\sec\alpha_0\,d\alpha_0\right) \\ &= \frac{\rho_{L0}}{2\pi\epsilon_0}\{[\ln(\sec\alpha+\tan\alpha)]_{\alpha=0}^{\pi/2} - [\ln(\sec\alpha_0+\tan\alpha_0)]_{\alpha_0=0}^{\pi/2}\} \\ &= \frac{\rho_{L0}}{2\pi\epsilon_0}\left[\ln\frac{(\sqrt{r^2+z^2}+z)\,r_0}{(\sqrt{r_0^2+z^2}+z)\,r}\right]_{z=0}^{\infty} \\ &= -\frac{\rho_{L0}}{2\pi\epsilon_0}\ln\frac{r}{r_0} \end{aligned} \tag{2-119}$$

In view of the cylindrical symmetry about the line charge, (2-119) is the general expression in cylindrical coordinates for the potential field of the infinitely long line charge of uniform density. It can be seen from (2-119) that a choice of $r_0 = \infty$ is not a good choice, since then the potential would be infinity at all points. The difficulty lies in the fact that infinity plus a finite number is still infinity. We also note from (2-119) that the equipotential surfaces are $\ln r/r_0 =$ constant or $r =$ constant, that is, surfaces of cylinders with the line charge as their axis. In Ex. 2-4, we found that the electric field intensity due to the line charge is directed radially away from the line charge. Thus the direction lines of **E** and the equipotential surfaces are indeed orthogonal to each other. ▮

Generalizing the expression for the computation of potential for a line charge distribution of density $\rho_L(\mathbf{r}')$ occupying a contour C', we have

$$V(\mathbf{r}) = \int_{C'} \frac{\rho_L(\mathbf{r}')}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|} \, dl' \tag{2-120a}$$

This is the Coulomb potential of line charge distribution $\rho_L(\mathbf{r}')$. Similarly, for a surface charge distribution of density $\rho_s(\mathbf{r}')$ occupying a surface S', we have

$$V(\mathbf{r}) = \int_{S'} \frac{\rho_s(\mathbf{r}')}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|} \, dS' \tag{2-120b}$$

and for a volume charge distribution of density $\rho(\mathbf{r}')$ occupying a volume V', we have

$$V(\mathbf{r}) = \int_{V'} \frac{\rho(\mathbf{r}')}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|} \, dv' \tag{2-120c}$$

EXAMPLE 2-18. A cube charge of sides 1 m² is situated with one corner at the origin and three edges coinciding with the coordinate axes. The charge density ρ within the cube is given by

$$\rho = (x + y + z) \text{ C}$$

Find the potential field of the cube charge at large distances from it correct to the first two significant terms.

This problem is an extension of Example 2-16 to a continuous charge distribution. For a continuous charge distribution, the summations of (2-115) have to be replaced by integrals so that we have

$$V = \frac{\int_{\text{vol}} dQ}{4\pi\epsilon_0 r} + \frac{\left(\int_{\text{vol}} dQ \, \mathbf{r}' \cdot \mathbf{r}\right)}{4\pi\epsilon_0 r^3} + \cdots \tag{2-121}$$

For the specified charge distribution

$$\int_{\text{vol}} dQ = \int_{\text{vol}} \rho\, dv$$
$$= \int_{x=0}^{1} \int_{y=0}^{1} \int_{z=0}^{1} (x + y + z)\, dx\, dy\, dz \qquad (2\text{-}122)$$
$$= \tfrac{3}{2}\,\text{C}$$

$$\int_{\text{vol}} (dQ\, \mathbf{r}') = \int_{\text{vol}} \rho\, dv\, \mathbf{r}'$$
$$= \int_{x=0}^{1} \int_{y=0}^{1} \int_{z=0}^{1} (x + y + z)\, dx\, dy\, dz\, (x\mathbf{i}_x + y\mathbf{i}_y + z\mathbf{i}_z)$$
$$= \tfrac{5}{6}(\mathbf{i}_x + \mathbf{i}_y + \mathbf{i}_z)\text{C-m} \qquad (2\text{-}123)$$

Substituting (2-122) and (2-123) into (2-121), we obtain

$$V = \frac{3}{8\pi\epsilon_0 r} + \frac{5(\mathbf{i}_x + \mathbf{i}_y + \mathbf{i}_z) \cdot \mathbf{r}}{24\pi\epsilon_0 r^3}$$
$$= \frac{3}{8\pi\epsilon_0 r} + \frac{5}{24\pi\epsilon_0 r^2}(\sin\theta\cos\phi + \sin\theta\sin\phi + \cos\theta) \qquad (2\text{-}124)$$

correct to the first two significant terms. ▮

2.11 Maxwell's Curl Equation for the Static Electric Field

In Section 2.9 we showed that the potential difference between two points A and B, that is, the quantity $\int_A^B \mathbf{E} \cdot d\mathbf{l}$, in the field of a point charge is independent of the path followed from A to B to evaluate it. Suppose we now consider two different paths ACB and ADB in the field of a point charge as shown in Fig. 2.28. We then have

$$\int_{ACB} \mathbf{E} \cdot d\mathbf{l} = \int_{ADB} \mathbf{E} \cdot d\mathbf{l} \qquad (2\text{-}125)$$

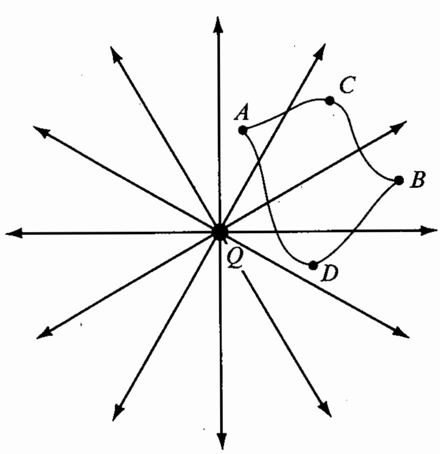

Fig. 2.28. Two different paths between points A and B in the electric field of a point charge.

where

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2}\mathbf{i}_r$$

Equation (2-125) can be rearranged to read

$$\int_{ACB} \mathbf{E} \cdot d\mathbf{l} - \int_{ADB} \mathbf{E} \cdot d\mathbf{l} = 0$$

or

$$\int_{ACB} \mathbf{E} \cdot d\mathbf{l} + \int_{BDA} \mathbf{E} \cdot d\mathbf{l} = 0 \tag{2-126}$$

or

$$\oint_{ACBDA} \mathbf{E} \cdot d\mathbf{l} = 0 \tag{2-127}$$

where we have introduced a circle in the integral sign to indicate that the integral is evaluated around a closed path.

