1

VECTOR ANALYSIS

Vector analysis is a shorthand notation by means of which we perform mathematical manipulations with quantities which have associated with them not only magnitude but also direction in space. Such quantities are known as vectors, in contrast to scalars which have only magnitude associated with them. Force and velocity are examples of vectors. Mass and length are examples of scalars. The electric and magnetic fields are examples of vectors. Voltage and current are examples of scalars. Since this book is concerned with electric and magnetic fields, it is necessary that we first learn the notation and certain rules of vector analysis. To distinguish vector quantities from scalar quantities, we use boldface type: **A**. Graphically, the vector **A** is represented by a line whose length is equal to the magnitude of **A**, denoted $|\mathbf{A}|$ or simply A, and with an arrowhead at the end of the line pointing toward the direction of **A**. If the top of the page is taken to be pointing toward the north, then Figs. 1.1(a), (b), and (c) represent vectors **A**, **B**, and **C** directed north, northeast, and west-northwest, respectively.

1.1 Some Simple Rules

a. Equality of Vectors.

Two vectors **A** and **B** are equal if and only if their magnitudes as well as directions are the same.

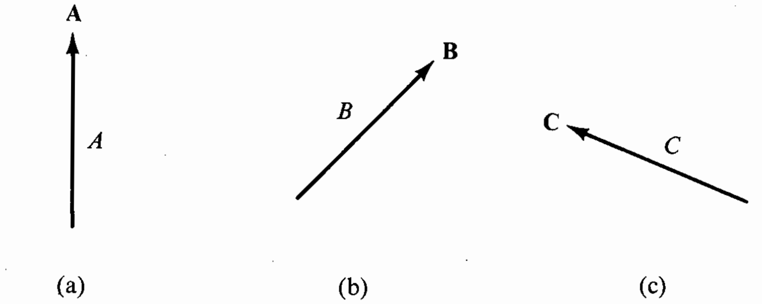


Fig. 1.1. Graphical representation of vectors.

b. Addition and Subtraction of Vectors.

Two vectors **A** and **B** are added by placing the beginning of one vector at the tip of the other as shown in Figs. 1.2(a) and (b). The sum vector is then obtained by joining the beginning of the first vector to the tip of the second vector. This rule is also known as the parallelogram law since, if we consider the two vectors as the adjacent sides of a parallelogram with their beginnings at a common point O as shown in Fig. 1.2(c), the sum vector is then given by the diagonal of the parallelogram drawn from the corner O to the opposite corner. From Figs. 1.2(a) and (b), it is clear that vector addition is commutative, that is,

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \tag{1-1}$$

Subtraction is a special case of addition. If we want to subtract a vector **B** from a vector **A**, we first construct the vector (−**B**), which has the same magnitude as that of **B** but opposite direction, and then add it to **A**, that is,

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}) \tag{1-2}$$

The graphical construction pertinent to (1-2) is shown in Fig. 1.3(a). If we decide to obtain **A** − **B** from the construction of a parallelogram with **A** and **B** as the adjacent sides emanating from the common point O similar to that in Fig. 1.2(c), then the construction of Fig. 1.3(b) indicates that

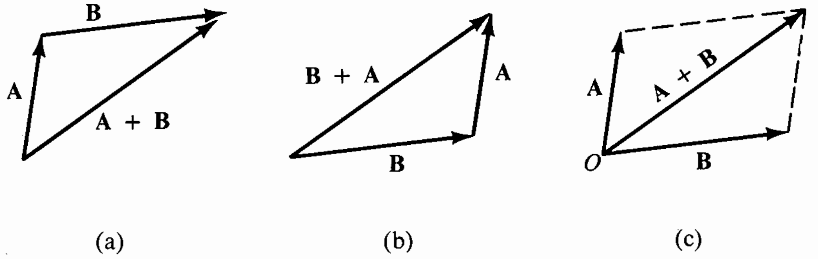


Fig. 1.2. Addition of two vectors.

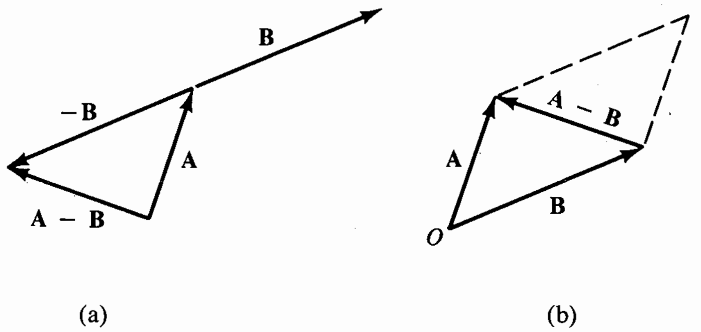


Fig. 1.3. Vector subtraction.

A − **B** is given by the diagonal of the parallelogram drawn from the tip of **B** to the tip of **A**. Finally, the constructions of Fig. 1.4 illustrate that vector addition is associative, that is,

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C} \tag{1-3}$$

c. Multiplication and Division by a Scalar.

When a vector **A** is multiplied by a scalar m, it is equivalent to adding **A** or (−**A**) a total of m times, depending upon whether m is positive or negative. Hence the direction of $m(\mathbf{A})$ is the same as or opposite to that of **A**, depending upon whether m is positive or negative, whereas the magnitude of $m(\mathbf{A})$ is $|m|$ times the magnitude of **A**. Thus

$$|m(\mathbf{A})| = |m||\mathbf{A}| = |m|A \tag{1-4}$$

$$\text{Direction of } m(\mathbf{A}) = \begin{cases} \text{direction of } \mathbf{A} & \text{if } m > 0 \\ \text{direction of } (-\mathbf{A}) & \text{if } m < 0 \end{cases} \tag{1-5}$$

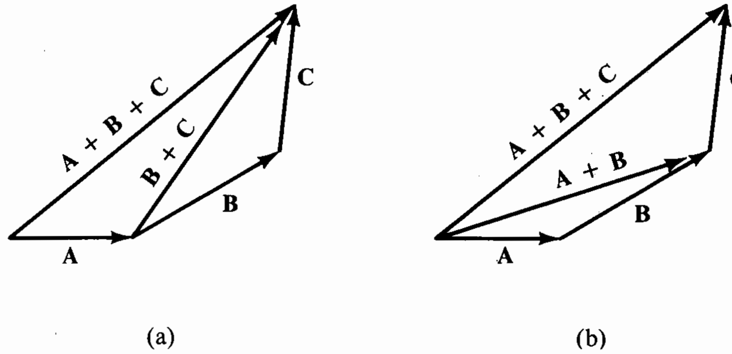


Fig. 1.4. Illustrating the associative property of vector addition.

Division by a scalar is, of course, a special case of multiplication, that is, to divide a vector by m we multiply it by $1/m$.

d. Unit Vector.

If we divide a vector $\mathbf{A}$ by its magnitude A, we obtain a vector whose magnitude is unity and whose direction is the same as the direction of $\mathbf{A}$. The resulting vector is called the "unit vector" in the direction of $\mathbf{A}$ and is denoted $\mathbf{i}_A$. Thus

$$\mathbf{i}_A = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{\mathbf{A}}{A} \tag{1-6}$$

Unit vectors play a very important role in vector analysis, as we will find throughout this book.

e. Scalar or Dot Product of Two Vectors.

The scalar or dot product of two vectors $\mathbf{A}$ and $\mathbf{B}$ is a scalar quantity of value equal to the product of the magnitudes of $\mathbf{A}$ and $\mathbf{B}$ and the cosine of the angle between $\mathbf{A}$ and $\mathbf{B}$. It is represented by a dot between $\mathbf{A}$ and $\mathbf{B}$. Thus

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos \alpha = AB \cos \alpha \tag{1-7}$$

where α is the angle between $\mathbf{A}$ and $\mathbf{B}$. Noting that

$$\mathbf{A} \cdot \mathbf{B} = AB \cos \alpha = A(B \cos \alpha) = B(A \cos \alpha) \tag{1-8}$$

we see from the constructions of Fig. 1.5 that the dot-product operation consists of multiplying the magnitude of one vector by the scalar obtained by projecting the second vector onto the first vector. This suggests that the dot product is useful for problems such as finding the work done in displacing a mass. The dot-product operation is commutative since

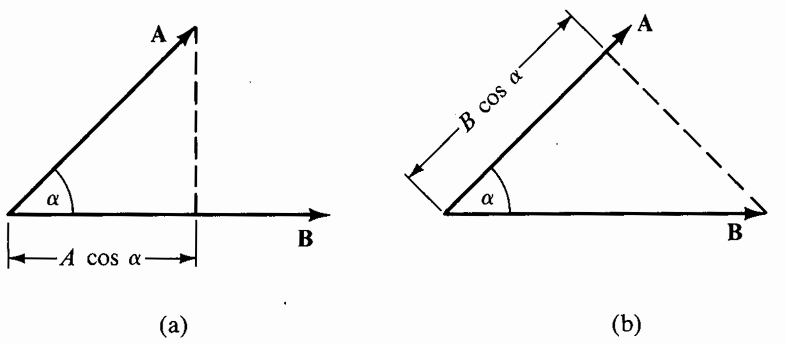


Fig. 1.5. Showing that the dot product of **A** and **B** is the product of the magnitude of one vector and the projection of the second vector onto the first vector.

$$\mathbf{B} \cdot \mathbf{A} = |\mathbf{B}||\mathbf{A}| \cos \alpha = |\mathbf{A}||\mathbf{B}| \cos \alpha = \mathbf{A} \cdot \mathbf{B} \tag{1-9}$$

Furthermore, the distributive property also holds, that is,

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C} \tag{1-10}$$

To prove the distributive property, we note from the construction shown in Fig. 1.6 that the projection of $(\mathbf{B} + \mathbf{C})$ onto $\mathbf{A}$ is equal to the sum of the projections of $\mathbf{B}$ and $\mathbf{C}$ onto $\mathbf{A}$. It follows from this that (1-10) is correct.

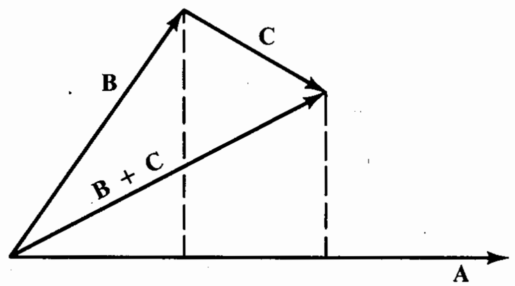


Fig. 1.6. For proving the distributive property of the dot-product operation.

f. Vector or Cross Product of Two Vectors.

In contrast to the dot product, the vector or cross product of two vectors $\mathbf{A}$ and $\mathbf{B}$ is another vector whose magnitude is the product of the magnitudes of $\mathbf{A}$ and $\mathbf{B}$ and the sine of the angle α between $\mathbf{A}$ and $\mathbf{B}$ and whose direction is the direction of advance of a right-hand screw as it is turned from $\mathbf{A}$ towards $\mathbf{B}$ through the angle α, as shown in Fig. 1.7(a). Thus

$$\mathbf{A} \times \mathbf{B} = |\mathbf{A}||\mathbf{B}| \sin \alpha \, \mathbf{i}_N = AB \sin \alpha \, \mathbf{i}_N \tag{1-11}$$

where $\mathbf{i}_N$ is the unit vector in the direction of advance of a right-hand screw as it is turned from $\mathbf{A}$ towards $\mathbf{B}$ through α. For example, if vector $\mathbf{A}$ is a unit vector directed eastward and vector $\mathbf{B}$ is a unit vector directed northward, then a right-hand screw advances upward as it is turned from east towards north through the 90° angle so that $\mathbf{A} \times \mathbf{B}$ has a magnitude (1)(1)(sin 90°) or unity and is directed upward. Alternatively, if we decide to turn the right-hand screw from east toward north through the 270° angle, we note that the screw advances downward. There is no inconsistency, however, since the product $|\mathbf{A}||\mathbf{B}| \sin \alpha$ is then equal to (1)(1)(sin 270°) or -1. When the minus sign is associated with the direction of advance of the screw, the direction of $\mathbf{A} \times \mathbf{B}$ becomes upward.

From the constructions of Figs. 1.7(a) and (b) it follows that

$$\mathbf{B} \times \mathbf{A} = -\mathbf{A} \times \mathbf{B} \tag{1-12}$$

so that the commutative law does not hold for the cross product. Similarly,

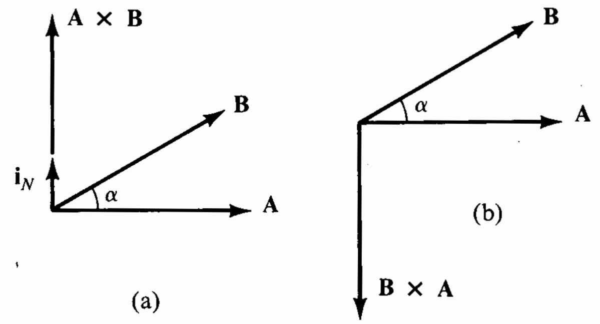


Fig. 1.7. Cross-product operations for two vectors **A** and **B**.

the associative law does not hold for the cross product, that is,

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} \neq \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \tag{1-13}$$

This can be demonstrated very easily by considering a particular case in which the three vectors **A**, **B**, and **C** are unit vectors directed eastward, northward, and southward, respectively, as shown in Fig. 1.8. Then (**A** × **B**) is the unit vector directed upward. (**A** × **B**) × **C** is the unit vector directed eastward, that is, **A**. On the other hand, (**B** × **C**) is equal to zero and hence **A** × (**B** × **C**) is equal to zero. Thus the associative law does not hold. That the distributive law,

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C} \tag{1-14}$$

holds will be proved in an example after we discuss the scalar triple product.

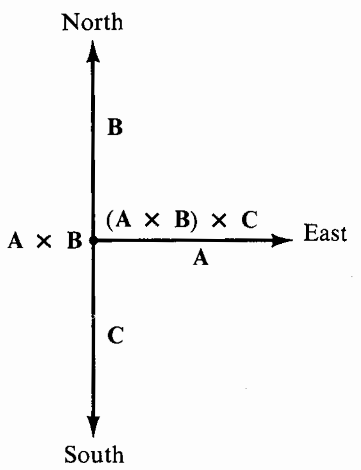


Fig. 1.8. For demonstrating that the associative law does not hold for the cross-product operation.

The cross-product operation is very convenient to define unit vectors. Thus a unit vector perpendicular to both **A** and **B** is, according to (1-11), given by

$$\mathbf{i}_N = \frac{\mathbf{A} \times \mathbf{B}}{|\mathbf{A}||\mathbf{B}|\sin\alpha} \tag{1-15}$$

g. Scalar Triple Product.

Another useful operation but of less importance is the scalar triple product $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$. Using the definitions of dot and cross products we have

$$\begin{aligned} \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) &= |\mathbf{A}||\mathbf{B} \times \mathbf{C}| \cos(\text{angle between } \mathbf{A} \text{ and } \mathbf{B} \times \mathbf{C}) \\ &= |\mathbf{A}||\mathbf{B}||\mathbf{C}| \sin(\text{angle between } \mathbf{B} \text{ and } \mathbf{C}) \\ &\quad \times \cos(\text{angle between } \mathbf{A} \text{ and } \mathbf{B} \times \mathbf{C}) \end{aligned} \tag{1-16}$$

From the construction of Fig. 1.9, we note that

$$\begin{aligned} \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) &= ABC \sin\beta \cos\alpha = (A\cos\alpha)(BC\sin\beta) \\ &= \text{volume of the parallelepiped formed by } \mathbf{A}, \mathbf{B}, \text{ and } \mathbf{C} \end{aligned} \tag{1-17}$$

Thus the scalar triple product has the geometric meaning that it represents the volume of the parallelepiped formed by the three vectors. From con-

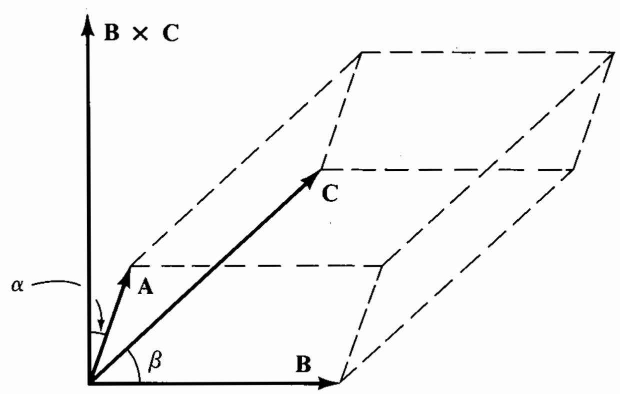


Fig. 1.9. Parallelepiped formed by **A**, **B**, and **C**.

structions similar to Fig. 1.9, it can be shown that $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$ or $(\mathbf{C} \times \mathbf{A}) \cdot \mathbf{B}$ represent the same volume so that

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) \tag{1-18}$$

Also, the parentheses in the scalar triple product are unnecessary since, for example, $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}$ can mean only $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ and not $(\mathbf{A} \cdot \mathbf{B}) \times \mathbf{C}$. This is so because $\mathbf{A} \cdot \mathbf{B}$ is a scalar and for a vector product, we need two vectors. Hence $(\mathbf{A} \cdot \mathbf{B}) \times \mathbf{C}$ is meaningless. It is therefore customary to omit the parentheses when writing a scalar triple product.

EXAMPLE 1-1. Vector **A** has a magnitude of 4 units and is directed towards the east. Vector **B** has a magnitude of 4 units and is oriented in a direction making an angle of 120° toward north from east. Vector **C** has a magnitude of 3 units and is directed 30° south of east. Find

(a) $\mathbf{A} + \mathbf{B}$

(b) $3\mathbf{A} - 4\mathbf{C}$

(c) $\mathbf{A} + \mathbf{B} - \mathbf{C}$

(d) $\mathbf{A} \cdot \mathbf{B}$

(e) $\mathbf{B} \times \mathbf{C}$

(f) $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}$

(g) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$

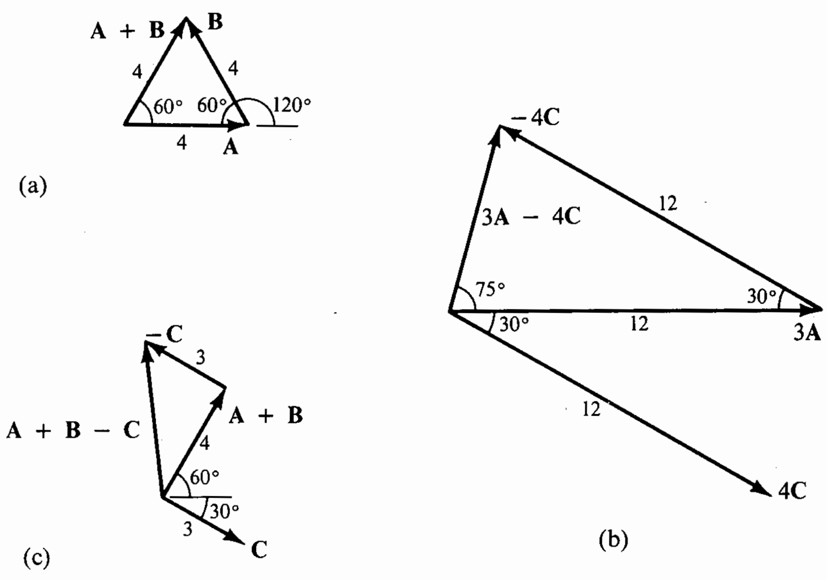


Fig. 1.10. For Example 1-1.

(a) From the construction of Fig. 1.10(a), $(\mathbf{A} + \mathbf{B})$ has a magnitude of 4 units and is directed 60° north of east.

(b) $3\mathbf{A} = 12$ units towards the east; $4\mathbf{C} = 12$ units directed 30° south of east. From the construction of Fig. 1.10(b), $3\mathbf{A} - 4\mathbf{C}$ has a magnitude of $24 \cos 75°$ or 6·21 units and is directed 75° north of east.

(c) From the construction of Fig. 1.10(c), $\mathbf{A} + \mathbf{B} - \mathbf{C}$ has a magnitude of 5 units and is directed $(60° + \tan^{-1} \frac{3}{4})$ or 96°52′ north of east.

(d) $\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos$ (angle between **A** and **B**) $= (4)(4)(\cos 120°) = -8$.

(e) $|\mathbf{B} \times \mathbf{C}| = |\mathbf{B}||\mathbf{C}||\sin(\text{angle between } \mathbf{B} \text{ and } \mathbf{C})| = (4)(3)(\sin 150°) = 6$. The direction of $\mathbf{B} \times \mathbf{C}$ is the direction in which a right-hand screw advances when it is turned from $\mathbf{B}$ toward $\mathbf{C}$ through the angle 150°. This direction is downward. Thus, $\mathbf{B} \times \mathbf{C}$ has a magnitude of 6 units and is directed downward.

(f) $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = |\mathbf{A}||\mathbf{B} \times \mathbf{C}| \cos(\text{angle between } \mathbf{A} \text{ and } \mathbf{B} \times \mathbf{C}) = (4)(6)(\cos 90°) = 0$. This is consistent with the reasoning that, since all three vectors are in a plane, the area of the parallelogram formed by them is zero.

(g) $|\mathbf{A} \times (\mathbf{B} \times \mathbf{C})| = |\mathbf{A}||\mathbf{B} \times \mathbf{C}||\sin(\text{angle between } \mathbf{A} \text{ and } \mathbf{B} \times \mathbf{C})| = (4)(6)(\sin 90°) = 24$. The direction of $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ is the direction in which a right-hand screw advances if it is turned from $\mathbf{A}$ toward $\mathbf{B} \times \mathbf{C}$ through the angle 90°, that is, from east to downward through the angle 90°. The screw advances towards the north. Thus $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ has a magnitude of 24 units and is directed northward. ▮

EXAMPLE 1-2. Show that $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$.

We will prove this equality by showing that

$$\mathbf{D} = \mathbf{A} \times (\mathbf{B} + \mathbf{C}) - \mathbf{A} \times \mathbf{B} - \mathbf{A} \times \mathbf{C} = 0$$

Taking the dot product of an arbitrary vector $\mathbf{E}$ and the vector $\mathbf{D}$ and using (1-10) and (1-18), we have

$$\begin{aligned}
\mathbf{E} \cdot \mathbf{D} &= \mathbf{E} \cdot [\mathbf{A} \times (\mathbf{B} + \mathbf{C}) - \mathbf{A} \times \mathbf{B} - \mathbf{A} \times \mathbf{C}] \\
&= \mathbf{E} \cdot \mathbf{A} \times (\mathbf{B} + \mathbf{C}) - \mathbf{E} \cdot \mathbf{A} \times \mathbf{B} - \mathbf{E} \cdot \mathbf{A} \times \mathbf{C} \\
&= (\mathbf{B} + \mathbf{C}) \cdot \mathbf{E} \times \mathbf{A} - \mathbf{B} \cdot \mathbf{E} \times \mathbf{A} - \mathbf{C} \cdot \mathbf{E} \times \mathbf{A} \\
&= \mathbf{B} \cdot \mathbf{E} \times \mathbf{A} + \mathbf{C} \cdot \mathbf{E} \times \mathbf{A} - \mathbf{B} \cdot \mathbf{E} \times \mathbf{A} - \mathbf{C} \cdot \mathbf{E} \times \mathbf{A} = 0
\end{aligned}$$

This result implies that $\mathbf{D}$ is either zero or perpendicular to $\mathbf{E}$. However, since $\mathbf{E}$ is an arbitrary vector, it can be chosen such that it is not perpendicular to $\mathbf{D}$, in which case $\mathbf{D}$ has to be zero for $\mathbf{E} \cdot \mathbf{D}$ to be zero. Thus $\mathbf{D}$ is equal to zero and hence the equality $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$ is correct. ▮

EXAMPLE 1-3. Two unit vectors $\mathbf{i}_A$ and $\mathbf{i}_B$ drawn at a point are perpendicular to each other. A vector $\mathbf{C}$ is also drawn from the same point. Express $\mathbf{C}$ in terms of its component vectors along $\mathbf{i}_A$ and $\mathbf{i}_B$.

From Fig. 1.11, the projection of $\mathbf{C}$ onto the line along $\mathbf{i}_A$ is equal to $C \cos\alpha = \mathbf{C} \cdot \mathbf{i}_A$. Hence the component vector of $\mathbf{C}$ along $\mathbf{i}_A$ is $(\mathbf{C} \cdot \mathbf{i}_A)\mathbf{i}_A$. Similarly, the component vector of $\mathbf{C}$ along $\mathbf{i}_B$ is $(\mathbf{C} \cdot \mathbf{i}_B)\mathbf{i}_B$. Since the component vectors form two adjacent sides of a rectangle whose diagonal is $\mathbf{C}$, as in Fig. 1.11, we have

$$\mathbf{C} = (\mathbf{C} \cdot \mathbf{i}_A)\mathbf{i}_A + (\mathbf{C} \cdot \mathbf{i}_B)\mathbf{i}_B$$

It also follows from Fig. 1.11 that

$$(\mathbf{C} \cdot \mathbf{i}_A)^2 + (\mathbf{C} \cdot \mathbf{i}_B)^2 = C^2$$

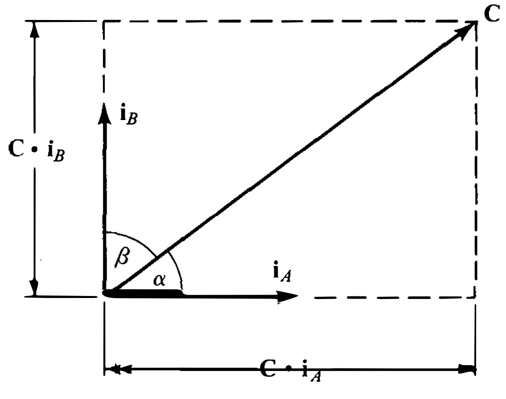


Fig. 1.11. Components of a vector along mutually perpendicular unit vectors.

Likewise, if we have three mutually perpendicular unit vectors $\mathbf{i}_A$, $\mathbf{i}_B$, and $\mathbf{i}_C$ drawn from a point, then the component vectors of a vector $\mathbf{D}$ along the unit vectors are $(\mathbf{D} \cdot \mathbf{i}_A)\mathbf{i}_A$, $(\mathbf{D} \cdot \mathbf{i}_B)\mathbf{i}_B$, and $(\mathbf{D} \cdot \mathbf{i}_C)\mathbf{i}_C$, respectively, so that

$$\mathbf{D} = (\mathbf{D} \cdot \mathbf{i}_A)\mathbf{i}_A + (\mathbf{D} \cdot \mathbf{i}_B)\mathbf{i}_B + (\mathbf{D} \cdot \mathbf{i}_C)\mathbf{i}_C$$

Furthermore,

$$(\mathbf{D} \cdot \mathbf{i}_A)^2 + (\mathbf{D} \cdot \mathbf{i}_B)^2 + (\mathbf{D} \cdot \mathbf{i}_C)^2 = D^2 \quad \blacksquare$$

1.2 Coordinate Systems

In the previous section we discussed some simple rules of vector analysis without involving any coordinate system. In physical problems, we cannot simply go on describing vectors by symbols **A**, **B**, **C**, and so on, if we wish to simplify the geometry associated with the mathematical operations using these vectors. We need to describe a vector in terms of component vectors along a set of reference directions such as east, north, and upward. Although several different coordinate systems are in existence, we will be interested only in three: (a) the cartesian, (b) the circular cylindrical or simply cylindrical, and (c) the spherical coordinate systems. Each coordinate system involves three surfaces which are mutually orthogonal. At any particular point, unit vectors can be drawn tangential to the curves of intersection of pairs of the three orthogonal surfaces. The three unit vectors drawn in this manner will be mutually perpendicular and will define the reference directions at that point. Once such reference directions are defined everywhere in space, we can represent vectors in terms of their component vectors along the reference directions and use them for performing vector operations. We will discuss each coordinate system separately and then summarize the details in the form of a table.

a. Cartesian Coordinate System.