If we now have a collection of point charges discrete or continuous, we can apply superposition in the usual manner and arrive at the result that, for any static electric field **E**,

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0 \tag{2-128}$$

Equation (2-128) states that the line integral of the electric field intensity vector of any static charge distribution evaluated around a closed path, or the circulation of the static electric field, is equal to zero. Multiplying both sides of (2-128) by a test charge q, we obtain

$$\oint q\mathbf{E} \cdot d\mathbf{l} = 0 \tag{2-129}$$

which states that the work involved in moving a test charge around a closed path in a static electric field is equal to zero. If a certain amount of work is done by an external agent during a portion of the closed path, the same amount of work must be done by the field during the remainder of the closed path. It is now evident that (2-129) is simply a statement of conservation of energy, which is so familiar in the case of the gravitational field as the work done in moving a mass around a closed path is zero. Fields which satisfy this property are known as conservative fields. The static electric field is thus a conservative field. In Chapter 4 we will learn that a time-varying electric field does not satisfy this property.

From the definition of the curl of a vector, we have

$$\nabla \times \mathbf{E} = \lim_{\Delta S \to 0} \frac{\oint_{\Delta C} \mathbf{E} \cdot d\mathbf{l}}{\Delta S}\mathbf{i}_n \tag{2-130}$$

where ΔS is the area bounded by the closed path ΔC and $\mathbf{i}_n$ is the normal vector to the area which should be oriented such that $\oint_{\Delta C} \mathbf{E} \cdot d\mathbf{l}$ is a maximum. However, for the static electric field, $\oint \mathbf{E} \cdot d\mathbf{l} = 0$ for any closed path and hence the right side of (2-130) is identically zero, thus giving

$$\nabla \times \mathbf{E} = 0 \tag{2-131}$$

Equation (2-131) is Maxwell's curl equation for the static electric field. It states that the curl of the static electric field intensity vector is everywhere equal to zero. Fields which satisfy the property of zero curl are known as irrotational fields; that is, such fields cannot rotate the paddle wheel discussed in Section 1.9. Together with Maxwell's divergence equation for the electric field given by (2-82), (2-131) completely defines the properties of the static electric field. Equation (2-131) determines whether or not a given vector field is realizable as a static electric field whereas Eq. (2-82) relates the field to the charge distribution responsible for producing the field. As an alternative approach to that which we followed in this chapter, it is possible to accept these two equations as a starting point and obtain the electric field intensity of a point charge and other charge distributions.

EXAMPLE 2-19. Determine if the following fields are realizable as static electric fields.

(a) $\mathbf{F}_a = -y\mathbf{i}_x + x\mathbf{i}_y$ cartesian coordinates

(b) $\mathbf{F}_b = (p_L/2\pi\epsilon_0 r^2)(\cos\phi\,\mathbf{i}_r + \sin\phi\,\mathbf{i}_\phi)$ cylindrical coordinates

(c) $\mathbf{F}_c = \sin\theta\,\mathbf{i}_r + \cos\theta\,\mathbf{i}_\theta$ spherical coordinates

(a)

$$\nabla \times \mathbf{F}_a = \begin{vmatrix} \mathbf{i}_x & \mathbf{i}_y & \mathbf{i}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} \neq 0$$

Hence $\mathbf{F}_a$ cannot be realized as a static electric field.

(b)

$$\nabla \times \mathbf{F}_b = \begin{vmatrix} \dfrac{\mathbf{i}_r}{r} & \mathbf{i}_\phi & \dfrac{\mathbf{i}_z}{r} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ \dfrac{p_L \cos\phi}{2\pi\epsilon_0 r^2} & \dfrac{p_L \sin\phi}{2\pi\epsilon_0 r} & 0 \end{vmatrix} = 0$$

Hence $\mathbf{F}_b$ is realizable as a static electric field. It is left as an exercise (Problem 2.15) for the student to show that $\mathbf{F}_b$ is the field of a two-dimensional electric dipole of moment p_L.

(c)

$$\nabla \times \mathbf{F}_c = \begin{vmatrix} \dfrac{\mathbf{i}_r}{r^2 \sin\theta} & \dfrac{\mathbf{i}_\theta}{r \sin\theta} & \dfrac{\mathbf{i}_\phi}{r} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ \sin\theta & r\cos\theta & 0 \end{vmatrix} = 0$$

Hence $\mathbf{F}_c$ can be realized as a static electric field. In fact, if we note that, in cylindrical coordinates, $\mathbf{F}_c = \mathbf{i}_r$, the irrotational nature of $\mathbf{F}_c$ becomes obvious. ▮

2.12 The Relationship Between Electric Field Intensity and Potential

In Section 1.9, we learned that the curl of any vector which can be expressed as the gradient of a scalar is zero. Conversely, if the curl of a vector is equal to zero, the vector can be expressed as the gradient of a scalar. From (2-131), we can say therefore, that the static electric field vector **E** can be expressed as the gradient of a scalar, say, Φ. The question that arises now is: What is this scalar function Φ? For a hint, let us compare the direction of the gradient of the potential V with the direction of **E**. The direction of the gradient of a scalar function at any point is the normal to the surface passing through that point and on which the scalar function has a constant value. Hence the direction of ∇V is normal to the equipotential surfaces. But we found in Section 2.9 that **E** is normal to the equipotential surfaces. Thus the directions of ∇V and **E** at a point have to be either the same or opposite.

To determine which of these is correct and to probe the relationship between **E** and V further, let us consider two equipotential surfaces in a static electric field as shown in Fig. 2.29. Let the potentials on these surfaces

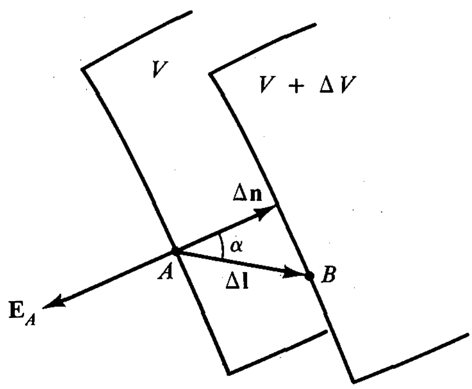

Fig. 2.29. For the determination of the relationship between **E** and V.

be V and $V + \Delta V$, where ΔV is infinitesimal. Since ΔV is infinitesimal, the two surfaces are infinitesimally close so that we can assume that the electric field intensity between the two surfaces in the neighborhood of point A is uniform and equal to the electric field intensity $\mathbf{E}_A$ at point A. We know from previous discussion that $\mathbf{E}_A$ is normal to the equipotential surface V at A. To decide whether $\mathbf{E}_A$ is directed towards the equipotential surface $V + \Delta V$ or away from it, we note that, if a test charge is moved along the direction of **E**, the field does the work; that is, the charge accelerates and hence loses potential energy. This is the same as stating that the charge moves from a higher potential to a lower potential. Thus $\mathbf{E}_A$ is directed away from the equipotential surface $V + \Delta V$ as shown in Fig. 2.29. Now, the potential difference between point A and another point B on the equipotential surface $V + \Delta V$ can be written, using (2-93), as

$$V_{AB} = \int_A^B \mathbf{E} \cdot d\mathbf{l} = \mathbf{E}_A \cdot \Delta\mathbf{l} \tag{2-132}$$

But

$$V_{AB} = V - (V + \Delta V) = -\Delta V \tag{2-133}$$

Also, if $\Delta\mathbf{n}$ is the normal vector from the surface V up to the surface $V + \Delta V$, we have

$$\mathbf{E}_A \cdot \Delta\mathbf{l} = -E_A\,\Delta l \cos\alpha = -E_A\,\Delta n \tag{2-134}$$

Substituting (2-133) and (2-134) into (2-132), we obtain

$$-\Delta V = -E_A\,\Delta n \tag{2-135}$$

or

$$E_A = \frac{\Delta V}{\Delta n} \tag{2-136}$$

and

$$\mathbf{E}_A = -\frac{\Delta V}{\Delta n}\mathbf{i}_n \tag{2-137}$$

where $\mathbf{i}_n$ is the unit vector along $\Delta\mathbf{n}$. If we now let Δn tend to zero, $(\Delta V/\Delta n)\mathbf{i}_n$ becomes ∇V. Dropping the subscript A in (2-137), since the same arguments can be applied to any other point in the field, we obtain a relationship between the static electric field intensity vector and the potential at a point as

$$\mathbf{E} = -\nabla V \tag{2-138}$$

Equation (2-138) permits us to compute $\mathbf{E}$ from a knowledge of V using differentiation.

Substituting (2-138) into Maxwell's divergence equation for the electric field, $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$, we have

$$\nabla \cdot (-\nabla V) = \frac{\rho}{\epsilon_0} \tag{2-139}$$

Recalling that $\nabla \cdot \nabla V$ is the Laplacian of V, denoted as $\nabla^2 V$, we see that Eq. (2-139) becomes

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \tag{2-140}$$

This is known as Poisson's equation. It is a differential equation which relates the potential at a point to the volume charge density at that point. If the volume charge density in a region is zero, then the right side of (2-140) is zero for that region so that (2-140) reduces to

$$\nabla^2 V = 0 \tag{2-141}$$

This is known as Laplace's equation. It states that the Laplacian of the electrostatic potential in a region devoid of charge is equal to zero. We will discuss the solutions of Poisson's and Laplace's equations in Chapter 6.

PROBLEMS

2.1. Find the electric field intensity required to counteract the earth's gravitational force on a charge of q C having a mass m kg. Compute the value of this electric field intensity if the charge is an electron.

2.2. A radial electric field given by

$$\mathbf{E} = \frac{E_0}{r}\,\mathbf{i}_r$$

where E_0 is a constant exists between two cylindrical surfaces $r = a$ and $r = b$. A test charge q having a mass m enters the electric field region at a radius r_0 with a velocity $\mathbf{v} = v_0\,\mathbf{i}_\phi$. Find the value of E_0 for which the test charge follows a circular orbit of radius r_0.

2.3. An electric field given by

$$\mathbf{E} = E_0\mathbf{i}_y$$

where E_0 is a constant exists in the space between two parallel metallic plates of length L as shown in Fig. 2.30. A small test charge q having a mass m enters the region between the plates at $t = 0$ with a velocity $\mathbf{v} = v_0\,\mathbf{i}_x$ as shown in the figure.

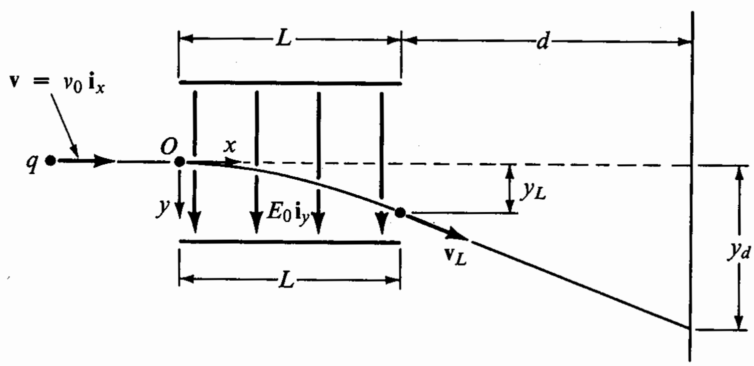

Fig. 2.30. For Problem 2.3.

(a) Show that the path of the test charge between the plates is parabolic.
(b) Find the position y_L along the y direction and velocity $\mathbf{v}_L$ of the test charge just after it emerges from the field region.
(c) Find the deflection y_d undergone by the test charge along the y direction at a distance d from the plates along the x direction.

2.4. Three point charges Q, kQ, and kQ are arranged as shown in Fig. 2.31. Find k in terms of x if a test charge placed at a point on $y = x$ in the plane of the charges is to experience no force. Compute k for $x = 1$ and $x = \frac{1}{4}$.

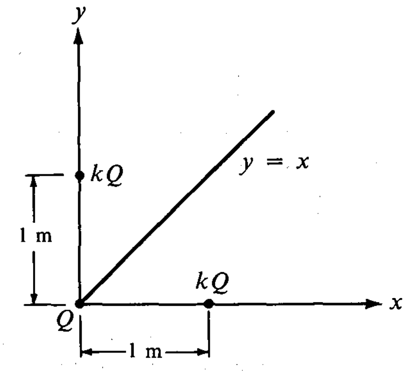

Fig. 2.31. For Problem 2.4.