For the cartesian coordinate system, the three mutually orthogonal surfaces are three planes. Let us consider three orthogonal planes which intersect at a particular point O which we will call the origin, as shown in Fig. 1.12(a). The three planes also define three straight lines which are the intersections of pairs of planes. These three straight lines are mutually perpendicular and form a set of coordinate axes which are denoted x, y, and z axes. Values of x, y, and z are measured from the origin so that the origin is taken as the reference point. We say that the coordinates of the origin are $(0, 0, 0)$, that is, $x = 0$, $y = 0$, and $z = 0$. Thus, if we consider the x axis, values of x on one side of the origin are positive and on the other side, they are negative. The direction of increasing values of x is indicated by an arrowhead. We will direct a unit vector $\mathbf{i}_x$ drawn from the origin in the direction of increasing values of x. By doing the same with the y and z axes, we define unit vectors $\mathbf{i}_y$ and $\mathbf{i}_z$ at O. Now, we note that we can choose the directions of increasing values of x, y, and z in two ways: (a) such that $\mathbf{i}_x \times \mathbf{i}_y = \mathbf{i}_z$ as in Fig. 1.12(a); or (b) such that $\mathbf{i}_y \times \mathbf{i}_x = \mathbf{i}_z$. The first is known as a right-hand coordinate system since, if a right-hand screw is turned from the direction of increasing values of x towards the direction of increasing values of y through the smaller angle 90°, it advances in the direction of increasing values of z. The second choice is known as a left-hand coordinate system since it requires a left-hand screw to advance in the direction of increasing values of z when turned from the direction of increasing values of x towards the direction of increasing values of y through the smaller angle 90°. By convention, the right-hand coordinate system is used.

Movement on the yz plane requires no displacement along the x direction; hence the value of x is constant on this plane. In particular, since the value of x at the origin is zero, this constant is zero. Also, the unit vector $\mathbf{i}_x$ is in the increasing x direction and hence is normal to this plane. Similarly, for the xz plane, $y =$ constant $= 0$ and $\mathbf{i}_y$ is normal to this plane; for the xy plane, $z =$ constant $= 0$ and $\mathbf{i}_z$ is normal to this plane. Any other point in space is now defined by the intersection of three planes parallel to the three planes defining the origin. Alternatively, we can displace the three planes $x = 0$, $y = 0$, and $z = 0$ along the coordinate axes (or unit vectors) perpendicular to them and obtain a new point of intersection. For example, by moving the $x = 0$ plane by one unit along the x axis, the $y = 0$ plane by three units along the y axis, and the $z = 0$ plane by four units along the z axis, we obtain a point of intersection whose coordinates are $(1, 3, 4)$, as shown in Fig. 1.12(b). On any plane parallel to the $x = 0$ plane, the value of x is constant and equal to its displacement from the $x = 0$ plane; on any plane parallel to the $y = 0$ plane, the value of y is constant and equal to its displacement from the $y = 0$ plane; and on any plane parallel to the $z = 0$ plane, the value of z is constant and equal to its displacement from the $z = 0$ plane. Thus the point $(1, 3, 4)$ is the intersection of the three planes $x = 1$, $y = 3$, and $z = 4$. These three planes also define three straight lines which are intersections of pairs of planes. Unit vectors $\mathbf{i}_x$, $\mathbf{i}_y$, and $\mathbf{i}_z$ can be drawn along these lines of intersections. These unit vectors are parallel to the

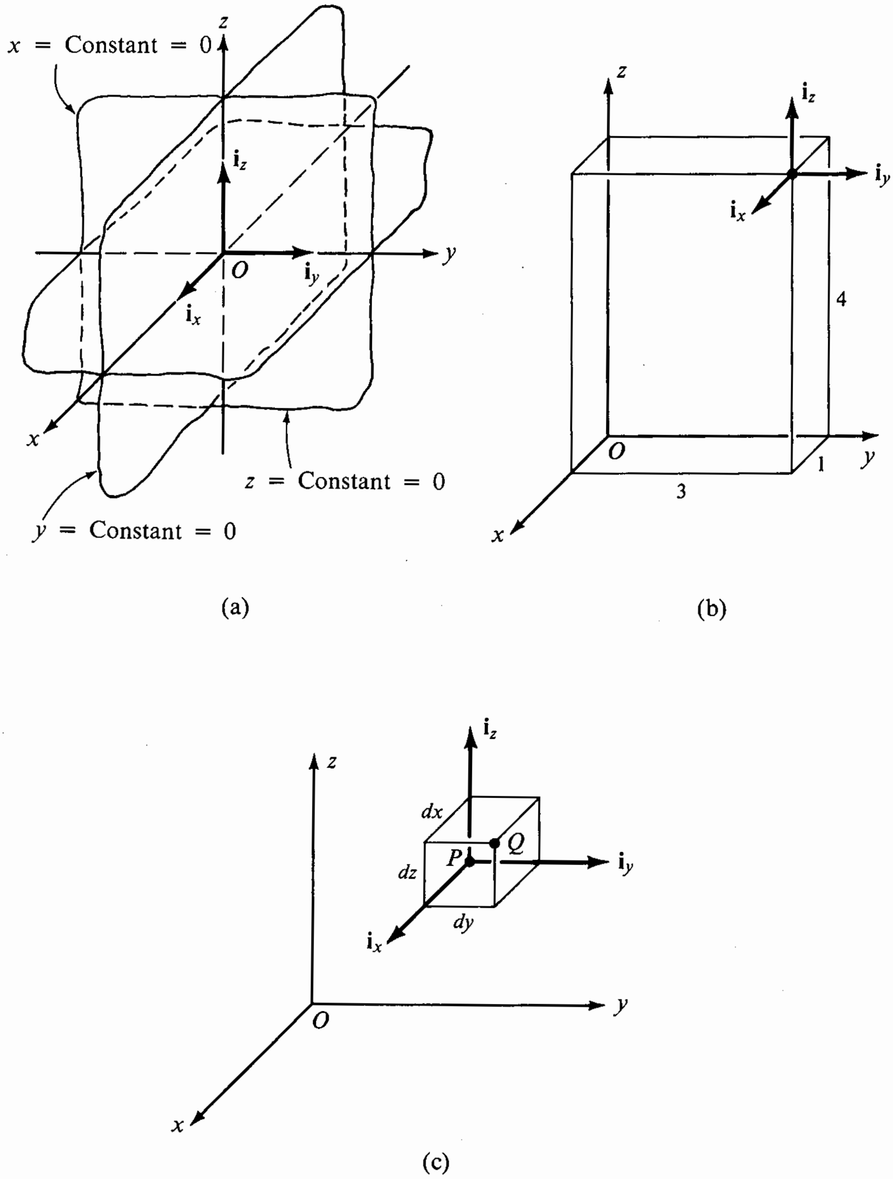


Fig. 1. 12. Cartesian coordinate system. (a) The three orthogonal planes defining the coordinate system. (b) Unit vectors at an arbitrary point. (c) Differential volume formed by incrementing the coordinates.

corresponding unit vectors at the origin since the lines of intersection are parallel to the x, y, and z axes. In general, an arbitrary point (a, b, c) is defined by the intersection of the three planes $x = a$, $y = b$, and $z = c$. Unit vectors $\mathbf{i}_x$, $\mathbf{i}_y$, and $\mathbf{i}_z$ are directed normal to these planes along increasing values of x, y, and z, respectively, and are parallel to the corresponding unit vectors at the origin. Thus, in the cartesian coordinate system, the directions of the unit vectors $\mathbf{i}_x$, $\mathbf{i}_y$, and $\mathbf{i}_z$ are everywhere the same as their directions at the origin.

Let us now consider two points $P(x, y, z)$ and $Q(x + dx, y + dy, z + dz)$, where Q is obtained by incrementing infinitesimally each coordinate from its value at P. The three orthogonal planes intersecting at P and the three orthogonal planes intersecting at Q define a rectangular box of edges dx, dy, and dz in the $\mathbf{i}_x$, $\mathbf{i}_y$, and $\mathbf{i}_z$ directions, respectively, as shown in Fig. 1.12(c). The differential displacements (or length elements) along the unit vectors $\mathbf{i}_x$, $\mathbf{i}_y$, and $\mathbf{i}_z$ in going from P to Q are therefore the same as the differential increments dx, dy, and dz of the coordinates x, y, and z, respectively. The vector displacement $d\mathbf{l}$ from P to Q is given by

$$d\mathbf{l} = dx\,\mathbf{i}_x + dy\,\mathbf{i}_y + dz\,\mathbf{i}_z \tag{1-19a}$$

The magnitude of this displacement is

$$dl = \sqrt{(dx)^2 + (dy)^2 + (dz)^2} \tag{1-19b}$$

The three displacements $dx\,\mathbf{i}_x$, $dy\,\mathbf{i}_y$, and $dz\,\mathbf{i}_z$ also define three surfaces of infinitesimal areas in the three planes intersecting at P. To take into account the orientation of the surface area, it is convenient to represent the area by a vector quantity whose magnitude is equal to the area and whose direction is that of the normal to the area. The three infinitesimal surfaces are then $\pm dy\,dz\,\mathbf{i}_x$, $\pm dz\,dx\,\mathbf{i}_y$, and $\pm dx\,dy\,\mathbf{i}_z$, where the $\pm$ sign takes into account two possible directions of normal to the surface. The infinitesimal volume of the box is $dx\,dy\,dz$. We will use the differential length elements, surface areas, and volume introduced here in later sections.

Any arbitrary surface is defined by an equation of the type

$$f(x, y, z) = 0 \tag{1-20}$$

where f denotes a function. Since an arbitrary curve is an intersection of two appropriate surfaces, it is defined by a pair of equations

$$f(x, y, z) = 0 \qquad \text{and} \qquad g(x, y, z) = 0 \tag{1-21}$$

where f and g are two different functions. Alternatively, a curve may be defined by three parametric equations

$$x = x(t) \qquad y = y(t) \qquad z = z(t) \tag{1-22}$$

where t is an independent parameter.

A vector drawn from the origin to an arbitrary point $P(x, y, z)$ is called the position vector defining the point P. It is denoted by the symbol $\mathbf{r}$.

Thus, in the cartesian coordinate system,

$$\mathbf{r} = x\mathbf{i}_x + y\mathbf{i}_y + z\mathbf{i}_z \tag{1-23}$$

b. Cylindrical Coordinate System.

For the cylindrical coordinate system, the three mutually orthogonal surfaces are a cylinder and two planes, as shown in Fig. 1.13(a). One of the planes is the same as the $z =$ constant plane in the cartesian coordinate

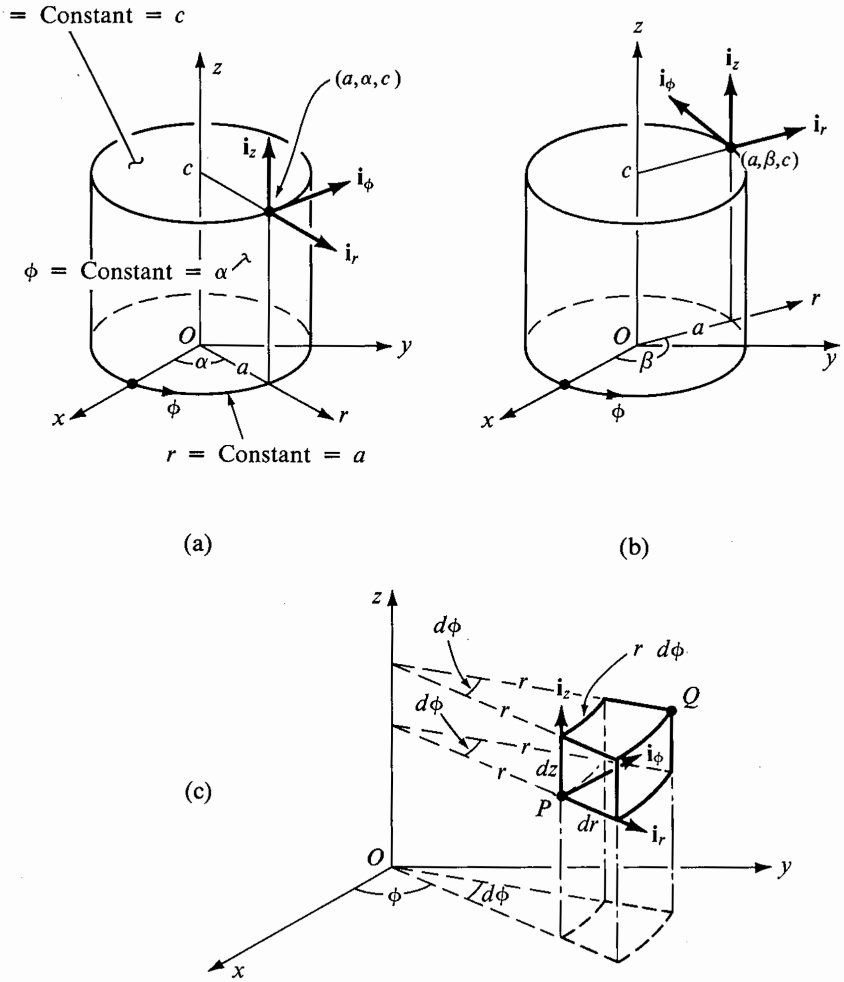


Fig. 1.13. Cylindrical coordinate system. (a) The three orthogonal surfaces defining the coordinate system (b) Unit vectors at an arbitrary point. (c) Differential volume formed by incrementing the coordinates.

system. The second plane is orthogonal to the z = constant plane and hence contains the z axis. It makes an angle ϕ with a reference plane, conveniently chosen as the xz plane of the cartesian coordinate system. This plane is therefore defined by ϕ = constant. The third orthogonal surface, which is cylindrical, has the z axis as its axis. On such a cylindrical surface, the radial distance r from the z axis is a constant. Thus the three orthogonal surfaces defining the cylindrical coordinates of a point are given by r = constant, ϕ = constant, and z = constant. In particular, the origin is defined by $r = 0$, $\phi = 0$, and $z = 0$. Note that only two of the coordinates (r and z) are distances, whereas the third coordinate (ϕ) is an angle. Since the radius of a cylinder cannot be negative, the coordinate r varies only from 0 to ∞. Since one revolution of the ϕ = constant plane about the z axis sweeps the entire space, the coordinate ϕ varies from 0 to 2π. The coordinate z varies from $-\infty$ to $+\infty$ as in the cartesian coordinate system.

Through any arbitrary point (a, α, c) we can pass a cylinder $r = a$, a plane $\phi = \alpha$, and another plane $z = c$. These three orthogonal surfaces define three curves, mutually perpendicular at (a, α, c), two of which are straight lines and the third is a circle. We draw unit vectors $\mathbf{i}_r$, $\mathbf{i}_\phi$, and $\mathbf{i}_z$ tangential to these curves at (a, α, c) and directed toward increasing values of r, ϕ, and z, respectively, as shown in Fig. 1.13(a). It follows that $\mathbf{i}_r$, $\mathbf{i}_\phi$, and $\mathbf{i}_z$ are mutually perpendicular and normal to the surfaces $r = a$, $\phi = \alpha$, and $z = c$, respectively, at the point (a, α, c). If we now consider a point (a, β, c), this point is defined by the intersection of the surfaces $r = a$, $\phi = \beta$, and $z = c$. Three mutually perpendicular unit vectors $\mathbf{i}_r$, $\mathbf{i}_\phi$, and $\mathbf{i}_z$ can be drawn at the point (a, β, c) tangential to the curves of intersection of pairs of these surfaces and in the directions of increasing r, ϕ, and z, respectively, as shown in Fig. 1.13(b). However, we note that the unit vectors $\mathbf{i}_r$ and $\mathbf{i}_\phi$ at this point are not parallel to the corresponding unit vectors at the point (a, α, c). Thus, unlike the unit vectors in the cartesian coordinate system, the unit vectors $\mathbf{i}_r$ and $\mathbf{i}_\phi$ do not have the same directions at all points; that is, the directions of $\mathbf{i}_r$ and $\mathbf{i}_\phi$ are functions of the coordinates r and ϕ, whereas $\mathbf{i}_z$ remains uniform. We also note that a right-hand coordinate system defined by $\mathbf{i}_r \times \mathbf{i}_\phi = \mathbf{i}_z$ and a left-hand coordinate system defined by $\mathbf{i}_\phi \times \mathbf{i}_r = \mathbf{i}_z$ are possible. However, we will work with the right-hand coordinate system.

Let us now consider two points $P(r, \phi, z)$ and $Q(r + dr, \phi + d\phi, z + dz)$, where Q is obtained by incrementing infinitesimally each coordinate from its value at P. The three orthogonal surfaces intersecting at P and the three orthogonal surfaces intersecting at Q define a box which can be considered as a rectangular box since dr, $d\phi$, and dz are infinitesimally small. The sides of this box are made up of the differential length elements dr, $r\,d\phi$, and dz along the $\mathbf{i}_r$, $\mathbf{i}_\phi$, and $\mathbf{i}_z$ directions, respectively, as shown in Fig. 1.13(c). Thus the differential displacements along the unit vectors $\mathbf{i}_r$, $\mathbf{i}_\phi$, and $\mathbf{i}_z$ in going from P to Q are dr, $r\,d\phi$, and dz, respectively. We note that the differential displacement in the ϕ direction is not $d\phi$ but $r\,d\phi$. The vector displacement $d\mathbf{l}$ from P to Q is given by

$$d\mathbf{l} = dr\,\mathbf{i}_r + r\,d\phi\,\mathbf{i}_\phi + dz\,\mathbf{i}_z \tag{1-24a}$$

The magnitude of this displacement is

$$dl = \sqrt{(dr)^2 + (r\,d\phi)^2 + (dz)^2} \tag{1-24b}$$

The infinitesimal areas in the three surfaces intersecting at P are $\pm(r\,d\phi)(dz)\mathbf{i}_r$, $\pm(dr)(dz)\,\mathbf{i}_\phi$, and $\pm(r\,d\phi)(dr)\mathbf{i}_z$. Finally, the infinitesimal volume of the box is $(dr)(r\,d\phi)(dz) = r\,dr\,d\phi\,dz$. Equations similar to (1-20), (1-21), and (1-22) define arbitrary surfaces and curves. The position vector defining an arbitrary point $P(r, \phi, z)$ is given by

$$\mathbf{r} = r\mathbf{i}_r + z\mathbf{i}_z \tag{1-25}$$

c. Spherical Coordinate System.

For the spherical coordinate system, the three mutually orthogonal surfaces are a sphere, a cone, and a plane, as shown in Fig. 1.14(a). The plane is the same as the ϕ = constant plane in the cylindrical coordinate system. The sphere is centered at the origin. On the surface of such a sphere, the radial distance r from the origin is constant and hence the sphere is defined by r = constant. The spherical coordinate r should not be confused with the cylindrical coordinate r. When these two coordinates appear in the same expression, we will use subscripts c and s to distinguish between

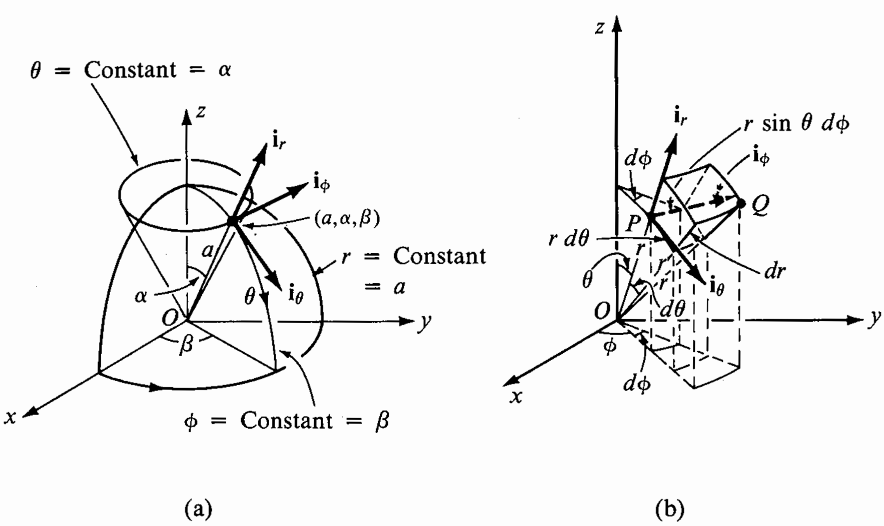


Fig. 1.14. Spherical coordinate system. (a) The three orthogonal surfaces defining the coordinate system. (b) Differential volume formed by incrementing the coordinates.

cylindrical and spherical. The cone has its vertex at the origin and its surface is symmetrical about the z axis, so that the angle θ which the conical surface makes with the z axis is constant. Thus the three orthogonal surfaces defining the spherical coordinates are given by $r =$ constant, $\theta =$ constant, and $\phi =$ constant. In particular, the origin is defined by $r = 0$, $\theta = 0$, and $\phi = 0$. Note that only one coordinate (r) is distance whereas the other two (θ and ϕ) are angles. Since the radius of a sphere cannot be negative, the coordinate r varies only from 0 to ∞. Likewise, it is sufficient if the coordinate θ is allowed to vary from 0 to π to cover the entire space. The coordinate ϕ varies from 0 to 2π as in the cylindrical coordinate system.

Through any arbitrary point (a, α, β) we can pass a sphere $r = a$, a cone $\theta = \alpha$, and a plane $\phi = \beta$. These three orthogonal surfaces define three curves, mutually perpendicular at (a, α, β). We draw unit vectors $\mathbf{i}_r$, $\mathbf{i}_\theta$, and $\mathbf{i}_\phi$ tangential to these curves at (a, α, β) and directed towards increasing values of r, θ, and ϕ, respectively, as shown in Fig. 1.14(a). It follows that $\mathbf{i}_r$, $\mathbf{i}_\theta$, and $\mathbf{i}_\phi$ are mutually perpendicular and normal to the surfaces $r = a$, $\theta = \alpha$, and $\phi = \beta$, respectively, at the point (a, α, β). By doing the same at another point, it may be seen that the directions of all three unit vectors $\mathbf{i}_r$, $\mathbf{i}_\theta$, and $\mathbf{i}_\phi$ are functions r, θ, and ϕ. We also note that a right-hand coordinate system defined by $\mathbf{i}_r \times \mathbf{i}_\theta = \mathbf{i}_\phi$ and a left-hand coordinate system defined by $\mathbf{i}_\theta \times \mathbf{i}_r = \mathbf{i}_\phi$ are possible. We will, however, work with the right-hand coordinate system.

Let us now consider two points $P(r, \theta, \phi)$ and $Q(r + dr, \theta + d\theta, \phi + d\phi)$, where Q is obtained by incrementing infinitesimally each coordinate from its value at P. The three orthogonal surfaces intersecting at P and the three orthogonal surfaces intersecting at Q define a box which can be considered as a rectangular box since dr, $d\theta$, and $d\phi$ are infinitesimally small. The sides of this box are made up of the differential length elements dr, $r\,d\theta$, and $r \sin\theta\, d\phi$ along the $\mathbf{i}_r$, $\mathbf{i}_\theta$, and $\mathbf{i}_\phi$ directions, respectively. Thus the differential displacements along the unit vectors $\mathbf{i}_r$, $\mathbf{i}_\theta$, and $\mathbf{i}_\phi$ in going from P to Q are dr, $r\,d\theta$, and $r \sin\theta\, d\phi$, respectively, as shown in Fig. 1.14(b). We note that the differential displacements in the θ and ϕ directions are $r\,d\theta$ and $r \sin\theta\, d\phi$ and not $d\theta$ and $d\phi$. The vector displacement $d\mathbf{l}$ from P to Q is given by

$$d\mathbf{l} = dr\,\mathbf{i}_r + r\,d\theta\,\mathbf{i}_\theta + r\sin\theta\,d\phi\,\mathbf{i}_\phi \tag{1-26a}$$

The magnitude of this displacement is

$$dl = \sqrt{(dr)^2 + (r\,d\theta)^2 + (r\sin\theta\,d\phi)^2} \tag{1-26b}$$

The infinitesimal areas in the three surfaces intersecting at P are $\pm(r\,d\theta)(r\sin\theta\,d\phi)\mathbf{i}_r$, $\pm(dr)(r\sin\theta\,d\phi)\mathbf{i}_\theta$, and $\pm(dr)(r\,d\theta)\mathbf{i}_\phi$. Finally, the infinitesimal volume of the box is $(dr)(r\,d\theta)(r\sin\theta\,d\phi) = r^2\sin\theta\,dr\,d\theta\,d\phi$. Equations similar to (1-20), (1-21), and (1-22) define arbitrary surfaces and curves. The position vector defining an arbitrary point $P(r, \theta, \phi)$ is given by

$$\mathbf{r} = r\mathbf{i}_r \tag{1-27}$$

The various details discussed thus far in this section are summarized in Table 1.1.

TABLE 1.1. Summary of Details Pertinent to the Cartesian, Cylindrical, and Spherical Coordinate Systems

	Cartesian	*Cylindrical*	*Spherical*
Orthogonal Surfaces	three planes	a cylinder and two planes	a sphere, a cone, and a plane
Geometry	Fig. 1.12	Fig. 1.13	Fig. 1.14
Coordinates	x, y, z	r, ϕ, z	r, θ, ϕ
Unit Vectors	$\mathbf{i}_x, \mathbf{i}_y, \mathbf{i}_z$	$\mathbf{i}_r, \mathbf{i}_\phi, \mathbf{i}_z$	$\mathbf{i}_r, \mathbf{i}_\theta, \mathbf{i}_\phi$
Limits of Coordinates	$-\infty < x < \infty$ $-\infty < y < \infty$ $-\infty < z < \infty$	$0 < r < \infty$ $0 < \phi < 2\pi$ $-\infty < z < \infty$	$0 < r < \infty$ $0 < \theta < \pi$ $0 < \phi < 2\pi$
Differential Length Elements	$dx\,\mathbf{i}_x$, $dy\,\mathbf{i}_y$, $dz\,\mathbf{i}_z$	$dr\,\mathbf{i}_r$, $r\,d\phi\,\mathbf{i}_\phi$, $dz\,\mathbf{i}_z$	$dr\,\mathbf{i}_r$, $r\,d\theta\,\mathbf{i}_\theta$, $r\sin\theta\,d\phi\,\mathbf{i}_\phi$
Differential Areas	$dx\,dy\,\mathbf{i}_z$ $dy\,dz\,\mathbf{i}_x$ $dz\,dx\,\mathbf{i}_y$	$r\,dr\,d\phi\,\mathbf{i}_z$ $r\,d\phi\,dz\,\mathbf{i}_r$ $dr\,dz\,\mathbf{i}_\phi$	$r\,dr\,d\theta\,\mathbf{i}_\phi$ $r^2\sin\theta\,d\theta\,d\phi\,\mathbf{i}_r$ $r\sin\theta\,dr\,d\phi\,\mathbf{i}_\theta$
Differential Volume	$dx\,dy\,dz$	$r\,dr\,d\phi\,dz$	$r^2\sin\theta\,dr\,d\theta\,d\phi$

Since any particular point in space can be defined by its coordinates in any one of the three coordinate systems, it is possible to derive relationships between the different sets of coordinates from simple considerations of geometry.

EXAMPLE 1-4. Express the cylindrical coordinates of a point in terms of its spherical coordinates.