2.5. Three point charges, each of mass m and charge Q, are suspended by strings of length L from a common point. It is found that the common point and the points occupied by the three charges form the corners of a tetrahedron. Find the relationship between Q, m, L, and the acceleration due to gravity, g.

2.6. Eight point charges, each of value 1 C, are situated at the corners of a cube of edges 2 m with one corner placed at the origin and three edges lying along the coordinate axes. (a) Find the force experienced by each charge. (b) Find the electric field intensity at the point (2, 2, 2). (c) Find the electric field intensity at the point (0, 0, 2).

2.7. Point charges Q, $-2Q$, and Q are located at $(0, 0, d)$, $(0, 0, 0)$, and $(0, 0, -d)$, respectively. Such an arrangement is known as a linear quadrupole. (a) Find the electric field intensity at distances large compared to d along the line joining the charges. (b) Find the electric field intensity at distances large compared to d normal to the line joining the charges.

2.8. A line charge is situated along the z axis. Consider the charge density ρ_L to be arbitrary function of z and show that the components of the electric field intensity at any point in the xy plane are given in cylindrical coordinates by

$$E_r = \frac{r}{4\pi\epsilon_0}\int_{z=-\infty}^{\infty} \frac{\rho_L\,dz}{(r^2 + z^2)^{3/2}}$$

$$E_\phi = 0$$

$$E_z = -\frac{1}{4\pi\epsilon_0}\int_{z=-\infty}^{\infty} \frac{\rho_L z\,dz}{(r^2 + z^2)^{3/2}}$$

Evaluate the field components for the following charge distributions:

(a) $\rho_L = \rho_{L0} \qquad -\infty < z < \infty$

(b) $\rho_L = \rho_{L0} \qquad -z_0 < z < z_0$

(c) $\rho_L = |z| \qquad -z_0 < z < z_0$

(d) $\rho_L = z \qquad -z_0 < z < z_0$

where ρ_{L0} is a constant. Discuss your results from considerations of symmetry. Verify your results by considering limiting cases wherever appropriate.

2.9. A ring charge of radius a is situated in the xy plane with its center at the origin. Consider the charge density ρ_L to be an arbitrary function of ϕ and show that the components of the electric field intensity at a point $(0, 0, z)$ are given in cartesian coordinates by

$$E_x = \frac{-a^2}{4\pi\epsilon_0(a^2+z^2)^{3/2}} \int_{\phi=0}^{2\pi} \rho_L \cos\phi \, d\phi$$

$$E_y = \frac{-a^2}{4\pi\epsilon_0(a^2+z^2)^{3/2}} \int_{\phi=0}^{2\pi} \rho_L \sin\phi \, d\phi$$

$$E_z = \frac{az}{4\pi\epsilon_0(a^2+z^2)^{3/2}} \int_{\phi=0}^{2\pi} \rho_L \, d\phi$$

Evaluate the field components for the following charge distributions:

(a) $\rho_L = \rho_{L0} \qquad 0 < \phi < 2\pi$

(b) $\rho_L = \begin{cases} \rho_{L0} & 0 < \phi < \pi \\ -\rho_{L0} & \pi < \phi < 2\pi \end{cases}$

(c) $\rho_L = \rho_{L0} \cos\phi \qquad 0 < \phi < 2\pi$

(d) $\rho_L = \rho_{L0} \sin\phi \qquad 0 < \phi < 2\pi$

where ρ_{L0} is a constant. Discuss your results from considerations of symmetry. Verify your results by considering limiting cases wherever appropriate.

2.10. A sheet charge is situated in the xy plane. Consider the charge density ρ_s to be an arbitrary function of r and ϕ and show that the components of the electric field intensity at a point $(0, 0, z)$ are given by

$$E_x = -\frac{1}{4\pi\epsilon_0} \int_{r=0}^{\infty} \int_{\phi=0}^{2\pi} \frac{\rho_s r^2 \cos\phi \, dr \, d\phi}{(r^2+z^2)^{3/2}}$$

$$E_y = -\frac{1}{4\pi\epsilon_0} \int_{r=0}^{\infty} \int_{\phi=0}^{2\pi} \frac{\rho_s r^2 \sin\phi \, dr \, d\phi}{(r^2+z^2)^{3/2}}$$

$$E_z = \frac{z}{4\pi\epsilon_0} \int_{r=0}^{\infty} \int_{\phi=0}^{2\pi} \frac{\rho_s r \, dr \, d\phi}{(r^2+z^2)^{3/2}}$$

Evaluate the field components for the following charge distributions:

(a) $\rho_s = \rho_{s0} \qquad 0 < r < \infty, 0 < \phi < 2\pi$

(b) $\rho_s = \begin{cases} \rho_{s0} & 0 < r < r_0, 0 < \phi < 2\pi \\ 0 & r_0 < r < \infty, 0 < \phi < 2\pi \end{cases}$

(c) $\rho_s = \begin{cases} 0 & 0 < r < r_0, 0 < \phi < 2\pi \\ \rho_{s0} & r_0 < r < \infty, 0 < \phi < 2\pi \end{cases}$

(d) $\rho_s = \dfrac{\rho_{s0} \cos\phi}{r} \qquad 0 < r < \infty, 0 < \phi < 2\pi$

(e) $\rho_s = \dfrac{\rho_{s0} \sin\phi}{r} \qquad 0 < r < \infty, 0 < \phi < 2\pi$

where ρ_{s0} is a constant. Discuss your results from considerations of symmetry. Verify your results by considering limiting cases wherever appropriate.

2.11. A surface charge is distributed over a spherical surface of radius a and centered at the origin. Consider the charge density ρ_s to be uniform in ϕ but not necessarily in θ and show that the electric field intensity at a point (0, 0, z) has only a z-component given by

$$E_z = \frac{a^2}{2\epsilon_0}\int_{\theta=0}^{\pi} \frac{\rho_s(z - a\cos\theta)\sin\theta\, d\theta}{(a^2 + z^2 - 2az\cos\theta)^{3/2}}$$

Evaluate E_z both for $|z| < a$ and for $|z| > a$ for the following charge distributions:

(a) $\rho_s = \rho_{s0} \qquad 0 < \theta < \pi$

(b) $\rho_s = \rho_{s0}\cos\theta \qquad 0 < \theta < \pi$

where ρ_{s0} is a constant.

2.12. A volume charge is distributed throughout an infinite slab of thickness $2a$ symmetrically placed about the xy plane. Consider the charge density ρ to be uniform in x and y but not necessarily in z and show that the electric field intensity at any point (x, y, z) has only a z component given by

$$E_z = \begin{cases} \dfrac{1}{2\epsilon_0}\displaystyle\int_{z=-a}^{a} \rho\, dz & z > a \\ \dfrac{1}{2\epsilon_0}\left(\displaystyle\int_{z=-a}^{a} \rho\, dz - \int_{z=z}^{a} \rho\, dz\right) & -a < z < a \\ -\dfrac{1}{2\epsilon_0}\displaystyle\int_{z=-a}^{a} \rho\, dz & z < a \end{cases}$$

Evaluate E_z as a function of z for $-\infty < z < \infty$ for the following charge distributions:

(a) $\rho = \rho_0 \qquad -a < z < a$

(b) $\rho = \begin{cases} \rho_0 & 0 < z < a \\ -\rho_0 & -a < z < 0 \end{cases}$

(c) $\rho = |z| \qquad -a < z < a$

(d) $\rho = z \qquad -a < z < a$

where ρ_0 is a constant. Discuss your results from considerations of symmetry.

2.13. A volume charge is distributed with uniform density ρ_0 C/m^3 throughout an infinitely long cylinder of radius a m. Obtain the electric field intensity at points both inside and outside the cylinder by dividing the cylindrical charge into several infinitesimal parts each of which can be considered as a point charge.

2.14. A small hole is drilled through the center of the spherical volume charge of Example 2-6., as shown in Fig. 2.32. The size of the hole is negligible compared to the size of the sphere. A point charge $q(< 0)$ is placed at one end of the hole and released from rest at $t = 0$. Assume that the magnitude of q is very small compared to the total charge Q (> 0) contained in the sphere. (a) Derive the equation of motion of the point charge. (b) Solve the equation for the position and velocity of the point charge as functions of time. (c) What is the frequency of oscillation of the point charge?

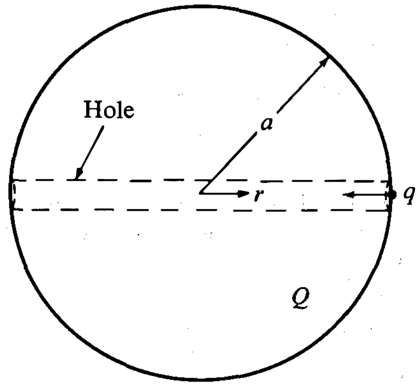

Fig. 2.32. For Problem 2.14.

2.15. Two infinitely long line charges of uniform but opposite densities ρ_{L0} and $-\rho_{L0}$ are situated parallel to the z axis and passing through $(d/2, 0, 0)$ and $(-d/2, 0, 0)$, respectively. The arrangement is known as a two-dimensional electric dipole, in contrast to the three-dimensional electric dipole made up of two equal but opposite point charges. (a) Obtain the electric field intensity due to the two-dimensional electric dipole in the limit that $d \longrightarrow 0$, keeping the dipole moment $\rho_{L0}d$ constant. (b) Find and sketch the direction lines.

2.16. Two infinitely long line charges of uniform densities ρ_{L1} and ρ_{L2}, respectively, are situated parallel to each other at a distance d apart. Show that the equation for the direction lines of **E** is

$$\alpha_1 \rho_{L1} + \alpha_2 \rho_{L2} = \text{constant}$$

in the plane normal to the line charges, where α_1 and α_2 are the angles made by the lines drawn from any point P to the line charges with the line joining the charges as shown in Fig. 2.33. Obtain and sketch the direction lines for the following cases:

(a) $\rho_{L1} = \rho_{L2} = \rho_{L0}$

(b) $\rho_{L1} = \rho_{L0}, \rho_{L2} = -\rho_{L0}$ (two-dimensional dipole)

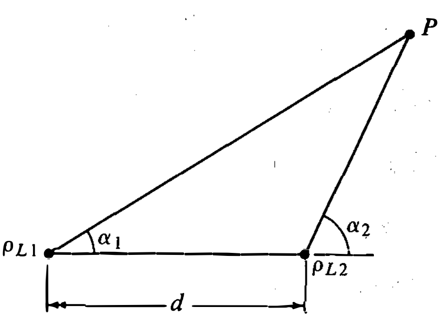

Fig. 2.33. For Problem 2.16.

2.17. Obtain the electric field intensity of a finitely long line charge of uniform density ρ_{L0} and length $2a$ at an arbitrary point. Show that the direction lines are hyperbolas with the ends of the line charge as their focii.

2.18. Carry out the mathematical proof to show that the net electric field flux emanating from an arbitrary surface not enclosing a point charge is zero.

2.19. Find the solid angle subtended by

(a) One face of a regular tetrahedron at the center of the tetrahedron
(b) One face of a cube at the center of the cube
(c) One face of a cube at one of the corners of the opposite face
(d) A hemispherical surface at a point on the base of the hemisphere other than its center
(e) The first quadrant of the xy plane at a point on the z axis
(f) The portion of any plane in the first octant at the origin.

2.20. An infinitely long line charge of uniform density ρ_{L0} C/m is situated along the z axis. Find the electric field flux cutting the portion of the plane $x + y = 1$ m lying in the first octant and bounded by the planes $z = 0$ and $z = 1$ m by evaluating $\int \mathbf{E} \cdot d\mathbf{S}$. Check your answer from considerations of symmetry of electric field flux emanating from the line charge.

2.21. A point charge Q C is located at the origin. Find the electric field flux cutting the portion of the plane $x + y = 1$ m lying in the first octant by evaluating $\int \mathbf{E} \cdot d\mathbf{S}$. Check your answer from considerations of symmetry of electric field flux emanating from the line charge.

2.22. Charges are located, in cartesian coordinates, as follows: (a) point charge, 1 C, at (0.23, 0.73, 0); (b) infinitely long line charge of uniform density 1 C/m parallel to the z axis and passing through (0.6, 0, 0); and (c) an infinite sheet charge of uniform density 1 C/m^2 in the $z = 0.5$ plane. Determine the total electric field flux cutting the upper half of the spherical surface of radius unity and centered at the origin.