From the construction of Fig. 1.15, the distance of point P (r_s, θ, ϕ) from the z axis is $r_s \sin\theta$. This is the radius of the cylinder passing through P and having the z axis as its axis. The height of point P above the xy plane is $r_s \cos\theta$. This is the value of z on the constant z plane passing through P. Thus the cylindrical coordinates (r_c, ϕ, z) of point P are

$$r_c = r_s \sin\theta$$

$$\phi = \phi$$

$$z = r_s \cos\theta$$

The various relationships between the different sets of coordinates obtained in this manner are summarized in Table 1.2. ▮

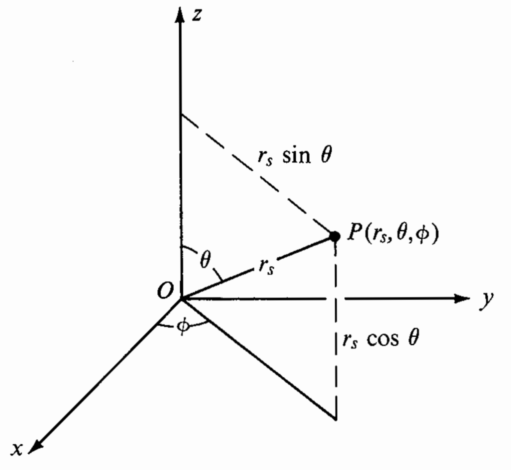


Fig. 1. 15. Conversion from spherical coordinates to cylindrical coordinates.

TABLE 1.2. Relationships Between Different Sets of Coordinates

	Cartesian x, y, z	*Cylindrical* r, ϕ, z	*Spherical* r, θ, ϕ
Cartesian x, y, z		$x = r\cos\phi$ $y = r\sin\phi$ $z = z$	$x = r\sin\theta\cos\phi$ $y = r\sin\theta\sin\phi$ $z = r\cos\theta$
Cylindrical r, ϕ, z	$r = \sqrt{x^2 + y^2}$ $\phi = \tan^{-1}\dfrac{y}{x}$ $z = z$		$r_c = r_s \sin\theta$ $\phi = \phi$ $z = r_s\cos\theta$
Spherical r, θ, ϕ	$r = \sqrt{x^2 + y^2 + z^2}$ $\theta = \tan^{-1}\dfrac{\sqrt{x^2+y^2}}{z}$ $\phi = \tan^{-1}\dfrac{y}{x}$	$r_s = \sqrt{r_c^2 + z^2}$ $\theta = \tan^{-1}\dfrac{r_c}{z}$ $\phi = \phi$	

1.3 Components of Vectors

Once we set up a coordinate system and define unit vectors pertinent to that coordinate system, we can express vectors at any point in terms of their components along the unit vectors at that point and perform vector operations using the components. Let $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ be a set of mutually perpendicular vectors at a point P such that $\mathbf{i}_1 \times \mathbf{i}_2 = \mathbf{i}_3$, so that they can represent any one of the sets of unit vectors $(\mathbf{i}_x, \mathbf{i}_y, \mathbf{i}_z)$, $(\mathbf{i}_r, \mathbf{i}_\phi, \mathbf{i}_z)$, and $(\mathbf{i}_r, \mathbf{i}_\theta, \mathbf{i}_\phi)$ in the three different coordinate systems. Let **A**, **B**, and **C** be three vectors at the point

P. Then we have, from Example 1-3,

$$\begin{aligned}\mathbf{A} &= (\mathbf{A}\cdot\mathbf{i}_1)\mathbf{i}_1 + (\mathbf{A}\cdot\mathbf{i}_2)\mathbf{i}_2 + (\mathbf{A}\cdot\mathbf{i}_3)\mathbf{i}_3 \\ &= A_1\mathbf{i}_1 + A_2\mathbf{i}_2 + A_3\mathbf{i}_3\end{aligned} \tag{1-28}$$

where A_1, A_2, and A_3 are equal to $(\mathbf{A}\cdot\mathbf{i}_1)$, $(\mathbf{A}\cdot\mathbf{i}_2)$, and $(\mathbf{A}\cdot\mathbf{i}_3)$, respectively; that is, A_1, A_2, and A_3 are the components of $\mathbf{A}$ along $\mathbf{i}_1$, $\mathbf{i}_2$, and $\mathbf{i}_3$, respectively. Similarly,

$$\mathbf{B} = B_1\mathbf{i}_1 + B_2\mathbf{i}_2 + B_3\mathbf{i}_3 \tag{1-29a}$$

$$\mathbf{C} = C_1\mathbf{i}_1 + C_2\mathbf{i}_2 + C_3\mathbf{i}_3 \tag{1-29b}$$

Now, we can perform the vector operations discussed in Section 1.1 as follows:

(a) *Equality of vectors:* Two vectors $\mathbf{A}$ and $\mathbf{B}$ are equal if and only if their respective components are equal; that is,

$$B_i = A_i, \qquad i = 1, 2, 3 \tag{1-30}$$

(b) *Magnitude of a vector:*

$$|\mathbf{A}| = A = \sqrt{A_1^2 + A_2^2 + A_3^2} \tag{1-31}$$

(c) *Addition and subtraction of vectors:*

$$\mathbf{A} + \mathbf{B} = (A_1 + B_1)\mathbf{i}_1 + (A_2 + B_2)\mathbf{i}_2 + (A_3 + B_3)\mathbf{i}_3 \tag{1-32a}$$

$$\mathbf{B} - \mathbf{C} = (B_1 - C_1)\mathbf{i}_1 + (B_2 - C_2)\mathbf{i}_2 + (B_3 - C_3)\mathbf{i}_3 \tag{1-32b}$$

(d) *Multiplication and division by a scalar:*

$$m(\mathbf{A}) = mA_1\mathbf{i}_1 + mA_2\mathbf{i}_2 + mA_3\mathbf{i}_3 \tag{1-33a}$$

$$\frac{1}{m}(\mathbf{B}) = \frac{B_1}{m}\mathbf{i}_1 + \frac{B_2}{m}\mathbf{i}_2 + \frac{B_3}{m}\mathbf{i}_3 \tag{1-33b}$$

(e) *Unit vector:* The unit vector along the direction of a vector $\mathbf{A}$ is given by

$$\mathbf{i}_A = \frac{A_1\mathbf{i}_1 + A_2\mathbf{i}_2 + A_3\mathbf{i}_3}{\sqrt{A_1^2 + A_2^2 + A_3^2}} \tag{1-34}$$

(f) *Scalar or dot product of two vectors:*

$$\begin{aligned}\mathbf{A}\cdot\mathbf{B} &= (A_1\mathbf{i}_1 + A_2\mathbf{i}_2 + A_3\mathbf{i}_3)\cdot(B_1\mathbf{i}_1 + B_2\mathbf{i}_2 + B_3\mathbf{i}_3) \\ &= A_1B_1 + A_2B_2 + A_3B_3\end{aligned} \tag{1-35}$$

(g) *Vector or cross product of two vectors:*

$$\begin{aligned}\mathbf{A}\times\mathbf{B} &= (A_1\mathbf{i}_1 + A_2\mathbf{i}_2 + A_3\mathbf{i}_3)\times(B_1\mathbf{i}_1 + B_2\mathbf{i}_2 + B_3\mathbf{i}_3) \\ &= A_1B_2\mathbf{i}_3 - A_1B_3\mathbf{i}_2 - A_2B_1\mathbf{i}_3 + A_2B_3\mathbf{i}_1 + A_3B_1\mathbf{i}_2 - A_3B_2\mathbf{i}_1 \\ &= (A_2B_3 - A_3B_2)\mathbf{i}_1 + (A_3B_1 - A_1B_3)\mathbf{i}_2 + (A_1B_2 - A_2B_1)\mathbf{i}_3 \\ &= \begin{vmatrix}\mathbf{i}_1 & \mathbf{i}_2 & \mathbf{i}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3\end{vmatrix}\end{aligned} \tag{1-36}$$

(h) *Scalar triple product:*

$$\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = (A_1\mathbf{i}_1 + A_2\mathbf{i}_2 + A_3\mathbf{i}_3) \cdot \begin{vmatrix} \mathbf{i}_1 & \mathbf{i}_2 & \mathbf{i}_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} \tag{1-37}$$

EXAMPLE 1-5. Find the dot and cross products of the unit vector $\mathbf{i}_{rc}$ at the point $P(r_c, \phi_c, z)$ and the unit vector $\mathbf{i}_\theta$ at the point $Q(r_s, \theta, \phi_s)$.

Since the unit vectors $\mathbf{i}_x$, $\mathbf{i}_y$, and $\mathbf{i}_z$ are uniform everywhere, we express $\mathbf{i}_{rc}$ at P and $\mathbf{i}_\theta$ at Q in terms of their components along $\mathbf{i}_x$, $\mathbf{i}_y$, and $\mathbf{i}_z$ and then perform the dot- and cross-product operations.

From the construction of Fig. 1.16 we have

$$\mathbf{i}_{rc} = \cos\phi_c\,\mathbf{i}_x + \sin\phi_c\,\mathbf{i}_y$$
$$\mathbf{i}_\theta = \cos\theta\cos\phi_s\,\mathbf{i}_x + \cos\theta\sin\phi_s\,\mathbf{i}_y - \sin\theta\,\mathbf{i}_z$$

Using (1-35) and (1-36) and simplifying, we then obtain

$$\mathbf{i}_{rc} \cdot \mathbf{i}_\theta = \cos\theta\cos(\phi_s - \phi_c)$$
$$\mathbf{i}_{rc} \times \mathbf{i}_\theta = \sin\theta\,\mathbf{i}_{\phi c} + \cos\theta\sin(\phi_s - \phi_c)\,\mathbf{i}_z$$

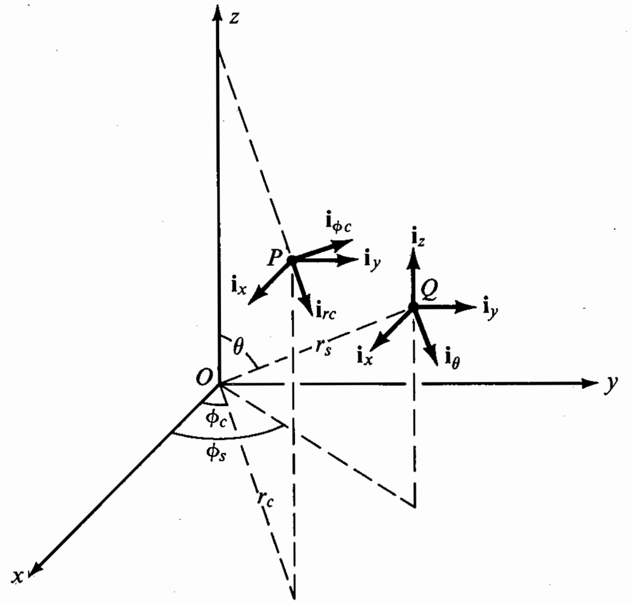


Fig. 1. 16. To find the dot product of $\mathbf{i}_{rc}$ at $P(r_c, \phi_c, z)$ and $\mathbf{i}_\theta$ at $Q(r_s, \theta, \phi_s)$.

If P and Q are the same points (r, θ, ϕ), then these results reduce to

$$\mathbf{i}_{rc} \cdot \mathbf{i}_\theta = \cos\theta$$
$$\mathbf{i}_{rc} \times \mathbf{i}_\theta = \sin\theta\, \mathbf{i}_\phi$$

Dot products and cross products between different unit vectors at the same point (r, θ, ϕ) are listed in Tables 1.3 and 1.4, respectively. ▮

TABLE 1.3. Dot Products of Unit Vectors at a Point (r, θ, ϕ)

	$\mathbf{i}_x$	$\mathbf{i}_y$	$\mathbf{i}_z$	$\mathbf{i}_{rc}$	$\mathbf{i}_\phi$	$\mathbf{i}_{rs}$	$\mathbf{i}_\theta$
$\mathbf{i}_x \cdot$	1	0	0	$\cos\phi$	$-\sin\phi$	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$
$\mathbf{i}_y \cdot$		1	0	$\sin\phi$	$\cos\phi$	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$
$\mathbf{i}_z \cdot$			1	0	0	$\cos\theta$	$-\sin\theta$
$\mathbf{i}_{rc} \cdot$				1	0	$\sin\theta$	$\cos\theta$
$\mathbf{i}_\phi \cdot$					1	0	0
$\mathbf{i}_{rs} \cdot$						1	0
$\mathbf{i}_\theta \cdot$							1

Table 1.4. Cross Products of Unit Vectors at a Point (r, θ, ϕ)

	$\mathbf{i}_x$	$\mathbf{i}_y$	$\mathbf{i}_z$	$\mathbf{i}_{rc}$	$\mathbf{i}_\phi$	$\mathbf{i}_{rs}$	$\mathbf{i}_\theta$
$\mathbf{i}_x \times$	0	$\mathbf{i}_z$	$-\mathbf{i}_y$	$\sin\phi\,\mathbf{i}_z$	$\cos\phi\,\mathbf{i}_z$	$\sin\theta\sin\phi\,\mathbf{i}_z - \cos\theta\,\mathbf{i}_y$	$\cos\theta\sin\phi\,\mathbf{i}_z + \sin\theta\,\mathbf{i}_y$
$\mathbf{i}_y \times$		0	$\mathbf{i}_x$	$-\cos\phi\,\mathbf{i}_z$	$\sin\phi\,\mathbf{i}_z$	$-\sin\theta\cos\phi\,\mathbf{i}_z + \cos\theta\,\mathbf{i}_x$	$-\cos\theta\cos\phi\,\mathbf{i}_z - \sin\theta\,\mathbf{i}_x$
$\mathbf{i}_z \times$			0	$\mathbf{i}_\phi$	$-\mathbf{i}_{rc}$	$\sin\theta\,\mathbf{i}_\phi$	$\cos\theta\,\mathbf{i}_\phi$
$\mathbf{i}_{rc} \times$				0	$\mathbf{i}_z$	$-\cos\theta\,\mathbf{i}_\phi$	$\sin\theta\,\mathbf{i}_\phi$
$\mathbf{i}_\phi \times$					0	$-\sin\theta\,\mathbf{i}_z + \cos\theta\,\mathbf{i}_{rc}$	$-\cos\theta\,\mathbf{i}_z - \sin\theta\,\mathbf{i}_{rc}$
$\mathbf{i}_{rs} \times$						0	$\mathbf{i}_\phi$
$\mathbf{i}_\theta \times$							0

Since any vector drawn at a point can be expressed in terms of its components along any one of the three sets of unit vectors, it is possible to derive relationships between the components of a vector in one coordinate system and the components of the same vector in another coordinate system.

EXAMPLE 1-6. Express the component A_θ of a vector $\mathbf{A}$ in terms of its components A_x, A_y, and A_z.

$$\begin{aligned} A_\theta &= \mathbf{A} \cdot \mathbf{i}_\theta \\ &= (A_x\mathbf{i}_x + A_y\mathbf{i}_y + A_z\mathbf{i}_z) \cdot (\cos\theta\cos\phi\,\mathbf{i}_x + \cos\theta\sin\phi\,\mathbf{i}_y - \sin\theta\,\mathbf{i}_z) \\ &= A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta \end{aligned}$$

The various relationships derived in this manner between different components of a vector are summarized in Table 1.5. ▮

EXAMPLE 1-7. Show that $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{B} \cdot \mathbf{C})\mathbf{A}$.

First, we note that the vector $\mathbf{A} \times \mathbf{B}$ is perpendicular to both vectors $\mathbf{A}$ and $\mathbf{B}$ and hence is normal to the plane containing $\mathbf{A}$ and $\mathbf{B}$. But the vector

TABLE 1.5. Relationships Between Components of a Vector in Different Coordinate Systems

	Cartesian A_x, A_y, A_z	*Cylindrical* A_r, A_ϕ, A_z	*Spherical* A_r, A_θ, A_ϕ
Cartesian A_x, A_y, A_z		$A_x = \dfrac{A_r x - A_\phi y}{\sqrt{x^2 + y^2}}$	$A_x = \dfrac{A_r x\sqrt{x^2 + y^2} + A_\theta xz - A_\phi y\sqrt{x^2 + y^2 + z^2}}{\sqrt{(x^2 + y^2)(x^2 + y^2 + z^2)}}$
		$A_y = \dfrac{A_r y + A_\phi x}{\sqrt{x^2 + y^2}}$	$A_y = \dfrac{A_r y\sqrt{x^2 + y^2} + A_\theta yz + A_\phi x\sqrt{x^2 + y^2 + z^2}}{\sqrt{(x^2 + y^2)(x^2 + y^2 + z^2)}}$
		$A_z = A_z$	$A_z = \dfrac{A_r z - A_\theta\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 + z^2}}$
Cylindrical A_r, A_ϕ, A_z	$A_r = A_x \cos\phi + A_y \sin\phi$		$A_{rc} = \dfrac{A_{rs} r_c + A_\theta z}{\sqrt{r_c^2 + z^2}}$
	$A_\phi = -A_x \sin\phi + A_y \cos\phi$		$A_\phi = A_\phi$
	$A_z = A_z$		$A_z = \dfrac{A_{rs} z - A_\theta r_c}{\sqrt{r_c^2 + z^2}}$
Spherical A_r, A_θ, A_ϕ	$A_r = A_x \sin\theta\cos\phi + A_y \sin\theta\sin\phi + A_z\cos\theta$	$A_{rs} = A_{rc}\sin\theta + A_z\cos\theta$	
	$A_\theta = A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta$	$A_\theta = A_{rc}\cos\theta - A_z\sin\theta$	
	$A_\phi = -A_x\sin\phi + A_y\cos\phi$	$A_\phi = A_\phi$	

$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$ is perpendicular to the vector $(\mathbf{A} \times \mathbf{B})$ as well as to the vector $\mathbf{C}$. Hence $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$ lies in the plane containing $\mathbf{A}$ and $\mathbf{B}$. In view of this, $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$ can be written as a superposition of two vectors proportional to $\mathbf{A}$ and $\mathbf{B}$; that is,

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = m\mathbf{A} + n\mathbf{B}$$

To find m and n, we expand $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$. Thus

$$\begin{aligned}
(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} &= \begin{vmatrix} \mathbf{i}_1 & \mathbf{i}_2 & \mathbf{i}_3 \\ (A_2B_3 - A_3B_2) & (A_3B_1 - A_1B_3) & (A_1B_2 - A_2B_1) \\ C_1 & C_2 & C_3 \end{vmatrix} \\
&= (A_2C_2B_1 + A_3C_3B_1 - B_2C_2A_1 - B_3C_3A_1)\mathbf{i}_1 \\
&\quad + (A_1C_1B_2 + A_3C_3B_2 - B_1C_1A_2 - B_3C_3A_2)\mathbf{i}_2 \\
&\quad + (A_1C_1B_3 + A_2C_2B_3 - B_1C_1A_3 - B_2C_2A_3)\mathbf{i}_3 \\
&= (A_1C_1B_1 + A_2C_2B_1 + A_3C_3B_1 - B_1C_1A_1 \\
&\qquad - B_2C_2A_1 - B_3C_3A_1)\mathbf{i}_1 \\
&\quad + (A_1C_1B_2 + A_2C_2B_2 + A_3C_3B_2 - B_1C_1A_2 \\
&\qquad - B_2C_2A_2 - B_3C_3A_2)\mathbf{i}_2 \\
&\quad + (A_1C_1B_3 + A_2C_2B_3 + A_3C_3B_3 - B_1C_1A_3 \\
&\qquad - B_2C_2A_3 - B_3C_3A_3)\mathbf{i}_3 \\
&= (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{B} \cdot \mathbf{C})\mathbf{A}
\end{aligned}$$

Similarly, it can be shown that

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C} \quad \blacksquare$$

EXAMPLE 1-8. Given

$$\begin{aligned}
\mathbf{A} &= 2\mathbf{i}_x - \mathbf{i}_z \\
\mathbf{B} &= 2\mathbf{i}_x - \mathbf{i}_y + 2\mathbf{i}_z \\
\mathbf{C} &= 2\mathbf{i}_x - 3\mathbf{i}_y + \mathbf{i}_z
\end{aligned}$$

We wish to perform several operations with these vectors as follows:

(a) $\mathbf{A} + \mathbf{B} = (2\mathbf{i}_x - \mathbf{i}_z) + (2\mathbf{i}_x - \mathbf{i}_y + 2\mathbf{i}_z) = 4\mathbf{i}_x - \mathbf{i}_y + \mathbf{i}_z$

(b) $\mathbf{B} - \mathbf{C} = (2\mathbf{i}_x - \mathbf{i}_y + 2\mathbf{i}_z) - (2\mathbf{i}_x - 3\mathbf{i}_y + \mathbf{i}_z) = 2\mathbf{i}_y + \mathbf{i}_z$

(c) $\mathbf{A} + \mathbf{B} - \mathbf{C} = \mathbf{A} + (\mathbf{B} - \mathbf{C}) = (2\mathbf{i}_x - \mathbf{i}_z) + (2\mathbf{i}_y + \mathbf{i}_z) = 2\mathbf{i}_x + 2\mathbf{i}_y$

(d) $|\mathbf{B}| = \sqrt{2^2 + (-1)^2 + 2^2} = 3$

(e) $\mathbf{i}_{\mathbf{B}} = \dfrac{\mathbf{B}}{|\mathbf{B}|} = \dfrac{2\mathbf{i}_x - \mathbf{i}_y + 2\mathbf{i}_z}{3} = \dfrac{2}{3}\mathbf{i}_x - \dfrac{1}{3}\mathbf{i}_y + \dfrac{2}{3}\mathbf{i}_z$

(f) $\mathbf{A} \cdot \mathbf{B} = (2\mathbf{i}_x - \mathbf{i}_z) \cdot (2\mathbf{i}_x - \mathbf{i}_y + 2\mathbf{i}_z) = 4 + 0 - 2 = 2$

(g) Cosine of the angle between $\mathbf{A}$ and $\mathbf{B} = \dfrac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}||\mathbf{B}|} = \dfrac{2}{3\sqrt{5}}$

(h) $\mathbf{B} \times \mathbf{C} = \begin{vmatrix} \mathbf{i}_x & \mathbf{i}_y & \mathbf{i}_z \\ 2 & -1 & 2 \\ 2 & -3 & 1 \end{vmatrix} = 5\mathbf{i}_x + 2\mathbf{i}_y - 4\mathbf{i}_z$

(i) Sine of the angle between $\mathbf{B}$ and $\mathbf{C} = \dfrac{|\mathbf{B} \times \mathbf{C}|}{|\mathbf{B}||\mathbf{C}|} = \dfrac{\sqrt{45}}{3\sqrt{14}} = \sqrt{\dfrac{5}{14}}$

(j) $\mathbf{B} \cdot \mathbf{C} \times \mathbf{A} = \begin{vmatrix} 2 & -1 & 2 \\ 2 & -3 & 1 \\ 2 & 0 & -1 \end{vmatrix} = 14$

(k) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} \mathbf{i}_x & \mathbf{i}_y & \mathbf{i}_z \\ 2 & 0 & -1 \\ 5 & 2 & -4 \end{vmatrix} = 2\mathbf{i}_x + 3\mathbf{i}_y + 4\mathbf{i}_z$

(l) Components of $\mathbf{B}$ in spherical coordinates at $(1, \pi/2, \pi)$

$$\begin{aligned} B_r &= B_x \sin\theta \cos\phi + B_y \sin\theta \sin\phi + B_z \cos\theta \\ &= 2 \sin\frac{\pi}{2} \cos\pi - 1 \sin\frac{\pi}{2} \sin\pi + 2\cos\frac{\pi}{2} = -2 \\ B_\theta &= B_x \cos\theta \cos\phi + B_y \cos\theta \sin\phi - B_z \sin\theta \\ &= 2\cos\frac{\pi}{2}\cos\pi - 1\cos\frac{\pi}{2}\cos\pi - 2\sin\frac{\pi}{2} = -2 \\ B_\phi &= -B_x \sin\phi + B_y \cos\phi = -2\sin\pi - 1\cos\pi = 1 \end{aligned}$$

(m) By using a vector product, find any vector perpendicular to $\mathbf{B}$. We can consider the unit vector $\mathbf{i}_x$ for simplicity. Then

$$\mathbf{D} = \mathbf{B} \times \mathbf{i}_x = \begin{vmatrix} \mathbf{i}_x & \mathbf{i}_y & \mathbf{i}_z \\ 2 & -1 & 2 \\ 1 & 0 & 0 \end{vmatrix} = 2\mathbf{i}_y + \mathbf{i}_z$$

We can verify that $\mathbf{D}$ is indeed perpendicular to $\mathbf{B}$ by showing that

$$\mathbf{B} \cdot \mathbf{D} = (2\mathbf{i}_x - \mathbf{i}_y + 2\mathbf{i}_z) \cdot (2\mathbf{i}_y + \mathbf{i}_z) = 0 - 2 + 2 = 0 \quad \blacksquare$$

1.4 Scalar and Vector Fields

A mathematical function or a graphical sketch constructed so as to describe the variation of a quantity in a given region is said to represent the "field" of that quantity associated with that region. We distinguish between scalar and vector fields, depending upon whether the quantity of interest is a scalar or a vector. We will first discuss scalar fields or functions. A simple example of scalar function is one by means of which we attempt to describe how the depth d of water in a lake varies from point to point on the lake surface. Assuming the lake surface to be plane, we first set up a two-dimensional coordinate system to define each and every point on the surface by a set of coordinates (x, y) with respect to a chosen origin. To each set of coordinates (x, y) we assign a number for d, which represents the depth of water beneath the point defined by that set of coordinates. The coordinates (x, y) are the independent variables and the depth d is the dependent variable. The function $d(x, y)$ represents the depth field associated with points on the surface of the lake.

If we join points in the xy plane for which the depth is equal to a particular constant, we obtain a curve known as a constant-depth contour. Similarly, by joining the points which have associated with them the same depth value but different from the previous constant, we obtain a different constant-depth contour. In this manner we can draw several constant-depth contours with convenient increments ranging from zero depth to the greatest depth, as shown in Fig. 1.17. The constant-depth contours provide a graphical representation of the depth field $d(x, y)$.

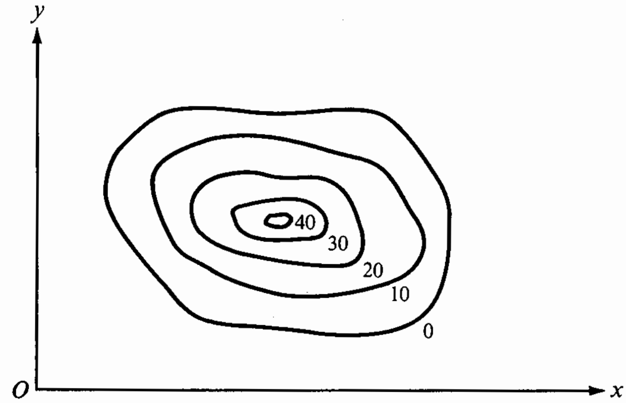


Fig. 1.17. Sketch of a two-dimensional scalar field $d(x, y)$ showing contours of constant values of d.

To add one more dimension to the scalar field, let us consider the temperature field associated with points inside a room. We can set up a coordinate system to define the location of each and every point inside the room with respect to a chosen origin. However, we will need all three coordinates (x, y, z) in this case instead of just two coordinates as in the previous example. To each set of coordinates (x, y, z) we assign a number which represents the temperature T at the point defined by that set of (x, y, z). The coordinates (x, y, z) are the independent variables and the temperature T is the dependent variable. The function $T(x, y, z)$ represents the temperature field. If we join points in the coordinate system for which the temperature is equal to a particular constant, we obtain a surface which is known as a constant-temperature or isothermal surface. Similarly, by joining the points which have associated with them the same temperature value but different from the previous constant, we obtain a different isothermal surface. In this manner we describe the temperature field in the room by a set of isothermal surfaces.

The addition of time t as an independent variable introduces one more dimension to the problem. The temperature at each and every point in the room varies with time in general so that the discussion in the preceding paragraph is valid only for fixed times or for the special case in which the temperature does not vary with time. In the latter case the temperature field in the room is said to be "static." In the general case, however, the temperature distributions measured throughout the room at two times t_1 and t_2 can be different so that the shapes of the isothermal surfaces representing the same constant temperatures at the two times can be different. Mathematically, we need two different functions of (x, y, z) to describe the temperature fields at these two times. To generalize this statement, since t is a continuous independent variable, T is a function of four independent variables x, y, z, and t. Thus we describe the time-varying temperature field in the room by a function $T(x, y, z, t)$.