2.23. Using Gauss' law in integral form, obtain the electric fields due to the following volume charge distributions, in cartesian coordinates:

(a) $\rho = \begin{cases} \rho_0 & |z| < a \\ 0 & |z| > a \end{cases}$

(b) $\rho = \begin{cases} \rho_0 & 0 < z < a \\ -\rho_0 & -a < z < 0 \end{cases}$

(c) $\rho = \begin{cases} |z| & |z| < a \\ 0 & |z| > a \end{cases}$

(d) $\rho = \begin{cases} z & |z| < a \\ 0 & |z| > a \end{cases}$

(e) $\rho = \begin{cases} a - |z| & |z| < a \\ 0 & |z| > a \end{cases}$

where ρ_0 is a constant.

2.24. Using Gauss' law in integral form, obtain the electric fields due to the following volume charge distributions, in cylindrical coordinates:

(a) $\rho = \begin{cases} \rho_0 & 0 < r < a \\ 0 & a < r < \infty \end{cases}$

$$(b)\ \rho = \begin{cases} 0 & 0 < r < a \\ \rho_0 & a < r < b \\ 0 & b < r < \infty \end{cases}$$

$$(c)\ \rho = \begin{cases} \rho_0 \dfrac{r}{a} & 0 < r < a \\ 0 & a < r < \infty \end{cases}$$

where ρ_0 is a constant.

2.25. Using Gauss' law in integral form, obtain the electric fields due to the following volume charge distributions, in spherical coordinates:

$$(a)\ \rho = \begin{cases} 0 & 0 < r < a \\ \rho_0 & a < r < b \\ 0 & b < r < \infty \end{cases}$$

$$(b)\ \rho = \begin{cases} \rho_0 \dfrac{r}{a} & 0 < r < a \\ 0 & a < r < \infty \end{cases}$$

$$(c)\ \rho = \begin{cases} \rho_0 \left(1 - \dfrac{r^2}{a^2}\right) & 0 < r < a \\ 0 & a < r < \infty \end{cases}$$

where ρ_0 is a constant.

2.26. Using Gauss' law in integral form, obtain the electric fields due to the following surface charge distributions:

$$(a)\ \rho_s = \left.\begin{cases} \rho_{s0} & z = a \\ -\rho_{s0} & z = -a \end{cases}\right\} \quad \text{cartesian coordinates}$$

$$(b)\ \rho_s = \rho_{s0} \qquad r = a \qquad \text{cylindrical coordinates}$$

$$(c)\ \rho_s = \left.\begin{cases} \rho_{s0} & r = a \\ -\rho_{s0}\dfrac{a}{b} & r = b \end{cases}\right\} \quad \text{cylindrical coordinates}$$

$$(d)\ \rho_s = \rho_{s0} \qquad r = a \qquad \text{spherical coordinates}$$

$$(e)\ \rho_s = \left.\begin{cases} \rho_{s0} & r = a \\ -\rho_{s0}\dfrac{a^2}{b^2} & r = b \end{cases}\right\} \quad \text{spherical coordinates}$$

where ρ_{s0} in a constant.

2.27. Volume charge of uniform density ρ_0 C/m³ is distributed in the region between two infinitely long, parallel cylindrical surfaces of radii a and b ($< a$) and with their axes separated by distance c ($< a - b$) as shown in Fig. 2.34. Find the electric field intensity in the charge-free region inside the cylindrical surface of radius b.

2.28. Verify your answers to Problem 2.23 by using Gauss' law in differential form.

2.29. Verify your answers to Problem 2.24 by using Gauss' law in differential form.

2.30. Verify your answers to Problem 2.25 by using Gauss' law in differential form.

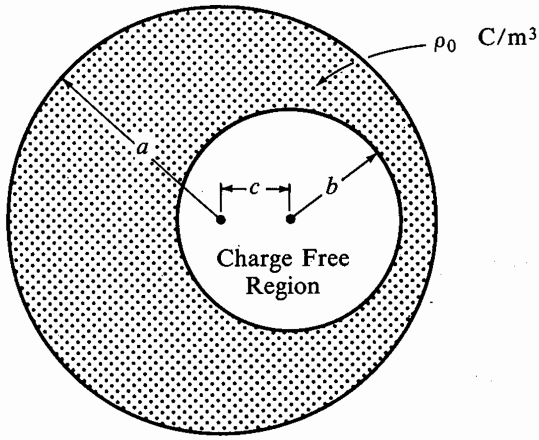

Fig. 2.34. For Problem 2.27.

2.31. For each of the following electric fields, find the charge distribution which produces the field, using Gauss' law in differential form:

(a) $$\mathbf{E} = \begin{cases} -\dfrac{\rho_{s0}}{\epsilon_0}\mathbf{i}_z & -\infty < z < 0 \\ -\dfrac{\rho_{s0}}{3\epsilon_0}\mathbf{i}_z & 0 < z < a \\ \dfrac{\rho_{s0}}{\epsilon_0}\mathbf{i}_z & a < z < \infty \end{cases} \quad \text{cartesian coordinates}$$

(b) $$\mathbf{E} = \frac{1 - e^r}{\epsilon_0 r}\mathbf{i}_r \qquad 0 < r < \infty \qquad \text{cylindrical coordinates}$$

(c) $$\mathbf{E} = \begin{cases} 0 & 0 < r < a \\ \dfrac{Q}{4\pi\epsilon_0 r^2}\mathbf{i}_r & a < r < b \\ 0 & b < r < \infty \end{cases} \quad \text{spherical coordinates}$$

where ρ_{s0} and Q are constants.