The same concepts can be used to describe vector fields. However, in the case of vector quantities, we need to describe not only how the magnitude of the vector varies as a function of the independent variables but also how the direction of the vector varies. Hence, if we wish to describe the variation of a vector as a function of position in three-dimensional space and also of time, we associate a set of two numbers to each possible combination (x, y, z, t) or (r, ϕ, z, t) or (r, θ, ϕ, t), depending upon the coordinate system used, where one of the two numbers represents the magnitude and the other, the direction of the vector. More conveniently, since the variation of the unit vectors in each coordinate system is completely known (the unit vectors are independent of time), it is sufficient if we describe how each component of the vector of interest varies with (x, y, z, t) or (r, ϕ, z, t) or (r, θ, ϕ, t). Thus we have reduced the problem of describing a vector field to one of describing the component scalar fields. Mathematically, we write

$$\mathbf{F}(x, y, z, t) = F_x(x, y, z, t)\mathbf{i}_x + F_y(x, y, z, t)\mathbf{i}_y + F_z(x, y, z, t)\mathbf{i}_z \tag{1-38}$$

$$\mathbf{F}(r, \phi, z, t) = F_r(r, \phi, z, t)\mathbf{i}_r + F_\phi(r, \phi, z, t)\mathbf{i}_\phi + F_z(r, \phi, z, t)\mathbf{i}_z \tag{1-39}$$

$$\mathbf{F}(r, \theta, \phi, t) = F_r(r, \theta, \phi, t)\mathbf{i}_r + F_\theta(r, \theta, \phi, t)\mathbf{i}_\theta + F_\phi(r, \theta, \phi, t)\mathbf{i}_\phi \tag{1-40}$$

where $\mathbf{F}$ is the vector of interest and remembering that the unit vectors $\mathbf{i}_r$ and $\mathbf{i}_\phi$ in cylindrical coordinates and $\mathbf{i}_r$, $\mathbf{i}_\theta$, and $\mathbf{i}_\phi$ in spherical coordinates are themselves known functions of the coordinates.

EXAMPLE 1-9. Consider a circular disk of radius a rotating with a constant angular velocity ω about an axis passing normally through its center. It is desired to describe the linear velocity vector field associated with the points on the disk.

We can choose the center of the disk as the origin and set up a two-dimensional coordinate system. We have a choice of the coordinates (x, y) or the coordinates (r, ϕ). Since the linear velocity of a point is equal to the product of the angular velocity and the distance from the axis about which the disk rotates, we note that points equidistant from the center have the same magnitude of velocity. Also, the velocity is directed everywhere in the angular direction. This suggests the use of (r, ϕ) coordinate system. Then, at a point (r, ϕ) the velocity magnitude is $r\omega$ and its direction is $\mathbf{i}_\phi$, as shown in Fig. 1.18(a). Thus the expression for the linear velocity vector field is given as

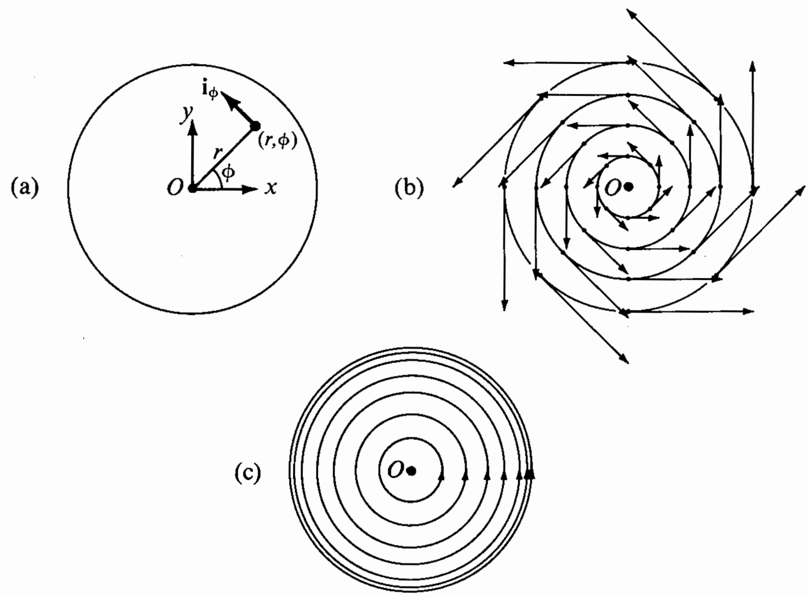


Fig. 1.18. (a) Rotating disk. (b) Field of the linear velocity vector associated with the points on the rotating disk. (c) Same as (b) with the arrows omitted and the density of direction lines used to indicate the magnitude variation.

$$\mathbf{v}(r, \phi) = v_r(r, \phi)\mathbf{i}_r + v_\phi(r, \phi)\mathbf{i}_\phi = r\omega\mathbf{i}_\phi \qquad \text{for } r < a \qquad (1\text{-}41)$$

The constant-magnitude contours are circles centered at the origin and having radii proportional to the magnitudes. The velocity direction is everywhere tangential to these circles. One way of pictorially representing the vector field is by drawing at several points vectors whose lengths are equal to the radii of the circles passing through those points and hence proportional to the velocity magnitudes at those points and whose directions are everywhere along $\mathbf{i}_\phi$, as shown in Fig. 1.18(b). For this field, these vectors are everywhere tangential to the constant-magnitude contours (circles) passing through those points; that is, the constant-magnitude contours are also the curves along which points on the disk move as the disk rotates. Such curves are known as direction lines since they indicate the direction of the vector field. The constant-magnitude contours and direction lines are not the same curves for a general vector field. The pictorial representation of Fig. 1.18(b) can be simplified by omitting the vectors and simply placing arrowheads along the circles, that is, the direction lines, as shown in Fig. 1.18(c). Also, by decreasing the spacing between the direction lines as r increases, the density of direction lines is used to indicate the magnitude variation. This is the common procedure adapted for graphically depicting a vector field. In Chapter 2 we will discuss a procedure for obtaining the equations for the direction lines from the field expressions. ▮

1.5 Differentiation of Vectors

In calculus, we have learned the rules for differentiation of scalar functions. If f is a function of x, then the derivative of f with respect to x is

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{1-42}$$

If f is a function of (x, y, z), then the partial derivative of f with respect to x is

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x} \tag{1-43}$$

and the differential increase in f from a point (x, y, z) to a neighboring point $(x + dx, y + dy, z + dz)$ is

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \tag{1-44}$$

where $\partial f/\partial y$ and $\partial f/\partial z$ are given by expressions similar to (1-43).

Differentiation of vector functions is defined in the same manner as differentiation of scalar functions. Let us consider a vector function $\mathbf{A}(x, y, z)$. The differential increment in $\mathbf{A}$ from a point (x, y, z) to a neighboring point $(x + dx, y + dy, z + dz)$ is

$$d\mathbf{A} = \frac{\partial \mathbf{A}}{\partial x} dx + \frac{\partial \mathbf{A}}{\partial y} dy + \frac{\partial \mathbf{A}}{\partial z} dz \tag{1-45}$$

where

$$\frac{\partial \mathbf{A}}{\partial x} = \lim_{\Delta x \to 0} \frac{\mathbf{A}(x + \Delta x, y, z) - \mathbf{A}(x, y, z)}{\Delta x} \tag{1-46}$$

$$\frac{\partial \mathbf{A}}{\partial y} = \lim_{\Delta y \to 0} \frac{\mathbf{A}(x, y + \Delta y, z) - \mathbf{A}(x, y, z)}{\Delta y} \tag{1-47}$$

$$\frac{\partial \mathbf{A}}{\partial z} = \lim_{\Delta z \to 0} \frac{\mathbf{A}(x, y, z + \Delta z) - \mathbf{A}(x, y, z)}{\Delta z} \tag{1-48}$$

Since $[\mathbf{A}(x + \Delta x, y, z) - \mathbf{A}(x, y, z)]$ is a vector, the derivative $\partial \mathbf{A}/\partial x$ is a vector which, in general, is oriented in a direction different from that of $\mathbf{A}$. Similarly, $\partial\mathbf{A}/\partial y$ and $\partial\mathbf{A}/\partial z$ are vectors which, in general, are oriented in directions different from that of $\mathbf{A}$. If we express a vector function in terms of its component vector functions in cartesian coordinates, that is, if

$$\mathbf{A}(x, y, z) = A_x(x, y, z)\mathbf{i}_x + A_y(x, y, z)\mathbf{i}_y + A_z(x, y, z)\mathbf{i}_z \tag{1-49}$$

then

$$d\mathbf{A} = dA_x\,\mathbf{i}_x + dA_y\,\mathbf{i}_y + dA_z\,\mathbf{i}_z \tag{1-50}$$

since $\mathbf{i}_x$, $\mathbf{i}_y$ and $\mathbf{i}_z$ are independent of x, y and z.

EXAMPLE 1-10. The unit vector $\mathbf{i}_r$ in cylindrical coordinates is independent of r and z but not ϕ. Hence $\partial\mathbf{i}_r/\partial r = d\mathbf{i}_r/\partial z = 0$ but $\partial\mathbf{i}_r/\partial\phi \neq 0$. We wish to find $\partial\mathbf{i}_r/\partial\phi$ in two ways: (a) from first principles, and (b) by using (1-50).

(a) By definition,

$$\frac{\partial\mathbf{i}_r}{\partial\phi} = \lim_{\Delta\phi\to 0} \frac{\mathbf{i}_r(r, \phi + \Delta\phi, z) - \mathbf{i}_r(r, \phi, z)}{\Delta\phi} \tag{1-51}$$

To deduce the right side of (1-51), consider the unit vectors $\mathbf{i}_{r1}$ and $\mathbf{i}_{r2}$ at the points $P(r, \phi, z)$ and $Q(r, \phi + \Delta\phi, z)$, as shown in Fig. 1.19. Then we can write

$$\mathbf{i}_{r2} = (\mathbf{i}_{r2} \cdot \mathbf{i}_{r1})\mathbf{i}_{r1} + (\mathbf{i}_{r2} \cdot \mathbf{i}_\phi)\mathbf{i}_\phi = \cos\Delta\phi\,\mathbf{i}_{r1} + \sin\Delta\phi\,\mathbf{i}_\phi \tag{1-52}$$

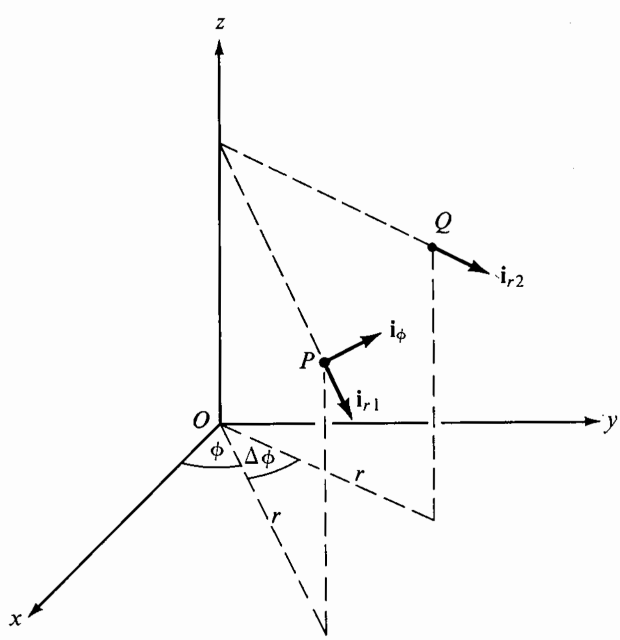


Fig. 1.19. For the evaluation of $\partial\mathbf{i}_r/\partial\phi$.

where $\mathbf{i}_\phi$ is the unit vector in the ϕ direction at point P. Using (1-52), we have

$$\begin{aligned}\mathbf{i}_r(r, \phi + \Delta\phi, z) - \mathbf{i}_r(r, \phi, z) &= \mathbf{i}_{r2} - \mathbf{i}_{r1} \\ &= (\cos\Delta\phi - 1)\mathbf{i}_{r1} + \sin\Delta\phi\,\mathbf{i}_\phi \end{aligned} \tag{1-53}$$

Substituting (1-53) into (1-51), we get

$$\frac{\partial \mathbf{i}_r}{\partial \phi} = \lim_{\Delta\phi \to 0} \frac{(\cos\Delta\phi - 1)\mathbf{i}_{r1} + \sin\Delta\phi\,\mathbf{i}_\phi}{\Delta\phi} = \mathbf{i}_\phi \tag{1-54}$$

(b) To use (1-50), we first note that since $\mathbf{i}_r$ is only a function of ϕ, $\partial \mathbf{i}_r/\partial\phi$ is the same as $d\mathbf{i}_r/d\phi$. Expressing $\mathbf{i}_r$ in terms of $\mathbf{i}_x$, $\mathbf{i}_y$, and $\mathbf{i}_z$, we have

$$\mathbf{i}_r = \cos\phi\,\mathbf{i}_x + \sin\phi\,\mathbf{i}_y \tag{1-55}$$

Then, from (1-50), we obtain

$$\begin{aligned} d\mathbf{i}_r &= d(\cos\phi)\,\mathbf{i}_x + d(\sin\phi)\,\mathbf{i}_y \\ &= (-\sin\phi\,\mathbf{i}_x + \cos\phi\,\mathbf{i}_y)d\phi = d\phi\,\mathbf{i}_\phi \end{aligned} \tag{1-56}$$

or

$$\frac{\partial \mathbf{i}_r}{\partial \phi} = \frac{d\mathbf{i}_r}{d\phi} = \mathbf{i}_\phi$$

which agrees with (1-54). Partial derivatives of unit vectors obtained in this manner are listed in Table 1.6. ▮

TABLE 1.6. Partial Derivatives of Unit Vectors; All Derivatives Not Listed in the Table are Zero

	∂x	∂y	∂z	∂r_c	$\partial \phi$	$\partial \theta$
$\partial \mathbf{i}_{rc}/$	$\frac{-\sin\phi}{r_c}\mathbf{i}_\phi$	$\frac{\cos\phi}{r_c}\mathbf{i}_\phi$	0	0	$\mathbf{i}_\phi$	0
$\partial \mathbf{i}_\phi/$	$\frac{\sin\phi}{r_c}\mathbf{i}_{rc}$	$\frac{-\cos\phi}{r_c}\mathbf{i}_{rc}$	0	0	$-\mathbf{i}_{rc}$	0
$\partial \mathbf{i}_{rs}/$	$\frac{1}{r_s}(-\sin\phi\,\mathbf{i}_\phi + \cos\theta\cos\phi\,\mathbf{i}_\theta)$	$\frac{1}{r_s}(\cos\phi\,\mathbf{i}_\phi + \cos\theta\sin\phi\,\mathbf{i}_\theta)$	$\frac{-\sin\theta}{r_s}\mathbf{i}_\theta$	$\frac{\cos\theta}{r_s}\mathbf{i}_\theta$	$\sin\theta\,\mathbf{i}_\phi$	$\mathbf{i}_\theta$
$\partial \mathbf{i}_\theta/$	$\frac{\cot\theta}{r_s}(-\sin\phi\,\mathbf{i}_\phi - \sin\theta\cos\phi\,\mathbf{i}_{rs})$	$\frac{\cot\theta}{r_s}(\cos\phi\,\mathbf{i}_\phi - \sin\theta\sin\phi\,\mathbf{i}_{rs})$	$\frac{\sin\theta}{r_s}\mathbf{i}_{rs}$	$\frac{-\cos\theta}{r_s}\mathbf{i}_{rs}$	$\cos\theta\,\mathbf{i}_\phi$	$-\mathbf{i}_{rs}$

Expressions similar to (1-50) are not true in cylindrical and spherical coordinates; that is,

$$d\mathbf{A} \neq dA_r\,\mathbf{i}_r + dA_\phi\,\mathbf{i}_\phi + dA_z\,\mathbf{i}_z \tag{1-57}$$

$$d\mathbf{A} \neq dA_r\,\mathbf{i}_r + dA_\theta\,\mathbf{i}_\theta + dA_\phi\,\mathbf{i}_\phi \tag{1-58}$$

To derive the correct expressions for these two coordinate systems, we make use of the differentiation rule,

$$d(f\mathbf{A}) = f\,d\mathbf{A} + df\,\mathbf{A} \tag{1-59}$$

where f is a scalar function. Thus if

$$\mathbf{A} = A_r\mathbf{i}_r + A_\phi\mathbf{i}_\phi + A_z\mathbf{i}_z$$

we have

$$\begin{aligned} d\mathbf{A} &= d(A_r\mathbf{i}_r) + d(A_\phi\mathbf{i}_\phi) + d(A_z\mathbf{i}_z) \\ &= A_r\,d\mathbf{i}_r + dA_r\,\mathbf{i}_r + A_\phi\,d\mathbf{i}_\phi + dA_\phi\,\mathbf{i}_\phi + dA_z\,\mathbf{i}_z \end{aligned} \tag{1-60}$$

Similarly, if

$$\mathbf{A} = A_r\mathbf{i}_r + A_\theta\mathbf{i}_\theta + A_\phi\mathbf{i}_\phi$$

we have

$$d\mathbf{A} = A_r\,d\mathbf{i}_r + dA_r\,\mathbf{i}_r + A_\theta\,d\mathbf{i}_\theta + dA_\theta\,\mathbf{i}_\theta + A_\phi\,d\mathbf{i}_\phi + dA_\phi\,\mathbf{i}_\phi \tag{1-61}$$

Finally, if $\mathbf{A}$ is also a function of t in addition to x, y, z, we have

$$d\mathbf{A} = dA_x\,\mathbf{i}_x + dA_y\,\mathbf{i}_y + dA_z\,\mathbf{i}_z \tag{1-50}$$

where

$$dA_i = \frac{\partial A_i}{\partial x}\,dx + \frac{\partial A_i}{\partial y}\,dy + \frac{\partial A_i}{\partial z}\,dz + \frac{\partial A_i}{\partial t}\,dt \qquad i = x, y, z \tag{1-62}$$

Rules for the differentiation of dot and cross products of vectors are as follows:

$$d(\mathbf{A} \cdot \mathbf{B}) = d\mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot d\mathbf{B} \tag{1-63}$$

$$d(\mathbf{A} \times \mathbf{B}) = d\mathbf{A} \times \mathbf{B} + \mathbf{A} \times d\mathbf{B} \tag{1-64}$$

1.6 The Gradient

Gradient is an operation performed on a scalar function which results in a vector function. The magnitude of this vector function at any point in the region of the scalar field is the maximum rate of increase of the scalar function at that point. The direction of the vector function at that point is the direction in which this maximum rate of increase occurs. To illustrate this concept mathematically, let us consider a scalar function $V(x, y, z)$ which is single-valued everywhere so that it is differentiable. From calculus, we can express the change in V from a point (x, y, z) to another point $(x + dx, y + dy, z + dz)$ an infinitesimal distance away from it as

$$\begin{aligned} dV &= \frac{\partial V}{\partial x}\,dx + \frac{\partial V}{\partial y}\,dy + \frac{\partial V}{\partial z}\,dz \\ &= \left(\frac{\partial V}{\partial x}\,\mathbf{i}_x + \frac{\partial V}{\partial y}\,\mathbf{i}_y + \frac{\partial V}{\partial z}\,\mathbf{i}_z\right) \cdot (dx\,\mathbf{i}_x + dy\,\mathbf{i}_y + dz\,\mathbf{i}_z) \\ &= \boldsymbol{\nabla} V \cdot d\mathbf{l} \end{aligned} \tag{1-65}$$

where the symbol $\boldsymbol{\nabla}$ stands for "del" and is a vector operator defined as

$$\boldsymbol{\nabla} = \frac{\partial}{\partial x}\,\mathbf{i}_x + \frac{\partial}{\partial y}\,\mathbf{i}_y + \frac{\partial}{\partial z}\,\mathbf{i}_z \tag{1-66}$$

When "del" operates on a scalar function, the operation is known as evaluating the gradient of the scalar function; that is, $\boldsymbol{\nabla} V$ is the gradient of V. Thus

$$\nabla V = \frac{\partial V}{\partial x}\mathbf{i}_x + \frac{\partial V}{\partial y}\mathbf{i}_y + \frac{\partial V}{\partial z}\mathbf{i}_z \tag{1-67}$$

The vector $d\mathbf{l}$ is the infinitesimal displacement vector drawn from the point (x, y, z) to the point $(x + dx, y + dy, z + dz)$.

To discuss the physical significance of ∇V, let us consider a surface containing a point $P(x_0, y_0, z_0)$ on which the scalar function is constant and equal to $V(x_0, y_0, z_0) = V_0$. If we now consider another point $Q(x_0 + dx, y_0 + dy, z_0 + dz)$ on the same surface and an infinitesimal distance away from P, dV between these two points is zero since V remains constant throughout this surface. It follows from (1-65) that for the vector $d\mathbf{l}$ drawn from P to Q on this surface,

$$\nabla V \cdot d\mathbf{l} = 0 \tag{1-68}$$

and hence ∇V is perpendicular to $d\mathbf{l}$. But since (1-68) is true for all points Q on the constant V surface surrounding P, ∇V must be normal to all possible infinitesimal displacement vectors drawn away from P on the constant V surface, and hence it is normal to that surface. Actually, it is sufficient for ∇V to be normal to any two different infinitesimal displacement vectors drawn away from P on the constant V surface to conclude that ∇V is normal to that surface. Thus we can reach the general conclusion that the gradient of a scalar function at any point is directed normal to the surface passing through that point and on which the value of the scalar function is a constant. Designating $\mathbf{i}_n$ as the unit vector normal to the constant V surface, we then have

$$\nabla V = |\nabla V|\mathbf{i}_n \tag{1-69}$$

Let us now consider two adjacent surfaces of constant V equal to V_0 and $V_0 + dV$, respectively, as shown in Fig. 1.20. Let P and Q be points on the V_0 and $V_0 + dV$ surfaces, respectively, and let $d\mathbf{l}$ be the displacement vector drawn from P to Q. Then, since dV is infinitesimally small and hence the two surfaces are infinitesimally close to each other, we have, according to (1-65),

$$dV = (\nabla V)_{\text{at } P} \cdot d\mathbf{l} = |\nabla V|\mathbf{i}_n \cdot dl\,\mathbf{i}_l \tag{1-70}$$

where we have substituted the right side of (1-69) for $(\nabla V)_{\text{at } P}$ and expressed $d\mathbf{l}$ as $dl\,\mathbf{i}_l$. From (1-70), we have

$$\left(\frac{dV}{dl}\right)_{\text{at } P} = |\nabla V|\mathbf{i}_n \cdot \mathbf{i}_l = |\nabla V|\cos\alpha \tag{1-71}$$

where α is the angle between $\mathbf{i}_n$ and $\mathbf{i}_l$. But since the maximum value of $\cos\alpha$ is unity, the maximum value of $(dV/dl)_{\text{at } P}$ is equal to $|\nabla V|$ and it occurs for α equal to zero, that is, for the case in which $\mathbf{i}_l = \mathbf{i}_n$. Thus $|\nabla V|$ is indeed the maximum rate of increase of V and it occurs in the direction normal to the constant V surface, consistent with the conclusion of the previous paragraph.

EXAMPLE 1-11. Consider the scalar function $V(x, y, z) = xy$. Obtain a unit vector normal to the constant V surface of value 2 at the point $(2, 1, 0)$ in two ways:

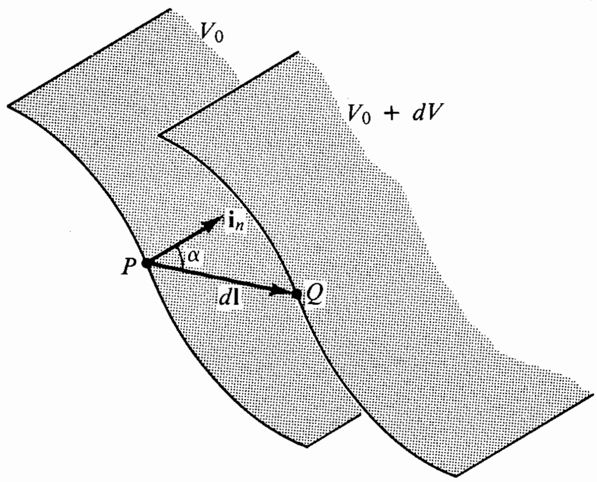


Fig. 1.20. Surfaces of constant V equal to V_0 and $V_0 + dV$, respectively, for evaluating ∇V.

(a) by using the cross product of two vectors which are tangential to the surface at that point; (b) by using the concept of the gradient of a scalar function. What is the maximum rate of increase of the scalar function at the point (2, 1, 0)?