2.32. A surface charge of density ρ_s C/m^2 occupies the spherical surface of radius r_0 and centered at the origin. Show that

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho_s\,\delta(r - r_0)$$

2.33. An infinitely long line charge of density ρ_{L0} C/m is situated parallel to the z axis and passes through the point (r_0, ϕ_0) in the $z = 0$ plane. Show that

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho_{L0}\frac{\delta(r - r_0)\,\delta(\phi - \phi_0)}{r_0}$$

where

$$\int_{r=0}^{\infty}\int_{\phi=0}^{2\pi} f(r, \phi)\frac{\delta(r - r_0)\,\delta(\phi - \phi_0)}{r_0}\,r\,dr\,d\phi = f(r_0, \phi_0)$$

2.34. A point charge Q C is located at the point (r_0, θ_0, ϕ_0). Show that

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} Q \frac{\delta(r - r_0)\,\delta(\theta - \theta_0)\,\delta(\phi - \phi_0)}{r_0^2 \sin \theta_0}$$

where

$$\int_{r=0}^{\infty} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} f(r, \theta, \phi) \frac{\delta(r - r_0)\,\delta(\theta - \theta_0)\,\delta(\phi - \phi_0)}{r_0^2 \sin \theta_0} r^2 \sin \theta \, dr \, d\theta \, d\phi$$
$$= f(r_0, \theta_0, \phi_0)$$

2.35. The electric field intensity is given in cylindrical coordinates by

$$\mathbf{E} = \frac{\cos \phi}{r^2} \mathbf{i}_r + \frac{\sin \phi}{r^2} \mathbf{i}_\phi$$

Find the work associated with the movement of a test charge from the point $(1, 0, -22.7)$ to the point $(0.5, \pi/2, 43.8)$. Is this work done by the field or by an external agent?

2.36. The electric field intensity is given, in cartesian coordinates, by

$$\mathbf{E} = \begin{cases} 0 & -\infty < x < 0 \\ 2x\mathbf{i}_x & 0 < x < 1 \\ \dfrac{2}{x^2}\mathbf{i}_x & 1 < x < \infty \end{cases}$$

Obtain and draw a graph of the potential difference between $x = 1$ and an arbitrary value of x.

2.37. For the three-dimensional electric dipole, show that the equipotential surfaces, $r^2 \sec \theta =$ constant, are orthogonal to the direction lines of $\mathbf{E}$ given by the intersections of $r \operatorname{cosec}^2 \theta =$ constant and $\phi =$ constant.

2.38. For the linear quadrupole consisting of an arrangement of point charges Q, $-2Q$, and Q at $(0, 0, d)$, $(0, 0, 0)$, and $(0, 0, -d)$, respectively, obtain the expression for the potential at distances large compared to d.

2.39. For the rectangular quadrupole consisting of an arrangement of four point charges as shown in Fig. 2.35, obtain the potential at distances large compared to the dimensions of the quadrupole.

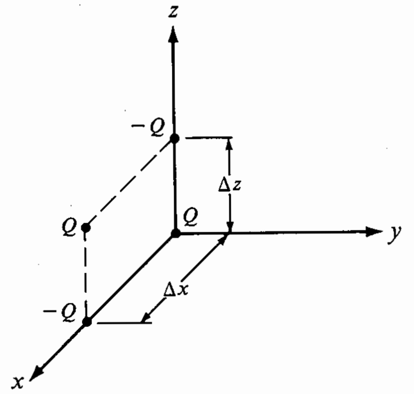

Fig. 2.35. For Problem 2.39.

2.40. For the arrangement of point charges shown in Fig. 2.36, obtain the expression for the potential at distances large compared to d.

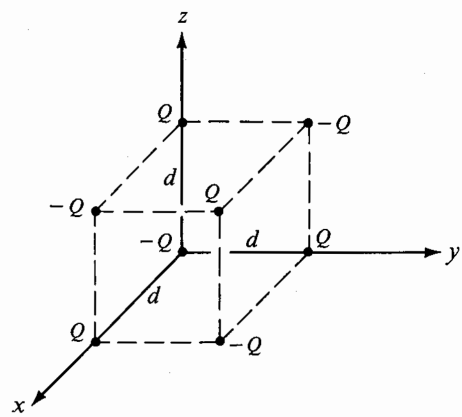

Fig. 2.36. For Problem 2.40.

2.41. For each of the arrangements of point charges shown in Fig. 2.37, find the first two significant terms in the potential at large distances from the origin.

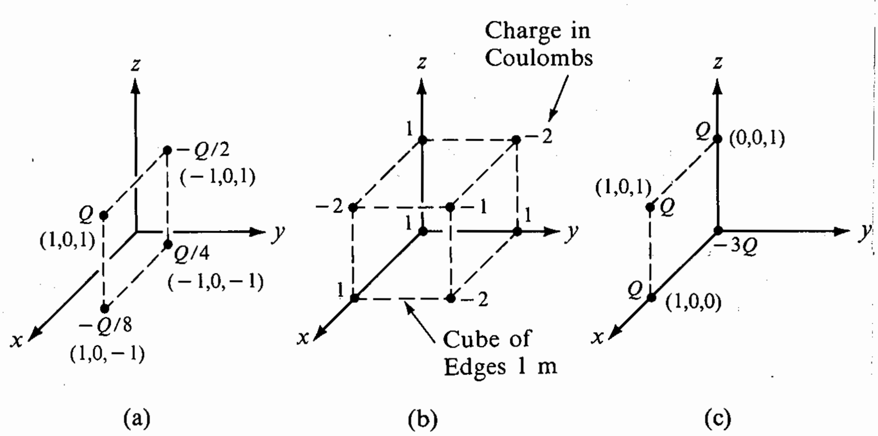

Fig. 2.37. For Problem 2.41.

2.42. For the arrangement of point charges shown in Fig. 2.37(c), $\sum Q = 0$. When $\sum Q = 0$ and $\sum Q\mathbf{r}' \neq 0$, $\sum Q\mathbf{r}'$ is independent of the point about which it is computed. Show that this is indeed true by computing the dipole moment for the arrangement of Fig. 2.37(c) about an arbitrary point (x, y, z).

2.43. For a line charge of finite length situated along the z axis between $z = -z_0$ and $z = +z_0$, consider the charge density ρ_L to be an arbitrary function of z and show that the potential at any point in the xy plane at a distance r from the origin is given by

$$V = \frac{1}{4\pi\epsilon_0} \int_{z=-z_0}^{z_0} \frac{\rho_L \, dz}{\sqrt{r^2 + z^2}}$$

Evaluate the integral for the following charge distributions:

(a) $\rho_L = \rho_{L0}$, a constant $\quad -z_0 < z < z_0$

(b) $\rho_L = |z|$ $\quad -z_0 < z < z_0$

(c) $\rho_L = z$ $\quad -z_0 < z < z_0$

Discuss your results from considerations of symmetry. Verify your results by considering limiting cases wherever appropriate.