(a) The constant V surface of value 2 is given by

$$xy = 2$$

A cut section of this surface is shown in Fig. 1.21(a) and its cross section in the xy plane is repeated in Fig. 1.21(b). The unit vector $\mathbf{i}_z$ is tangential to the surface everywhere. Hence it is sufficient if we find another vector tangential to the surface at (2, 1, 0) so that we can take the cross product of these two vectors to find a unit vector normal to the surface. For simplicity, we can find the tangential unit vector $\mathbf{i}_t$ lying in the xy plane. To find the components of $\mathbf{i}_t$ along $\mathbf{i}_x$ and $\mathbf{i}_y$, we need the angle α which the tangent to the curve $xy = 2$ in Fig. 1.21(b) makes with the x axis. Noting that the curve is defined by $y = 2/x$, we obtain

$$\left(\frac{dy}{dx}\right)_{2,1} = \left(-\frac{2}{x^2}\right)_{2,1} = -\frac{1}{2}$$

Hence $\tan \alpha = \frac{1}{2}$ or $\alpha = 26.6°$. Now,

$$\mathbf{i}_t = \cos \alpha \, \mathbf{i}_x - \sin \alpha \, \mathbf{i}_y = \frac{2}{\sqrt{5}}\mathbf{i}_x - \frac{1}{\sqrt{5}}\mathbf{i}_y$$

The unit vector normal to the surface $xy = 2$ at (2, 1, 0) is then given by

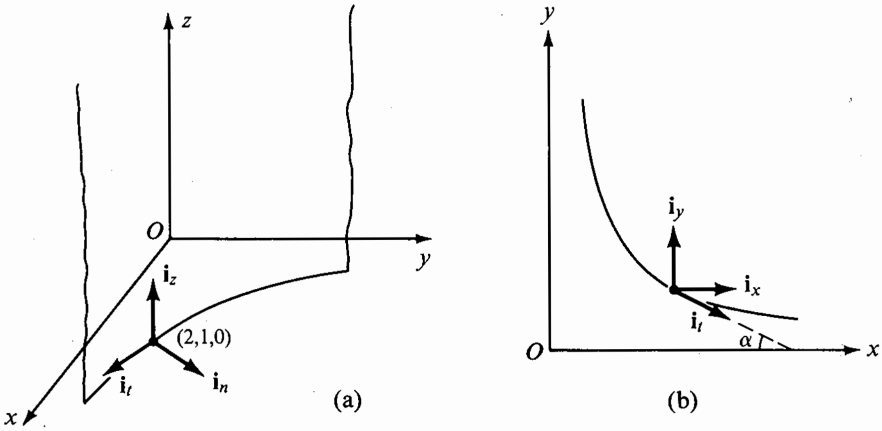


Fig. 1.21. For Example 1-11.

$$\mathbf{i}_n = \mathbf{i}_z \times \mathbf{i}_t = \begin{vmatrix} \mathbf{i}_x & \mathbf{i}_y & \mathbf{i}_z \\ 0 & 0 & 1 \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \end{vmatrix} = \frac{1}{\sqrt{5}}\mathbf{i}_x + \frac{2}{\sqrt{5}}\mathbf{i}_y$$

(b) The direction of the gradient of a scalar is normal to the surface of constant value of the scalar. Hence by evaluating the gradient of the given scalar function at the point (2, 1, 0) we can find the required unit vector.

$$\nabla V = \frac{\partial(xy)}{\partial x}\mathbf{i}_x + \frac{\partial(xy)}{\partial y}\mathbf{i}_y = y\mathbf{i}_x + x\mathbf{i}_y$$

$$(\nabla V)_{2,1,0} = \mathbf{i}_x + 2\mathbf{i}_y$$

This vector is normal to the surface $xy = 2$. To find the unit vector we divide it by its magnitude which is $\sqrt{5}$. Thus

$$\mathbf{i}_n = \frac{1}{\sqrt{5}}\mathbf{i}_x + \frac{2}{\sqrt{5}}\mathbf{i}_y$$

which agrees with the result of part (a). The maximum rate of increase of the scalar function at (2, 1, 0) is the magnitude of the gradient. Hence it is equal to $\sqrt{5}$. ▮

Equation (1-67) gives the gradient of a scalar in cartesian coordinates. We can similarly consider cylindrical coordinates and write the following steps:

$$\begin{aligned} dV &= \frac{\partial V}{\partial r}\,dr + \frac{\partial V}{\partial \phi}\,d\phi + \frac{\partial V}{\partial z}\,dz \\ &= \left(\frac{\partial V}{\partial r}\mathbf{i}_r + \frac{\partial V}{\partial \phi}\mathbf{i}_\phi + \frac{\partial V}{\partial z}\mathbf{i}_z\right) \cdot (dr\,\mathbf{i}_r + d\phi\,\mathbf{i}_\phi + dz\,\mathbf{i}_z) \end{aligned} \tag{1-72}$$

But, in cylindrical coordinates,

$$d\mathbf{l} = dr\,\mathbf{i}_r + r\,d\phi\,\mathbf{i}_\phi + dz\,\mathbf{i}_z \tag{1-24a}$$

Hence we have to modify (1-72) as

$$\begin{aligned} dV &= \frac{\partial V}{\partial r}dr + \frac{1}{r}\frac{\partial V}{\partial \phi}r\,d\phi + \frac{\partial V}{\partial z}dz \\ &= \left(\frac{\partial V}{\partial r}\mathbf{i}_r + \frac{1}{r}\frac{\partial V}{\partial \phi}\mathbf{i}_\phi + \frac{\partial V}{\partial z}\mathbf{i}_z\right)\cdot(dr\,\mathbf{i}_r + r\,d\phi\,\mathbf{i}_\phi + dz\,\mathbf{i}_z) \qquad (1\text{-}73) \\ &= \nabla V \cdot d\mathbf{l} \end{aligned}$$

Thus, in cylindrical coordinates,

$$\nabla V = \frac{\partial V}{\partial r}\mathbf{i}_r + \frac{1}{r}\frac{\partial V}{\partial \phi}\mathbf{i}_\phi + \frac{\partial V}{\partial z}\mathbf{i}_z \tag{1-74}$$

Similarly, in spherical coordinates,

$$\nabla V = \frac{\partial V}{\partial r}\mathbf{i}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{i}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{i}_\phi \tag{1-75}$$

EXAMPLE 1-12. Find the rate at which the scalar function $V = r^2 \sin 2\phi$, in cylindrical coordinates, increases in the direction of the vector $\mathbf{A} = \mathbf{i}_r + \mathbf{i}_\phi$ at the point $(2, \pi/4, 0)$.

Evidently, the required quantity is $\nabla V \cdot \mathbf{A}/|\mathbf{A}|$ evaluated at $(2, \pi/4, 0)$.

$$\begin{aligned} \nabla V &= \frac{\partial}{\partial r}(r^2 \sin 2\phi)\,\mathbf{i}_r + \frac{1}{r}\frac{\partial}{\partial \phi}(r^2 \sin 2\phi)\,\mathbf{i}_\phi + \frac{\partial}{\partial z}(r^2 \sin 2\phi)\,\mathbf{i}_z \\ &= 2r \sin 2\varphi\,\mathbf{i}_r + 2r\cos 2\phi\,\mathbf{i}_\phi \\ \nabla V \cdot \mathbf{A} &= (2r\sin 2\phi\,\mathbf{i}_r + 2r\cos 2\phi\,\mathbf{i}_\phi)\cdot(\mathbf{i}_r + \mathbf{i}_\phi) \\ &= 2r\sin 2\phi + 2r\cos 2\phi \\ \frac{\nabla V \cdot \mathbf{A}}{|\mathbf{A}|} &= \sqrt{2}\,r\sin 2\phi + \sqrt{2}\,r\cos 2\phi \end{aligned}$$

Finally, the rate of increase of V along $\mathbf{A}$ at the point $(2, \pi/4, 0)$ is equal to $2\sqrt{2}$. ▮

1.7 Volume, Surface, and Line Integrals

In the study of electromagnetic fields, we repeatedly encounter three types of integrals: (a) the volume integral, (b) the surface integral, and (c) the line integral. We will discuss each of these separately and provide some examples for evaluating them.

a. The Volume Integral.

If the density of a quantity is specified throughout a certain volume, we make use of volume integral to evaluate the amount of that quantity in that volume. For example, let us assume that the density of mass ρ of a body is known as a function of the coordinates (x, y, z), (r, ϕ, z), or (r, θ, ϕ). To obtain the total mass contained in the volume occupied by the body, we divide the volume into a number of infinitesimal volumes $dv_1, dv_2, dv_3, \ldots$. Within each infinitesimal volume, the density may be considered to be constant so that the mass contained within each volume is given by the product of the infinitesimal volume and the density in that volume. The total mass m is then the sum of these several masses, that is,

$$m = \rho_1\, dv_1 + \rho_2\, dv_2 + \rho_3\, dv_3 + \cdots = \sum_i \rho_i\, dv_i \tag{1-76}$$

where ρ_i is the density associated with the volume dv_i. Equation (1-76) gives only the approximate value of m since the density within each infinitesimal volume is not quite constant. However, it becomes exact in the limit that dv_i tends to zero (i.e., shrinks to a point) in which case the summation becomes an integral

$$m = \int_V \rho\, dv \tag{1-77}$$

where the integration is performed throughout the volume V of the body, as indicated by the letter V associated with the integral sign. The integral on the right side of (1-77) is known as a volume integral. The volume integral is a triple integral since dv is the product of three differential lengths.

EXAMPLE 1-13. The density of mass of a spherical body of radius a centered at the origin is given by

$$\rho(r, \theta, \phi) = \frac{\rho_0}{r}$$

where ρ_0 is a constant. It is desired to find the mass m of the spherical body.

The differential volume dv in spherical coordinates is $r^2 \sin\theta\, dr\, d\theta\, d\phi$. Substituting for ρ and dv in (1-77) and introducing the limits for the three variables r, θ, and ϕ, we have

$$\begin{aligned} m &= \int_{r=0}^{a} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{\rho_0}{r} r^2 \sin\theta\, dr\, d\theta\, d\phi \\ &= 2\pi\rho_0 a^2 \quad \blacksquare \end{aligned}$$

b. The Surface Integral.

If the density of flow of a fluid or, in general, the flux density of any physical quantity is specified over a certain surface, we make use of surface integrals to evaluate the amount of the flux of that quantity crossing that surface. For example, let us assume that the density of current at all points on a particular surface S is known. Since current is due to the flow of charges, the current density at a point has magnitude and direction and hence is a vector. Let us denote the current density vector as $\mathbf{J}$. To obtain the current crossing the surface S, we first divide the surface into several infinitesimal areas of magnitudes $dS_1, dS_2, dS_3, \ldots$. Since each of these areas is very,

very small in magnitude, we can treat them as plane surfaces and define their orientations by their corresponding normal vectors $\mathbf{i}_{n1}, \mathbf{i}_{n2}, \mathbf{i}_{n3}, \ldots$. Furthermore, we can consider the current density vector associated with each area to be constant.

Let us then consider one infinitesimal area $dS_1\, \mathbf{i}_{n1}$ and its associated current density vector $\mathbf{J}_1$ as shown in Fig. 1.22. Let the angle between $\mathbf{i}_{n1}$

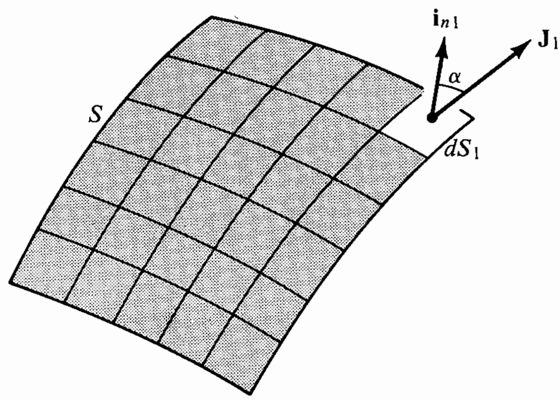


Fig. 1.22. Division of a surface S into several infinitesimal areas to evaluate the flux of a vector $\mathbf{J}$ crossing the surface.

and $\mathbf{J}_1$ be α. Then the projection of the area dS_1 onto a plane normal to the current density vector $\mathbf{J}_1$ has an area $dS_1 \cos \alpha$. The current crossing this projected area and hence the current crossing the surface $dS_1\, \mathbf{i}_{n1}$ is equal to $|\mathbf{J}_1|\, dS_1 \cos \alpha$, or $\mathbf{J}_1 \cdot dS_1\, \mathbf{i}_{n1}$. Similarly, we can obtain the currents flowing through all the other infinitesimal surfaces and add them up to give the total current I as

$$\begin{aligned} I &= \mathbf{J}_1 \cdot dS_1\, \mathbf{i}_{n1} + \mathbf{J}_2 \cdot dS_2\, \mathbf{i}_{n2} + \mathbf{J}_3 \cdot dS_3\, \mathbf{i}_{n3} + \cdots \\ &= \sum_k \mathbf{J}_k \cdot dS_k\, \mathbf{i}_{nk} = \sum_k \mathbf{J}_k \cdot d\mathbf{S}_k \end{aligned} \tag{1-78}$$

where $d\mathbf{S}_k = dS\, \mathbf{i}_{nk}$. Equation (1-78) is approximate since the assumption of constant current density vector for any infinitesimal surface is true only in the limit that the area of that surface tends to zero (i.e., shrinks to a point). In this limit, the summation in (1-78) becomes an integral, giving us

$$I = \int_S \mathbf{J} \cdot d\mathbf{S} = \int_S \mathbf{J} \cdot \mathbf{i}_n\, dS \tag{1-79}$$

where the integration is performed over the entire surface S. The integral on the right side of (1-79) is known as a surface integral. The surface integral is a double integral since dS is the product of two differential lengths. If the surface is closed, we call it a closed surface integral and write it with a small circle associated with the integral sign, as follows:

$$I = \oint_S \mathbf{J} \cdot d\mathbf{S} \tag{1-80}$$

Also, the normal vectors to the differential areas comprising the closed surface are then usually chosen to be pointing away from the volume bounding that surface so that the closed surface integral represents the flux emanating from the volume.

EXAMPLE 1-14. In a certain region, the current density vector is given by

$$\mathbf{J} = 3x\mathbf{i}_x + (y - 3)\mathbf{i}_y + (2 + z)\mathbf{i}_z \quad \text{amp/m}^2$$

Find the total current flowing out of the surface of the box bounded by the five planes $x = 0$, $y = 0$, $y = 2$, $z = 0$, and $3x + z = 3$.

With reference to Fig. 1.23, we will consider the normal vector to be always pointing out of the box so that $\int \mathbf{J} \cdot d\mathbf{S}$ gives the current flowing out of the surface.

For the surface $x = 0$, $d\mathbf{S} = -dy\,dz\,\mathbf{i}_x$, $\mathbf{J} = (y - 3)\mathbf{i}_y + (2 + z)\mathbf{i}_z$.

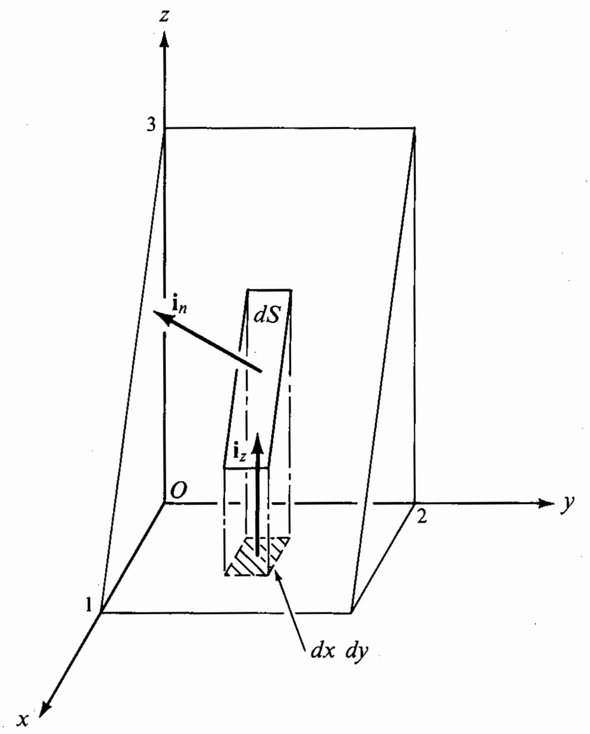


Fig. 1.23. For Example 1-14.

$$\mathbf{J} \cdot d\mathbf{S} = 0$$

$$\int \mathbf{J} \cdot d\mathbf{S} = 0$$

For the surface $y = 0$, $d\mathbf{S} = -dz\,dx\,\mathbf{i}_y$, $\mathbf{J} = 3x\mathbf{i}_x - 3\mathbf{i}_z + (2 + z)\mathbf{i}_z$.

$$\mathbf{J} \cdot d\mathbf{S} = 3\,dz\,dx$$

$$\int \mathbf{J} \cdot d\mathbf{S} = \int_{x=0}^{1} \int_{z=0}^{3-3x} 3\,dz\,dx = \frac{9}{2}$$

For the surface $y = 2$, $d\mathbf{S} = dz\,dx\,\mathbf{i}_y$, $\mathbf{J} = 3x\mathbf{i}_x - \mathbf{i}_y + (2 + z)\mathbf{i}_z$.

$$\mathbf{J} \cdot d\mathbf{S} = -1\,dz\,dx$$

$$\int \mathbf{J} \cdot d\mathbf{S} = \int_{x=0}^{1} \int_{z=0}^{3-3x} (-1)\,dz\,dx = -\frac{3}{2}$$

For the surface $z = 0$, $d\mathbf{S} = -dx\,dy\,\mathbf{i}_z$, $\mathbf{J} = 3x\mathbf{i}_x + (y - 3)\mathbf{i}_y + 2\mathbf{i}_z$.

$$\mathbf{J} \cdot d\mathbf{S} = -2\,dx\,dy$$

$$\int \mathbf{J} \cdot d\mathbf{S} = \int_{x=0}^{1} \int_{y=0}^{2} (-2)\,dx\,dy = -4$$

For the surface $3x + z = 3$,

$$\mathbf{i}_n = \frac{\nabla(3x + z)}{|\nabla(3x + z)|} = \frac{3\mathbf{i}_x + \mathbf{i}_z}{\sqrt{3^2 + 1^2}} = \frac{3}{\sqrt{10}}\mathbf{i}_x + \frac{1}{\sqrt{10}}\mathbf{i}_z$$

From $dS\,\mathbf{i}_n \cdot \mathbf{i}_z = dx\,dy$, we have

$$dS = \frac{dx\,dy}{\mathbf{i}_n \cdot \mathbf{i}_z} = \sqrt{10}\,dx\,dy$$

$$d\mathbf{S} = (3\mathbf{i}_x + \mathbf{i}_z)\,dx\,dy$$

$$\mathbf{J} = 3x\mathbf{i}_x + (y - 3)\mathbf{i}_y + (5 - 3x)\mathbf{i}_z$$

$$\mathbf{J} \cdot d\mathbf{S} = (9x + 5 - 3x)\,dx\,dy = (6x + 5)\,dx\,dy$$

$$\int \mathbf{J} \cdot d\mathbf{S} = \int_{x=0}^{1} \int_{y=0}^{2} (6x + 5)\,dx\,dy = 16$$

Finally, adding the values of $\int \mathbf{J} \cdot d\mathbf{S}$ for the five surfaces, we obtain the total current flowing out of the box to be $0 + \frac{9}{2} - \frac{3}{2} - 4 + 16 = 15$ amp. ▮

c. Line Integral.

The line integral consists of evaluating along a specified path the integral of the dot product of a vector and the differential displacement vector tangential to the path. For example, let us consider a path from point a to point b, as shown in Fig. 1.24 in a region of a known force vector field $\mathbf{F}$. To find the total work done by the force from a to b, we divide the path from a to

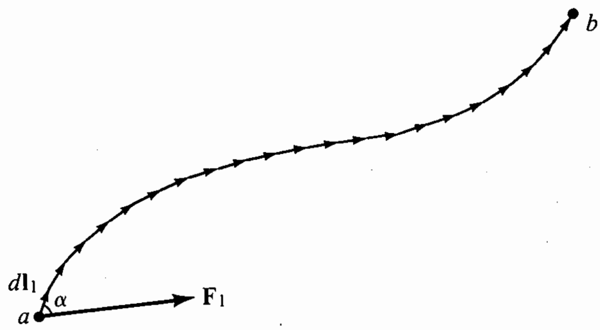


Fig. 1.24. Division of a curve into several infinitesimal segments to evaluate the work done by the force vector **F** along the curve.

b into a number of segments of infinitesimal lengths $dl_1, dl_2, dl_3, \ldots$. Since the length of each segment is very, very small, we can treat these segments as straight lines and define their orientations by the corresponding differential vectors $d\mathbf{l}_1, d\mathbf{l}_2, d\mathbf{l}_3, \ldots$. Within each segment, we can consider the force vector to be constant.

Let us then consider one infinitesimal segment $d\mathbf{l}_1$ and its associated force vector $\mathbf{F}_1$. Let the angle between $d\mathbf{l}_1$ and $\mathbf{F}_1$ be α. The component of the force $\mathbf{F}_1$ along the direction of $d\mathbf{l}_1$ is equal to $F_1 \cos\alpha$. Hence the work done by $\mathbf{F}_1$ along $d\mathbf{l}_1$ is equal to $(F_1 \cos\alpha)(dl_1)$, or $\mathbf{F}_1 \cdot d\mathbf{l}_1$. Similarly, we can obtain the work done by the forces for all the other infinitesimal segments and add them up to give the total work W done from a to b as

$$W = \mathbf{F}_1 \cdot d\mathbf{l}_1 + \mathbf{F}_2 \cdot d\mathbf{l}_2 + \mathbf{F}_3 \cdot d\mathbf{l}_3 + \cdots = \sum_i \mathbf{F}_i \cdot d\mathbf{l}_i \qquad (1\text{-}81)$$

Equation (1-81) is approximate since the assumption of constant force vector for any infinitesimal segment is true only in the limit that the length of that segment tends to zero (i.e., shrinks to a point). In this limit, the summation in (1-81) becomes an integral, giving us

$$W = \int_a^b \mathbf{F} \cdot d\mathbf{l} \qquad (1\text{-}82)$$

where the integration is performed along the path from a to b. The integral on the right side of (1-82) is known as a line integral. For the case of the force vector, it represents the work done by the force field. For other vectors it will have different meanings. When the line integral is evaluated around a closed path C, it is known as the "circulation" around that path and we write it with a small circle associated with the integral sign, as follows:

$$W = \oint_C \mathbf{F} \cdot d\mathbf{l} \qquad (1\text{-}83)$$

EXAMPLE 1-15. Find the work done by the force vector

$$\mathbf{F} = y\mathbf{i}_x - x\mathbf{i}_y$$

around the closed path *abcdefga* shown in Fig. 1.25.

The work done by the force vector is $\oint_{abcdefga} \mathbf{F} \cdot d\mathbf{l}$. This integral consists of seven parts which will be evaluated independently. First, we note that

$$\mathbf{F} \cdot d\mathbf{l} = (y\mathbf{i}_x - x\mathbf{i}_y) \cdot (dx\,\mathbf{i}_x + dy\,\mathbf{i}_y) = y\,dx - x\,dy$$

Along path *ab*, $y = x^2$, $dy = 2x\,dx$, $\mathbf{F} \cdot d\mathbf{l} = -x^2\,dx$.

$$\int_a^b \mathbf{F} \cdot d\mathbf{l} = -\int_{x=0}^{-1} x^2\,dx = \frac{1}{3}$$

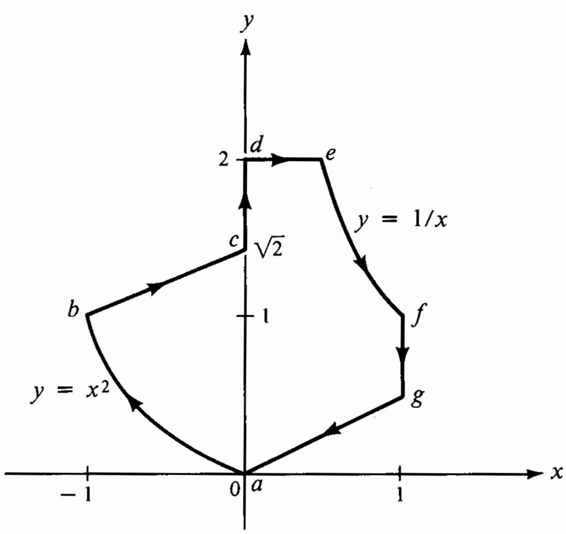


Fig. 1.25. For Example 1-15.

Along path *bc*, $y = (\sqrt{2} - 1)x + \sqrt{2}$, $dy = (\sqrt{2} - 1)\,dx$

$$\mathbf{F} \cdot d\mathbf{l} = \sqrt{2}\,dx.$$

$$\int_b^c \mathbf{F} \cdot d\mathbf{l} = \int_{x=-1}^{0} \sqrt{2}\,dx = \sqrt{2}$$

Along path *cd*, $x = 0$, $dx = 0$, $\mathbf{F} \cdot d\mathbf{l} = 0$.

$$\int_c^d \mathbf{F} \cdot d\mathbf{l} = 0$$

Along path *de*, $y = 2$, $dy = 0$, $\mathbf{F} \cdot d\mathbf{l} = 2\,dx$.

$$\int_d^e \mathbf{F} \cdot d\mathbf{l} = \int_{x=0}^{1/2} 2\,dx = 1$$

Along path *ef*, $y = 1/x$, $dy = -(1/x^2)\,dx$, $\mathbf{F} \cdot d\mathbf{l} = (2/x)dx$.

$$\int_e^f \mathbf{F} \cdot d\mathbf{l} = \int_{x=1/2}^{1} \frac{2}{x} dx = 2 \ln 2$$

Along path *fg*, $x = 1$, $dx = 0$, $\mathbf{F} \cdot d\mathbf{l} = -dy$.

$$\int_f^g \mathbf{F} \cdot d\mathbf{l} = -\int_{y=1}^{1/2} dy = \frac{1}{2}$$

Along path *ga*, $x = 2y$, $dx = 2\,dy$, $\mathbf{F} \cdot d\mathbf{l} = 0$.

$$\int_g^a \mathbf{F} \cdot d\mathbf{l} = 0$$

Finally, adding the values of $\int \mathbf{F} \cdot d\mathbf{l}$ for the seven paths, we obtain the total work done to be $\frac{1}{3} + \sqrt{2} + 0 + 1 + 2 \ln 2 + \frac{1}{2} + 0 \approx 4.634$. The fact that the integrals along the paths *cd* and *ga* are zero is obvious if we note that $\mathbf{F} = y\mathbf{i}_x - x\mathbf{i}_y = -r\mathbf{i}_\phi$. Thus the force vector is everywhere tangential to the circle with the center at the origin and, since *cd* and *ga* are radial to the origin, $\mathbf{F} \cdot d\mathbf{l} \equiv 0$ for these paths. Hence $\int \mathbf{F} \cdot d\mathbf{l}$ is zero for the paths *cd* and *ga*. ▮

Integration of vectors is performed by expressing the integrand in terms of its components in cartesian coordinates, thereby reducing the problem to one of evaluating three scalar integrals. Thus, for example,

$$\begin{aligned}\int \mathbf{A}\, dm &= \int (A_x\mathbf{i}_x + A_y\mathbf{i}_y + A_z\mathbf{i}_z)\, dm \\ &= \left(\int A_x\, dm\right)\mathbf{i}_x + \left(\int A_y\, dm\right)\mathbf{i}_y + \left(\int A_z\, dm\right)\mathbf{i}_z \qquad \text{(1-84)}\end{aligned}$$

where *dm* stands for *dv*, *dS*, or *dl*, depending upon whether the integration is over a volume, surface, or along a line, respectively. Similar expressions using the components in cylindrical and spherical coordinate systems are not correct since some or all of the unit vectors in these coordinate systems are functions of the coordinates. For example, the magnitude of the sum of two component vectors along the unit vector $\mathbf{i}_r$ at two different points is not, in general, the sum of the magnitudes of the two vectors since the two components are directed in different directions. For that matter, the direction of the sum of the two component vectors is not the direction of either of the component vectors. Thus

$$\int \mathbf{A}\, dm \neq \left(\int A_r\, dm\right)\mathbf{i}_r + \left(\int A_\phi\, dm\right)\mathbf{i}_\phi + \left(\int A_z\, dm\right)\mathbf{i}_z \qquad \text{(1-85a)}$$

$$\int \mathbf{A}\, dm \neq \left(\int A_r\, dm\right)\mathbf{i}_r + \left(\int A_\theta\, dm\right)\mathbf{i}_\theta + \left(\int A_\phi\, dm\right)\mathbf{i}_\phi \qquad \text{(1-85b)}$$

The integrand must, in general, be expressed as the sum of its component vectors along $\mathbf{i}_x$, $\mathbf{i}_y$, and $\mathbf{i}_z$ for correct results.

1.8 Divergence and the Divergence Theorem

In Section 1.7 we introduced the concept of the surface integral. Let us consider a closed surface S enclosing a volume V in a region in which the current density vector $\mathbf{J}$ is specified. Then the amount of current emanating from this volume is given by

$$I = \oint_S \mathbf{J} \cdot d\mathbf{S} \tag{1-86}$$

where the integration is performed over the closed surface S. If we let this volume shrink to an infinitesimal value Δv, we obtain an infinitesimal amount of current flowing out of the surface ΔS bounding Δv. In the limit that we let the volume shrink to a point, the current emanating from the point may tend to zero. On the other hand, since the volume occupied by the point is zero, the ratio of the current emanating from the point to the volume occupied by the point can be nonzero; that is, although the quantity $\oint_{\Delta S} \mathbf{J} \cdot d\mathbf{S}$ may tend to zero in the limit $\Delta v \longrightarrow 0$, the quantity

$$\frac{\oint_{\Delta S} \mathbf{J} \cdot d\mathbf{S}}{\Delta v}$$

can approach a nonzero value in the limit $\Delta v \longrightarrow 0$. The quantity

$$\frac{\oint_{\Delta S} \mathbf{J} \cdot d\mathbf{S}}{\Delta v}$$

is the amount of current, or the flux of the quantity whose density vector is represented by $\mathbf{J}$, per unit volume emanating from the infinitesimal volume Δv. The value that this quantity approaches as Δv tends to zero (i.e., shrinks to a point) is known as the divergence of the vector $\mathbf{J}$. The divergence of $\mathbf{J}$ is represented as the dot product of the vector operator $\boldsymbol{\nabla}$ and the vector $\mathbf{J}$, that is, as $\boldsymbol{\nabla} \cdot \mathbf{J}$. Thus

$$\boldsymbol{\nabla} \cdot \mathbf{J} = \lim_{\Delta v \to 0} \frac{\oint_{\Delta S} \mathbf{J} \cdot d\mathbf{S}}{\Delta v} \tag{1-87}$$

Since the surface integral results in a scalar, the divergence of a vector is a scalar. It is the flux emanating per unit volume as the volume shrinks to a point. Hence the concept of divergence is valid at a point.

To make use of the concept of divergence of a vector, we need to derive expressions for it in terms of the components of the vector in different coordinate systems. Let us choose the cylindrical coordinate system for this purpose. The method of deriving the required expressions consists of following exactly the steps involved in the definition of divergence. First we choose an infinitesimal volume at an arbitrary point $P(r_0, \phi_0, z_0)$, as shown in Fig. 1.26. The infinitesimal volume is formed by the surfaces $r = r_0$, $r = r_0 + dr$,

$\phi = \phi_0$, $\phi = \phi_0 + d\phi$, $z = z_0$, and $z = z_0 + dz$. The resulting differential surfaces 1, 2, 3, 4, 5, and 6 are given by $-r_0\, d\phi\, dz\, \mathbf{i}_r$, $(r_0 + dr)\, d\phi\, dz\, \mathbf{i}_r$, $-dr\, dz\, \mathbf{i}_\phi$, $dr\, dz\, \mathbf{i}_\phi$, $-r_0\, d\phi\, dr\, \mathbf{i}_z$, and $r_0\, d\phi\, dr\, \mathbf{i}_z$, respectively. Expressing **J** in terms of its components in cylindrical coordinates, we have

$$\mathbf{J} = J_r\mathbf{i}_r + J_\phi\mathbf{i}_\phi + J_z\mathbf{i}_z \tag{1-88}$$

The next step is to evaluate the integral of $\mathbf{J} \cdot d\mathbf{S}$ over the surface bounding the differential volume. We do this by evaluating the surface integrals over the six surfaces separately and then adding them up. Over

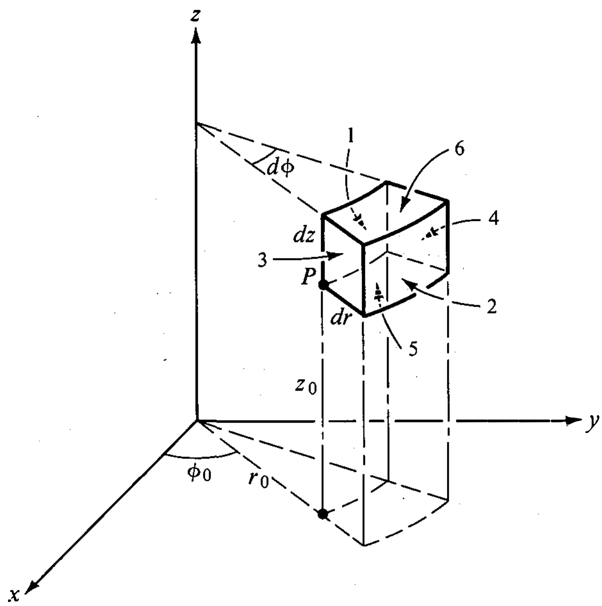


Fig. 1.26. For obtaining the expression for the divergence of a vector in cylindrical coordinates.

each surface, we can assume that **J** is constant since the surface area is infinitesimal. Only one of the three components of **J** will contribute to the flux crossing a particular surface since the other two components are tangential. Thus the flux leaving the volume from any surface is simply the product of the surface area and the normal component of the **J** vector evaluated on that surface or its negative, depending upon whether that component is directed out of or into the volume. In this manner we obtain

$$\text{flux leaving the volume from surface 1} = -[J_r]_{r=r_0}\, r_0\, d\phi\, dz \tag{1-89}$$

and

$$\text{flux leaving the volume from surface 2} = [J_r]_{r=r_0+dr}(r_0 + dr)\, d\phi\, dz \tag{1-90}$$

From (1-89) and (1-90) we have

$$\begin{aligned}&\text{net flux out of the volume due to surfaces 1 and 2}\\&= [J_r]_{r=r_0+dr}(r_0 + dr)\,d\phi\,dz - [J_r]_{r=r_0}\,r_0\,d\phi\,dz\\&=\{[rJ_r]_{r=r_0+dr} - [rJ_r]_{r=r_0}\}\,d\phi\,dz\end{aligned} \tag{1-91a}$$

Similarly,

$$\begin{aligned}&\text{net flux out of the volume due to surfaces 3 and 4}\\&= \{[J_\phi]_{\phi=\phi_0+d\phi} - [J_\phi]_{\phi=\phi_0}\}\,dr\,dz\end{aligned} \tag{1-91b}$$

and

$$\begin{aligned}&\text{net flux out of the volume due to surfaces 5 and 6}\\&= \{[J_z]_{z=z_0+dz} - [J_z]_{z=z_0}\}\,r_0\,dr\,d\phi\end{aligned} \tag{1-91c}$$

The total flux emanating from the differential volume is the sum of the expressions on the right sides of (1-91a), (1-91b), and (1-91c). Adding these three expressions and dividing by the differential volume,

$$\Delta v = r_0\,dr\,d\phi\,dz \tag{1-92}$$

we obtain

$$\begin{aligned}\frac{\oint_{\Delta S} \mathbf{J}\cdot d\mathbf{S}}{\Delta v} &= \frac{[rJ_r]_{r=r_0+dr} - [rJ_r]_{r=r_0}}{r_0\,dr}\\&+ \frac{[J_\phi]_{\phi=\phi_0+d\phi} - [J_\phi]_{\phi=\phi_0}}{r_0\,d\phi}\\&+ \frac{[J_z]_{z=z_0+dz} - [J_z]_{z=z_0}}{dz}\end{aligned} \tag{1-93}$$

By taking the limit of (1-93) as $\Delta v \to 0$, we obtain $\nabla \cdot \mathbf{J}$ at $P(r_0, \phi_0, z_0)$ as

$$\begin{aligned}[\nabla\cdot\mathbf{J}]_{(r_0,\phi_0,z_0)} &= \lim_{\Delta v\to 0}\frac{\oint_{\Delta S}\mathbf{J}\cdot d\mathbf{S}}{\Delta v}\\&= \lim_{dr\to 0}\frac{[rJ_r]_{r=r_0+dr} - [rJ_r]_{r=r_0}}{r_0\,dr} + \lim_{d\phi\to 0}\frac{[J_\phi]_{\phi=\phi_0+d\phi} - [J_\phi]_{\phi=\phi_0}}{r_0\,d\phi}\\&\quad + \lim_{dz\to 0}\frac{[J_z]_{z=z_0+dz} - [J_z]_{z=z_0}}{dz}\\&= \frac{1}{r_0}\left[\frac{\partial}{\partial r}(rJ_r)\right]_{r=r_0} + \frac{1}{r_0}\left[\frac{\partial J_\phi}{\partial\phi}\right]_{\phi=\phi_0} + \left[\frac{\partial J_z}{\partial z}\right]_{z=z_0}\\&= \left[\frac{1}{r}\frac{\partial}{\partial r}(rJ_r) + \frac{1}{r}\frac{\partial J_\phi}{\partial\phi} + \frac{\partial J_z}{\partial z}\right]_{(r_0,\phi_0,z_0)}\end{aligned} \tag{1-94}$$

Now, since (1-94) is valid for any (r_0, ϕ_0, z_0), we can generalize (1-94) by stating that at any point (r, ϕ, z),

$$\nabla\cdot\mathbf{J} = \frac{1}{r}\frac{\partial}{\partial r}(rJ_r) + \frac{1}{r}\frac{\partial J_\phi}{\partial\phi} + \frac{\partial J_z}{\partial z} \tag{1-95}$$

Similar expressions for the divergence can be derived in the cartesian and spherical coordinate systems by repeating the procedure followed for the cylindrical coordinate system. The resulting expressions are as follows:

Cartesian coordinates:

$$\nabla \cdot \mathbf{J} = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} \tag{1-96}$$

Spherical coordinates:

$$\nabla \cdot \mathbf{J} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 J_r) + \frac{1}{r \sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, J_\theta) + \frac{1}{r\sin\theta}\frac{\partial J_\phi}{\partial\phi} \tag{1-97}$$

By rewriting (1-96) as

$$\nabla \cdot \mathbf{J} = \left(\frac{\partial}{\partial x}\mathbf{i}_x + \frac{\partial}{\partial y}\mathbf{i}_y + \frac{\partial}{\partial z}\mathbf{i}_z\right) \cdot (J_x\mathbf{i}_x + J_y\mathbf{i}_y + J_z\mathbf{i}_z) \tag{1-98}$$

we note why the divergence of $\mathbf{J}$ is written as $\nabla \cdot \mathbf{J}$.

We will now derive a theorem which relates the closed surface integral $\oint_S \mathbf{J} \cdot d\mathbf{S}$ to a volume integral evaluated in the volume V bounded by S. To do this, let us divide the volume V into a large number of infinitesimal volumes $dv_1, dv_2, dv_3, \ldots$ having surfaces $\Delta S_1, \Delta S_2, \Delta S_3, \ldots$, respectively. For each infinitesimal volume, we can assume $\nabla \cdot \mathbf{J}$ to be uniform and equal to the value it approaches in the limit the volume shrinks to a point. According to definition, $\nabla \cdot \mathbf{J}$ is the flux of the quantity, represented by $\mathbf{J}$, per unit volume in the limit that the volume shrinks to a vanishingly small value. Now, let us consider one of the infinitesimal volumes dv_1 with its associated surface ΔS_1 and vector $\mathbf{J}_1$. The total flux emanating from this volume is equal to $(\nabla \cdot \mathbf{J})_1\, dv_1$ where $(\nabla \cdot \mathbf{J})_1$ is the value of $\nabla \cdot \mathbf{J}$ evaluated in that volume. But, from the concept of surface integral, this flux is also equal to $\oint_{\Delta S_1} \mathbf{J}_1 \cdot d\mathbf{S}$. Thus

$$(\nabla \cdot \mathbf{J})_1\, dv_1 = \oint_{\Delta S_1} \mathbf{J}_1 \cdot d\mathbf{S} \tag{1-99}$$

By writing similar expressions for all the other infinitesimal volumes and adding them up, we obtain

$$\begin{aligned}(\nabla \cdot \mathbf{J})_1\, dv_1 &+ (\nabla \cdot \mathbf{J})_2\, dv_2 + (\nabla \cdot \mathbf{J})_3\, dv_3 + \cdots \\ &= \oint_{\Delta S_1} \mathbf{J}_1 \cdot d\mathbf{S} + \oint_{\Delta S_2} \mathbf{J}_2 \cdot d\mathbf{S} + \oint_{\Delta S_3} \mathbf{J}_3 \cdot d\mathbf{S} + \cdots\end{aligned} \tag{1-100}$$

But the right side of (1-100) is equal to $\oint_S \mathbf{J} \cdot d\mathbf{S}$, since contributions from all the surfaces and portions of the surfaces inside the boundary of the volume V cancel, leaving a net integral over the surface bounding the volume V. Equation (1-100) then becomes

$$\sum_i (\nabla \cdot \mathbf{J})_i\, dv_i = \oint_S \mathbf{J} \cdot d\mathbf{S} \tag{1-101}$$

Equation (1-101) is approximate since the assumption of uniform $\nabla \cdot \mathbf{J}$

inside any infinitesimal volume is true only in the limit that the volume shrinks to a point. In this limit, the summation in (1-101) becomes an integral, giving us

$$\int_V (\nabla \cdot \mathbf{J})\, dv = \oint_S \mathbf{J} \cdot d\mathbf{S} \tag{1-102}$$

where the integration is performed throughout the volume V bounded by S. The result represented by (1-102) is known as the divergence theorem. It permits the replacement of a surface integration by a volume integration and vice versa.

EXAMPLE 1-16. In Example 1-14 we evaluated a surface integral to find the current flowing out of a box. It is now desired to compute the same quantity by using the divergence theorem and performing a volume integration.

According to the divergence theorem (1-102), the current flowing out of the box of Fig. 1.23 is $\int_V (\nabla \cdot \mathbf{J})\, dv$, where V is the volume of the box and $\mathbf{J}$ is the current density vector specified in Example 1-14. For this current density vector, the divergence is equal to 5. Hence

$$\begin{aligned}\oint_S \mathbf{J} \cdot d\mathbf{S} = \int_V (\nabla \cdot \mathbf{J})\, dv = \int_V 5\, dv = 5 \int_V dv \\ = 5(\text{volume of the box}) \\ = 5 \times \left(\frac{1 \times 2 \times 3}{2}\right) = 15 \text{ amp}\end{aligned}$$

This result agrees with the result of Example 1-14. ∎

1.9 Curl and Stokes' Theorem

In Section 1.7 we introduced the concept of circulation or line integral around a closed path. Let us consider an infinitesimal area ΔS in the field of a vector $\mathbf{F}$ and orient it such that the circulation $\oint \mathbf{F} \cdot d\mathbf{l}$ around the periphery ΔC of the area is a maximum. Let $\mathbf{i}_n$ be the unit vector normal to the area for that particular orientation. Then we define a vector quantity known as the "curl" of $\mathbf{F}$, having the symbol "del cross" as

$$\nabla \times \mathbf{F} = \lim_{\Delta S \to 0} \frac{\oint_{\Delta C} \mathbf{F} \cdot d\mathbf{l}}{\Delta S} \mathbf{i}_n \tag{1-103}$$

We note that in the limit $\Delta S \longrightarrow 0$, although $\oint_{\Delta C} \mathbf{F} \cdot d\mathbf{l}$ may tend to zero, $\nabla \times \mathbf{F}$ can be nonzero. The line integral in (1-103) is evaluated by traversing the perimeter of the area ΔS on the side of the unit vector $\mathbf{i}_n$ in such a direction that the area is on the left, as shown in Fig. 1.27. This is the same as the direction in which a right-hand screw turns as it advances in the direction of the normal vector. Just as the divergence of a vector is associated with a point in space, the curl of a vector is also associated with a point in space, in view of the limit on the right side of (1-103). Whereas the divergence of a vector is a scalar, the curl of a vector is another vector. The magnitude of this vector is the circulation per unit area as the area shrinks to a point, maintaining in this process an orientation of the area such that the circulation around its periphery is a maximum. The direction of the vector is the direction which the normal vector to the area assumes as the area shrinks to the point.

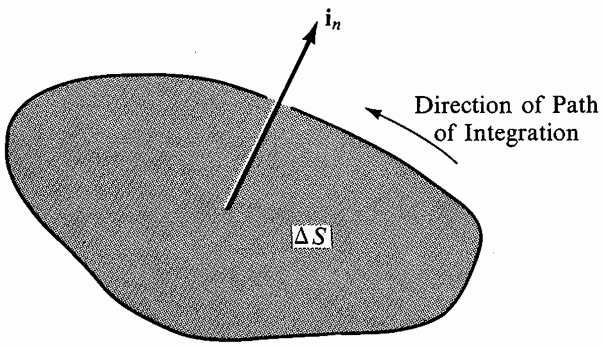


Fig. 1.27. Illustrating the sense of traversal around the periphery of area ΔS to evaluate the line integral in (1-103).

Later in this section we will explore the physical significance of curl, but first let us obtain the expressions for the curl of a vector in terms of the components of the vector. To do this, we first wish to show that the components of the curl of a vector at a point are simply the circulations per unit area at that point with the areas oriented normal to the corresponding coordinate axes.

Let us consider an infinitesimal plane area ABC, as shown in Fig. 1.28, such that its normal vector $\mathbf{i}_n$ is oriented in an arbitrary direction in the field of the vector $\mathbf{F}$. We can write

$$\oint_{ABCA} \mathbf{F} \cdot d\mathbf{l} = \oint_{ABOA} \mathbf{F} \cdot d\mathbf{l} + \oint_{BCOB} \mathbf{F} \cdot d\mathbf{l} + \oint_{CAOC} \mathbf{F} \cdot d\mathbf{l} \qquad \text{(1-104)}$$

since the contribution to the integrals on the right side from the paths between O and A, O and B, and O and C cancel. Dividing both sides of (1-104) by the area ABC, we get

$$\frac{\oint_{ABCA} \mathbf{F} \cdot d\mathbf{l}}{\text{area } ABC} = \frac{\oint_{ABOA} \mathbf{F} \cdot d\mathbf{l}}{\text{area } ABC} + \frac{\oint_{BCOB} \mathbf{F} \cdot d\mathbf{l}}{\text{area } ABC} + \frac{\oint_{CAOC} \mathbf{F} \cdot d\mathbf{l}}{\text{area } ABC} \qquad \text{(1-105)}$$

With the relationships

$$\text{area } AOB = (\text{area } ABC)\mathbf{i}_n \cdot \mathbf{i}_z \qquad \text{(1-106a)}$$

$$\text{area } BOC = (\text{area } ABC)\mathbf{i}_n \cdot \mathbf{i}_x \qquad \text{(1-106b)}$$

$$\text{area } COA = (\text{area } ABC)\mathbf{i}_n \cdot \mathbf{i}_y \qquad \text{(1-106c)}$$

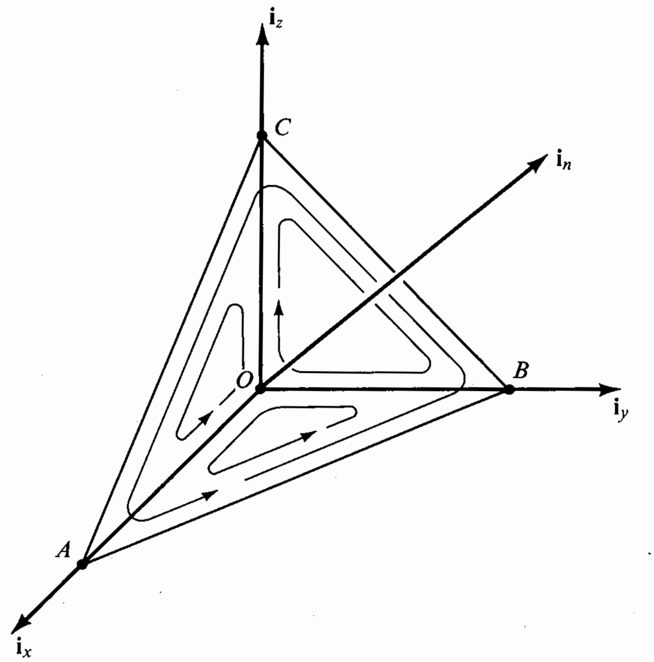


Fig. 1.28. For showing that the components of the curl of a vector at a point are the circulations per unit area at that point with the areas oriented normal to the corresponding coordinate axes.

(1-105) can be written as

$$\frac{\oint_{ABCA} \mathbf{F} \cdot d\mathbf{l}}{\text{area } ABC} = \frac{\oint_{ABOA} \mathbf{F} \cdot d\mathbf{l}}{\text{area } AOB} \mathbf{i}_n \cdot \mathbf{i}_z + \frac{\oint_{BCOB} \mathbf{F} \cdot d\mathbf{l}}{\text{area } BOC} \mathbf{i}_n \cdot \mathbf{i}_x + \frac{\oint_{CAOC} \mathbf{F} \cdot d\mathbf{l}}{\text{area } COA} \mathbf{i}_n \cdot \mathbf{i}_y$$
$$= \mathbf{i}_n \cdot \left(\frac{\oint_{ABOA} \mathbf{F} \cdot d\mathbf{l}}{\text{area } AOB} \mathbf{i}_z + \frac{\oint_{BCOB} \mathbf{F} \cdot d\mathbf{l}}{\text{area } BOC} \mathbf{i}_x + \frac{\oint_{CAOC} \mathbf{F} \cdot d\mathbf{l}}{\text{area } COA} \mathbf{i}_y \right) \tag{1-107}$$

Taking the limit of both sides of (1-107) as the area $ABC \longrightarrow 0$, we have

$$\lim_{ABC \to 0} \frac{\oint_{ABCA} \mathbf{F} \cdot d\mathbf{l}}{\text{area } ABC} = \mathbf{i}_n \cdot \left(\lim_{AOB \to 0} \frac{\oint_{ABOA} \mathbf{F} \cdot d\mathbf{l}}{\text{area } AOB} \mathbf{i}_z + \lim_{BOC \to 0} \frac{\oint_{BCOB} \mathbf{F} \cdot d\mathbf{l}}{\text{area } BOC} \mathbf{i}_x + \lim_{COA \to 0} \frac{\oint_{CAOC} \mathbf{F} \cdot d\mathbf{l}}{\text{area } COA} \mathbf{i}_y \right) \tag{1-108}$$

The magnitude of $\nabla \times \mathbf{F}$ is the maximum possible value of

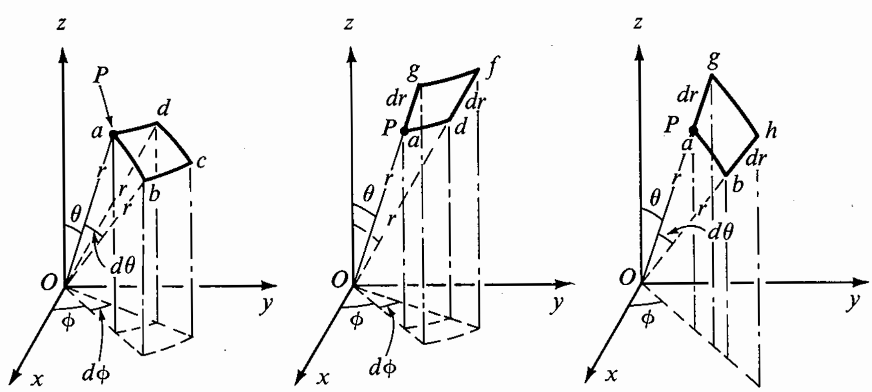
$$\lim_{ABC \to 0} \frac{\oint_{ABCA} \mathbf{F} \cdot d\mathbf{l}}{\text{area } ABC}$$

that is, the maximum possible value of the quantity on the left side of (1-108). The maximum value of this quantity occurs when the orientation of $\mathbf{i}_n$ coincides with the direction of the vector inside the parentheses on the right side of (1-108). It then follows that this maximum value is the magnitude of the vector inside the parentheses. Hence the vector inside the parentheses on the right side of (1-108) is indeed $\nabla \times \mathbf{F}$. Thus the components of $\nabla \times \mathbf{F}$ are simply the circulations per unit area at the point of interest with the areas oriented normal to the corresponding unit vectors and as these areas are shrunk to the point. Although the foregoing proof is carried out for the cartesian coordinate system, it is obvious that it is valid for any orthogonal coordinate system since the unit vectors in Fig. 1.28 can be replaced by any orthogonal set of unit vectors.

We will now derive the expressions for the components of $\nabla \times \mathbf{F}$. Let us choose the spherical coordinate system for this purpose. To obtain the r component, we consider an infinitesimal area $abcd$ normal to the unit vector $\mathbf{i}_r$ at point $P(r, \theta, \phi)$, as shown in Fig. 1.29(a). From our experience in deriving the expressions for the divergence in Section 1.8, there is no need to consider a point (r_0, θ_0, ϕ_0) and then generalize the result. Expressing $\mathbf{F}$ in terms of its components in spherical coordinates, we have

$$\mathbf{F} = F_r\mathbf{i}_r + F_\theta\mathbf{i}_\theta + F_\phi\mathbf{i}_\phi \tag{1-109}$$

Then we evaluate the circulation of $\mathbf{F}$ around the path $abcd$ in Fig. 1.29(a), divide the circulation by the area of $abcd$, and find the limit of the resulting



(a) (b) (c)

Fig. 1.29. For obtaining the expression for the curl of a vector in spherical coordinates.

quantity as the area *abcd* tends to zero. The circulation of **F** around *abcd* is the sum of four line integrals, evaluated along the four sides of the area *abcd*. For each side, we can assume that **F** is constant since the lengths are infinitesimal. Only one of the three components of **F** contribute to the line integral involving any particular side since the other two components are normal to the path. Thus the line integral along any side is simply the product of the length of the side and the tangential component of the **F** vector evaluated along that side or its negative, depending upon whether the component is directed along or opposite to the path of integration. In this manner, we obtain

$$\int_a^b \mathbf{F} \cdot d\mathbf{l} = [F_\theta]_\phi \, r \, d\theta \tag{1-110}$$

$$\int_b^c \mathbf{F} \cdot d\mathbf{l} = [F_\phi]_{\theta+d\theta} \, r \sin(\theta + d\theta) \, d\phi \tag{1-111}$$

$$\int_c^d \mathbf{F} \cdot d\mathbf{l} = -[F_\theta]_{\phi+d\phi} \, r \, d\theta \tag{1-112}$$

$$\int_d^a \mathbf{F} \cdot d\mathbf{l} = -[F_\phi]_\theta \, r \sin\theta \, d\phi \tag{1-113}$$

From (1-110)–(1-113) we have

$$\begin{aligned}
\oint_{abcda} \mathbf{F} \cdot d\mathbf{l} &= \int_a^b \mathbf{F} \cdot d\mathbf{l} + \int_b^c \mathbf{F} \cdot d\mathbf{l} + \int_c^d \mathbf{F} \cdot d\mathbf{l} + \int_d^a \mathbf{F} \cdot d\mathbf{l} \\
&= \{[F_\theta]_\phi - [F_\theta]_{\phi+d\phi}\} r \, d\theta \\
&\quad + \{[F_\phi]_{\theta+d\theta} \sin(\theta + d\theta) - [F_\phi]_\theta \sin\theta\} r \, d\phi \\
&= \{[F_\theta]_\phi - [F_\theta]_{\phi+d\phi}\} r \, d\theta \\
&\quad + \{[F_\phi \sin\theta]_{\theta+d\theta} - [F_\phi \sin\theta]_\theta\} r \, d\phi
\end{aligned} \tag{1-114}$$

Dividing both sides of (1-114) by area $abcd = r^2 \sin\theta \, d\theta \, d\phi$ and taking the limit as the area tends to zero, we have

$$\begin{aligned}
\lim_{abcd \to 0} \frac{\oint_{abcda} \mathbf{F} \cdot d\mathbf{l}}{\text{area } abcd} &= \lim_{\substack{d\theta \to 0 \\ d\phi \to 0}} \frac{\{[F_\theta]_\phi - [F_\theta]_{\phi+d\phi}\} r \, d\theta}{r^2 \sin\theta \, d\theta \, d\phi} \\
&\quad + \lim_{\substack{d\theta \to 0 \\ d\phi \to 0}} \frac{\{[F_\phi \sin\theta]_{\theta+d\theta} - [F_\phi \sin\theta]_\theta\} r \, d\phi}{r^2 \sin\theta \, d\theta \, d\phi} \\
&= -\frac{1}{r \sin\theta} \frac{\partial F_\theta}{\partial \phi} + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta}(\sin\theta \, F_\phi)
\end{aligned} \tag{1-115}$$

Similarly, to derive the θ component of $\nabla \times \mathbf{F}$, we consider an infinitesimal area *adfg* normal to the unit vector $\mathbf{i}_\theta$ at point $P(r, \theta, \phi)$, as shown in Fig. 1.29(b), and evaluate the circulation around the periphery of this area. Following in the same manner as for the r component of $(\nabla \times \mathbf{F})$, we have

$$\begin{aligned}\oint_{adfga} \mathbf{F} \cdot d\mathbf{l} &= [F_\phi]_r\, r \sin\theta\, d\phi + [F_r]_{\phi+d\phi}\, dr \\ &\quad - [F_\phi]_{r+dr}(r + dr)\sin\theta\, d\phi - [F_r]_\phi\, dr \\ &= \{[rF_\phi]_r - [rF_\phi]_{r+dr}\}\sin\theta\, d\phi \\ &\quad + \{[F_r]_{\phi+d\phi} - [F_r]_\phi\}\, dr\end{aligned} \tag{1-116}$$

Noting that area $adfg = r \sin\theta\, dr\, d\phi$, we obtain

$$\begin{aligned}\lim_{adfg\to 0} \frac{\oint_{adfga} \mathbf{F} \cdot d\mathbf{l}}{\text{area } adfg} &= \lim_{\substack{dr\to 0\\ d\phi\to 0}} \frac{\{[rF_\phi]_r - [rF_\phi]_{r+dr}\}\sin\theta\, d\phi}{r \sin\theta\, dr\, d\phi} \\ &\quad + \lim_{\substack{dr\to 0\\ d\phi\to 0}} \frac{\{[F_r]_{\phi+d\phi} - [F_r]_\phi\}\, dr}{r \sin\theta\, dr\, d\phi} \\ &= -\frac{1}{r}\frac{\partial}{\partial r}(rF_\phi) + \frac{1}{r\sin\theta}\frac{\partial F_r}{\partial \phi}\end{aligned} \tag{1-117}$$

Finally, to obtain the ϕ component of $\nabla \times \mathbf{F}$, we choose an infinitesimal area *aghb* normal to the unit vector $\mathbf{i}_\phi$ at point $P(r, \theta, \phi)$, as shown in Fig. 1.29(c), and evaluate the circulation around the periphery of this area. Following in the same manner as for the r and θ components of $\nabla \times \mathbf{F}$, we have

$$\begin{aligned}\oint_{aghba} \mathbf{F} \cdot d\mathbf{l} &= [F_r]_\theta\, dr + [F_\theta]_{r+dr}(r + dr)\, d\theta - [F_r]_{\theta+d\theta}\, dr - [F_\theta]_r r\, d\theta \\ &= \{[F_r]_\theta - [F_r]_{\theta+d\theta}\}\, dr \\ &\quad + \{[rF_\theta]_{r+dr} - [rF_\theta]_r\}\, d\theta\end{aligned} \tag{1-118}$$

Noting that area $aghb = r\, dr\, d\theta$, we obtain

$$\begin{aligned}\lim_{aghb\to 0} \frac{\oint_{aghba} \mathbf{F} \cdot d\mathbf{l}}{\text{area } aghb} &= \lim_{\substack{dr\to 0\\ d\theta\to 0}} \frac{\{[F_r]_\theta - [F_r]_{\theta+d\theta}\}\, dr}{r\, dr\, d\theta} \\ &\quad + \lim_{\substack{dr\to 0\\ d\theta\to 0}} \frac{\{[rF_\theta]_{r+dr} - [rF_\theta]_r\}\, d\theta}{r\, dr\, d\theta} \\ &= -\frac{1}{r}\frac{\partial F_r}{\partial \theta} + \frac{1}{r}\frac{\partial}{\partial r}(rF_\theta)\end{aligned} \tag{1-119}$$

Thus, in the spherical coordinate system, we note that

$$\begin{aligned}\nabla \times \mathbf{F} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, F_\phi) - \frac{\partial F_\theta}{\partial \phi}\right]\mathbf{i}_r \\ &\quad + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial F_r}{\partial \phi} - \frac{\partial}{\partial r}(rF_\phi)\right]\mathbf{i}_\theta + \frac{1}{r}\left[\frac{\partial}{\partial r}(rF_\theta) - \frac{\partial F_r}{\partial \theta}\right]\mathbf{i}_\phi \\ &= \begin{vmatrix} \dfrac{\mathbf{i}_r}{r^2\sin\theta} & \dfrac{\mathbf{i}_\theta}{r\sin\theta} & \dfrac{\mathbf{i}_\phi}{r} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ F_r & rF_\theta & r\sin\theta\, F_\phi \end{vmatrix}\end{aligned} \tag{1-120}$$

Similar expressions for the curl can be derived in the cartesian and cylindrical coordinate systems by repeating the procedure followed for the spherical coordinate system. The resulting expressions are as follows:

Cartesian coordinates:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i}_x & \mathbf{i}_y & \mathbf{i}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \tag{1-121}$$

Cylindrical coordinates:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \dfrac{\mathbf{i}_r}{r} & \mathbf{i}_\phi & \dfrac{\mathbf{i}_z}{r} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ F_r & rF_\phi & F_z \end{vmatrix} \tag{1-122}$$

The form of the right side of (1-121) explains why the curl of $\mathbf{F}$ is written as $\nabla \times \mathbf{F}$.