2.44. For the ring charge of Problem 2.9, show that the potential at a point (0, 0, z) is given by

$$V = \frac{a}{4\pi\epsilon_0(a^2 + z^2)^{1/2}} \int_{\phi=0}^{2\pi} \rho_L \, d\phi$$

Evaluate V for the charge distributions specified in Problem 2.9, and discuss the results from considerations of symmetry. Verify your results by considering limiting cases wherever appropriate.

2.45. For the sheet charge of Problem 2.10, show that the potential at a point (0, 0, z) is given by

$$V = \frac{1}{4\pi\epsilon_0} \int_{r=0}^{\infty} \int_{\phi=0}^{2\pi} \left[\frac{\rho_s r \, dr \, d\phi}{(r^2 + z^2)^{1/2}} - \frac{\rho_s r \, dr \, d\phi}{(r^2 + z_0^2)^{1/2}}\right]$$

where (0, 0, z_0) is the reference point for zero potential. Evaluate V for the charge distributions specified in Problem 2.10, and discuss the results from considerations of symmetry. Verify your results by considering limiting cases wherever appropriate.

2.46. For the surface charge of Problem 2.11, show that the potential at a point (0, 0, z) is given by

$$V = \frac{a^2}{2\epsilon_0} \int_{\theta=0}^{\pi} \frac{\rho_s \sin\theta \, d\theta}{(a^2 + z^2 - 2az\cos\theta)^{1/2}}$$

Evaluate V both for $|z| < a$ and for $|z| > a$ for the charge distributions specified in Problem 2.11 and discuss your results from considerations of symmetry.

2.47. Obtain the potential field of a finitely long line charge of uniform density ρ_{L0} and length $2a$ at an arbitrary point. Show that the equipotential surfaces are ellipsoids with the ends of the line as their foci. Establish their orthogonality with the direction lines deduced in Problem 2.17.

2.48. For the two-dimensional electric dipole of Problem 2.15, (a) obtain the potential field and (b) show that the equipotential surfaces are orthogonal to the direction lines deduced in Problem 2.15.

2.49. A volume charge is distributed throughout a sphere of radius a, and centered at the origin, with uniform density ρ_0 C/m^3. Find the potential field of the volume charge distribution.

2.50. For the volume charge distributions specified in Problem 2.23, obtain the potential fields by evaluating $\int \mathbf{E} \cdot d\mathbf{l}$.

2.51. For the volume charge distributions specified in Problem 2.24, obtain the potential fields by evaluating $\int \mathbf{E} \cdot d\mathbf{l}$.

2.52. For the volume charge distributions specified in Problem 2.25, obtain the potential fields by evaluating $\int \mathbf{E} \cdot d\mathbf{l}$.

2.53. For the following surface charge distributions, obtain the potential fields:

(a) $\rho_s = \begin{cases} \rho_{s0} & z = a \\ -\rho_{s0} & z = -a \end{cases}$ cartesian coordinates

(b) $\rho_s = \begin{cases} \rho_{s0} & r = a \\ -\rho_{s0}\dfrac{a}{b} & r = b \end{cases}$ cylindrical coordinates

(c) $\rho_s = \begin{cases} \rho_{s0} & r = a \\ -\rho_{s0}\dfrac{a^2}{b^2} & r = b \end{cases}$ spherical coordinates

where ρ_{s0} is a constant.

2.54. A volume charge is distributed with uniform density ρ_0 C/m³ in the portion of a sphere of radius a centered at the origin and lying in the first octant. Find the potential field at large distances from the charge distribution correct to the first two significant terms.

2.55. A ring charge of radius a is situated in the xy plane with its center at the origin. Find the dipole moments about the origin for the following charge densities, where ρ_{L0} is constant:

(a) $\rho_L = \rho_{L0} \cos \phi \qquad 0 < \phi < 2\pi$
(b) $\rho_L = \rho_{L0} \sin 2\phi \qquad 0 < \phi < 2\pi$
(c) $\rho_L = \rho_{L0}\, \phi \sin \phi \qquad 0 < \phi < 2\pi$

What are the dipole moments for cases (a), and (b) about any point other than the origin? Explain.

2.56. Determine if the following fields are realizable as static electric fields:

(a) $\mathbf{A} = \dfrac{1}{y^2}(y\mathbf{i}_x - x\mathbf{i}_y)$ cartesian coordinates

(b) $\mathbf{B} = \dfrac{1}{r}\mathbf{i}_\phi$ cylindrical coordinates

(c) $\mathbf{C} = \left(1 + \dfrac{1}{r^2}\right)\cos\phi\, \mathbf{i}_r - \left(1 - \dfrac{1}{r^2}\right)\sin\phi\, \mathbf{i}_\phi$ cylindrical coordinates

(d) $\mathbf{D} = \left(1 + \dfrac{2}{r^3}\right)\cos\theta\, \mathbf{i}_r - \left(1 - \dfrac{1}{r^3}\right)\sin\theta\, \mathbf{i}_\theta$ spherical coordinates

2.57. Check that $\mathbf{E} = -\nabla V$ by substituting independently obtained expressions for $\mathbf{E}$ and V for the following charge distributions:

(a) An infinitely long line charge of uniform density.
(b) A finitely long line charge of uniform density.
(c) A three-dimensional dipole.
(d) A two-dimensional dipole.
(e) A spherical surface charge of radius a and uniform density ρ_{s0}.
(f) A spherical volume charge of radius a and uniform density ρ_0.

2.58. In shorthand notation, the three-dimensional Dirac delta function situated at the origin is written as $\delta(\mathbf{r})$, and is defined as

$$\delta(\mathbf{r}) = \lim_{\substack{r_0 \to 0 \\ \theta_0 \to 0 \\ \phi_0 \to 0}} \frac{\delta(r - r_0)\,\delta(\theta - \theta_0)\,\delta(\phi - \phi_0)}{r_0^2 \sin^2 \theta_0}$$

$$\int_V \delta(\mathbf{r})\,dv = \begin{cases} 1 & \text{if the volume } V \text{ contains the origin} \\ 0 & \text{if the volume } V \text{ does not contain the origin} \end{cases}$$

By performing volume integration of $\nabla^2 (1/r) = \nabla \cdot \nabla(1/r)$ throughout a sphere of radius a and centered at the origin and then letting $a \longrightarrow 0$, show that

$$\nabla^2 \left(\frac{1}{r}\right) = -4\pi\,\delta(\mathbf{r})$$

Hence, show that the potential field of a point charge Q located at the origin is $Q/4\pi\epsilon_0 r$.