Let us now discuss briefly the physical significance of curl. To do this, we will use the concept of the curl meter or the paddle wheel as suggested by Skilling (see bibliography). Consider a stream of rectangular cross section carrying water in the z direction, as shown in Fig. 1.30(a). Assume the velocity $\mathbf{v}$ of the water to be independent of height but increasing uniformly from a value of zero at the banks to a maximum value of v_0 at the center. Thus

$$\mathbf{v} = \begin{cases} v_0 \dfrac{y}{a}\mathbf{i}_z & \text{for } 0 < y < a \\ v_0 \dfrac{2a - y}{a}\mathbf{i}_z & \text{for } a < y < 2a \end{cases} \tag{1-123}$$

The curl of the velocity vector is given by

$$\begin{aligned} \nabla \times \mathbf{v} &= \begin{vmatrix} \mathbf{i}_x & \mathbf{i}_y & \mathbf{i}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & 0 & v_z \end{vmatrix} \\ &= \frac{\partial v_z}{\partial y}\mathbf{i}_x - \frac{\partial v_z}{\partial x}\mathbf{i}_y \\ &= \begin{cases} \dfrac{v_0}{a}\mathbf{i}_x & 0 < y < a \\ -\dfrac{v_0}{a}\mathbf{i}_x & a < y < 2a \end{cases} \end{aligned} \tag{1-124}$$

Sketches of v_z and $(\nabla \times \mathbf{v})_x$ are shown in Figs. 1.30(b) and (c), respectively. Now, let us consider a frictionless paddle wheel having negligible influence on the velocity of the water and introduce it into the water with its shaft

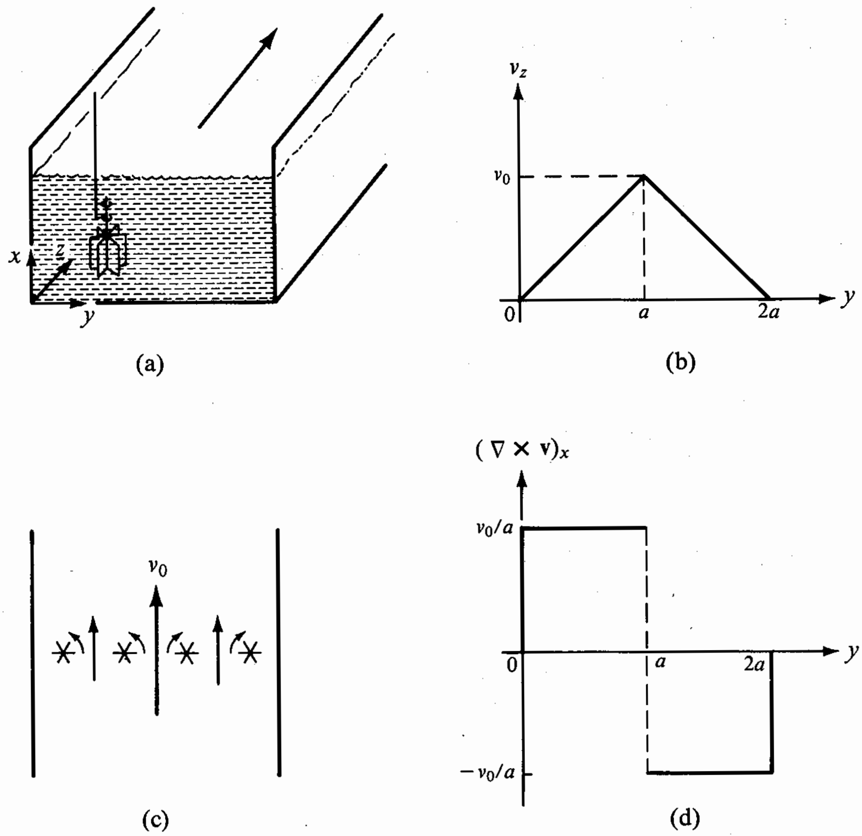


Fig. 1.30. For explaining the physical significance of curl using the paddle-wheel device.

vertical, that is, parallel to the x axis. It will turn in the counterclockwise direction on the left side of the center of the stream and in the clockwise direction on the right side of the center, as shown in Fig. 1.30(d). Moreover, since the velocity differential is independent of y, it will turn at the same rate independent of y. In exactly the midstream, it will not turn since the velocities on either side are equal and are in the same direction. Now, if we examine the graph of $(\nabla \times \mathbf{v})_x$ and compare it with the action of the paddle wheel, the physical meaning of curl is apparent. It signifies the ability of the vector field to rotate the paddle wheel. If we insert the paddle wheel horizontally, that is, along the z axis or along the y axis or in any other direction parallel to the yz plane, it will not rotate since the top and bottom plates are hit with the same force, thus indicating that the curl for this field has no horizontal component, as indeed the expression (1-124) shows. The curl has nothing to do with curvature or curling flow as the name might imply. We have already seen in the example just discussed that a vector field whose direction lines are straight lines has a nonzero curl. Likewise, it is possible to have vector fields whose direction lines are curved but with zero curl. As an example, consider the field given in cylindrical coordinates by

$$\mathbf{F} = \frac{1}{r}\mathbf{i}_\phi \tag{1-125}$$

For this vector field, (1-122) gives

$$\nabla \times \mathbf{F} = \begin{cases} 0 & \text{everywhere except at } r = 0 \\ \infty\, \mathbf{i}_z & \text{at } r = 0 \end{cases} \tag{1-126}$$

This can be explained by referring to Fig. 1.31. Although the magnitude of the force on the right side of the center of the paddle wheel is less than on the left side, there are more blades hit by the force on the right side, thereby keeping the paddle wheel still. At $r = 0$, however, there is circular motion of the fluid which turns the paddle wheel.

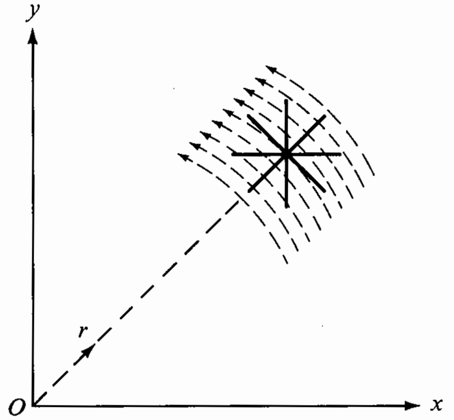


Fig. 1.31. An exaggerated picture of a paddle wheel in the field $(1/r)\mathbf{i}_\phi$.

Two important identities involving curl are

$$\nabla \cdot \nabla \times \mathbf{F} \equiv 0 \tag{1-127}$$

$$\nabla \times \nabla V \equiv 0 \tag{1-128}$$

The first identity states that the divergence of any vector which can be expressed as the curl of another vector is zero, whereas the second identity states that the curl of any vector which can be expressed as the gradient of a scalar is zero. These relations can be derived simply by carrying out the vector operations indicated by the left sides of (1-127) and (1-128). Conversely, if the divergence of a vector is zero, it can be expressed as the curl of a vector and if the curl of a vector is zero, it can be expressed as the gradient of a scalar.

We will now derive a theorem which relates the closed line integral

$\oint \mathbf{F} \cdot d\mathbf{l}$ to a surface integral evaluated over any surface bounded by the closed path. To do this, let us consider in the field of the vector $\mathbf{F}$ a contour C which is the boundary of a surface S, not necessarily plane, as shown in Fig. 1.32. Let us divide the surface S into a large number of infinitesimal areas $dS_1, dS_2, dS_3, \ldots$ bounded by contours $\Delta C_1, \Delta C_2, \Delta C_3, \ldots$, respectively. For each infinitesimal area, we can assume $\nabla \times \mathbf{F}$ to be uniform and equal to the value it approaches in the limit the area shrinks to a point. According to definition, $\nabla \times \mathbf{F}$ is the maximum circulation of $\mathbf{F}$ per unit area at a point. If an infinitesimal area dS is oriented such that its normal vector is in the direction of $\nabla \times \mathbf{F}$, the circulation around the periphery of that infinitesimal area is $(\nabla \times \mathbf{F})\, dS$. If the infinitesimal area has some other orientation, we have to take the component of $\nabla \times \mathbf{F}$ along the normal vector to that area and multiply by the area to obtain the circulation.

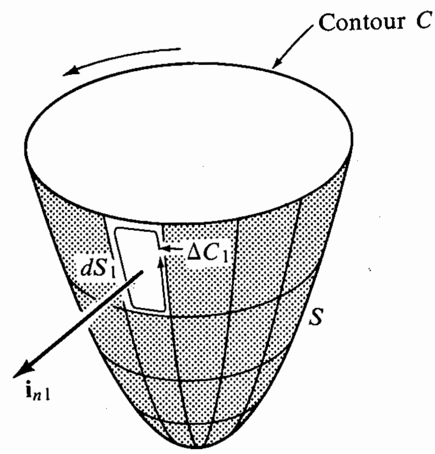


Fig. 1.32. Division of a surface S bounded by a contour C into a number of infinitesimal areas to derive Stokes' theorem.

Let us then consider one of the infinitesimal areas dS_1 with its associated vector $\mathbf{F}_1$. The circulation around the contour ΔC_1 bounding this infinitesimal area is equal to $(\nabla \times \mathbf{F})_1 \cdot \mathbf{i}_{n1}\, dS_1$, where $\mathbf{i}_{n1}$ is the unit normal vector to dS_1 oriented in accordance with the convention shown in Fig. 1.27 and $(\nabla \times \mathbf{F})_1$ is the value of $\nabla \times \mathbf{F}$ evaluated over that area. But, from the concept of line integral, this circulation is also equal to $\oint_{\Delta C_1} \mathbf{F}_1 \cdot d\mathbf{l}$. Thus

$$(\nabla \times \mathbf{F})_1 \cdot \mathbf{i}_{n1}\, dS_1 = \oint_{\Delta C_1} \mathbf{F}_1 \cdot d\mathbf{l} \tag{1-129}$$

By writing similar expressions for all the other infinitesimal areas and adding them up, we obtain

$$\begin{aligned}(\nabla \times \mathbf{F})_1 \cdot \mathbf{i}_{n1}\, dS_1 &+ (\nabla \times \mathbf{F})_2 \cdot \mathbf{i}_{n2}\, dS_2 + (\nabla \times \mathbf{F})_3 \cdot \mathbf{i}_{n3}\, dS_3 + \cdots \\ &= \oint_{\Delta C_1} \mathbf{F}_1 \cdot d\mathbf{l} + \oint_{\Delta C_2} \mathbf{F}_2 \cdot d\mathbf{l} + \oint_{\Delta C_3} \mathbf{F}_3 \cdot d\mathbf{l} + \cdots\end{aligned} \tag{1-130}$$

But the right side of (1-130) is equal to $\oint_C \mathbf{F} \cdot d\mathbf{l}$, since contributions from all the contours and portions of the contours inside the periphery of the surface S cancel, leaving a net integral around the periphery. Equation (1-130) then becomes

$$\sum_j (\nabla \times \mathbf{F})_j \cdot \mathbf{i}_{nj}\, dS_j = \oint_C \mathbf{F} \cdot d\mathbf{l} \tag{1-131}$$

Equation (1-131) is approximate since the assumption of uniform $\nabla \times \mathbf{F}$ over any infinitesimal area is true only in the limit that the area shrinks to zero. In this limit, the summation in (1-131) becomes an integral, giving us

$$\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{i}_n\, dS = \oint_C \mathbf{F} \cdot d\mathbf{l} \tag{1-132}$$

or

$$\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{l} \tag{1-133}$$

where we have absorbed the unit vector $\mathbf{i}_n$ into the vector $d\mathbf{S}$. The result represented by (1-132) is known as Stokes' theorem. It permits the replacement of a line integration by a surface integration and vice versa. In (1-132) and (1-133), the sense of traversal around C must be such that the area on the side of the normal vector $\mathbf{i}_n$ is on the left.

EXAMPLE 1-17. In Example 1-15 we used line integration to evaluate the work done by a force vector around a closed path. It is now desired to compute the same quantity by performing a surface integration.

According to Stokes' theorem, the work done by the force vector is $\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{i}_n\, dS$, where S is the surface bounded by the closed path and $\mathbf{F}$ is the force vector specified in Example 1-15. For this force vector, the curl is equal to $-2\mathbf{i}_z$. The normal vector $\mathbf{i}_n$ must be chosen such that it is on the left side while traversing the path specified in Example 1-15. Hence $\mathbf{i}_n = -\mathbf{i}_z$, and

$$\begin{aligned}\int_{abcdefg} (\nabla \times \mathbf{F}) \cdot \mathbf{i}_n\, dS &= \int_{abcdefg} (-2\mathbf{i}_z) \cdot (-\mathbf{i}_z)\, dS \\ &= 2(\text{area } abcdefg)\end{aligned}$$

$$\text{area } abcdefg = \frac{2}{3} + \frac{\sqrt{2}-1}{2} + \frac{1}{2} + \frac{1}{4} + \ln 2 = 2.317$$

Thus

$$\oint_{abcdefga} \mathbf{F} \cdot d\mathbf{l} = \int_{abcdefg} (\nabla \times \mathbf{F}) \cdot \mathbf{i}_n\, dS = 2(2.317) = 4.634$$

This result agrees with the result of Example 1-15. ∎

1.10 The Laplacian

In Sections 1.6, 1.8, and 1.9 we introduced gradient, divergence, and curl, respectively. Gradient is an operation performed only on scalar functions, whereas divergence and curl are operations performed only on vector functions. In this section we will introduce another operation, known as the Laplacian, which is performed both on scalar and vector functions.

a. The Laplacian of a Scalar.

The Laplacian of a scalar function V is defined as the divergence of the gradient of V. The gradient of V is a vector and the divergence of a vector is a scalar. Hence the Laplacian of a scalar results in a scalar. The Laplacian operation has the symbol ∇^2. Thus

$$\nabla^2 V = \nabla \cdot \nabla V \tag{1-134}$$

In cartesian coordinates,

$$\begin{aligned}\nabla^2 V &= \left(\frac{\partial}{\partial x}\mathbf{i}_x + \frac{\partial}{\partial y}\mathbf{i}_y + \frac{\partial}{\partial z}\mathbf{i}_z\right) \cdot \left(\frac{\partial V}{\partial x}\mathbf{i}_x + \frac{\partial V}{\partial y}\mathbf{i}_y + \frac{\partial V}{\partial z}\mathbf{i}_z\right) \\ &= \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}\end{aligned} \tag{1-135}$$

Similarly, expressions for $\nabla^2 V$ can be derived in other coordinate systems. These expressions are as follows:

Cylindrical coordinates:

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} \tag{1-136}$$

Spherical coordinates:

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} \tag{1-137}$$

b. The Laplacian of a Vector.

The Laplacian of a vector $\mathbf{A}$ is defined as the gradient of divergence of $\mathbf{A}$ minus the curl of curl of $\mathbf{A}$; that is,

$$\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times \nabla \times \mathbf{A} \tag{1-138}$$

Expansions for $\nabla^2 \mathbf{A}$ in different coordinate systems can be obtained by carrying out the operations on the right side of (1-138) and simplifying the resulting expressions. The results are as follows:

Cartesian coordinates:

$$\nabla^2 \mathbf{A} = (\nabla^2 A_x)\mathbf{i}_x + (\nabla^2 A_y)\mathbf{i}_y + (\nabla^2 A_z)\mathbf{i}_z \tag{1-139}$$

Cylindrical coordinates:

$$\begin{aligned}\nabla^2\mathbf{A} = &\left(\nabla^2 A_r - \frac{A_r}{r^2} - \frac{2}{r^2}\frac{\partial A_\phi}{\partial \phi}\right)\mathbf{i}_r \\ &+ \left(\nabla^2 A_\phi - \frac{A_\phi}{r^2} + \frac{2}{r^2}\frac{\partial A_r}{\partial \phi}\right)\mathbf{i}_\phi \\ &+ (\nabla^2 A_z)\mathbf{i}_z\end{aligned} \tag{1-140}$$

Spherical coordinates:

$$\begin{aligned}\nabla^2\mathbf{A} = &\left[\nabla^2 A_r - \frac{2}{r^2}A_r - \frac{2}{r^2 \sin\theta}\frac{\partial}{\partial\theta}(A_\theta \sin\theta) - \frac{2}{r^2\sin\theta}\frac{\partial A_\phi}{\partial\phi}\right]\mathbf{i}_r \\ &+ \left(\nabla^2 A_\theta - \frac{A_\theta}{r^2\sin^2\theta} + \frac{2}{r^2}\frac{\partial A_r}{\partial\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\phi}{\partial\phi}\right)\mathbf{i}_\theta \\ &+ \left(\nabla^2 A_\phi - \frac{A_\phi}{r^2\sin^2\theta} + \frac{2}{r^2\sin\theta}\frac{\partial A_r}{\partial\phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\theta}{\partial\phi}\right)\mathbf{i}_\phi\end{aligned} \tag{1-141}$$

1.11 Some Useful Vector Relations

In this section we will summarize the important vector relations discussed in this chapter and present additional useful vector identities. The following notation is used:

$\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ = set of mutually perpendicular unit vectors forming a right-hand coordinate system.
u_1, u_2, u_3 = set of three orthogonal coordinates.
dl_1, dl_2, dl_3 = differential displacements along the direction of the unit vectors $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$, respectively.
h_1, h_2, h_3 = $dl_1/du_1, dl_2/du_2, dl_3/du_3$ known as the metric coefficients.

Table 1.7 summarizes u_1, u_2, u_3 and h_1, h_2, h_3 for the three coordinate systems.

TABLE 1.7. Coordinates and Metric Coefficients for the Three Coordinate Systems

	u_1	u_2	u_3	h_1	h_2	h_3
Cartesian	x	y	z	1	1	1
Cylindrical	r	ϕ	z	1	r	1
Spherical	r	θ	ϕ	1	r	$r\sin\theta$

We will denote the components of a vector $\mathbf{A}$ as A_1, A_2, and A_3 so that

$$\mathbf{A} = A_1\mathbf{i}_1 + A_2\mathbf{i}_2 + A_3\mathbf{i}_3$$

The following general relations can be written:

$$\mathbf{A}\cdot\mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3 \tag{1-142}$$

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix}\mathbf{i}_1 & \mathbf{i}_2 & \mathbf{i}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3\end{vmatrix} \tag{1-143}$$

$$\nabla V = \frac{1}{h_1}\frac{\partial V}{\partial u_1}\mathbf{i}_1 + \frac{1}{h_2}\frac{\partial V}{\partial u_2}\mathbf{i}_2 + \frac{1}{h_3}\frac{\partial V}{\partial u_3}\mathbf{i}_3 \tag{1-144}$$

$$\nabla \cdot \mathbf{J} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(h_2 h_3 J_1) + \frac{\partial}{\partial u_2}(h_3 h_1 J_2) + \frac{\partial}{\partial u_3}(h_1 h_2 J_3)\right] \tag{1-145}$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \dfrac{\mathbf{i}_1}{h_2 h_3} & \dfrac{\mathbf{i}_2}{h_3 h_1} & \dfrac{\mathbf{i}_3}{h_1 h_2} \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix} \tag{1-146}$$

$$\nabla^2 V = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial V}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial V}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial V}{\partial u_3}\right)\right] \tag{1-147}$$

We will now list some useful vector identities. U and V are scalar functions whereas **A**, **B**, **C**, and **D** are vectors.

$$\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \mathbf{B} \cdot \mathbf{C} \times \mathbf{A} = \mathbf{C} \cdot \mathbf{A} \times \mathbf{B}$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{A}(\mathbf{B} \cdot \mathbf{C})$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{B} \times (\mathbf{C} \times \mathbf{A}) + \mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = 0$$

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{B} \cdot \mathbf{C})(\mathbf{A} \cdot \mathbf{D})$$

$$(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \times \mathbf{B} \cdot \mathbf{D})\mathbf{C} - (\mathbf{A} \times \mathbf{B} \cdot \mathbf{C})\mathbf{D}$$

$$\nabla(U + V) = \nabla U + \nabla V$$

$$\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B}$$

$$\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}$$

$$\nabla(UV) = U\,\nabla V + V\,\nabla U$$

$$\nabla \cdot (U\mathbf{A}) = \mathbf{A} \cdot \nabla U + U\,\nabla \cdot \mathbf{A}$$

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B}$$

$$\nabla \times (U\mathbf{A}) = \nabla U \times \mathbf{A} + U\,\nabla \times \mathbf{A}$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}\,\nabla \cdot \mathbf{B} - \mathbf{B}\,\nabla \cdot \mathbf{A} + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$$

$$\nabla \cdot \nabla \times \mathbf{A} = 0$$

$$\nabla \times \nabla U = 0$$

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

$$\frac{d}{dt}(U\mathbf{A}) = U\frac{d\mathbf{A}}{dt} + \frac{dU}{dt}\mathbf{A}$$

$$\frac{d}{dt}(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \cdot \frac{d\mathbf{B}}{dt} + \mathbf{B} \cdot \frac{d\mathbf{A}}{dt}$$

$$\frac{d}{dt}(\mathbf{A} \times \mathbf{B}) = \mathbf{A} \times \frac{d\mathbf{B}}{dt} + \mathbf{B} \times \frac{d\mathbf{A}}{dt}$$

PROBLEMS

1.1. For the vectors of Example 1-1, perform the following operations:

(a) $\mathbf{A} - \mathbf{B}$.

(b) $\mathbf{B}/4 + \mathbf{C}/3$.

(c) $\mathbf{A} - \mathbf{B} + \mathbf{C}$.

(d) $\mathbf{B} \cdot \mathbf{C}$.

(e) $\mathbf{C} \times \mathbf{A}$.

(f) $(\mathbf{A} + \mathbf{B}) \cdot \mathbf{C}$.

(g) $(\mathbf{A} - \mathbf{B}) \times \mathbf{C}$.

(h) $(\mathbf{B}/4 + \mathbf{C}/3) \cdot (3\mathbf{A} - 4\mathbf{C})$.

(i) $(\mathbf{B}/4 + \mathbf{C}/3) \times (3\mathbf{A} - 4\mathbf{C})$.

(j) $\mathbf{B} \cdot \mathbf{C} \times \mathbf{A}$.

(k) $\mathbf{C} \cdot \mathbf{A} \times \mathbf{B}$.

(l) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$.

(m) $\mathbf{B} \times (\mathbf{C} \times \mathbf{A})$.

1.2. (a) Show that the area of a triangle having vectors **A** and **B** as two of its sides is equal to $\frac{1}{2}|\mathbf{A} \times \mathbf{B}|$.

(b) Show that the volume of a tetrahedron formed by three vectors **A**, **B**, and **C** originating from a point is equal to $\frac{1}{6}|\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}|$.

1.3. A triangle is formed by three vectors **A**, **B**, and **C** such that $\mathbf{C} = \mathbf{B} - \mathbf{A}$ and hence $\mathbf{C} \cdot \mathbf{C} = (\mathbf{B} - \mathbf{A}) \cdot (\mathbf{B} - \mathbf{A})$. Obtain the law of cosines relating C to A, B, and the angle between **A** and **B**.

1.4. The tips of three vectors **A**, **B**, and **C** drawn from a point determine a plane.

(a) Show that $(\mathbf{A} \times \mathbf{B} + \mathbf{B} \times \mathbf{C} + \mathbf{C} \times \mathbf{A})$ is normal to the plane.

(b) Show that the minimum distance from the point to the plane is

$$\frac{|\mathbf{A} \cdot [(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{C})]|}{|(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{C})|}$$

(c) Obtain an equation for the plane in terms of **A**, **B**, and **C**.

1.5. (a) In the expression for the differential displacement vector $d\mathbf{l}$ in the cartesian coordinate system given by (1-19a), substitute for x, y, z in terms of the cylindrical coordinates r, ϕ, z and obtain the expression for $d\mathbf{l}$ in the cylindrical coordinate system.

(b) Repeat (a) by substituting for x, y, z in terms of the spherical coordinates r, θ, ϕ to obtain the expression for $d\mathbf{l}$ in the spherical coordinate system.

(c) The parabolic cylindrical coordinates u, v, z are related to x, y, z as

$$x = \tfrac{1}{2}(u^2 - v^2) \qquad y = uv \qquad z = z$$

Obtain the expression for $d\mathbf{l}$ in the parabolic cylindrical coordinate system.

(d) What is the expression for the differential volume dv in the parabolic cylindrical coordinate system?

1.6. Derive the relationships listed in Table 1.2 between the different sets of coordinates.

1.7. In Fig. 1.33, a point of observation T on the surface of the earth is defined by a spherical coordinate system with the origin at the center of the earth. The spherical coordinates of T are its distance r_0 from the center of the earth, its colatitude θ_T and its east longitude ϕ_T. The colatitude is 90° minus the latitude, with south latitudes being negative. N is the north pole. A point P in space is now defined by a coordinate system centered at the point of observation T. The coordinates of P in this new coordinate system are the azimuthal angle α, which is the angle between the great circle path TN and the great circle path TR, where R is the projection of P onto the earth's surface, the elevation angle Δ in the plane TPR and the range S. The colatitude and east longitude of R are θ_R and ϕ_R, respectively.

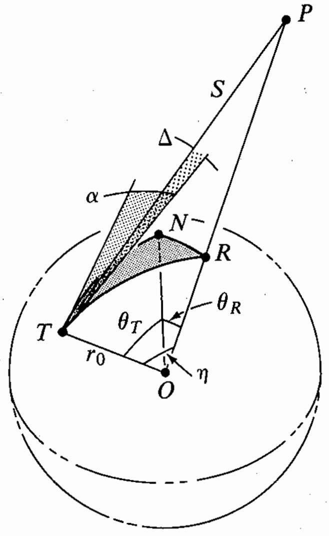


Fig. 1.33. For Problem 1.7.

(a) Show that

$$\cos \eta = \sin \theta_T \sin \theta_R \cos (\phi_T - \phi_R) + \cos \theta_T \cos \theta_R$$

(b) Show that

$$\cos \alpha = \frac{\cos \theta_R - \cos \eta \cos \theta_T}{\sin \eta \sin \theta_T}$$

(c) Find α, Δ, and S if T is at Urbana, Illinois (40.069° N latitude, 88.225° W longitude) and P represents a geostationary satellite parked above the equator at 50° W longitude. The earth radius r_0 is equal to 6370 km and the height h of the geostationary satellite above the earth's surface is equal to 35800 km.

(d) Find α if T represents Bondville, Illinois (40.1° N latitude, 88.4° W longitude) and R is located at Houston, Texas (29.4° N latitude, 95.0° W longitude). Repeat for the locations of T and R reversed.

1.8. Derive the expressions listed in Tables 1.3 and 1.4 for the dot products and cross products of unit vectors in the different coordinate systems.

1.9. Derive the relationships listed in Table 1.5 between the components of a vector in the different coordinate systems.

1.10. Which of the following pairs of vectors are equal?

(a) $\mathbf{i}_x + 2\mathbf{i}_y + 3\mathbf{i}_z$ at (1, 2, 3) and $\mathbf{i}_x + 2\mathbf{i}_y + 3\mathbf{i}_z$ at (5.6, 9.8, 3.7) in cartesian coordinates.

(b) $\mathbf{i}_r + \mathbf{i}_\phi + 3\mathbf{i}_z$ at $(2, \pi/2, 3)$ and $\mathbf{i}_r + \mathbf{i}_\phi + 3\mathbf{i}_z$ at $(3.6, 3\pi/4, 9.4)$ in cylindrical coordinates.

(c) $\mathbf{i}_r + \mathbf{i}_\phi + 3\mathbf{i}_z$ at $(2, \pi/2, 3)$ and $\sqrt{2}\,\mathbf{i}_r + 3\mathbf{i}_z$ at $(3.6, 3\pi/4, 9.4)$ in cylindrical coordinates.

(d) $3\mathbf{i}_r + \sqrt{3}\,\mathbf{i}_\theta - 2\mathbf{i}_\phi$ at $(1, \pi/3, \pi/6)$ and $3\mathbf{i}_r + \sqrt{3}\,\mathbf{i}_\theta - 2\mathbf{i}_\phi$ at $(5.4, \pi/6, \pi/3)$ in spherical coordinates.

(e) $3\mathbf{i}_r + \sqrt{3}\,\mathbf{i}_\theta - 2\mathbf{i}_\phi$ at $(1, \pi/3, \pi/6)$ and $\mathbf{i}_r + \sqrt{3}\,\mathbf{i}_\theta - 2\sqrt{3}\,\mathbf{i}_\phi$ at $(5.4, \pi/6, \pi/3)$ in spherical coordinates.

1.11. Show that

(a) $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{B} \cdot \mathbf{C})(\mathbf{A} \cdot \mathbf{D})$.

(b) $(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \times \mathbf{B} \cdot \mathbf{D})\mathbf{C} - (\mathbf{A} \times \mathbf{B} \cdot \mathbf{C})\mathbf{D}$.

(c) $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{B} \times \mathbf{C}) \times (\mathbf{C} \times \mathbf{A}) = (\mathbf{A} \times \mathbf{B} \cdot \mathbf{C})^2$.

(d) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{B} \times (\mathbf{C} \times \mathbf{A}) + \mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = 0$.

1.12. Four vectors are given by

$$\mathbf{A} = \mathbf{i}_x + 2\mathbf{i}_y + 3\mathbf{i}_z$$
$$\mathbf{B} = 3\mathbf{i}_x + 2\mathbf{i}_y + \mathbf{i}_z$$
$$\mathbf{C} = \mathbf{i}_x - 2\mathbf{i}_y + \mathbf{i}_z$$
$$\mathbf{D} = -2\mathbf{i}_x + \mathbf{i}_y$$

Find

(a) $\mathbf{A} + \mathbf{B} - \mathbf{C}$, $\mathbf{C} - \mathbf{D} - \mathbf{A}$, $\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D}$.

(b) $2\mathbf{A} + 3\mathbf{C}$, $\mathbf{A} - 3\mathbf{C} + 2\mathbf{D}$.

(c) $|\mathbf{C} - \mathbf{D}|$, $|\mathbf{C} - \mathbf{D} - \mathbf{A}|$.

(d) The unit vector along $(\mathbf{C} - \mathbf{D} - \mathbf{A})$.

(e) $\mathbf{A} \cdot \mathbf{B}$, $\mathbf{A} \cdot (\mathbf{C} - \mathbf{D})$, $\mathbf{B} \cdot (\mathbf{C} - \mathbf{D} - \mathbf{A})$.

(f) The cosines of the angles and the angles between $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{A}$ and $(\mathbf{C} - \mathbf{D})$, $\mathbf{B}$ and $(\mathbf{C} - \mathbf{D} - \mathbf{A})$.

(g) $\mathbf{A} \times \mathbf{B}$, $\mathbf{B} \times \mathbf{C}$, $\mathbf{C} \times \mathbf{A}$, $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$, $\mathbf{B} \times (\mathbf{C} \times \mathbf{A})$, $\mathbf{C} \times (\mathbf{A} \times \mathbf{B})$.

(h) The sines of the angles and the angles between $\mathbf{A}$ and $(\mathbf{B} \times \mathbf{C})$, $\mathbf{B}$ and $(\mathbf{C} \times \mathbf{A})$, $\mathbf{C}$ and $(\mathbf{A} \times \mathbf{B})$.

(i) $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D})$; verify by using the identity of Problem 1.11(a).

(j) $\mathbf{A} \times \mathbf{B} \cdot \mathbf{D}$, $\mathbf{A} \times \mathbf{B} \cdot \mathbf{C}$, $\mathbf{B} \times \mathbf{C} \cdot \mathbf{A}$, $\mathbf{C} \times \mathbf{B} \cdot \mathbf{A}$.

(k) $(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D})$; verify by using the identity of Problem 1.11(b).

(l) $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{B} \times \mathbf{C}) \times (\mathbf{C} \times \mathbf{A})$; verify by using the identity of Problem 1.11(c).

(m) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{B} \times (\mathbf{C} \times \mathbf{A}) + \mathbf{C} \times (\mathbf{A} \times \mathbf{B})$.

(n) The components of $\mathbf{C}$ in cylindrical and spherical coordinates.

(o) A vector perpendicular to $(\mathbf{A} + \mathbf{B})$ by using a vector product; verify by using a dot product.

1.13. Let A and **B** be vectors in the xy plane making angles α and β with the x axis. With the aid of dot and cross products, prove the following trigonometric identities:

(a) $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$.
(b) $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$.
(c) $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.
(d) $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.

1.14. Write an expression for the component of a vector A along the direction of another vector **B** without the use of a coordinate system. Then find the component of $\mathbf{A} = 2\mathbf{i}_x - 3\mathbf{i}_y + \mathbf{i}_z$ along the direction of $\mathbf{B} = 3\mathbf{i}_x - \mathbf{i}_y - 2\mathbf{i}_z$.

1.15. Using two vectors in the plane $x + 2y + 3z = 3$, find the unit vector normal to that plane.

1.16. Show that the equation of the plane passing through the point (x_0, y_0, z_0) and normal to the vector $a\mathbf{i}_x + b\mathbf{i}_y + c\mathbf{i}_z$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

1.17. For the following scalar functions, describe the shapes of the constant-magnitude surfaces:

(a) $T(x, y, z) = x^2 + 4y^2 + 9z^2$.
(b) $U(r, \phi, z) = (\cos\phi)/r$.
(c) $V(r, \theta, \phi) = (\sin\theta)/r$.

1.18. Using a spherical coordinate system with the origin at the center of the earth, write a vector function for the linear velocity of points inside the earth due to its spin motion. Describe the constant-magnitude surfaces and direction lines.

1.19. Using a spherical coordinate system with the origin at the center of the earth, write a vector function for the force experienced by a mass m in the gravitational field of the earth. Describe the constant-magnitude surfaces and direction lines.

1.20. Discuss the following vector fields with the aid of sketches:

(a) $\mathbf{A}(x, y, z) = (x - 2)\mathbf{i}_x$.
(b) $\mathbf{B}(r, \phi, z) = r(r - 1)\mathbf{i}_\phi$.
(c) $\mathbf{C}(r, \theta, \phi) = (1/r)\mathbf{i}_\theta$.
(d) $\mathbf{D}(r, \theta, \phi) = r\,\mathbf{i}_r$.

1.21. Derive the expressions listed in Table 1.6 for the partial derivatives of unit vectors with respect to the coordinates.

1.22. Let $\mathbf{r} = x\mathbf{i}_x + y\mathbf{i}_y + z\mathbf{i}_z = r_c\mathbf{i}_{rc} + z\mathbf{i}_z = r_s\mathbf{i}_{rs}$ be the position vector of a point P moving in three dimensions. Obtain the expressions for the velocity **v** and acceleration **a** of the point in all three coordinate systems.

1.23. (a) A point P moves along a curve in two dimensions such that its coordinates are given by $r = at$ and $\phi = bt$, where a and b are constants. Find the velocity and acceleration of the point.

(b) A point P moves along a curve in three dimensions such that its coordinates are given by $x = a\cos\omega t$, $y = b\sin\omega t$, and $z = ct$, where a, b, c, and ω are constants. Find the velocity and acceleration of the point.

1.24. Verify Eqs. (1-63) and (1-64) by expansion in cartesian coordinates.

1.25. Find a unit vector normal to the surface $r^2 \cos 2\phi = 1$ at the point $(\sqrt{2}, \pi/6, 0)$ in the cylindrical coordinate system in two ways: (a) by using two vectors which are tangential to the surface at that point; and (b) by using the concept of the gradient of a scalar function.

1.26. Find the scalar functions whose gradients are given by the following vector functions:

(a) $\nabla T(x, y, z) = yz\mathbf{i}_x + zx\mathbf{i}_y + xy\mathbf{i}_z$.

(b) $\nabla U(x, y, z) = 3x^2yz^2\mathbf{i}_x + x^3z^2\mathbf{i}_y + 2x^3yz\mathbf{i}_z$.

(c) $\nabla V(r, \phi, z) = (1/r^2)(\cos \phi\, \mathbf{i}_r + \sin \phi\, \mathbf{i}_\phi)$.

(d) $\nabla W(r, \theta, \phi) = -n\mathbf{r}/r^{n+2}$, where $\mathbf{r}$ is the position vector.

1.27. Make up a table of gradients of the scalar functions defining the orthogonal surfaces in the three different coordinate systems.

1.28. Find the component of the unit vector normal to the surface $x^2 - y^2 = 3$ at the point (2, 1, 1) in the direction of the vector joining the point (1, −2, 0) to the point (0, 0, 2).

1.29. Find the rate of change of $V = x^2y + yz^2 + zy^2$ in the direction normal to the surface $x^2y - yz + xz^2 = 5$ at the point (1, 2, 3).

1.30. Find the equation of the plane tangential to the surface $xyz = 1$ at the point $(\frac{1}{2}, \frac{1}{4}, 8)$.

1.31. Evaluate the following volume integrals:

(a) $\int_V xyz\, dv$, where V is the volume enclosed by the planes $x = 0, y = 0, z = 0$, and $x + y + z = 1$.

(b) $\int_V \frac{1}{r}\, dv$, where V is the volume of a cylinder of radius a with the z axis as its axis and of length l.

(c) $\int_V x\, dv$, where V is that part of the volume of a sphere of radius unity lying in the first octant.

1.32. Given $\mathbf{A} = x^2yz\mathbf{i}_x + y^2zx\mathbf{i}_y + z^2xy\mathbf{i}_z$, evaluate $\oint \mathbf{A} \cdot d\mathbf{S}$ over the following closed surfaces:

(a) The surface of the cubical box bounded by the planes

$$x = 0,\ x = 1$$
$$y = 0,\ y = 1$$
$$z = 0,\ z = 1$$

(b) The surface of the box bounded by the planes

$$x = 0,\ y = 0,\ z = 0$$
$$x + 2y + 3z = 3$$

1.33. Given $\mathbf{A} = r \cos \phi\, \mathbf{i}_r - r \sin \phi\, \mathbf{i}_\phi$ in cylindrical coordinates, evaluate $\oint \mathbf{A} \cdot d\mathbf{S}$ over the following surfaces:

(a) The surface of the box bounded by the planes $z = 0$, $z = l$, and the cylinder $r = a$.

(b) The surface of the box bounded by the planes $x = 0$, $y = 0$, $z = 0$, $z = l$, and the cylinder $r = a$.

1.34. Given $\mathbf{A} = r^2\mathbf{i}_r + r \sin\theta\, \mathbf{i}_\theta$ in spherical coordinates, evaluate $\oint \mathbf{A} \cdot d\mathbf{S}$ over the following:

(a) The surface of that part of the spherical volume of radius unity lying in the first octant.

(b) The surface of a solid spherical shell lying between $r = a$ and $r = b$, where $b > a$ (note that this surface consists of two disconnected surfaces; the normal vectors to the surfaces must both be chosen to be away from or into the volume bounded by the surfaces).

1.35. For the force vector $\mathbf{F} = y\mathbf{i}_x + x\mathbf{i}_y$, find the work done by the force vector from the origin to the point $(\pi/2, 1, 0)$ along the following paths:

(a) $y = \sin^2 x$, $z = 0$.

(b) $y = (4/\pi^2)x^2$, $z = 0$.

(c) $x = (\pi/2)y^2$, $z = 0$.

(d) Any other path of your choice not necessarily in the $z = 0$ plane.

1.36. A certain vector field is given by

$$\mathbf{A} = a^2 y\mathbf{i}_x - b^2 x\mathbf{i}_y$$

where a and b are constants. Evaluable $\int \mathbf{A} \cdot d\mathbf{l}$ from the origin to the point $(1, 1, 1)$ along the following paths:

(a) $y = x = z^2$.

(b) The path given by $y = 0$, $z = 0$, then $x = 1$, $z = 0$, and then $x = y = 1$.

(c) The path given by $y = x$, $z = 0$, and then $x = y = 1$.

(d) The path given by $x = 0$, $z = 0$, then $y = 1$, $z = 0$, and then $x = y = 1$.

(e) $x = y = z$.

1.37. Given $\mathbf{A} = xy\mathbf{i}_x + yz\mathbf{i}_y + zx\mathbf{i}_z$, evaluate the circulation $\oint \mathbf{A} \cdot d\mathbf{l}$ around the contour *abcda* shown in Fig. 1.34.

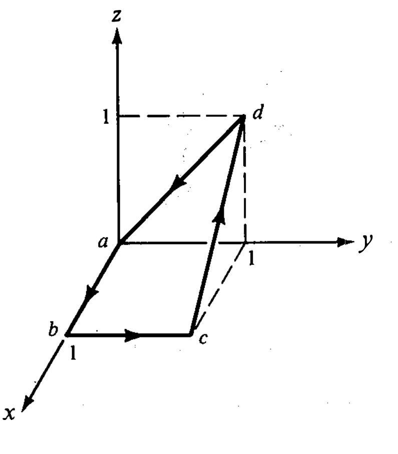


Fig. 1.34. For Problem 1.37.

1.38 Given $\mathbf{A} = 2r \cos \phi \, \mathbf{i}_r + r\mathbf{i}_\phi$ in cylindrical coordinates, find:

(a) $\oint_C \mathbf{A} \cdot d\mathbf{l}$, where C is the contour shown in Fig. 1.35(a).

(b) $\oint_{C_1} \mathbf{A} \cdot d\mathbf{l} + \oint_{C_2} \mathbf{A} \cdot d\mathbf{l}$, where C_1 and C_2 are the contours shown in Fig. 1.35(b).

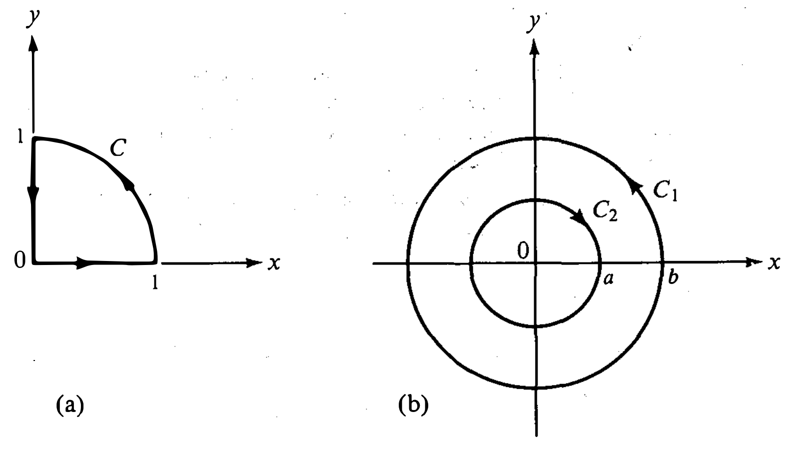


Fig. 1.35. For Problem 1.38.

1.39. Given $\mathbf{A} = (e^{-r}/r)\mathbf{i}_\theta$ in spherical coordinates, evaluate $\oint \mathbf{A} \cdot d\mathbf{l}$ around the contour *abca* shown in Fig. 1.36.

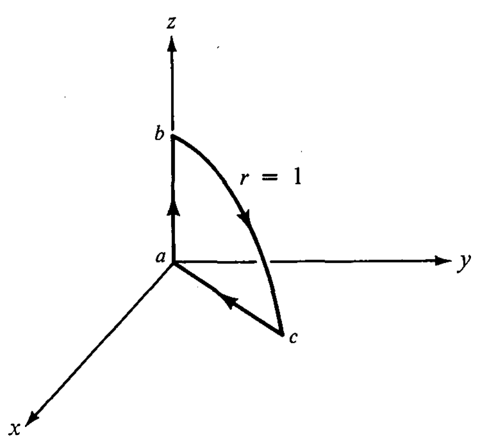


Fig. 1.36. For Problem 1.39.

1.40. Evaluate the following vector integrals:

(a) $\oint_C d\mathbf{l}$, where C is any closed path of your choice.

(b) $\oint_S d\mathbf{S}$, where S is the surface of the hemispherical volume of radius a above the xy plane and with center at the origin.

(c) $\int_V \mathbf{i}_\theta \, dv$, where V is the volume of the sphere of radius a centered at the origin.

1.41. Derive the expression for the divergence of a vector in cartesian coordinates given by (1-96).

1.42. Derive the expression for the divergence of a vector in spherical coordinates given by (1-97).

1.43. Make up a table of divergences of the unit vectors in the three coordinate systems.

1.44. Find the divergences of the following vectors:

(a) $\mathbf{A} = x^2yz\mathbf{i}_x + y^2zx\mathbf{i}_y + z^2xy\mathbf{i}_z$.

(b) $\mathbf{B} = 3x\mathbf{i}_x + (y - 3)\mathbf{i}_y + (2 + z)\mathbf{i}_z$.

(c) $\mathbf{C} = r \cos \phi \, \mathbf{i}_r - r \sin \phi \, \mathbf{i}_\phi$, cylindrical coordinates.

(d) $\mathbf{D} = (1/r^2)\mathbf{i}_r$, spherical coordinates.

(e) $\mathbf{E} = r^2\mathbf{i}_r + r \sin \theta \, \mathbf{i}_\theta$.

1.45. Using the position vector $\mathbf{r} = r\mathbf{i}_r$ in three dimensions, verify the divergence theorem by considering a sphere of radius a, and centered at the origin.

1.46. Verify your answers to Problem 1.32 by evaluating the appropriate volume integrals and using the divergence theorem.

1.47. Verify your answers to Problem 1.33 by evaluating the appropriate volume integrals and using the divergence theorem.

1.48. Verify your answers to Problem 1.34 by evaluating the appropriate volume integrals and using the divergence theorem.

1.49. For the vector $\mathbf{A} = yz\mathbf{i}_x + zx\mathbf{i}_y + xy\mathbf{i}_z$, use the divergence theorem to show that $\oint_S \mathbf{A} \cdot d\mathbf{S}$ is zero, where S is any closed surface. Then evaluate $\int \mathbf{A} \cdot d\mathbf{S}$ over the following surfaces:

(a) That part of the plane $x + 2y + 3z = 3$ lying in the first octant.

(b) That part of the cylindrical surface $r = 1$ lying in the first octant and between the planes $z = 0$ and $z = 1$.

(c) The upper half of the spherical surface $r = 1$.

(d) That part of the conical surface $\theta = \pi/4$ lying below the plane $z = 1$.

1.50. Derive the expression for the curl of a vector in cartesian coordinates given by (1-121).

1.51. Derive the expression for the curl of a vector in cylindrical coordinates given by (1-122).

1.52. Make up a table of curls of the unit vectors in the three coordinate systems.

1.53. Find the curls of the following vectors:

(a) $\mathbf{A} = xy\mathbf{i}_x + yz\mathbf{i}_y + zx\mathbf{i}_z$.

(b) $\mathbf{B} = y\mathbf{i}_x - x\mathbf{i}_y$.

(c) $\mathbf{C} = 2r \cos\phi\, \mathbf{i}_r + r\, \mathbf{i}_\phi$, cylindrical coordinates.
(d) $\mathbf{D} = (1/r)\mathbf{i}_\phi$, cylindrical coordinates.
(e) $\mathbf{E} = (e^{-r}/r)\mathbf{i}_\theta$.

1.54. Discuss the curls of the following vector fields by using the "paddle-wheel" device and also by expansion in the appropriate coordinate system:

(a) The velocity vector field associated with points inside the earth due to its spin motion.
(b) The position vector field associated with points in three-dimensional space.
(c) The velocity vector field associated with the flow of water in the stream of Fig. 1-30(a) such that the velocity varies uniformly from zero at the bottom of the stream to a maximum at the top surface.
(d) the vector field $\mathbf{F} = \mathbf{i}_\phi$.

1.55. By expansion in cartesian coordinates, verify

$$\nabla \cdot \nabla \times \mathbf{F} \equiv 0$$
$$\nabla \times \nabla V \equiv 0$$

1.56. Determine which of the following vectors can be expressed as the curl of another vector and which of them can be expressed as the gradient of a scalar:

(a) $\mathbf{A} = yz\mathbf{i}_x + zx\mathbf{i}_y + xy\mathbf{i}_z$.
(b) $\mathbf{B} = xy\mathbf{i}_x + yz\mathbf{i}_y + zx\mathbf{i}_z$.
(c) $\mathbf{C} = (x^2 - y^2)\mathbf{i}_x - 2xy\mathbf{i}_y + 4\mathbf{i}_z$.
(d) $\mathbf{D} = (e^{-r}/r)\mathbf{i}_\phi$, cylindrical coordinates.
(e) $\mathbf{E} = (1/r^2)(\cos\phi\, \mathbf{i}_r + \sin\phi\, \mathbf{i}_\phi)$, cylindrical coordinates.
(f) $\mathbf{F} = (1/r^3)(2\cos\theta\, \mathbf{i}_r + \sin\theta\, \mathbf{i}_\theta)$, spherical coordinates.

1.57. Verify your answer to Problem 1.37 by evaluating the appropriate surface integral and using Stokes' theorem.

1.58. Verify your answers to Problem 1.38 by evaluating the appropriate surface integrals and using Stokes' theorem.

1.59. Verify your answer to Problem 1.39 by evaluating the appropriate surface integral and using Stoke's theorem.

1.60. For the vector $\mathbf{A} = yz\mathbf{i}_x + zx\mathbf{i}_y + xy\mathbf{i}_z$, use Stokes' theorem to show that $\oint_C \mathbf{A} \cdot d\mathbf{l}$ is zero, where C is any closed path. Then evaluate $\int \mathbf{A} \cdot d\mathbf{l}$ along the following paths:

(a) From the origin to the point $(1, \pi/2, 0)$ along the curve $r = t$, $\phi = (\pi/2)t$, $z = \sin \pi t$, in cylindrical coordinates.
(b) From the origin to the point $(1, 1, 1)$ along the curve $x = \sqrt{2} \sin t$, $y = \sqrt{2} \sin t$, $z = (4/\pi)t$.
(c) From the origin to the point $(22.34, 5.68, -6.93)$ in cartesian coordinates along any path of your choice.

1.61. Use Stokes' theorem and the divergence theorem to prove that $\nabla \cdot \nabla \times \mathbf{A} \equiv 0$, without the implication of a coordinate system.

1.62. From the definition of ∇V, show that $\oint_C \nabla V \cdot d\mathbf{l} \equiv 0$, where C is any closed path. Then use this result and Stoke's theorem to prove that $\nabla \times \nabla V \equiv 0$, without the implication of a coordinate system.

1.63. Find the Laplacians of the following scalar and vector functions:

(a) $T(x, y, z) = x^3yz^2$.

(b) $U(r, \phi, z) = (\cos \phi)/r$.

(c) $V(r, \theta, \phi) = e^{-r}/r$.

(d) $\mathbf{A}(x, y, z) = x^2yz\mathbf{i}_x + xy^2z\mathbf{i}_y + xyz^2\mathbf{i}_z$.

1.64. Derive the expansion for the Laplacian of a vector in cartesian coordinates given by (1-139).

1.65. Derive the expansion for the Laplacian of a vector in cylindrical coordinates given by (1-140).

1.66. Derive the expansion for the Laplacian of a vector in spherical coordinates given by (1-141).

1.67. Verify the general expressions for ∇V, $\nabla \cdot \mathbf{J}$, $\nabla \times \mathbf{F}$ and $\nabla^2 V$ given by (1-144), (1-145), (1-146), and (1-147), respectively.

1.68. By expansion in cartesian coordinates, show that

(a) $\nabla \cdot U\mathbf{A} = \mathbf{A} \cdot \nabla U + U \nabla \cdot \mathbf{A}$.

(b) $\nabla \times U\mathbf{A} = \nabla U \times \mathbf{A} + U \nabla \times \mathbf{A}$.

(c) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B}$.

(d) $\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}\, \nabla \cdot \mathbf{B} - \mathbf{B}\, \nabla \cdot \mathbf{A} + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$.